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LOCAL-GLOBAL PRINCIPLE FOR OVER SEMIGLOBAL FIELDS

PHILIPPE GILLE AND RAMAN PARIMALA

ABSTRACT. We compare different local-global principles for torsors under a reductive group G defined over a semiglobal field F . In particular if the F -group G is a retract rational F -variety, we prove that the local global principle holds for the completions with respect to divisorial valuations of F .

Keywords: Local-global principle, curves over local fields, torsors, reductive groups.

MSC 2000: 11G99, 14G99, 14G05, 11E72, 11E12, 20G35

1. INTRODUCTION

The theme is the arithmetic theory of torsors over a semiglobal field. It was initiated by Hartmann, Harbater and Krashen [H-H-K] and developed further [C-T-H-H-K-P-S2, C-T-P-S, H-H-K2, P-S, R-S].

Let B be a complete discrete valuation ring with fraction field K and residue field k . Let $p \geq 1$ be the characteristic exponent of k . Let X be a smooth, projective, geometrically integral curve over K . Let $F = K(X)$ be the function field of X and let t be an uniformizing parameter of B . Let \mathfrak{X} be a normal model of F , i.e. a normal connected projective B -curve with function field F . We denote by Y the closed fiber of \mathfrak{X} and fix a separable closure F_s of F .

Given an affine algebraic F -group G , we can study the local-global principle for G -torsors with respect to the infinite set of overfields F_P that arise from completions at the points P of Y . This gives rise to the Tate-Shafarevich set

$$\text{III}_{\mathfrak{X}}(F, G) = \ker \left(H^1(F, G) \rightarrow \prod_{P \in Y} H^1(F_P, G) \right);$$

it is denoted simply by $\text{III}_X(F, G)$ in [H-H-K2]. Using Lipman's resolution of singularities permits to consider the inductive limit

$$\text{III}_{\text{patch}}(F, G) = \varinjlim_{\mathfrak{X}} \text{III}_{\mathfrak{X}}(F, G)$$

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taken on the models (or equivalently on the regular models). On the other hand, we consider the completions associated to the set $\Omega_{F,div}$ of divisorial valuations on F arising of various regular projective models over B of the curve X . It gives rise to the

$$\mathbb{I}\mathbb{I}\mathbb{I}_{div}(F, G) = \ker\left(H^1(F, G) \rightarrow \prod_{v \in \Omega_{F,div}} H^1(F_v, G)\right);$$

In view of [H-H-K2, prop. 8.2], we have the following inclusion

$$\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(F, G) \subseteq \mathbb{I}\mathbb{I}\mathbb{I}_{div}(F, G).$$

The main result of the paper is to show (under some condition on p with respect to G), that the two Tate-Shafarevich sets coincide (Thm. 4.4). If furthermore G is an F -rational variety (or even retract rational F -variety), we have that $\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(F, G) = 1$ according to Harbater-Krashen-Hartmann [H-H-K2, Thm. 9.1] (see [K, Thm. 2.2.4] for the extension to the retract rational case) so that $\mathbb{I}\mathbb{I}\mathbb{I}_{div}(F, G) = 1$ in this case. It means that if G is a retract rational F -variety then the local-global principle holds for G -torsors with respect to divisorial valuations on F . A further related question is whether the set $\mathbb{I}\mathbb{I}\mathbb{I}_{div}(F, G)$ coincide with the set $\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(F, G)$ with respect to a single integral model and a single patch on the model. A first result in that direction is the case of inner type A_{n-1} (n prime to p) obtained recently by Suresh [Su, Thm. 1.4].

We discuss also a 0-cycle version of this result, see Theorem 6.2. This was known for certain classical groups [R-S, P-S, Su]. As in [G-P], the proof uses the study of loop torsors which requires group schemes defined over B . To do so, for a given Chevalley group G_0 over \mathbb{Z} , we explain in the first section why the $G_0 \rtimes \text{Aut}(G_0)$ -torsors permit to classify couples (E, G) where G is a form of G_0 and E a G -torsor.

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2. PAIRS OF A GROUP SCHEME TOGETHER WITH A TORSOR

The following is a complement to the ‘‘sorites’’ of [G3]. We work in the setting of fppf sheaves on a base scheme S . We consider the groupoid \mathfrak{P} whose objects are pairs (E, G) where G is a fppf S -sheaf in groups and E a G -torsor; the morphisms between two pairs (E_1, G_1) and (E_2, G_2) are pairs of isomorphisms $f = (f', f'')$ where $f'' : G_1 \xrightarrow{\sim} G_2$ is an S -isomorphism of S -sheaves in groups and $f' : E_1 \xrightarrow{\sim} E_2$ an S -isomorphism of sheaves such that the following diagram commutes

$$\begin{array}{ccc} E_1 \times_S G_1 & \longrightarrow & E_1 \\ \downarrow f' \times f'' & & \downarrow f' \\ E_2 \times_S G_2 & \longrightarrow & E_2. \end{array}$$

Note that it induces an isomorphism $f_* : \underline{\text{Aut}}_{G_1}(E_1) \xrightarrow{\sim} \underline{\text{Aut}}_{G_2}(E_2)$ or in other words an isomorphism $f_* : {}^{E_1}G_1 \xrightarrow{\sim} {}^{E_2}G_2$ between the associated twisted group sheaves.

Lemma 2.1. *Let G be an S -sheaf in groups and let E be a G -torsor over S . We denote by ${}^E G$ the twisted S -sheaf of G by E by inner automorphisms.*

(1) *We have a natural exact sequence of fppf S -sheaves in groups*

$$1 \rightarrow {}^E G \rightarrow \underline{\text{Aut}}_{\mathfrak{P}}(E, G) \rightarrow \underline{\text{Aut}}(G) \rightarrow 1$$

which is locally split for the flat topology.

(2) *If $E = G$ the sequence above splits and the S -functor $\underline{\text{Aut}}_{\mathfrak{P}}(G, G)$ is isomorphic to $G \rtimes \underline{\text{Aut}}(G)$.*

Proof. We prove the two assertions at the same time. The projection $\underline{\text{Aut}}_{\mathfrak{P}}(E, G) \rightarrow \underline{\text{Aut}}(G)$ is a homomorphism of S -sheaves in groups. On the other hand we have a monomorphism ${}^E G = \underline{\text{Aut}}_G(E) \rightarrow \underline{\text{Aut}}_{\mathfrak{P}}(E, G)$, $f'' \mapsto (f'', id)$. The fact that the sequence is exact at the middle term is the fact [Gd, cor. 2.3.8].

We consider now the case $E = G$. In this case the sequence above splits by $f'' \mapsto (f'', f'')$ hence an isomorphism $G \rtimes \underline{\text{Aut}}(G) \xrightarrow{\sim} \underline{\text{Aut}}_{\mathfrak{P}}(G, G)$. Coming back to the general case it implies that the projection $\underline{\text{Aut}}_{\mathfrak{P}}(E, G) \rightarrow \underline{\text{Aut}}(G)$ is an epimorphism of S -sheaves. \square

Given a fppf S -sheaf in groups G , we can apply [G3, lemme 2.6.3] to express the cohomology

$$(2.1) \quad H^1(S, G \rtimes \underline{\text{Aut}}(G)) \xrightarrow{\sim} \bigsqcup_{[G'] \in H^1(S, \underline{\text{Aut}}(G))} H^1(S, G') / \underline{\text{Aut}}(G')(S).$$

by using the fact that $H^1(S, \underline{\text{Aut}}(G))$ classifies the S -forms of G . The action on $\underline{\text{Aut}}(G')(S)$ on $H^1(S, G')$ is the natural one so that the map $H^1(S, G') \rightarrow H^1(S, G') / \underline{\text{Aut}}(G')(S)$ has trivial kernel so that the composite $H^1(S, G') \rightarrow H^1(S, G \rtimes \underline{\text{Aut}}(G))$ has trivial kernel. This map is nothing but the composition

$$H^1(S, G') \rightarrow H^1(S, G' \rtimes \underline{\text{Aut}}(G')) \xrightarrow[\sim]{\text{Torsion bijection}} H^1(S, G \rtimes \underline{\text{Aut}}(G))$$

where the torsion bijection is with respect to the $\underline{\text{Aut}}(G)$ -torsor $\underline{\text{Isom}}(G, G')$. Summarizing the decomposition (2.1), $H^1(S, G \rtimes \underline{\text{Aut}}(G))$ encodes the classes of torsors for all twisted forms of G . The map $\text{int} : G \rightarrow \underline{\text{Aut}}(G)$ extends to

$$\begin{array}{ccc}
u : G \rtimes \underline{\mathrm{Aut}}(G) & \longrightarrow & \underline{\mathrm{Aut}}(G) \rtimes^c \underline{\mathrm{Aut}}(G) & \xrightarrow{\sim} & \underline{\mathrm{Aut}}(G) \times \underline{\mathrm{Aut}}(G), \\
(g, f) & \mapsto & (\mathrm{int}(g), f) & & \\
& & (f_1, f_2) & \mapsto & (f_1 f_2, f_2)
\end{array}$$

where c stands for the conjugacy action. In terms of objects classified by the respective torsors, u_* applies a pair (E', G') to $(E'G', G')$.

3. REFINEMENTS ON THE REDUCTION OF TORSORS TO FINITE SUBGROUPS

We provide two variants of [C-G-R, Theorem 1.2.(2)] for reduction of torsors from certain group schemes to finite subgroup schemes. Let $1 \rightarrow G \rightarrow \tilde{G} \rightarrow J \rightarrow 1$ be an exact of \mathbb{Z} -group schemes where G is a Chevalley reductive \mathbb{Z} -group scheme and J a constant \mathbb{Z} -group scheme (not necessarily finite). Let $T \subset G$ be a maximal \mathbb{Z} -split subtorus and put $N = N_G(T)$, $W = N_G(T)/T$.

Lemma 3.1. (1) *The fppf sheaf normalizer $\tilde{N} = N_{\tilde{G}}(T)$ is representable by a smooth \mathbb{Z} -group scheme which fits in an exact sequence*

$$1 \rightarrow N \rightarrow \tilde{N} \rightarrow J \rightarrow 1.$$

(2) *The fppf quotient $\tilde{W} = \tilde{N}/T$ is representable by a constant \mathbb{Z} -group scheme. Furthermore the map $\tilde{N}(\mathbb{Z}) \rightarrow \tilde{W}(\mathbb{Z})$ is onto.*

Proof. (1) According to [G3, lemme 3.4.5.(1)], \tilde{N} is representable by a \mathbb{Z} -group scheme which is locally of finite presentation. We claim that the map $\tilde{N} \rightarrow J$ is an epimorphism of fppf sheaves. We are given a ring A and an element $x \in J(A)$ and want to find a fppf cover B of A such that $x_B \in J(B)$ lifts to $\tilde{G}(B)$. The preimage of x in \tilde{G}_A is a G_A -torsor so is an affine smooth A -scheme $\mathrm{Spec}(B_1)$. It follows that there exists $\tilde{g}_1 \in \tilde{G}(B_1)$ mapping to $x_{B_1} \in J(B_1)$. Next we consider the maximal B_1 -torus $\tilde{g}_1 T$ of G_{B_1} . We consider the strict transporter $\mathrm{Transp}_{G_{B_1}}(T_{B_1}, \tilde{g}_1 T)$ as defined in [SGA3, VI_B.6]. This is the B_1 -functor defined by

$$\mathrm{Transp}_{G_{B_1}}(T_{B_1}, \tilde{g}_1 T)(C) = \{g \in G(C) \mid gT_{B_1}(C')g^{-1} = (\tilde{g}_1 T)(C') \text{ for each } C\text{-ring } C'\}$$

for each B_1 -ring C . According to Grothendieck [SGA3, XI.5.2], it is representable by an N -torsor over $\mathrm{Spec}(B_1)$ so is an affine smooth B_1 -scheme $\mathrm{Spec}(B_2)$. It follows that there exists $g_2 \in G(B_2)$ such that $g_2 T_{B_1} = \tilde{g}_1 T$ so that $\tilde{n}_2 = g_2^{-1} g_{1, B_2} \in \tilde{N}(B_2)$. Thus \tilde{n}_2 maps to $x_{B_2} \in J(B_2)$ and the claim is established. We have then an exact sequence of \mathbb{Z} -group schemes $1 \rightarrow N \rightarrow \tilde{N} \rightarrow J \rightarrow 1$. Since N and J are smooth, so is \tilde{N} in view of [SGA3, VI_B.9.(xii)].

(2) A diagram chase provides an exact sequence of fppf \mathbb{Z} -sheaves $1 \rightarrow W \rightarrow \widetilde{W} \rightarrow J \rightarrow 1$. Since \mathbb{Z} is simply connected, we have $H^1(\mathbb{Z}, W) = 1$ whence an exact sequence

$$(3.1) \quad 1 \rightarrow W(\mathbb{Z}) \rightarrow \widetilde{W}(\mathbb{Z}) \rightarrow J(\mathbb{Z}) \rightarrow 1.$$

We denote by \widetilde{W}_0 the constant \mathbb{Z} -group scheme attached to the abstract group $\widetilde{W}(\mathbb{Z})$. It fits in the exact sequence of constant \mathbb{Z} -group schemes $1 \rightarrow W \rightarrow \widetilde{W}_0 \rightarrow J \rightarrow 1$. We have a homomorphism $\tilde{\phi} : \widetilde{W}_0 \rightarrow \widetilde{W}$ which is an isomorphism by diagram chasing. Thus \widetilde{W} is representable by a constant \mathbb{Z} -group scheme. \square

Our goal is to construct a family $(\widetilde{S}_{2n})_{n \geq 1}$ of \mathbb{Z} -subgroups schemes of \widetilde{N} . We have an action of \widetilde{W} on $T \cong \mathbb{G}_{m, \mathbb{Z}}^r$ whence a homomorphism of constant \mathbb{Z} -group schemes $h : \widetilde{W} \rightarrow \text{Aut}_{\mathbb{Z}}(T) \cong \text{GL}_r(\mathbb{Z})_{\mathbb{Z}}$. We define the constant \mathbb{Z} -group schemes $T_2 = (T(\mathbb{Z}))_{\mathbb{Z}}$, $\widetilde{N}_2 = (\widetilde{N}(\mathbb{Z}))_{\mathbb{Z}}$. Since $T(\mathbb{Z}) = \{\pm 1\}^r$, we observe $T_2 \cong (\mathbb{Z}/2\mathbb{Z})^r$ so that the map $T_2 \rightarrow T$ is induced by the \mathbb{Z} -map $\mathbb{Z}/2\mathbb{Z} \rightarrow \mu_2$, $1 \mapsto -1$. The sequence (3.1) gives rise to exact sequence of constant \mathbb{Z} -group schemes $1 \rightarrow T_2 \rightarrow \widetilde{N}_2 \rightarrow \widetilde{W} \rightarrow 1$ which fits in a commutative exact diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & T_2 & \longrightarrow & \widetilde{N}_2 & \longrightarrow & \widetilde{W} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & T & \longrightarrow & \widetilde{N} & \longrightarrow & \widetilde{W} \longrightarrow 1. \end{array}$$

For each $n \geq 1$, we define the \mathbb{Z} -group scheme \widetilde{S}_{2n} as the push-out of \widetilde{N}_2 by the homomorphism $T_2 \rightarrow {}_2T \rightarrow {}_{2n}T$, it fits in a commutative exact diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & T_2 & \longrightarrow & \widetilde{N}_2 & \longrightarrow & \widetilde{W} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & {}_2T & \longrightarrow & \widetilde{S}_{2n} & \longrightarrow & \widetilde{W} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & T & \longrightarrow & \widetilde{N} & \longrightarrow & \widetilde{W} \longrightarrow 1 \end{array}$$

where the map ${}_{2n}T \rightarrow T$ is the natural embedding. We observe that \widetilde{S}_{2n} is a \mathbb{Z} -subgroup scheme of \widetilde{N} . We denote by $d(\widetilde{G})$ the l.c.m. of the cardinals of finite subgroups of $\text{Im}(h) \subset \text{GL}_r(\mathbb{Z})$.

Theorem 3.2. *Let X be a scheme. We assume one of the following:*

- (i) X is a connected normal scheme which is locally noetherian;
- (ii) The image of $h : \widetilde{W}(\mathbb{Z}) \rightarrow \mathrm{GL}_r(\mathbb{Z})$ is finite.

We assume furthermore that there exists an integer $D \geq 1$ such that

$$D \mathrm{Pic}(X') = 0$$

for any finite étale cover X' of X whose degree divides $d(\widetilde{G})$. We consider the \mathbb{Z} -subgroup $\widetilde{S} = \widetilde{S}_{2d(\widetilde{G})D}$ of \widetilde{N} .

- (1) The map $H_{\mathrm{fppf}}^1(X, \widetilde{S}) \rightarrow H_{\mathrm{fppf}}^1(X, \widetilde{N})$ is onto.
- (2) Assume furthermore that $X = \mathrm{Spec}(R)$ with R an LG ring. Then the map $H_{\mathrm{fppf}}^1(X, \widetilde{S}_{2d(\widetilde{G})}) \rightarrow H_{\mathrm{fppf}}^1(X, \widetilde{G})$ is onto.

Recall that a ring R has the property LG if whenever a polynomial $f \in R[x_1, \dots, x_n]$ represents a unit over $R_{\mathfrak{m}}$ for each maximal ideal \mathfrak{m} of R , then f represents a unit over R [E-G]. We need the following auxiliary statement.

Lemma 3.3. *In the setting of Theorem 3.2, let F be a \widetilde{W} -torsor over X and consider the twisted X -torus $M = {}^F T$ with respect to the action of \widetilde{W} on T via h .*

- (1) M is split after a finite étale extension X'/X of degree d dividing $d(\widetilde{G})$.
- (2) We have $d(\widetilde{G})D H_{\mathrm{fppf}}^1(X, M) = 0$.
- (3) For each $n \geq 1$, the map

$$\mathrm{Ker}\left(H_{\mathrm{fppf}}^2(X, {}_n M) \rightarrow H_{\mathrm{fppf}}^2(X, M)\right) \rightarrow H_{\mathrm{fppf}}^2(X, {}_{nd(\widetilde{G})D} M)$$

is trivial.

Proof. (1) We need to distinguish the cases (i) and (ii) of Theorem 3.2.

Case (i). According to [G6, lemme 2.14.(2)], the \widetilde{W} -torsor F is isotrivial. Let $Y \rightarrow X$ be a finite connected Galois étale cover of group Γ which splits F . Then the class of $[F]$ is given by a homomorphism $\phi : \Gamma \rightarrow \widetilde{W}(\mathbb{Z}) = \widetilde{W}(Y)$ and the isomorphism class of M is given by the composite $\Gamma \rightarrow \widetilde{W}(\mathbb{Z}) \xrightarrow{h} \mathrm{GL}_r(\mathbb{Z})$. Modding out by the kernel of this homomorphism provides a finite connected Galois étale cover X' of X of degree d dividing $d(\widetilde{G})$ and which splits M .

Case (ii). The scheme X is not assumed connected anymore but $\mathrm{Im}(h)$ is assumed to be finite. It follows that the X -torus M is isotrivial, it splits after a finite connected Galois étale cover X'/X of degree d which divides $d(\widetilde{G})$.

(2) By restriction-corestriction, the kernel $\ker(H_{\mathrm{fppf}}^1(X, M) \rightarrow H_{\mathrm{fppf}}^1(X', M))$ is killed by d . On the other hand, we have $M_{X'} \cong T_{X'} \xrightarrow{\sim} \mathbb{G}_{m, X'}^r$ so that $H_{\mathrm{fppf}}^1(X', M) \cong \mathrm{Pic}(X')^r$. It follows that $DH_{\mathrm{fppf}}^1(X', M) = 0$ whence $d(\widetilde{G})D H_{\mathrm{fppf}}^1(X, M) = 0$.

(3) We consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 1 & \longrightarrow & {}_n M & \longrightarrow & M & \xrightarrow{\times n} & M \longrightarrow 1 \\
 & & \downarrow & & \downarrow \times d(\tilde{G})_D & & \parallel \\
 1 & \longrightarrow & {}_{nd(\tilde{G})_D} M & \longrightarrow & M & \xrightarrow{\times nd(\tilde{G})_D} & M \longrightarrow 1.
 \end{array}$$

It induces a commutative diagram of exact sequences

$$\begin{array}{ccccc}
 H_{\text{fppf}}^1(X, M) & \longrightarrow & H_{\text{fppf}}^2(X, {}_n M) & \longrightarrow & H_{\text{fppf}}^2(X, M) \\
 \times d(\tilde{G})_D \downarrow & & \downarrow & & \parallel \\
 H_{\text{fppf}}^1(X, M) & \longrightarrow & H_{\text{fppf}}^2(X, {}_{nd(\tilde{G})_D} M) & \longrightarrow & H_{\text{fppf}}^2(X, M).
 \end{array}$$

Since the left handside vertical map is trivial by (1), we obtain (3) by diagram chasing. \square

We proceed to the proof of Theorem 3.2.

Proof. (1) Let E be a \tilde{N} -torsor and consider the \tilde{W} -torsor $F = E \wedge^{\tilde{N}} \tilde{W}$. We consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 1 & \longrightarrow & {}_2 T & \longrightarrow & \tilde{S}_2 & \longrightarrow & \tilde{W} \longrightarrow 1 \\
 & & \parallel & & \parallel & & \parallel \\
 1 & \longrightarrow & {}_{2d(\tilde{G})_D} T & \longrightarrow & \tilde{S}_{2d(\tilde{G})_D} & \longrightarrow & \tilde{W} \longrightarrow 1 \\
 & & \parallel & & \parallel & & \parallel \\
 1 & \longrightarrow & T & \longrightarrow & \tilde{N} & \longrightarrow & \tilde{W} \longrightarrow 1.
 \end{array}$$

The obstruction to lift F to a \tilde{S}_2 -torsor is a class $\Delta([F]) \in H_{\text{fppf}}^2(X, {}_2 M)$ where the X -torus $M = {}^F T$ is the twist of T_X by F [Gd, IV.4.2]. The point is that its image in $H_{\text{fppf}}^2(X, M)$ vanishes since F lifts to the \tilde{N} -torsor E in view of the compatibilities of the obstructions [Gd, §IV.3.6]. According to Lemma 3.3.(3), we have $\Delta([F]) \in \ker \left(H_{\text{fppf}}^2(X, {}_2 M) \rightarrow H_{\text{fppf}}^2(X, {}_{2d(\tilde{G})_D} M) \right)$. Again by functoriality, it follows that the \tilde{W} -torsor F lifts to a $\tilde{S}_{2d(\tilde{G})_D}$ -torsor V . We consider the \tilde{N} -torsor $E_V = \tilde{N} \times^{\tilde{S}_{2d(\tilde{G})_D}} V$. Twisting by V and E_V , the first diagram provides the commutative diagram of exact sequence of X -group schemes

$$(3.2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & {}_{2d(\tilde{G})_D}M & \longrightarrow & {}^V\tilde{S}_{2d(\tilde{G})_D} & \longrightarrow & {}^F\tilde{W} \longrightarrow 1 \\ & & \parallel & & \parallel & & \parallel \\ 1 & \longrightarrow & M & \longrightarrow & {}^{E_V}\tilde{N} & \longrightarrow & {}^F\tilde{W} \longrightarrow 1. \end{array}$$

Next we consider the commutative diagram of torsion bijections (with respect to the torsors V, E_V and F)

$$\begin{array}{ccc} H_{\text{fppf}}^1(X, {}^V\tilde{S}_{2d(\tilde{G})_D}) & \xrightarrow[\sim]{\tau_2} & H_{\text{fppf}}^1(X, \tilde{S}_{2d(\tilde{G})_D}) \\ \downarrow & & \downarrow \\ H_{\text{fppf}}^1(X, {}^{E_V}\tilde{N}) & \xrightarrow[\sim]{\tau} & H_{\text{fppf}}^1(X, \tilde{N}) \\ \downarrow & & \downarrow \\ H_{\text{fppf}}^1(X, {}^F\tilde{W}) & \xrightarrow[\sim]{\tau'} & H_{\text{fppf}}^1(X, \tilde{W}). \end{array}$$

Since $[E]$ maps to $[F]$ by the map $H_{\text{fppf}}^1(X, \tilde{N}) \rightarrow H_{\text{fppf}}^1(X, \tilde{W})$, it follows that the class $\tau^{-1}([E])$ maps to $1 \in H_{\text{fppf}}^1(X, {}^F\tilde{W})$. On the other hand, the diagram (3.2) induces a commutative diagram of exact sequences of pointed sets

$$(3.3) \quad \begin{array}{ccccc} H_{\text{fppf}}^1(X, {}_{2d(\tilde{G})_D}M) & \longrightarrow & H_{\text{fppf}}^1(X, {}^V\tilde{S}_{2d(\tilde{G})_D}) & \longrightarrow & H_{\text{fppf}}^1(X, {}^F\tilde{W}) \\ \downarrow & & \downarrow & & \parallel \\ H_{\text{fppf}}^1(X, M) & \longrightarrow & H_{\text{fppf}}^1(X, {}^{E_V}\tilde{N}) & \longrightarrow & H_{\text{fppf}}^1(X, {}^F\tilde{W}) \end{array}$$

and the point is that the left vertical map is onto in view of Lemma 3.3.(2). A diagram chase shows that $\tau^{-1}([E])$ arises from a class of $H_{\text{fppf}}^1(X, {}^V\tilde{S}_{2d(\tilde{G})_D})$. By applying τ , we obtain that $[E]$ arises from a class of $H_{\text{fppf}}^1(X, \tilde{S}_{2d(\tilde{G})_D})$.

(2) We assume now that $X = \text{Spec}(R)$ with R a LG ring. According to [G-N, Thm. 2.6], all twisted forms of \tilde{G} admit a maximal R -torus so that the map $H^1(R, N_{\tilde{G}}(T)) \rightarrow H^1(R, \tilde{G})$ is onto. The ring R satisfies that $\text{Pic}(R') = 0$ for each finite extension R'/R [E-G, Thm. 2.10]. The condition on Picard groups is satisfied with $D = 1$ so that (1) implies the wished statement. \square

Let l be a prime. Our goal is to provide a localized version at l of Theorem 3.2 when J is a finite constant \mathbb{Z} -group so that \tilde{W} is finite as well in view of Lemma 3.1.(1). For $n \geq 1$, we need to construct a localized version at l of the \mathbb{Z} -group scheme \tilde{S}_{2n}

fitting in the sequence

$$(3.4) \quad 1 \rightarrow {}_{2n}T \rightarrow \widetilde{S}_{2n} \rightarrow \widetilde{W} \rightarrow 1.$$

We denote by $\widetilde{W}(\mathbb{Z})^{(l)}$ a l -Sylow subgroup of $\widetilde{W}(\mathbb{Z})$ and by $\widetilde{W}^{(l)} \subset \widetilde{W}(\mathbb{Z})$ the associated constant \mathbb{Z} -group scheme. The first step is to pull back the above sequence by $\widetilde{W}^{(l)} \rightarrow \widetilde{W}$ so that we obtain

$$(3.5) \quad 1 \rightarrow {}_{2n}T \rightarrow \widetilde{S}_{2n} \rightarrow \widetilde{W}^{(l)} \rightarrow 1.$$

Case $l = 2$. We write $n = 2^a m$ with m odd and define $\widetilde{S}_{2n}^{(2)}$ as the push-out of the extension (3.5) with respect to the map ${}_{2n}T \rightarrow {}_{2^{a+1}}T$. It is representable by an affine \mathbb{Z} -group scheme¹ which is a \mathbb{Z} -subgroup scheme of \widetilde{S}_{2n} and a fortiori of \widetilde{N} . According to [SGA3, VI_B.9.2.(viii)], the affine \mathbb{Z} -group scheme $\widetilde{S}_{2n}^{(2)}$ is flat and finite free.

$$(3.6) \quad 1 \rightarrow {}_{2^{a+1}}T \rightarrow \widetilde{S}_{2n}^{(2)} \rightarrow \widetilde{W}^{(2)} \rightarrow 1.$$

Case l is odd. We write $n = l^b m$ with $(m, l) = 1$ and define $\widetilde{S}_{2n}^{(l)}$ as the push-out of the extension (3.5) with respect to the map ${}_{2n}T \rightarrow {}_{l^b}T$. As above it is representable by a finite flat affine \mathbb{Z} -group scheme (which is a \mathbb{Z} -subgroup scheme of \widetilde{S}_{2n} and a fortiori of \widetilde{N}) and fits in an commutative diagram of exact sequences

$$(3.7) \quad \begin{array}{ccccccc} 1 & \longrightarrow & {}_{l^b}T & \longrightarrow & \widetilde{S}_{2n}^{(l)} & \longrightarrow & \widetilde{W}^{(l)} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & {}_{2n}T & \longrightarrow & \widetilde{S}_{2n} & \longrightarrow & \widetilde{W} \longrightarrow 1. \end{array}$$

We come back to the general case and consider the quotient \mathbb{Z} -scheme $\widetilde{S}_{2n}/\widetilde{S}_{2n}^{(l)}$ as defined in [D-G, §III.2.6]. It is an affine \mathbb{Z} -scheme and the quotient map $\widetilde{S}_{2n} \rightarrow \widetilde{S}_{2n}/\widetilde{S}_{2n}^{(l)}$ is an $\widetilde{S}_{2n}^{(l)}$ -torsor. According to [SGA3, VI_B.9.2.(xi), (xii)], $\widetilde{S}_{2n}/\widetilde{S}_{2n}^{(l)}$ is flat and quasi-finite over \mathbb{Z} . According to [Stacks, Tag 0AH6], $\widetilde{S}_{2n}/\widetilde{S}_{2n}^{(l)}$ is proper over \mathbb{Z} so is finite [Stacks, Tag 02OG]. Since a finite flat \mathbb{Z} -module is free, $\widetilde{S}_{2n}/\widetilde{S}_{2n}^{(l)}$ is finite free over \mathbb{Z} of degree $\deg(\widetilde{S}_{2n})/\deg(\widetilde{S}_{2n}^{(l)})$ which is prime to l .

Theorem 3.4. *In the above setting, we assume that \widetilde{W} is finite over \mathbb{Z} (equivalently J is finite). We assume furthermore that there exists an integer D such that*

$$D \operatorname{Pic}(X')\{l\} = 0$$

for any finite étale cover X' of X whose degree divides $d(\widetilde{G})$. We consider the \mathbb{Z} -subgroup $\widetilde{S}^{(l)} = \widetilde{S}_{2d(\widetilde{G})D}^{(l)}$ of \widetilde{N} .

¹We mod out $\widetilde{S}_{2n} \times_{\mathbb{Z}} {}_{2^{a+1}}T$ by a finite diagonalizable \mathbb{Z} -subgroup.

- (1) Let E be a \tilde{N} -torsor. Then there exists a finite locally free flat cover X' of R of constant degree prime to l such that the G -torsor $E_{X'}$ admits a reduction to $\tilde{S}^{(l)}$.
- (2) Assume that $X = \text{Spec}(R)$ where R is a LG ring. Let E be a \tilde{G} -torsor. Then there exists a finite locally free flat cover R' of R of constant degree prime to l such that the G -torsor $E_{R'}$ admits a reduction to $\tilde{S}_{2d(\tilde{G})}^{(l)}$.
- (3) If R is a field, we can require furthermore that R' is a field in (2).

We proceed to the proof of Theorem 3.4.

Proof. (1) According to Theorem 3.2.(1), the \tilde{N} -torsor \tilde{E} admits a reduction to a $\tilde{S}_{2d(\tilde{G})D}^{(l)}$ -torsor F . We consider the \mathbb{Z} -scheme

$$X' = F/\tilde{S}_{2d(\tilde{G})D}^{(l)} = F \wedge^{\tilde{S}_{2d(\tilde{G})D}} \tilde{S}_{2d(\tilde{G})D}/\tilde{S}_{2d(\tilde{G})D}^{(l)}$$

of reductions of F to $\tilde{S}_{2d(\tilde{G})D}^{(l)}$. By descent, it is finite locally free of degree $\deg(\tilde{S}_{2d(\tilde{G})D}/\tilde{S}_{2d(\tilde{G})D}^{(l)})$ which is prime to l . It follows that $E_{X'}$ admits a reduction to $\tilde{S}_{2d(\tilde{G})D}^{(l)}$.

(2) The argument is similar by applying this times Theorem 3.2.(2).

(3) We assume that R is a field k . Then the basic Lemma 3.5 below provides a field extension k'/k of degree prime to l such that $X'(k') \neq \emptyset$. This is the reason why we can improve (2) in this case. \square

Lemma 3.5. *Let l be a prime and let k be a field and let X be a finite k -scheme of degree d prime to l . Then there exists a field extension k'/k satisfying the three following facts:*

- (i) $([k' : k], l) = 1$;
- (ii) $[k' : k] \leq d$;
- (iii) $X(k') \neq \emptyset$.

Proof. According to [Stacks, Tag 0AAX], we have $k[X] = A_1 \times \cdots \times A_c$ where A_i is a (finite) Artinian k -algebra of residue field k_i . We have $X(k_i) \neq \emptyset$ for each i and $d = \sum_{i=1}^c [k_i : k]^{e_i}$. Since l and d are coprime, there exists i such that $[k_i : k]$ is prime to l . \square

4. PATCHING TECHNIQUES

4.1. Setting. We recall the setting. Let B be a complete (excellent) discrete valuation ring with fraction field K , residue field k and uniformizing parameter t . Let F be a one-variable function field over K and let \mathfrak{X} be a normal model of F , i.e. a normal connected projective B -curve with function field F . We denote by Y the closed fiber of \mathfrak{X} and fix a separable closure F_s of F .

For each point $P \in Y$, let R_P be the local ring of \mathfrak{X} at P ; its completion \widehat{R}_P is a domain with fraction field denoted by F_P .

For each subset U of Y that is contained in an irreducible component of Y and does not meet the other components, we define $R_U = \bigcap_{P \in U} R_P \subset F$. We denote by

\widehat{R}_U the t -adic completion of R_U . The rings R_U and \widehat{R}_U are excellent normal domains and we denote by F_U the fraction field of \widehat{R}_U [H-H-K3, Remark 3.2.(b)].

Each height one prime \mathfrak{p} in \widehat{R}_P that contains t defines a branch of Y at P lying on some irreducible component of Y . The t -adic completion $\widehat{R}_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$ of \widehat{R}_P at \mathfrak{p} is a complete DVR whose fraction field is denoted by $F_{\mathfrak{p}}$. The field $F_{\mathfrak{p}}$ contains also F_U if U is an irreducible open subset of Y such that $P \in \overline{U} \setminus U$. We have then a diagram of fields

$$\begin{array}{ccc} & F_{\mathfrak{p}} & \\ & \swarrow \quad \nwarrow & \\ F_P & & F_U. \end{array}$$

Setting 4.1. Let \mathcal{P} be a non-empty finite set of closed points of Y that contains all the closed points at which distinct irreducible components meet and such that each component contains at least one point of \mathcal{P} . Let \mathcal{U} be the set of connected components of $Y \setminus \mathcal{P}$ and let \mathcal{B} be the set of branches of Y at points of \mathcal{P} . This yields a finite inverse system of field $F_P, F_U, F_{\mathfrak{p}}$ (for $P \in \mathcal{P}; U \in \mathcal{U}, \mathfrak{p} \in \mathcal{B}$) where $F_P, F_U \subset F_{\mathfrak{p}}$ if \mathfrak{p} is a branch of Y at P lying in the closure of U .

4.2. Torsors and models.

Proposition 4.2. *Let $1 \rightarrow G \rightarrow \widetilde{G} \rightarrow J \rightarrow 1$ be an exact sequence of smooth B -group schemes where G is reductive and J is twisted constant². Let E_1 (resp. E_2) be a \widetilde{G} -torsor over F admitting a reduction to a finite étale B -subgroup of S_1 (resp. S_2) of \widetilde{G} with prime to p order.*

Assume that $E_{1,F_v} \cong E_{2,F_v}$ for any divisorial discrete valuation v on F . Then there exists a regular model Ω of F as above together with Y, \mathcal{P} , etc. such that $E_{1,F_P} \cong E_{2,F_P}$ for all $P \in \mathcal{P}$ and $E_{1,F_U} \cong E_{2,F_U}$ for all connected components U of $Y \setminus \mathcal{P}$.

Proof. We consider the F -scheme $Z = \text{Isom}_{\widetilde{G}}(E_1, E_2)$. Our assumption is that $Z(F_v) \neq \emptyset$ for all divisorial discrete valuations v on F .

Let V_i be an $S_{i,F}$ -torsor such that $V_i \wedge^{S_i, F} \widetilde{G} \cong E_i$ for $i = 1, 2$. According to [G-P, lemma 5.1] there exists a regular proper model \mathfrak{X} of X and a strict normal crossing divisor D containing the irreducible components of Y such that V_i extends to an $\mathfrak{X} \setminus D$ -torsor \mathfrak{V}_i under S_i . We are given a closed point $P \in Y$. If y is of codimension

²i.e. there exists an abstract group Γ such that J is locally isomorphic to the B -constant group scheme Γ_B .

one, by hypothesis, $Z(F_y)$ is not empty. We therefore look at a closed point $P \in Y$. We pick a height one prime \mathfrak{p} in \widehat{R}_P that contains t . It defines a branch of Y at P lying on some irreducible component Y_1 of Y .

We consider the regular local ring $A = \widehat{R}_P$ and denote by A_D its localization at D . Since $S_{i,B}$ is finite étale of degree prime to p , $H^1(A_D, S_i)$ consists in loop torsors as defined in [G5, §2.3, lemma 2.3.(2)], i.e. those arising from cocycles related to tame Galois covers of A_D . It follows that the A_D -torsors $\mathfrak{Y}_1, \mathfrak{Y}_2$ under \widetilde{G} are loop (*ibid*, lemma 5.3.(3)).

Let $F_{P,v}$ be the completion of the field F_P for the valuation associated to the blow-up of $\text{Spec}(A)$ at its closed point. Our assumption states in particular that $Z(F_{P,v}) \neq \emptyset$. According to [G6, Thm. 6.9], the map

$$H_{loop}^1(A_D, \widetilde{G}) \rightarrow H^1(F_{P,v}, \widetilde{G})$$

is injective. It follows that $\text{Isom}_{\widetilde{G}_{A_D}}(\mathfrak{Y}_1, \mathfrak{Y}_2)(A_D) \neq \emptyset$. A fortiori $Z(F_P) \neq \emptyset$. \square

Proposition 4.3. *Let G be a reductive F -algebraic group and let G_0 be the underlying Chevalley reductive \mathbb{Z} -group scheme. Let E be a G -torsor and assume that the $\text{Aut}(G_0)$ -torsor $\underline{\text{Isom}}(G_0, G)$ admits a reduction to a finite étale B -subgroup of $\text{Aut}(G_0)$ of prime to p order and that the $G_0 \rtimes \text{Aut}(G_0)$ -torsors (G, G) (resp. (E, G)) admits a reduction to a finite étale B -subgroup of $G_0 \rtimes \text{Aut}(G_0)$ of prime to p order. Then the two following assertions are equivalent:*

(i) *There exists a regular proper model \mathfrak{X} of X with special fiber $Y = \mathfrak{X}_k$ such that for every point $y \in Y$, then $E(F_y) \neq \emptyset$.*

(ii) *For each divisorial discrete valuation v on F , we have $E(F_v) \neq \emptyset$.*

Proof. The implication (i) \implies (ii) is a general fact, see [H-H-K-P, Thm. 3.4]. We prove then (ii) \implies (i) assuming that $E(F_v) \neq \emptyset$ for each discrete divisorial valuation v on F .

We denote by $s_0 : \text{Aut}(G_0) \rightarrow \text{Aut}_{\mathfrak{B}}(G_0, G_0)$ the splitting $\text{Aut}_{\mathfrak{B}}(G_0, G_0) \rightarrow \text{Aut}(G_0)$ and consider the $\text{Aut}_{\mathfrak{B}}(G_0, G_0)$ -torsor $E_1 = \underline{\text{Isom}}_{\mathfrak{B}}((G_0, G_0), (G, G))$ which is nothing but $s_{0,*}(\underline{\text{Isom}}_{gr}(G_0, G))$. Our assumption implies that E_1 admits a reduction to a finite étale B -subgroup of $\text{Aut}_{\mathfrak{B}}(G_0, G_0)$ of prime to p order.

We shall apply Proposition 4.2 to the $G_0 \rtimes \text{Aut}(G_0)$ -torsors $E_1 = \underline{\text{Isom}}_{\mathfrak{B}}((G_0, G_0), (G, G))$ and $E_2 = \underline{\text{Isom}}_{\mathfrak{B}}((G_0, G_0), (E, G))$.

Assume that $E_{1,F_v} \cong E_{2,F_v}$ for any divisorial discrete valuation v on F . Proposition 4.2 provides a regular model Ω of F as above together with Y, \mathcal{P} , etc. such that $E_{1,F_P} \cong E_{2,F_P}$ for all $P \in \mathcal{P}$ and $E_{1,F_U} \cong E_{2,F_U}$ for all connected components U of $Y \setminus \mathcal{P}$. It follows that for every point $y \in Y$, then $E(F_y) \neq \emptyset$. \square

Theorem 4.4. *Let G be a reductive F -algebraic group and let G_0 be the underlying Chevalley reductive \mathbb{Z} -group scheme. Assume that p does not divide the order of the*

Weyl group W_0 of G_0 and that G becomes an inner form of G_0 after a Galois prime to p extension of F .

- (1) We have $\text{III}_{\text{patch}}(F, G) = \text{III}_{\text{div}}(F, G)$;
- (2) If G is a retract rational F -variety, then $\text{III}_{\text{div}}(F, G) = 1$.

Proof. (1) We know that $\text{III}_{\text{patch}}(F, G) \subset \text{III}_{\text{div}}(F, G)$. To establish the reverse inclusion, we are given a G -torsor E such that $E(F_v) \neq \emptyset$ for each divisorial discrete valuation v . We need to check the conditions for applying Proposition 4.3. We start with the isomorphism class $[G] \in H^1(F, \text{Aut}(G_0))$ by using the sequence $1 \rightarrow G_{0,ad} \rightarrow \text{Aut}(G_0) \rightarrow \text{Out}(G_0) \rightarrow 1$ where $\text{Out}(G_0)$ is a constant \mathbb{Z} -group scheme. Since $H^1(F, \text{Out}(G_0)) = \text{Hom}_{ct}(\text{Gal}(F_s/F), \text{Out}(G_0)(\mathbb{Z}))$ modulo $\text{Out}(G_0)(\mathbb{Z})$ -conjugacy, the image of $[G]$ is represented by a continuous homomorphism $u : \text{Gal}(F_s/F) \rightarrow \text{Out}(G_0)(\mathbb{Z})$ whose kernel defines a Galois extension F'/F which is the minimal subextension of F_s making G an inner form. Our assumption implies that $\text{Gal}(F'/F)$ is of prime to p degree and so that $\Gamma = u(\text{Gal}(F_s/F))$ is a finite subgroup of $\text{Out}(G_0)(\mathbb{Z})$ of order prime to p . We denote by $\text{Aut}_\Gamma(G_0) = \text{Aut}(G_0) \times_{\text{Out}(G_0)} \Gamma$, we have an exact sequence of \mathbb{Z} -group schemes

$$(4.1) \quad 1 \rightarrow G_{0,ad} \rightarrow \text{Aut}_\Gamma(G_0) \rightarrow \Gamma_{\mathbb{Z}} \rightarrow 1.$$

Further the isomorphism class $[G]$ arises from a cohomology class $\gamma \in H^1(F, \text{Aut}_\Gamma(G_0))$. According to [C-G-R, Thm. 1.2] applied to the ring \mathbb{Z} , there exists a finite \mathbb{Z} -subgroup S of $\text{Aut}_\Gamma(G_0)$ such that the map $H^1(F, S) \rightarrow H^1(F, \text{Aut}_\Gamma(G_0))$ is onto. Let $T_0 \subset G_0$ be a maximal split \mathbb{Z} -subtorus and consider the \mathbb{Z} -group scheme $\text{Aut}_\Gamma(G_0, T_0)$ of $\text{Aut}_\Gamma(G_0)$ as defined in [SGA3, §XXIV.3]. For each ring C , we have

$$\text{Aut}_\Gamma(G_0, T_0)(C) = \{f \in \text{Aut}_\Gamma(G_0)(C) \mid f(T_{0,C}) = T_{0,C}\}.$$

Further by the same statement, we can take $S \subset \text{Aut}_\Gamma(G_0, T_0)$ fitting in an exact sequence $1 \rightarrow {}_n(T_0) \rightarrow S \rightarrow \text{Aut}_\Gamma(G_0, T_0)/T_0 \rightarrow 1$ where n divides the square of the order of the finite group $\text{Aut}_\Gamma(G_0, T_0)/T_0$. We have a sequence $1 \rightarrow W_0 \rightarrow \text{Aut}_\Gamma(G_0, T_0)/T_0 \rightarrow \Gamma \rightarrow 1$ so that $\text{Aut}_\Gamma(G_0, T_0)/T_0$ is finite constant of prime to p order. It implies that ${}_n(T_0)$ is a finite diagonalisable \mathbb{Z} -group scheme whose base change to B is étale. Altogether S_B is a finite étale B -subgroup of $\text{Aut}_\Gamma(G_0)_B$ of prime to p order such that γ admits a reduction to S_B . A fortiori, $[G] \in H^1(F, \text{Aut}(G_0))$ admits a reduction to S_B .

The same method works as well for the class $[(E, G)] \in H^1(F, G_0 \rtimes \text{Aut}(G_0))$ by using the exact sequence $1 \rightarrow G_0 \rtimes_{\mathbb{Z}} G_{0,ad} \rightarrow G_0 \rtimes_{\mathbb{Z}} \text{Aut}(G_0) \rightarrow \text{Out}(G_0) \rightarrow 1$ and provides a finite étale B -subgroup of S' of $(G_0 \rtimes_{\mathbb{Z}} \text{Aut}(G_0))_B$ of prime to p -order such that the $G_0 \rtimes_{\mathbb{Z}} \text{Aut}(G_0)$ -torsor over $\text{Spec}(F)$ arises from an S' -torsor.

Proposition 4.3, (ii) \implies (i), applies and shows that there exists a regular proper model \mathfrak{X} of X with special fiber $Y = \mathfrak{X}_k$ such that for every point $y \in Y$, then $E(F_y) \neq \emptyset$. In other words $[E]$ belongs in $\text{III}_{\mathfrak{X}}(F, G)$ and a fortiori to $\text{III}_{\text{patch}}(F, G)$.

(2) If G is a retract rational F -variety, we have that $\text{III}_{patch}(F, G) = 1$ [H-H-K2, Thm. 9.1] (see [K, Thm. 2.2.4] for the extension to the retract rational case). The first part of the statement enables us to conclude that $\text{III}_{div}(F, G) = 1$ in this case. \square

Remarks 4.5. (a) If G is a torus, the condition is that the minimal splitting extension of T is of prime to p degree. Note that for norm one tori with respect to a prime to p Galois extension, the main theorem had been established by Sumit Chandra Mishra [Mi, Thm. 5.1].

(b) The assumptions on p are not sharp. For example, if G is semisimple of type G_2 , the statement excludes the primes 2 and 3 but excluding 2 is enough. The point is that we can deal with the subgroup $\mu_2 \times \mu_2 \times \mathbb{Z}/2\mathbb{Z} \subset S$ in the proof.

Similarly for $n \geq 1$ such that $(p, n!) = 1$, Theorem 4.4.(2) shows that $\text{III}_{div}(F, \text{PGL}_n) = 1$. This will be strengthened in Corollary 6.5.

Corollary 4.6. *Let G be a semisimple quasi-split F -group which is quasi-split by a Galois prime to p field extension. Assume that p does not divide the order of W_0 . In each case*

(i) G is split;

(ii) G is simply connected or adjoint;

we have $\text{III}_{div}(F, G) = 1$.

Proof. Such an F -group is a rational K -variety so the result follows from Theorem 4.4. \square

Corollary 6.4 of section 6 will refine the statement for groups without factor of type E_8 . This applies as well to many other groups in view of the rationality results of Chernousov-Platonov [C-P] and others. This is the case for example for an isotropic F -group of type F_4 .

5. OTHER TATE-SCHAFAREVICH SETS

5.1. Various sets of valuations. Given an affine algebraic F -group G , we can also study the local-global principle for G -torsors with respect to the following set of completions of F :

(1) Completions associated to the set Ω_F of all non-trivial B -valuations on F (i.e., those whose valuation ring contains T).

(2) Completions associated to the set Ω_F^1 of all non-trivial rank one B -valuations on F .

(3) Completions associated to the set $\Omega_{F,dvr}^1$ of all non-trivial rank one discrete B -valuations on F .

We have $\Omega_{F,div} \subset \Omega_{F,dvr}^1 \subset \Omega_F^1 \subset \Omega_F$. This leads to three Tate-Shafarevich sets as for example $\text{III}_{\Omega_{dvr}^1}(F, G) = \ker\left(H^1(F, G) \rightarrow \prod_{v \in \Omega_{dvr}^1} H^1(F_v, G)\right)$. We have the

following inclusions

$$\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(F, G) \subset \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_F}(F, G) \subset \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_F^1}(F, G) \subset \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G) \subset \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G).$$

According to [H-H-K-P, Thm. 3.4], we have $\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(F, G) = \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_F^1}(F, G) = \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_F}(F, G)$. The fact that $\mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_F}(F, G) = \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_F^1}(F, G) = \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G)$ is known in two following situations [H-H-K2, Thm. 8.10, cases (i) and (iii)]:

- (A) G is a rational F -variety and the residue field k of T is algebraically closed of characteristic zero;
- (B) G is semisimple simply connected and the residue field k of T is algebraically closed of characteristic zero.

Coming back to the present paper, Theorem 4.4 has then the following consequence.

Corollary 5.1. *Under the assumptions of Theorem 4.4, we have then*

$$\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(F, G) = \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_F}(F, G) = \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_F^1}(F, G) = \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G) = \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G)$$

and those sets are trivial when G is a retract F -rational variety.

5.2. The constant case. This is the case when we deal with a reductive B -group scheme G . It was proven in [C-T-P-S, Thm. 4.2] (where only divisorial valuations are used in the proof) that $\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(F, G) = \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G)$. Note that there is no need of any condition on the characteristic exponent p of the residue field k . Other references are [H-H-K2, Thm. 8.10, case (ii)] and [C-T-H-H-K-P-S2, Thm. 3.2]) This applies in particular when G_F is a retract rational F -variety; since $\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(F, G) = 1$, we have that $\mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G) = 1$ in characteristic free. The interest of our result is then mostly for the non-constant case.

6. VARIANT WITH 0-CYCLES OF DEGREE ONE

The goal is to replace rational points by 0-cycles in order to improve Theorem 4.4.(2).

6.1. Serre's injectivity question. For a semisimple (absolutely k -simple) group G over a field k , Serre asked whether the map

$$H^1(k, G) \rightarrow \prod_{i=1}^c H^1(k_i, G)$$

is injective where the k_i 's are finite field extensions of k such that the *g.c.d.* of the $[k_i : k]$'s is prime to the torsion primes of G [Se, §2.4]. We can also ask the same question for G reductive assuming that the *g.c.d.* of the $[k_i : k]$'s is 1; note that the two questions are equivalent in the semisimple case in view of Lemma A.3.(2). If the above map is injective (resp. has trivial kernel), we will say that Serre's injectivity question (resp. Serre's 0-cycle question) has a positive answer. Partial results are the following.

- (1) The above questions reduce to the characteristic zero case by using the lifting method [G4, proof of Thm. 3.4.1].
- (2) The answer of Serre's 0-cycle question is positive for simply connected and adjoint classical groups and special orthogonal groups as established by Black [Bl, Thm. 0.2, prop. 3.6] and for semisimple classical groups involving types A, B, C (Bhaskhar, [Bh]).
- (3) The answer of Serre's 0-cycle question is positive for quasi-split semisimple groups without factors of type E_8 . This involves case by case results on other exceptional groups, see [Bl, §5], [Bh, Thm. 1.3] and [G4, Cor. 3.4.5].

6.2. A local variant.

Proposition 6.1. *Let G be a reductive F -algebraic group. Let l be a prime, $l \neq p$.*

- (1) *Let $\gamma \in \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G)$. Then there exists a finite extension L/F of degree prime to l such that $\gamma_L \in \mathbb{I}\mathbb{I}\mathbb{I}_{patch}(L, G)$.*
- (2) *Furthermore if $\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(L, G) = 1$, then $\gamma_L = 1$.*

Proof. (1) We are given a class $\gamma = [E] \in \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G)$. The proof is quite similar to that of Theorem 4.4 by using the localized version of the technique of reduction to torsors to finite subgroup schemes (established by Theorem 3.4).

We remind the reader that G_0 denotes the Chevalley form of G . We start with the isomorphism class $[G] \in H^1(F, \text{Aut}(G_0))$ by using the sequence [SGA3, XXIV.1.3]

$$(6.1) \quad 1 \rightarrow G_{0,ad} \rightarrow \text{Aut}(G_0) \rightarrow \text{Out}(G_0) \rightarrow 1$$

where $\text{Out}(G_0)$ is a constant \mathbb{Z} -group scheme. Since $H^1(F, \text{Out}(G_0)) = \text{Hom}_{ct}(\text{Gal}(F_s/F), \text{Out}(G_0)(\mathbb{Z})) / \sim$, the image of $[G]$ is represented by a continuous homomorphism $u : \text{Gal}(F_s/F) \rightarrow \text{Out}(G_0)(\mathbb{Z})$ whose kernel defines a Galois extension F'/F which is the minimal subextension of F_s making G an inner form. Our assumption implies that $\text{Gal}(F'/F)$ is of prime to p degree and so that $\Gamma = u(\text{Gal}(F_s/F))$ is a finite subgroup of $\text{Out}(G_0)(\mathbb{Z})$. We denote by $\text{Aut}_{\Gamma}(G_0) = \text{Aut}(G_0) \times_{\text{Out}(G_0)} \Gamma$; by construction $[G]$ arises from a cohomology class $\beta \in H^1(F, \text{Aut}_{\Gamma}(G_0))$. According to Theorem 3.4 applied to \mathbb{Z} -group scheme $\tilde{G}_0 = \text{Aut}_{\Gamma}(G_0)$ there exists a finite \mathbb{Z} -subgroup S^{\sharp} of $\text{Aut}_{\Gamma}(G_0)$ which is finite free of index l^c and such that there exists a finite prime to l field extension F'/F such that $\beta_{F'}$ belongs to the image of $H^1(F', S^{\sharp}) \rightarrow H^1(F', \text{Aut}_{\Gamma}(G_0))$. Once again, replacing F by F' permits to suppose that β belongs to the image if $H^1(F, S^{\sharp}) \rightarrow H^1(F, \text{Aut}_{\Gamma}(G_0))$.

The same method works as well for the class $[(E, G)] \in H^1(F, G_0 \rtimes \text{Aut}(G_0))$ by using the exact sequence $1 \rightarrow G_0 \rtimes_{\mathbb{Z}} G_{0,ad} \rightarrow G_0 \rtimes_{\mathbb{Z}} \text{Aut}(G_0) \rightarrow \text{Out}(G_0) \rightarrow 1$ and provides a finite B -subgroup of S^{\diamond} of $(G_0 \rtimes_{\mathbb{Z}} \text{Aut}(G_0))_B$ of l -power such that the $G_0 \rtimes_{\mathbb{Z}} \text{Aut}(G_0)$ -torsor over $\text{Spec}(F)$ arises from an S^{\diamond} -torsor (again up to taking a further finite prime to l -extension of F).

Proposition 4.3, $(ii) \implies (i)$, applies and shows that there exists a regular proper model \mathfrak{X} of X with special fiber $Y = \mathfrak{X}_k$ such that for every point $y \in Y$, then $E(F_y) \neq \emptyset$. In other words $[E]$ belongs in $\text{III}_{\mathfrak{X}}(F, G)$ and a fortiori to $\text{III}_{\text{patch}}(F, G)$.
(2) This readily follows of (1). \square

6.3. The 0-cycle variant.

Theorem 6.2. *Let G be a reductive F -algebraic group and let G_0 be the underlying Chevalley reductive \mathbb{Z} -group scheme. We assume that $\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(L, G) = 1$ for all finite extensions L of F . Let E be a G -torsor over $\text{Spec}(F)$ such that $[E] \in \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G)$.*

(1) *E admits a 0-cycle of degree p^c .*

(2) *Assume furthermore that p is not a torsion prime of DG_0 and that G becomes an inner form of G_0 after a Galois prime to p extension of F . Then E admits a 0-cycle of degree 1.*

(3) *Assume furthermore that Serre's 0-cycle question admits a positive answer for G . Then the G -torsor E is trivial.*

Proof. (1) Let $[E] \in \mathbb{I}\mathbb{I}\mathbb{I}_{\Omega_{div}^1}(F, G)$. For each prime $l \neq p$, we have to prove that there exists a prime to l finite field extension F'/F such that $E(F') \neq \emptyset$. This fact follows indeed of Proposition 6.1.(2).

(2) From (1), it remains to deal with the case $l = p$. Up to replace F by a suitable prime to p -extension of F , we can assume that G is an inner form of G_0 . In other words G is the twist of G_0 by a $G_{0,ad}$ -torsor Q . Let $\text{Ind}_F(Q)$ be the index of Q , that is, the *g.c.d.* of the degrees of the rational points of the F -variety Q . Since p is not a torsion prime of $G_{0,ad}$, Lemma A.3.(2) shows that p does not divide $\text{Ind}_F(Q)$ so that Q carries a point after a prime to p extension. Once again it is harmless to assume that $G = G_0$.

Next we use the exact sequence $1 \rightarrow C_0 \rightarrow G_0 \rightarrow G_0/C_0 \rightarrow 1$ where C_0 is the radical torus of G_0 . Then C_0 is split and Hilbert 90 theorem provides a commutative exact diagram of pointed sets

$$\begin{array}{ccccc} 1 & \longrightarrow & H^1(F, G_0) & \longrightarrow & H^1(F, G_0/C_0) \\ & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \prod_{v \in \Omega_{F,div}} H^1(F_v, G_0) & \longrightarrow & \prod_{v \in \Omega_{F,div}} H^1(F_v, G_0/C_0). \end{array}$$

The above diagram reduces the proof E admits a 0-cycle of degree 1 to the case of an F -torsor Q' under the semisimple split group $G'_0 = G_0/C_0$. The argument used for Q and $G_{0,ad}$ works and enables us to conclude that E becomes trivial after a prime to p extension of F .

(3) This is straightforward. □

Remarks 6.3. (a) By inspection of the proof, it is enough to assume that there exist finite fields extensions F_i 's of F such that $\text{g.c.d.}([F_i : F]) = 1$ and such that $\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(L_i, G) = 1$ for each finite field extension L_i/F_i .

(b) The assumption on the triviality of the sets $\mathbb{I}\mathbb{I}\mathbb{I}_{patch}$ is satisfied when G is a retract rational F -variety [K, Thm. 2.2.4].

(c) There are cases when G is not retract rational but for which the sets III_{patch} vanish. We provide an example below, see Remark 6.7.

6.4. Applications. Taking into account the results of §6.1, Theorem 6.2.(3) has the following consequence which slightly improves Corollary 4.6.

Corollary 6.4. *Let G be a semisimple quasi-split F -group with no E_8 factor. Assume that p is not a torsion prime of G . In each case*

- (i) G is split;
- (ii) G is simply connected or adjoint;

we have $\text{III}_{\Omega_{div}^1}(F, G) = 1$.

Proof. In both cases, we have seen at the end of §6.1 that Serre’s 0-cycle question admits a positive answer for G . Then Theorem 6.2.(3) implies that $\text{III}_{\Omega_{div}^1}(F, G) = 1$. \square

We deal now with special cases where G is a retract rational F -variety and when Serre’s 0-cycle question has a positive answer. We recover then by our method results obtained by using injectivity properties of cohomological invariants and local-global principles for Galois cohomology. Parimala-Suresh’s theorem [H-K-P, Cor. 4.4] states that the map

$$(6.2) \quad H^{q+1}(F, \mu_n^{\otimes q}) \rightarrow \prod_{v \in \Omega_{div}^1} H^{q+1}(F_v, \mu_n^{\otimes q})$$

is injective for each $q \geq 1$ and $(n, p) = 1$. For PGL_n , since $H^1(F, \text{PGL}_n)$ injects in $H^2(F, \mu_n)$ we obtain

Corollary 6.5. *Let n be an integer prime to p . Then $\text{III}_{\Omega_{div}^1}(F, \text{PGL}_n) = 1$.*

It is also a corollary of Corollary 6.4.(1). Similarly we have the following statement in view of the Merkurjev-Suslin’s theorem on the characterization of reduced norms of squarefree central simple algebra [G-Sz, Thm. 8.9.1].

Corollary 6.6. *(Inner type A) Let A be a central simple k -algebra with squarefree index prime to p . Then $\text{III}_{\Omega_{div}^1}(F, \text{SL}_1(A)) = 1$.*

Let us explain our way to prove it.

Proof. We write $A = M_r(D)$ for a division F -algebra D . We have $H^1(F, \text{SL}_1(A)) = F^\times / \text{Nrd}(A^\times)$ and as noticed in (2) of §6.1, the answer to Serre’s injectivity question is positive for $\text{SL}_1(A)$ -torsors. According to [G2, prop. 2.4], $\text{SL}_1(A) = \text{SL}_r(D)$ is retract F -rational. Theorem 6.2.(3) applies and yields $\text{III}_{\Omega_{div}^1}(F, \text{SL}_1(A)) = 1$. \square

Remark 6.7. Assume that K is a p -adic field and that A is of index n prime to p . Parimala-Preeti-Suresh have proven that $\text{III}_{\Omega_{div}^1}(F, \text{SL}_1(A)) = 1$ [P-P-S, Thm. 1.1]. If the index divides 4, we know that the F -group $\text{SL}_1(A)$ is not retract F -rational [Me].

Corollary 6.8. *Let A be a central simple F -algebra of degree $2n \geq 4$ equipped with an orthogonal involution σ . We assume that $p \neq 2$ and consider the special orthogonal F -group $O^+(A, \sigma)$ [K-M-R-T, §23]. Then we have $\text{III}_{\Omega_{div}^1}(F, O^+(A, \sigma)) = 1$.*

In the special case of $\text{SO}(q)$ this was known by applying [P-S, thm 4.3.(i)] to SO_{2n} and by inspecting the proof to check that only divisorial valuations are used; if q is isotropic we have furthermore that $\text{III}_{\Omega_{div}^1}(F, \text{Spin}(q)) = 1$ by using the fact that the map $H^1(F, \text{Spin}(q)) \rightarrow H^1(F, \text{SO}(q))$ has trivial kernel. We present to the proof of Corollary 6.8.

Proof. It uses the answer to Serre's injectivity question is positive for torsors; more precisely this is [Bl, prop. 3.6]. Next the F -group $O^+(A, \sigma)$ is F -rational in view of the Cayley parametrization [W, p. 599] so that Theorem 6.2.(3) applies. \square

In the case of tori, Serre's injectivity question has a positive answer in view of the classical corestriction-restriction argument.

Corollary 6.9. *Let T be an F -torus which is split by a prime to p Galois extension. Assume that T is a retract rational F -variety. We have $\text{III}_{\Omega_{div}^1}(F, T) = 1$.*

APPENDIX A. TORSION PRIMES

Let k be a field. If X is a non-empty k -scheme locally of finite type, we denote by $\text{Ind}_k(X)$ its index, i.e. the g.c.d. of the degrees of the finite field extensions k' of k such that $X(k') \neq \emptyset$. In other words $\text{Ind}_k(X)$ is the smallest positive degree of a 0-cycle on X . If k'/k is a finite field extension, it satisfies the divisibility properties $\text{Ind}_k(X) \mid \text{Ind}_{k'}(X_{k'}) \mid [k' : k] \text{Ind}_k(X)$.

Let M be a k -group of multiplicative type and let $\alpha \in H_{\text{fppf}}^i(k, M)$ with $i \geq 1$. The group $H_{\text{fppf}}^i(k, M)$ is torsion and the period of α is its exponent; it is denoted by $\text{per}_k(\alpha)$. Its index $\text{Ind}_k(\alpha)$ is the g.c.d. of the degrees of the finite field extensions k' of k such that $\alpha_{k'} = 0$. It satisfies similarly the above divisibility properties, i.e. $\text{Ind}_k(\alpha) \mid \text{Ind}_{k'}(\alpha_{k'}) \mid [k' : k] \text{Ind}_k(\alpha)$ by using the corestriction maps as defined in [G1, §0.4].

In the case of $\text{Br}(k) = H^2(k, \mathbb{G}_m)$, this is related with the index of central simple algebras, see [G-Sz, Prop. 4.5.1].

Lemma A.1. (1) *Assume that $\text{per}_k(\alpha) = l^r$ for a prime l . Then $\text{Ind}_k(\alpha)$ is a power of l .*

(2) *The period $\text{per}_k(\alpha)$ divides $\text{Ind}_k(\alpha)$ and these numbers share the same prime divisors.*

Proof. (1) We assume first that l is invertible in k .

Case $i = 1$ and M is finite. It is harmless to assume that M is l -primary and in particular étale. We see $\alpha \in H_{\text{fppf}}^1(k, M)$ as the class of an M -torsor X . Since M is

étale of degree l^a so is X . Then the identity point of $X(k[X])$ provides an effective (separable) 0-cycle of degree l^a on X .

Case $i = 1$ and M is a torus. By using the exact sequence $1 \rightarrow {}_l M \rightarrow M \xrightarrow{\times l^r} M \rightarrow 1$, we see that α comes from a class $\beta \in H_{\text{fppf}}^1(k, {}_l M)$. The preceding case shows that $\text{Ind}_k(\beta)$ is a power of l and so is $\text{Ind}_k(\alpha)$.

Case $i = 1$. We have an exact sequence $1 \rightarrow T \rightarrow M \xrightarrow{f} N \rightarrow 1$ where N is finite and T is a k -torus. From the first case, we know that $\text{Ind}_k(f_*\alpha)$ is an l -power, and more precisely that there are finite separable field extensions k_i 's of k killing $f_*\alpha$ such that $\text{g.c.d.}([k_i : k])$ is an l -power. For each i the preceding case provides separable field extensions $l_{i,j}$'s of k_i killing α_{k_i} (which arises from a class of $H^1(k_i, T)\{l\}$) and such that $\text{g.c.d.}([l_{i,j} : k_i])$ is an l -power l^{e_i} . Without loss of generality we can assume that we deal with $l_{i,j}$'s for $i = 1, \dots, n$ and $j = 1, \dots, m$.

The $l_{i,j}/l$'s kill α and we claim that $\text{g.c.d.}([l_{i,j} : k])$ is an l -power. Let $q \neq l$ be a prime dividing $[l_{i,j} : k] = [l_{i,j} : k_i][k_i : k]$ for all i, j . Then q divides $l^{e_i}[k_i : k]$ so divides each $[k_i : k]$ which is a contradiction. Thus $\text{Ind}_k(\alpha)$ is a power of l .

$i \geq 2$. It goes by the classical shifting argument. Let K/k be a finite separable such that $\alpha_K = 0$ and consider the exact sequence $1 \rightarrow M \rightarrow R_{K/k}(M) \rightarrow Q \rightarrow 1$ of k -groups of multiplicative type. Since $H_{\text{fppf}}^i(k, R_{K/k}(M)) = H_{\text{fppf}}^i(K, M)$, α comes from a class $\gamma \in H_{\text{fppf}}^{i-1}(k, Q)\{l\}$ with respect to the boundary map $\partial : H_{\text{fppf}}^{i-1}(k, Q) \rightarrow H_{\text{fppf}}^i(k, M)$. By induction on i , we obtain that $\text{Ind}_k(\gamma)$ is a power of l and so is $\text{Ind}_k(\alpha)$.

We come now to the case when $l = p$ is the characteristic of k .

Case M is finite. We can assume that M is p -primary so that $M(\bar{k}) = 1$. Since $H^i(\text{Gal}(\bar{k}/k^{\text{perf}}), M(\bar{k})) \xrightarrow{\sim} H_{\text{fppf}}^i(k^{\text{perf}}, M)$, it follows that $\alpha_{k^{\text{perf}}} = 1$. There exists then a finite purely inseparable field extension of k which kills α . Thus $\text{Ind}_k(\alpha)$ divides a power of p .

General case. The above arguments (by exact sequences and shifting) yield the torus case and the general case.

(2) For each finite field extension k'/k , we have a corestriction map $\text{Cor}_k^{k'} : H^i(k', M) \rightarrow H^i(k, M)$ which satisfies $\text{Cor}_k^{k'} \circ \text{Res}_k^{k'} = \times [k' : k]$, see [G1, §0.4]. If k' splits α , it follows that $[k' : k]\alpha = 0$ so that $[k' : k]$ divides $\text{per}_k(\alpha)$. It follows that $\text{per}_k(\alpha)$ divides $\text{Ind}_k(\alpha)$. It remains to show that a prime divisor l of $\text{Ind}_k(\alpha)$ divides $\text{per}_k(\alpha)$. We consider the prime decomposition $\text{per}_k(\alpha) = p_1^{r_1} \dots p_c^{r_c}$ and write $\alpha = \alpha_1 + \dots + \alpha_c$ with $\text{per}_k(\alpha_i) = p_i^{r_i}$. Given a finite field extension k'/k , k'/k kills α iff it kills each α_i . It follows that $\text{Ind}_k(\alpha)$ is the g.c.d. of the $\text{Ind}_k(\alpha_i)$'s. We are then reduced to the case $\text{per}_k(\alpha) = l^r$ for a prime l which has been handled in (1).

□

Remark A.2. By inspection of the proof, in (1), if l is invertible in k , the index is realized by separable field extensions. A contrario, if $l = p$ is the characteristic of k , the index is realized with a single purely inseparable field extension.

Given a semisimple algebraic group G defined over k , we remind the reader that we can attach to G a list of torsion primes [G4, §4.8, 5.1] which depends only of the isogeny class of G and which extend Serre's definition [Se, §2.4]. The index $\text{Ind}_k(G)$ of G is the g.c.d. of the degrees of the finite splitting fields extensions of G , that is, making G split. In other words, if G_0 is Chevalley form of G , this is the index of the $\text{Aut}(G_0)$ -torsor $\text{Isom}(G_0, G)$.

Lemma A.3. (1) *If G is almost k -simple, the primes dividing $\text{Ind}_k(G)$ are torsion primes.*

(2) *If E is a G -torsor, the primes dividing $\text{Ind}_k(E)$ are torsion primes.*

(3) *Let X be the variety of Borel k -subgroups of G . Then the primes dividing $\text{Ind}_k(X)$ are torsion primes.*

Proof. (1) This follows of [T2, prop. A1].

(2) We assume first that G is semisimple adjoint and quasi-split. Given $\gamma \in H^1(k, G)$, Steinberg's theorem provides a maximal k -torus T such that γ belongs to the image of $H^1(k, T) \rightarrow H^1(k, G)$ [St, Thm. 11.1]. Let d be the period of an antecedent of γ . Then γ belongs to the image $H^1(k, {}_d T) \rightarrow H^1(k, G)$ so that $\text{Ind}_k(\gamma)$ divides a power of d according to Lemma A.1. In view of a result of Harder [G4, Thm. 5.1.2], the primes dividing d are torsion primes. Thus the primes dividing $\text{Ind}_k(\gamma)$ are torsion primes.

The next case is when G is adjoint. We denote by G^{qs} the quasi-split form of G . Since G is inner form of G^{qs} , the first case shows that there exists finite field extensions k_i 's of k which quasi-split G and such that the prime divisors of $\text{g.c.d.}([k_i : k])$ are torsion primes. Let E be a G -torsor. For each i , the first case provides finite field extensions $l_{i,j}$'s of k_i which split E_{k_i} and such that the prime divisors of $\text{g.c.d.}([l_{i,j} : k_i])$ are torsion primes. It follows that the $l_{i,j}$'s split E and that $\text{g.c.d.}([l_{i,j} : k])$ is a product of torsion primes (the argument is similar with that in the proof of Lemma A.3.(2)).

For the general case, we consider the exact sequence $1 \rightarrow \mu \rightarrow G \rightarrow G_{ad} \rightarrow 1$. Let E be a G -torsor. The preceding case provides finite field extensions k 's of k which split $E_{ad} = E/\mu$ and such that the prime divisors of $\text{g.c.d.}([k_i : k])$ are torsion primes. For each i , there exists a μ -torsor F_i such that $E_{k_i} \cong F_i \wedge^\mu G$. Since the order of μ is a product of torsion primes, for each i , there are finite field extensions $l_{i,j}$'s of k_i which split F_{k_i} (and then a fortiori E_{k_i}) and such that the prime divisors of $\text{g.c.d.}([l_{i,j} : k_i])$ are torsion primes (Lemma A.1). It follows that the $l_{i,j}$'s split E and that $\text{g.c.d.}([l_{i,j} : k])$ is a product of torsion primes.

(3) It is proved along the same lines as (2). □

APPENDIX B. 0-CYCLES FOR TWISTED FLAG VARIETIES

We can also discuss the following variant of the main result of [G-P, Thm. 5.4]. The setting is that of §4.1. The ring B is a complete discrete valuation ring with fraction field K , residue field k and uniformizing parameter t ; the field F is a one-variable function field over K .

Theorem B.1. *Let G be a semisimple F -group. Let Z be a twisted flag F -variety of G such that $Z(F_v) \neq \emptyset$ for all discrete valuations of F arising from normal models of X .*

(1) *Z admits a 0-cycle of degree p^c .*

(2) *Assume that p is not a torsion prime of G and that G becomes quasi-split after a prime to p Galois extension of F . Then the F -variety Z admits a 0-cycle of degree 1.*

Proof. Without loss of generality we can assume that G is adjoint. Let G_0 be the Chevalley form of G and let (B_0, T_0) be a Killing couple for G_0 and let Δ_0 be the associated Dynkin diagram. The variety Z is a form of the variety $\text{Par}_I(G_0)$ of parabolic subgroups of type I where $I \subset \Delta_0$ is stable under the star action defined by the $\text{Aut}(G_0)$ -torsor $Q = \text{Isom}(G_0, G)$. In particular Q admits a reduction Q_I to the stabilizer $\text{Aut}_I(G_0)$ for the action $\text{Aut}(G_0)$ on Δ_0 through the morphism $\text{Aut}(G_0) \rightarrow \text{Out}(G_0) \xrightarrow{\sim} \text{Aut}(\Delta_0)$. Furthermore Z is isomorphic to ${}^Q\text{Par}_I(G_0)$.

(1) For each prime $l \neq p$, we have to show that there exists a finite prime to p field extension L/F such that $Z(L) \neq \emptyset$. We apply now Theorem 3.4.(2) to the \mathbb{Z} -group scheme $\text{Aut}_I(G_0)$. It provides a finite \mathbb{Z} -subgroup S^\sharp of degree l^m of $\text{Aut}_I(G_0)$ and a finite extension L of F of degree prime to l such that $Q_{I,L}$ admits a reduction to S^\sharp , that is, there exists a S^\sharp -torsor E over L such that $E \wedge^{S^\sharp} \text{Aut}_I(G_0) \cong Q_I$. It is harmless to replace L by F . We have that S_B^\sharp is finite étale of degree l^m .

The finite flat \mathbb{Z} -group scheme S^\sharp admits a faithful representation $S^\sharp \hookrightarrow \text{GL}_{N,\mathbb{Z}}$ [Br-T, §1.4.5] and the quotient $\text{GL}_{N,\mathbb{Z}}/S^\sharp$ is representable by an affine \mathbb{Z} -scheme [D-G, §III.2.6]. According to [G-P, Lemma 5.1], there exists a regular model \mathfrak{X} of X and a strict normal crossing divisor D containing the irreducible components of Y such that E extends to a $\mathfrak{X} \setminus D$ -torsor \mathfrak{E} under S^\sharp . We put $\mathfrak{Q}_I = \mathfrak{E} \wedge^{S^\sharp} \text{Aut}_I(G_0)$ and consider the $\mathfrak{X} \setminus D$ -group scheme $\mathfrak{G} = {}^{\mathfrak{Q}_I}G_0$ with generic fiber G .

If $y \in Y$ is of codimension one, by hypothesis, $Z(F_y)$ is not empty. We therefore look at a closed point $P \in Y$ and pick a height one prime \mathfrak{p} in \widehat{R}_P that contains t . It defines a branch of Y at P lying on some irreducible component Y_1 of Y .

We consider the local ring $A = R_{\mathfrak{p}}$ of \mathfrak{X} at P and denote by A_D its localization at D . Since S_B^\sharp is finite étale of degree prime to p , $H^1(A_D, S^\sharp)$ consists in loop torsors as defined in [G5, §2.3, lemma 2.3.(2)], i.e. those arising from cocycles related to tame Galois covers of A_D . It follows that the A_D -torsor Q_I is a loop $\text{Aut}_I(G)$ -torsor [G5, lemma 2.3.(3)], so that $\mathfrak{G} \times_{\mathfrak{X} \setminus D} A_D$ is by definition a loop reductive group scheme.

Let $F_{P,v}$ be the completion of the field F_P for the valuation associated to the blow-up of $\text{Spec}(A)$ at its closed point. Our assumption states in particular that $Z(F_{P,v}) \neq \emptyset$, that is, $G_{F_{P,v}}$ admits a parabolic subgroup of type I . According to [G5, th. 4.1, (iii) \implies (i)], $\mathfrak{G} \times_{\mathfrak{x} \setminus D} A_D$ admits a parabolic subgroup of type I . A fortiori G_{F_P} admits a parabolic subgroup of type I so that $Z(F_P) \neq \emptyset$. In view of [G-P, Cor. 4.5], we conclude that $Z(F)$ is not empty.

(2) After (1), it remains to deal with the case $l = p$. In the proof of Theorem 6.2, we have seen that our assumption on p implies that G becomes split after a prime to p extension L/F . It follows that $Z(L) \neq \emptyset$. \square

In several cases, we know that is the F -variety Z admits a 0-cycle of degree 1, then it admits F -point. This provides the following improvement of [G-P] which is close of Corollary 6.4.

Corollary B.2. *Let G be a semisimple F -group and assume that p is not torsion prime of G and that G becomes quasi-split after a prime to p Galois extension. Let Z be the variety of Borel subgroups of G . Assume that G has no factor of type E_8 . If $Z(F_v) \neq \emptyset$ for all discrete valuations of F arising from normal models of X , then $Z(F) \neq \emptyset$.*

An important case is the following.

Corollary B.3. *Assume that $p \neq 2$.*

(1) *Let q be a regular quadratic form of rank ≥ 3 . If q_{F_v} is isotropic for all discrete valuations of F arising from normal models of X , then q is isotropic.*

(2) *Let A be a central simple F -algebra of degree $2n$ equipped with an orthogonal involution σ . If σ_{F_v} is hyperbolic for all discrete valuations of F arising from normal models of X , then σ is hyperbolic.*

Proof. (1) In this case, we apply the result to the projective quadric Z which is projective homogeneous under $\text{SO}(q)$. Theorem B.1.(1) shows that Z admits a 0-cycle of degree p^c which is odd. It follows that q become isotropic on some finite field extension F'/F of odd degree. According to Springer's theorem q is isotropic.

(2) If A is split, σ is adjoint to a regular quadratic F -form q of dimension $2n$. Witt's decomposition and (1) yield that local hyperbolicity implies hyperbolicity for the quadratic form q .

If $A \cong M_r(D)$ is not split, we consider the F -variety of isotropic ideals of (A, σ) of dimension $2n$. This is a projective G -variety for the F -group $G = \text{SO}(A, \sigma)$. Theorem B.1.(1) shows that Z admits a 0-cycle of degree p^c which is odd. It follows that σ become hyperbolic on some finite field extension F'/F of odd degree. Bayer-Lenstra's theorem (see [K-M-R-T, Cor. 6.15]) enables us to conclude that σ is hyperbolic. \square

In general flag varieties may have a 0-cycle of degree one but no rational point [P]. However there are a few cases where we know that the existence of a 0-cycle of degree

is equivalent to the existence of a rational point. The paper [G-S] presents the case of some flag variety in type E_7 (resp. E_6) and we have then a local-global principle as in Corollary B.3 when $p \neq 2$ (resp. $p \neq 3$).

Finally it would be interesting to investigate local-global principles for 0-cycles on projective G -homogenous varieties in the spirit of the paper [C-T-H-H-K-P-S1] which deals mostly with the patching viewpoint. For us, after localization at a prime l , we are interested in local-global principles for 0-cycles for closed points (resp. closed separable points) of degree prime to l with respect with completions for divisorial valuations.

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