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LOCAL-GLOBAL PRINCIPLE FOR OVER SEMIGLOBAL FIELDS

PHILIPPE GILLE AND RAMAN PARIMALA

ABSTRACT. We compare different local-global principles for torsors under a reductive group G defined over a semiglobal field F . In particular if the F -group G is a retract rational F -variety, we prove that the local global principle holds for the completions with respect to divisorial valuations of F .

Keywords: Local-global principle, curves over local fields, torsors, reductive groups.

MSC 2000: 11G99, 14G99, 14G05, 11E72, 11E12, 20G35

1. INTRODUCTION

The theme is the arithmetic theory of torsors over a semiglobal field. It had been initiated by Hartmann, Harbater and Krashen [H-H-K] and developed further [C-T-O-H-H-K-P-S2, C-T-P-S, H-H-K2, P-S, R-S].

Let T be a complete discrete valuation ring with fraction field K and residue field k . Let $p \geq 1$ be the characteristic exponent of k . Let X be a smooth, projective, geometrically integral curve over K . Let $F = K(X)$ be the function field of X and let t be a uniformizing parameter of T . Let \mathfrak{X} be a normal model of F , i.e. a normal connected projective T -curve with function field F . We denote by Y the closed fiber of \mathfrak{X} and fix a separable closure F_s of F .

Given an affine algebraic F -group G , we can study the local-global principle for G -torsors with respect to the infinite set of overfields F_P that arise from completions at the points P of Y . This gives rise to the Tate-Shafarevich set

$$\text{III}_{\mathfrak{X}}(F, G) = \ker \left(H^1(F, G) \rightarrow \prod_{P \in Y} H^1(F_P, G) \right);$$

it is denoted simply by $\text{III}_X(F, G)$ in [H-H-K2]. Using Lipman's resolution of singularities permits to consider the inductive limit

$$\text{III}_{\text{patch}}(F, G) = \varinjlim_{\mathfrak{X}} \text{III}_{\mathfrak{X}}(F, G)$$

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taken on the models (or equivalently on the regular models). On the other hand, we consider the completions associated to the set $\Omega_{F,div}$ of divisorial valuations on F arising of various regular projective models over T of the curve X . It gives rise to the

$$\mathbb{I}\mathbb{I}\mathbb{I}_{div}(F, G) = \ker\left(H^1(F, G) \rightarrow \prod_{v \in \Omega_{F,div}} H^1(F_v, G)\right);$$

In view of [H-H-K2, prop. 8.2], we have the following inclusion

$$\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(F, G) \subseteq \mathbb{I}\mathbb{I}\mathbb{I}_{div}(F, G).$$

The main result of the paper is to show (under some condition on p with respect to G), that the two Tate-Shafarevich sets coincide (Thm. 3.4). If furthermore G is a F -rational variety (or even retract rational F -variety), we have that $\mathbb{I}\mathbb{I}\mathbb{I}_{patch}(F, G) = 1$ according to Harbater-Krashen-Hartmann [H-H-K2, Thm. 9.1] (see [K, Thm. 2.2.4] for the extension to the retract rational case) so that $\mathbb{I}\mathbb{I}\mathbb{I}_{div}(F, G) = 1$ in this case. It means that if G is a retract rational F -variety that the local-global principle holds for G -torsors with respect to divisorial valuations on F .

This was known for certain classical groups [R-S, P-S]. As in [G-P], the proof uses the technique of loop torsors which requires to deal with group schemes defined over T . To do so, for a given Chevalley group G_0 over \mathbb{Z} , we explain in the first section why the $G_0 \rtimes \text{Aut}(G_0)$ -torsors permit to classify couples (E, G) where G is a form of G_0 and E a G -torsor.

Acknowledgements.

2. PAIRS OF A GROUP SCHEME TOGETHER WITH A TORSOR

The following is a complement to the ‘‘sorites’’ of [G1]. We work in the setting of fppf sheaves on a base scheme S . We consider the groupoid \mathfrak{P} whose objects are pairs (E, G) where G is a fppf S -sheaf in groups and E a G -torsor; the morphisms between two pairs (E_1, G_1) and (E_2, G_2) are pairs of isomorphisms $f = (f', f'')$ where $f'' : G_1 \xrightarrow{\sim} G_2$ is an S -isomorphism of S -sheaves in groups and $f' : E_1 \xrightarrow{\sim} E_2$ an S -isomorphism of sheaves such that the following diagram commutes

$$\begin{array}{ccc} E_1 \times_S G_1 & \longrightarrow & E_1 \\ \downarrow f' \times f'' & & \downarrow f' \\ E_2 \times_S G_2 & \longrightarrow & E_2. \end{array}$$

Note that it induces an isomorphism $f_* : {}^{E_1}G_1 \xrightarrow{\sim} {}^{E_2}G_2$ between the associated twisted group sheaves.

Lemma 2.1. *Let G be an S -sheaf in groups and let E be a G -torsor over S . We denote by ${}^E G$ the twisted S -sheaf of G by E by inner automorphisms.*

(1) *We have a natural exact sequence of fppf S -sheaves in groups*

$$1 \rightarrow {}^E G \rightarrow \underline{\text{Aut}}_{\mathfrak{P}}(E, G) \rightarrow \underline{\text{Aut}}(G) \rightarrow 1$$

which is locally split for the flat topology.

(2) *If $E = G$ the sequence above splits and the S -functor $\underline{\text{Aut}}_{\mathfrak{P}}(G, G)$ is isomorphic to $G \rtimes \underline{\text{Aut}}(G)$.*

Proof. We prove the two assertions in the same time. The projection $\underline{\text{Aut}}_{\mathfrak{P}}(E, G) \rightarrow \underline{\text{Aut}}(G)$ is a homomorphism of S -sheaves in groups. On the other hand we have a monomorphism ${}^E G = \text{Aut}_G(E) \rightarrow \underline{\text{Aut}}_{\mathfrak{P}}(E, G)$, $f'' \mapsto (f'', id)$. The fact that the sequence is exact at the middle term is the fact [Gd, cor. 2.3.8].

We consider now the case $E = G$. In this case the sequence above splits by $f'' \mapsto (f'', f'')$ hence an isomorphism $G \rtimes \underline{\text{Aut}}(G) \xrightarrow{\sim} \underline{\text{Aut}}(G, G)$. Coming back to the general case it implies that the projection $\underline{\text{Aut}}_{\mathfrak{P}}(E, G) \rightarrow \underline{\text{Aut}}(G)$ is an epimorphism of S -sheaves. \square

Given a fppf S -sheaf in groups G , we can apply [G1, lemme 2.6.3] to express the cohomology

$$(2.1) \quad H^1(S, G \rtimes \underline{\text{Aut}}(G)) \xrightarrow{\sim} \bigsqcup_{[G'] \in H^1(S, \underline{\text{Aut}}(G))} H^1(S, G') / \underline{\text{Aut}}(G')(S).$$

by using the fact that $H^1(S, \underline{\text{Aut}}(G))$ classifies the S -forms of G . The action on $\underline{\text{Aut}}(G')(S)$ on $H^1(S, G')$ is the natural one so that the map $H^1(S, G') \rightarrow H^1(S, G') / \underline{\text{Aut}}(G')(S)$ has trivial kernel so that the composite $H^1(S, G') \rightarrow H^1(S, G \rtimes \underline{\text{Aut}}(G))$ has trivial kernel. This map is nothing but the composition

$$H^1(S, G') \rightarrow H^1(S, G' \rtimes \underline{\text{Aut}}(G')) \xrightarrow[\sim]{\text{Torsion bijection}} H^1(S, G \rtimes \underline{\text{Aut}}(G))$$

where the torsion bijection is with respect to the $\underline{\text{Aut}}(G)$ -torsor $\underline{\text{Isom}}(G, G')$. Summarizing the decomposition (2.1) shows that $H^1(S, G \rtimes \underline{\text{Aut}}(G))$ encodes the classes of torsors for all twisted forms of G . The map $\text{int} : G \rightarrow \underline{\text{Aut}}(G)$ extends to

$$u : G \rtimes \underline{\text{Aut}}(G) \longrightarrow \underline{\text{Aut}}(G) \rtimes^c \underline{\text{Aut}}(G) \xrightarrow{\sim} \text{Aut}(G) \times \text{Aut}(G),$$

$$(g, f) \quad \mapsto \quad (\text{int}(g), f)$$

$$(f_1, f_2) \quad \mapsto \quad (f_1 f_2, f_2)$$

where c stands for the conjugacy action. In terms of objects classified by the respective torsors, u_* applies a pair (E', G') to $({}^{E'}G', G')$

3. PATCHING TECHNIQUES

We recall the setting. Let T be a complete (excellent) discrete valuation ring with fraction field K , residue field k and uniformizing parameter t . Let F be a one-variable function field over K and let \mathfrak{X} be a normal model of F , i.e. a normal connected projective T -curve with function field F . We denote by Y the closed fiber of \mathfrak{X} and fix a separable closure F_s of F .

For each point $P \in Y$, let R_P be the local ring of \mathfrak{X} at P ; its completion \widehat{R}_P is a domain with fraction field denoted by F_P .

For each subset U of Y that is contained in an irreducible component of Y and does not meet the other components, we define $R_U = \bigcap_{P \in U} R_P \subset F$. We denote by \widehat{R}_U the t -adic completion of R_U . The rings R_U and \widehat{R}_U are excellent normal domains and we denote by F_U the fraction field of \widehat{R}_U [H-H-K3, Remark 3.2.(b)].

Each height one prime \mathfrak{p} in \widehat{R}_P that contains t defines a branch of Y at P lying on some irreducible component of Y . The t -adic completion $\widehat{R}_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$ of \widehat{R}_P at \mathfrak{p} is a complete DVR whose fraction field is denoted by $F_{\mathfrak{p}}$. The field $F_{\mathfrak{p}}$ contains also F_U if U is an irreducible open subset of Y such that $P \in \overline{U} \setminus U$. We have then a diagram of fields

$$\begin{array}{ccc} & F_{\mathfrak{p}} & \\ & \swarrow \quad \nwarrow & \\ F_P & & F_U. \end{array}$$

Setting 3.1. Let \mathcal{P} be a non-empty finite set of closed points of Y that contains all the closed points at which distinct irreducible components meet. Let \mathcal{U} be the set of connected components of $Y \setminus \mathcal{P}$ and let \mathcal{B} be the set of branches of Y at points of \mathcal{P} . This yields a finite inverse system of field $F_P, F_U, F_{\mathfrak{p}}$ (for $P \in \mathcal{P}$; $U \in \mathcal{U}$, $\mathfrak{p} \in \mathcal{B}$) where $F_P, F_U \subset F_{\mathfrak{p}}$ if \mathfrak{p} is a branch of Y at P lying in the closure of U .

Proposition 3.2. *Let $1 \rightarrow G \rightarrow \widetilde{G} \rightarrow J \rightarrow 1$ be an exact sequence of smooth T -group schemes where G is reductive and J is twisted constant. Let E_1 (resp. E_2) be a \widetilde{G} -torsor over F admitting a reduction to a finite étale T -subgroup of S_1 (resp. S_2) of \widetilde{G} with prime to p order.*

Assume that $E_{1,F_v} \cong E_{2,F_v}$ for any divisorial discrete valuation v on F . Then there exists a regular model Ω of F as above together with Y, \mathcal{P} , etc. such that $E_{1,F_P} \cong E_{2,F_P}$ for all $P \in \mathcal{P}$ and $E_{1,F_U} \cong E_{2,F_U}$ for all connected components U of $Y \setminus \mathcal{P}$.

Proof. We consider the F -scheme $Z = \text{Isom}_{\widetilde{G}}(E_1, E_2)$. Our assumption is that $Z(F_v) \neq \emptyset$ for all divisorial discrete valuations v on F .

Let V_i be an $S_{i,F}$ -torsor such that $V_i \wedge^{S_{i,F}} \tilde{G} \cong E_i$ for $i = 1, 2$. According to [G-P, lemma 5.1] there exists a regular proper model \mathfrak{X} of X and a strict normal crossing divisor D containing the irreducible components of Y such that V_i extends to a $\mathfrak{X} \setminus D$ -torsor \mathfrak{V}_i under S_i . We are given a closed point $P \in Y$. If y is of codimension one, by hypothesis, $Z(F_y)$ is not empty. We therefore look at a closed point $P \in Y$. We pick a height one prime \mathfrak{p} in \widehat{R}_P that contains t . It defines a branch of Y at P lying on some irreducible component Y_1 of Y .

We consider the local ring $A = R_P$ of \mathfrak{X} at P and denote by A_D its localization at D . Since $S_{i,T}$ is finite étale of degree prime to p , $H^1(A_D, S_i)$ consists in loop torsors as defined in [G2, §2.3, lemma 2.3.(2)], i.e. those arising from cocycles related to tame Galois covers of A_D . It follows that the A_D -torsors $\mathfrak{V}_1, \mathfrak{V}_2$ under \tilde{G} are loop (*ibid*, lemma 5.3.(3)).

Let $F_{P,v}$ be the completion of the field F_P for the valuation associated to the blow-up of $\text{Spec}(A)$ at its closed point. Our assumption states in particular that $Z(F_{P,v}) \neq \emptyset$. According to [G3, Thm. 4.5], the map

$$H_{loop}^1(A_D, \tilde{G}) \rightarrow H^1(F_{P,v}, \tilde{G})$$

is injective. It follows that $\text{Isom}_{\tilde{G}_{A_D}}(\mathfrak{V}_1, \mathfrak{V}_2)(A_D) \neq \emptyset$. A fortiori $Z(F_P) \neq \emptyset$. \square

Proposition 3.3. *Let G be a reductive F -algebraic group and let G_0 be the underlying Chevalley reductive \mathbb{Z} -group scheme. Let E be a G -torsor and assume that the $\text{Aut}(G_0)$ -torsor $\underline{\text{Isom}}(G_0, G)$ admits a reduction to a finite étale T -subgroup of $\text{Aut}(G_0)$ of prime to p order and that the $G_0 \rtimes \text{Aut}(G_0)$ -torsors (G, G) (resp. (E, G)) admits a reduction to a finite étale T -subgroup of $G_0 \rtimes \text{Aut}(G_0)$ of prime to p order. Then the two following assertions are equivalent:*

- (i) *There exists a regular proper model \mathfrak{X} of X with special fiber $Y = \mathfrak{X}_k$ such that for every point $y \in Y$, then $E(F_y) \neq \emptyset$.*
- (ii) *For each divisorial discrete valuation v on F , we have $E(F_v) \neq \emptyset$.*

Proof. The implication (i) \implies (ii) is a general fact, see [H-H-K-P, Thm. 3.4]. We prove then (ii) \implies (i) assuming that $E(F_v) \neq \emptyset$ for each discrete divisorial valuation v on F .

We shall apply Proposition 3.2 to the $G_0 \rtimes \text{Aut}(G_0)$ -torsors $E_1 = \underline{\text{Isom}}_{\mathfrak{p}}((G_0, G_0), (G, G))$ and $E_2 = \underline{\text{Isom}}_{\mathfrak{p}}((G_0, G_0), (E, G))$. By means of the splitting of $\text{Aut}_{\mathfrak{p}}(G_0, G_0) \rightarrow \text{Aut}(G_0)$, we observe that E_1 admits a reduction to a finite étale T -subgroup of $G_0 \rtimes \text{Aut}(G_0)$ of prime to p order.

Assume that $E_{1,F_v} \cong E_{2,F_v}$ for any divisorial discrete valuation v on F . Proposition 3.2 provides a regular model \mathfrak{Q} of F as above together with Y, \mathcal{P} , etc. such that $E_{1,F_P} \cong E_{2,F_P}$ for all $P \in \mathcal{P}$ and $E_{1,F_U} \cong E_{2,F_U}$ for all connected components U of $Y \setminus \mathcal{P}$. It follows that for every point $y \in Y$, then $E(F_y) \neq \emptyset$. \square

Theorem 3.4. *Let G be a reductive F -algebraic group and let G_0 be the underlying Chevalley reductive \mathbb{Z} -group scheme. Assume that p does not divide the order of the Weyl group W_0 of G_0 and that G becomes an inner form of G_0 after a Galois prime to p extension of F .*

(1) *We have $\text{III}_{\text{patch}}(F, G) = \text{III}_{\text{div}}(F, G)$;*

(2) *If G is a retract rational F -variety, then $\text{III}_{\text{div}}(F, G) = 1$.*

Proof. (1) We know that $\text{III}_{\text{patch}}(F, G) \subset \text{III}_{\text{div}}(F, G)$. To establish the converse inclusion, we are given a G -torsor E such that $E(F_v) \neq \emptyset$ for each divisorial discrete valuation v . We need to check the conditions for applying Proposition 3.3. We start with the isomorphism class $[G] \in H^1(F, \text{Aut}(G_0))$ by using the sequence $1 \rightarrow G_{0,ad} \rightarrow \text{Aut}(G_0) \rightarrow \text{Out}(G_0) \rightarrow 1$ where $\text{Out}(G_0)$ is a constant \mathbb{Z} -group scheme. Since $H^1(F, \text{Out}(G_0)) = \text{Hom}_{ct}(\text{Gal}(F_s/F), \text{Out}(G_0)(\mathbb{Z})) / \sim$, the image of $[G]$ is represented by a continuous homomorphism $u : \text{Gal}(F_s/F) \rightarrow \text{Out}(G_0)(\mathbb{Z})$ whose kernel defines a Galois extension F'/F which is the minimal subextension of F_s making G an inner form. Our assumption implies that $\text{Gal}(F'/F)$ is of prime to p degree and so that $\Gamma = u(\text{Gal}(F_s/F))$ is a finite subgroup of $\text{Out}(G_0)(\mathbb{Z})$ of order prime to p . We denote by $\text{Aut}_\Gamma(G_0) = \text{Aut}(G_0) \times_{\text{Out}(G_0)} \Gamma$; by construction $[G]$ arises from a cohomology class $\gamma \in H^1(F, \text{Aut}_\Gamma(G_0))$. According to [C-G-R, Thm. 1.2] applied to the ring \mathbb{Z} , there exists a finite \mathbb{Z} -subgroup S of $\text{Aut}_\Gamma(G_0)$ such that the map $H^1(F, S) \rightarrow H^1(F, \text{Aut}_\Gamma(G_0))$ is onto. Further by the same statement, we can take $S \subset \text{Aut}_\Gamma(G_0, T_0)$ where T_0 is a maximal split \mathbb{Z} -torus of G_0 fitting in an exact sequence $1 \rightarrow {}_n(T_0) \rightarrow S \rightarrow \text{Aut}_\Gamma(G_0, T_0)/T_0 \rightarrow 1$ where n divides the square of the order of the finite group $\text{Aut}_\Gamma(G_0, T_0)/T_0$. We have a sequence $1 \rightarrow W_0 \rightarrow \text{Aut}_\Gamma(G_0, T_0)/T_0 \rightarrow \Gamma \rightarrow 1$ so that $\text{Aut}_\Gamma(G_0, T_0)/T_0$ is finite constant of prime to p order. It implies that ${}_n(T_0)$ is a finite diagonalisable \mathbb{Z} -group scheme whose base change to T is étale. Altogether S_T is a finite étale T -subgroup of $\text{Aut}_\Gamma(G_0)_T$ of prime to p order such that γ admits a reduction to S_T . A fortiori, $[G] \in H^1(F, \text{Aut}(G_0))$ admits a reduction to S_T .

The same method works as well for the class $[(E, G)] \in H^1(F, G_0 \times \text{Aut}(G_0))$ by using the exact sequence $1 \rightarrow G_0 \rtimes_{\mathbb{Z}} G_{0,ad} \rightarrow G_0 \rtimes_{\mathbb{Z}} \text{Aut}(G_0) \rightarrow \text{Out}(G_0) \rightarrow 1$ and provides a finite étale T -subgroup of S' of $(G_0 \rtimes_{\mathbb{Z}} \text{Aut}(G_0))_T$ of prime to p -order such that the $G_0 \rtimes_{\mathbb{Z}} \text{Aut}(G_0)$ -torsor over $\text{Spec}(F)$ arises from a S' -torsor.

Proposition 3.3, (ii) \implies (i), applies and shows that there exists a regular proper model \mathfrak{X} of X with special fiber $Y = \mathfrak{X}_k$ such that for every point $y \in Y$, then $E(F_y) \neq \emptyset$. In other words $[E]$ belongs in $\text{III}_{\mathfrak{X}}(F, G)$ and a fortiori to $\text{III}_{\text{patch}}(F, G)$.

(2) If G is a retract rational F -variety, we have that $\text{III}_{\text{patch}}(F, G) = 1$ [H-H-K2, Thm. 9.1] (see [K, Thm. 2.2.4] for the extension to the retract rational case). The first part of the statement enables us to conclude that $\text{III}_{\text{div}}(F, G) = 1$ in this case. \square

Remarks 3.5. (a) If G is a torus, the condition is that the minimal splitting extension of T is of prime to p degree. Note that for norm one tori, the main theorem had been established by Sumit Chandra Mishra [M, Thm. 5.1].

(b) The assumptions are not p are not sharp. For example, if G is semisimple of type G_2 , the statement excludes the primes 2 and 3 but excluding 2 is enough. The point is that we can deal with the subgroup $\mu_2 \times \mu_2 \times \mathbb{Z}/2\mathbb{Z} \subset S$ in the proof.

Corollary 3.6. *Let G be a semisimple quasi-split F -group which is quasi-split by a Galois prime to p field extension. Assume that p does not divide the order of W_0 . In each case*

(i) G is split;

(ii) G is simply connected or adjoint;

we have $\text{III}_{\text{div}}(F, G) = 1$.

Proof. Such a F -group is a rational K -variety so the result follows from Theorem 3.4. \square

This applies as well to many other groups in view of the rationality results of Chernousov-Platonov [C-P] and others. This is the case for example for an isotropic k -group of type F_4 .

4. OTHER TATE-SCHAFAREVICH SETS

Given an affine algebraic F -group G , we can also study the local-global principle for G -torsors with respect to the following set of completions of F :

(1) Completions associated to the set Ω_F of all non-trivial T -valuations on F (i.e., those whose valuation ring contains T).

(2) Completions associated to the set Ω_F^1 of all non-trivial rank one T -valuations on F .

(3) Completions associated to the set $\Omega_{F, \text{dvr}}^1$ of all non-trivial rank one discrete T -valuations on F .

We have $\Omega_{F, \text{div}} \subset \Omega_{F, \text{dvr}}^1 \subset \Omega_{F, \text{dvr}}^1 \subset \Omega_F$. This leads to three Tate-Shafarevich sets as for example $\text{III}_{\text{dvr}}^1(F, G) = \ker\left(H^1(F, G) \rightarrow \prod_{v \in \Omega_{\text{dvr}}} H^1(F_v, G)\right)$ which is denoted simply by $\text{III}^1(F, G)$ in the references [C-T-P-S, H-H-K2]. We have the following inclusions

$$\text{III}_{\Omega_F}^1(F, G) \subset \text{III}_{\Omega_F^1}^1(F, G) \subset \text{III}_{\text{patch}}^1(F, G) \subset \text{III}_{\text{dvr}}^1(F, G) \subset \text{III}_{\text{div}}^1(F, G).$$

According to [H-H-K-P, Thm. 3.4], we have $\text{III}_{\Omega_F}^1(F, G) = \text{III}_{\Omega_F^1}^1(F, G) = \text{III}_{\text{patch}}^1(F, G)$.

The fact that $\text{III}_{\Omega_F}^1(F, G) = \text{III}_{\Omega_F^1}^1(F, G) = \text{III}_{\text{dvr}}^1(F, G)$ is known in two situations requiring the residue field k to be algebraically closed [H-H-K2, Thm. 8.10, cases (i) and (iii)].

Coming back to the present paper, the of Theorem 3.4 has then the following consequence.

Corollary 4.1. *under the assumptions of Theorem 3.4, we have then*

$$\mathbb{H}_{\Omega_F}^1(F, G) = \mathbb{H}_{\Omega_F^1}^1(F, G) = \mathbb{H}_{patch}^1(F, G) = \mathbb{H}_{div}^1(F, G) = \mathbb{H}_{div}^1(F, G)$$

and those sets are trivial when G is a retract F -rational variety.

The constant case. This the case when we deal with a reductive T -group scheme G . In this case, in view of [H-H-K2, Thm. 8.10, case (ii)] (see also of [C-T-O-H-H-K-P-S2, Thm. 3.2]), there is no need of any condition on the characteristic exponent p of the residue field k for having $\mathbb{H}_{patch}^1(F, G) = \mathbb{H}_{div}^1(F, G)$. Inspection of the proof shows that only divisorial valuations of F are involved so that $\mathbb{H}_{patch}^1(F, G) = \mathbb{H}_{div}^1(F, G)$. This applies in particular when G_F is a retract rational F -variety; since $\mathbb{H}_{patch}^1(F, G) = 1$, we have that $\mathbb{H}_{div}^1(F, G) = 1$ in characteristic free. The interest of our result is then mostly for the non-constant case.

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