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#### LOCAL-GLOBAL PRINCIPLE FOR OVER SEMIGLOBAL FIELDS

#### PHILIPPE GILLE AND RAMAN PARIMALA

ABSTRACT. We compare different local-global principles for torsors under a reductive group G defined over a semiglobal field F. In particular if the F-group G is a retract rational F-variety, we prove that the local global principle holds for the completions with respect to divisorial valuations of F.

Keywords: Local-global principle, curves over local fields, torsors, reductive groups.

MSC 2000: 11G99, 14G99, 14G05, 11E72, 11E12, 20G35

#### 1. INTRODUCTION

The theme is the arithmetic theory of torsors over a semiglobal field. It had been initiated by Hartmann, Harbater and Krashen [H-H-K] and developped further [C-T-O-H-H-K-P-S2, C-T-P-S, H-H-K2, P-S, R-S].

Let T be a complete discrete valuation ring with fraction field K and residue field k.Let  $p \ge 1$  be the characteristic exponent of k. Let X be a smooth, projective, geometrically integral curve over K. Let F = K(X) be the function field of X and let t be an uniformizing parameter of T. Let  $\mathfrak{X}$  be a normal model of F, i.e. a normal connected projective T-curve with function field F. We denote by Y the closed fiber of  $\mathfrak{X}$  and fix a separable closure  $F_s$  of F.

Given an affine algebraic F-group G, we can study the local-global principle for G-torsors with respect to the infinite set of overfields  $F_P$  that arise from completions at the points P of Y. This gives rise to the Tate-Shafarevich set

$$\operatorname{III}_{\mathfrak{X}}(F,G) = \operatorname{ker}\left(H^{1}(F,G) \to \prod_{P \in Y} H^{1}(F_{P},G)\right);$$

it is denoted simply by  $III_X(F, G)$  in [H-H-K2]. Using Lipman's resolution of singularities permits to consider the inductive limit

$$\operatorname{III}_{patch}(F,G) = \varinjlim_{\mathfrak{X}} \operatorname{III}_{\mathfrak{X}}(F,G)$$

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taken on the models (or equivalently on the regular models). On the other hand, we consider the completions associated to the set  $\Omega_{F,div}$  of divisorial valuations on Farising of various regular projective models over T of the curve X. It gives rise to the

$$\operatorname{III}_{div}(F,G) = \operatorname{ker}\left(H^1(F,G) \to \prod_{v \in \Omega_{F,div}} H^1(F_v,G)\right);$$

In view of [H-H-K2, prop. 8.2], we have the following inclusion

$$\mathrm{III}_{patch}(F,G) \subseteq \mathrm{III}_{div}(F,G).$$

The main result of the paper is to show (under some condition on p with respect to G), that the two Tate-Shafarevich sets coincide (Thm. 3.4). If furthermore G is a F-rational variety (or even retract rational F-variety), we have that  $\operatorname{III}_{patch}(F, G) = 1$  according to Harbater-Krashen-Hartmann [H-H-K2, Thm. 9.1] (see [K, Thm. 2.2.4] for the extension to the retract rational case) so that  $\operatorname{III}_{div}(F, G) = 1$  in this case. It means that if G is a retract rational F-variety that the local-global principle principle holds for G-torsors with respect to divisorial valuations on F.

This was known for certain classical groups [R-S, P-S]. As in [G-P], the proof uses the technique of loop torsors which requires to deal with group schemes defined over T. To do so, for a given Chevalley group  $G_0$  over  $\mathbb{Z}$ , we explain in the first section why the  $G_0 \rtimes \operatorname{Aut}(G_0)$ -torsors permit to classify couples (E, G) where G is a form of  $G_0$  and E a G-torsor.

#### Acknowledgements.

#### 2. Pairs of a group scheme together with a torsor

The following is a complement to the "sorites" of [G1]. We work in the setting of fppf sheaves on a base scheme S. We consider the groupoid  $\mathfrak{P}$  whose objects are pairs (E, G) where G is a fppf S-sheaf in groups and E a G-torsor; the morphisms between two pairs  $(E_1, G_1)$  and  $(E_2, G_2)$  are pairs of isomorphisms f = (f', f'') where  $f'' : G_1 \xrightarrow{\sim} G_2$  is an S-isomorphism of S-sheaves in groups and  $f' : E_1 \xrightarrow{\sim} E_2$  an S-isomorphism of sheaves such that the following diagram commutes

$$E_1 \times_S G_1 \longrightarrow E_1$$

$$\downarrow^{f' \times f''} \qquad \downarrow^{f'}$$

$$E_2 \times_S G_2 \longrightarrow E_2.$$

Note that it induces an isomorphism  $f_* : {}^{E_1}G_1 \xrightarrow{\sim} {}^{E_2}G_2$  between the associated twisted group sheaves.

**Lemma 2.1.** Let G be an S-sheaf in groups and let E be a G-torsor over S. We denote by  ${}^{E}G$  the twisted S-sheaf of G by E by inner automorphisms.

(1) We have a natural exact sequence of fppf S-sheaves in groups

 $1 \to {}^{E}G \to \underline{\operatorname{Aut}}_{\mathfrak{P}}(E,G) \to \underline{\operatorname{Aut}}(G) \to 1$ 

which is locally split for the flat topology.

(2) If E = G the sequence above splits and the S-functor  $\underline{Aut}_{\mathfrak{P}}(G, G)$  is isomorphic to  $G \rtimes \underline{Aut}(G)$ .

*Proof.* We prove the two assertions in the same time. The projection  $\underline{\operatorname{Aut}}_{\mathfrak{P}}(E,G) \to \underline{\operatorname{Aut}}_{G}(G)$  is a homomorphism of S-sheaves in groups. On the other hand we have a monomorphism  ${}^{E}G = \operatorname{Aut}_{G}(E) \to \underline{\operatorname{Aut}}_{\mathfrak{P}}(E,G), f'' \mapsto (f'',id)$ . The fact that the sequence is exact at the middle term is the fact [Gd, cor. 2.3.8].

We consider now the case E = G. In this case the sequence above splits by  $f'' \mapsto (f'', f'')$  hence an isomorphism  $G \rtimes \underline{\operatorname{Aut}}(G) \xrightarrow{\sim} \underline{\operatorname{Aut}}(G, G)$ . Coming back to the general case it implies that the projection  $\underline{\operatorname{Aut}}_{\mathfrak{P}}(E, G) \to \underline{\operatorname{Aut}}(G)$  is an epimorphism of S-sheaves.

Given a fppf S-sheaf in groups G, we can apply [G1, lemme 2.6.3] to express the cohomology

(2.1) 
$$H^{1}(S, G \rtimes \underline{\operatorname{Aut}}(G)) \xrightarrow{\sim} \bigsqcup_{[G'] \in H^{1}(S, \underline{\operatorname{Aut}}(G))} H^{1}(S, G') / \underline{\operatorname{Aut}}(G')(S)$$

by using the fact that  $H^1(S, \underline{\operatorname{Aut}}(G))$  classifies the *S*-forms of *G*. The action on  $\underline{\operatorname{Aut}}(G')(S)$  on  $H^1(S, G')$  is the natural one so that the map  $H^1(S, G') \to H^1(S, G') / \underline{\operatorname{Aut}}(G')(S)$ has trivial kernel so that the composite  $H^1(S, G') \to H^1(S, G \rtimes \underline{\operatorname{Aut}}(G))$  has trivial kernel. This map is nothing but the composition

$$H^1(S,G') \to H^1(S,G' \rtimes \underline{\operatorname{Aut}}(G')) \xrightarrow{\operatorname{Torsion bijection}} H^1(S,G \rtimes \underline{\operatorname{Aut}}(G))$$

where the torsion bijection is with respect to the  $\underline{\operatorname{Aut}}(G)$ -torsor  $\underline{\operatorname{Isom}}(G, G')$ . Summarizing the decomposition (2.1) shows that  $H^1(S, G \rtimes \underline{\operatorname{Aut}}(G))$  encodes the classes of torsors for all twisted forms of G. The map int :  $G \to \underline{\operatorname{Aut}}(G)$  extends to

$$u: G \rtimes \underline{\operatorname{Aut}}(G) \longrightarrow \underline{\operatorname{Aut}}(G) \rtimes^{c} \underline{\operatorname{Aut}}(G) \longrightarrow \overline{\operatorname{Aut}}(G) \longrightarrow \overline{\operatorname{Aut}}(G) \times \overline{\operatorname{Aut}}(G),$$

 $(g, f) \qquad \mapsto \qquad (\operatorname{int}(g), f)$ 

 $(f_1, f_2) \qquad \mapsto \qquad (f_1 f_2, f_2)$ 

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where c stands for the conjugacy action. In terms of objects classified by the respective torsors,  $u_*$  applies a pair (E', G') to  $({}^{E'}G', G')$ 

#### 3. PATCHING TECHNIQUES

We recall the setting. Let T be a complete (excellent) discrete valuation ring with fraction field K, residue field k and uniformizing parameter t. Let F be a one-variable function field over K and let  $\mathfrak{X}$  be a normal model of F, i.e. a normal connected projective T-curve with function field F. We denote by Y the closed fiber of  $\mathfrak{X}$  and fix a separable closure  $F_s$  of F.

For each point  $P \in Y$ , let  $R_P$  be the local ring of  $\mathfrak{X}$  at P; its completion  $\widehat{R}_P$  is a domain with fraction field denoted by  $F_P$ .

For each subset U of Y that is contained in an irreducible component of Y and does not meet the other components, we define  $R_U = \bigcap_{P \in U} R_P \subset F$ . We denote by  $\widehat{R}_U$  the *t*-adic completion of  $R_U$ . The rings  $R_U$  and  $\widehat{R}_U$  are excellent normal domains and we denote by  $F_U$  the fraction field of  $\widehat{R}_U$  [H-H-K3, Remark 3.2.(b)].

Each height one prime  $\mathfrak{p}$  in  $\widehat{R}_P$  that contains t defines a branch of Y at P lying on some irreducible component of Y. The t-adic completion  $\widehat{R}_{\mathfrak{p}}$  of the local ring  $R_{\mathfrak{p}}$ of  $\widehat{R}_P$  at  $\mathfrak{p}$  is a complete DVR whose fraction field is denoted by  $F_{\mathfrak{p}}$ . The field  $F_{\mathfrak{p}}$ contains also  $F_U$  if U is an irreducible open subset of Y such that  $P \in \overline{U} \setminus U$ . We have then a diagram of fields



Setting 3.1. Let  $\mathcal{P}$  be a non-empty finite set of closed points of Y that contains all the closed points at which distinct irreducible components meet. Let  $\mathcal{U}$  be the set of connected components of  $Y \setminus \mathcal{P}$  and let  $\mathcal{B}$  be the set of branches of Y at points of  $\mathcal{P}$ . This yields a finite inverse system of field  $F_P, F_U, F_{\mathfrak{p}}$  (for  $P \in \mathcal{P}$ ;  $U \in \mathcal{U}, \mathfrak{p} \in \mathcal{B}$ ) where  $F_P, F_U \subset F_{\mathfrak{p}}$  if  $\mathfrak{p}$  is a branch of Y at P lying in the closure of U.

**Proposition 3.2.** Let  $1 \to G \to \widetilde{G} \to J \to 1$  be an exact sequence of smooth T-group schemes where G is reductive and J is twisted constant. Let  $E_1$  (resp.  $E_2$ ) be a  $\widetilde{G}$ -torsor over F admitting a reduction to a finite étale T-subgroup of  $S_1$  (resp.  $S_2$ ) of  $\widetilde{G}$  with prime to p order.

Assume that  $E_{1,F_v} \cong E_{2,F_v}$  for any divisorial discrete valuation v on F. Then there exists a regular model  $\mathfrak{Q}$  of F as above together with  $Y, \mathcal{P}$ , etc. such that  $E_{1,F_P} \cong E_{2,F_P}$  for all  $P \in \mathcal{P}$  and  $E_{1,F_U} \cong E_{2,F_U}$  for all connected components U of  $Y \setminus \mathcal{P}$ .

*Proof.* We consider the *F*-scheme  $Z = \text{Isom}_{\widetilde{G}}(E_1, E_2)$ . Our assumption is that  $Z(F_v) \neq \emptyset$  for all divisorial discrete valuations v on F.

Let  $V_i$  be an  $S_{i,F}$ -torsor such that  $V_i \wedge^{S_{i,F}} \widetilde{G} \cong E_i$  for i = 1, 2. According to [G-P, lemma 5.1] there exists a regular proper model  $\mathfrak{X}$  of X and a strict normal crossing divisor D containing the irreducible components of Y such that  $V_i$  extends to a  $\mathfrak{X} \setminus D$ torsor  $\mathfrak{V}_i$  under  $S_i$ . We are given a closed point  $P \in Y$ . If y is of codimension one, by hypothesis,  $Z(F_y)$  is not empty. We therefore look at a closed point  $P \in Y$ . We pick a height one prime  $\mathfrak{p}$  in  $\widehat{R}_P$  that contains t. It defines a branch of Y at P lying on some irreducible component  $Y_1$  of Y.

We consider the local ring  $A = R_P$  of  $\mathfrak{X}$  at P and denote by  $A_D$  its localization at D. Since  $S_{i,T}$  is finite étale of degree prime to p,  $H^1(A_D, S_i)$  consists in loop torsors as defined in [G2, §2.3, lemma 2.3.(2)], i.e. those arising from cocycles related to tame Galois covers of  $A_D$ . It follows that the  $A_D$ -torsors  $\mathfrak{V}_1$ ,  $\mathfrak{V}_2$  under  $\tilde{G}$  are loop (*ibid*, lemma 5.3.(3)).

Let  $F_{P,v}$  be the completion of the field  $F_P$  for the valuation associated to the blow-up of Spec(A) at its closed point. Our assumption states in particular that  $Z(F_{P,v}) \neq \emptyset$ . According to [G3, Thm. 4.5], the map

$$H^1_{loop}(A_D, \widetilde{G}) \to H^1(F_{P,v}, \widetilde{G})$$

is injective. It follows that  $\operatorname{Isom}_{\widetilde{G}_{A_D}}(\mathfrak{V}_1,\mathfrak{V}_2)(A_D) \neq \emptyset$ . A fortiori  $Z(F_P) \neq \emptyset$ .  $\Box$ 

**Proposition 3.3.** Let G be a reductive F-algebraic group and let  $G_0$  be the underlying Chevalley reductive  $\mathbb{Z}$ -group scheme. Let E be a G-torsor and assume that the Aut $(G_0)$ -torsor Isom $(G_0, G)$  admits a reduction to a finite étale T-subgroup of Aut $(G_0)$  of prime to p order and that the  $G_0 \rtimes Aut(G_0)$ -torsors (G, G) (resp. (E, G)) admits a reduction to a finite étale T-subgroup of  $G_0 \rtimes Aut(G_0)$  of prime to p order. Then the two following assertions are equivalent:

(i) There exists a regular proper model  $\mathfrak{X}$  of X with special fiber  $Y = \mathfrak{X}_k$  such that for every point  $y \in Y$ , then  $E(F_y) \neq \emptyset$ .

(ii) For each divisorial discrete valuation v on F, we have  $E(F_v) \neq \emptyset$ .

*Proof.* The implication  $(i) \Longrightarrow (ii)$  is a general fact, see [H-H-K-P, Thm. 3.4]. We prove then  $(ii) \Longrightarrow (i)$  assuming that  $E(F_v) \neq \emptyset$  for each discrete divisorial valuation v on F.

We shall apply Proposition 3.2 to the  $G_0 \rtimes \operatorname{Aut}(G_0)$ -torsors  $E_1 = \operatorname{\underline{Isom}}_{\mathfrak{P}}((G_0, G_0), (G, G))$ and  $E_2 = \operatorname{\underline{Isom}}_{\mathfrak{P}}((G_0, G_0), (E, G))$ . By means of the splitting of  $\operatorname{Aut}_{\mathfrak{P}}(G_0, G_0) \rightarrow$  $\operatorname{Aut}(G_0)$ , we observe that  $E_1$  admits a reduction to a finite étale *T*-subgroup of  $G_0 \rtimes \operatorname{Aut}(G_0)$  of prime to *p* order.

Assume that  $E_{1,F_v} \cong E_{2,F_v}$  for any divisorial discrete valuation v on F. Proposition 3.2 provides a regular model  $\mathfrak{Q}$  of F as above together with  $Y, \mathcal{P}$ , etc. such that  $E_{1,F_P} \cong E_{2,F_P}$  for all  $P \in \mathcal{P}$  and  $E_{1,F_U} \cong E_{2,F_U}$  for all connected components U of  $Y \setminus \mathcal{P}$ . It follows that for every point  $y \in Y$ , then  $E(F_y) \neq \emptyset$ .  $\Box$  **Theorem 3.4.** Let G be a reductive F-algebraic group and let  $G_0$  be the underlying Chevalley reductive  $\mathbb{Z}$ -group scheme. Assume that p does not divide the order of the Weyl group  $W_0$  of  $G_0$  and that G becomes an inner form of  $G_0$  after a Galois prime to p extension of F.

- (1) We have  $\operatorname{III}_{patch}(F,G) = \operatorname{III}_{div}(F,G)$ ;
- (2) If G is a retract rational F-variety, then  $\coprod_{div}(F,G) = 1$ .

*Proof.* (1) We know that  $\coprod_{patch}(F,G) \subset \coprod_{div}(F,G)$ . To establish the converse inclusion, we are given a G-torsor E such that  $E(F_v) \neq \emptyset$  for each divisorial discrete valuation v. We need to check the conditions for applying Proposition 3.3. We start with the isomorphism class  $[G] \in H^1(F, \operatorname{Aut}(G_0))$  by using the sequence  $1 \to 1$  $G_{0,ad} \to \operatorname{Aut}(G_0) \to \operatorname{Out}(G_0) \to 1$  where  $\operatorname{Out}(G_0)$  is a constant  $\mathbb{Z}$ -group scheme. Since  $H^1(F, \operatorname{Out}(G_0)) = \operatorname{Hom}_{ct}(\operatorname{Gal}(F_s/F), \operatorname{Out}(G_0)(\mathbb{Z})) / \sim$ , the image of [G] is represented by a continuus homomorphism  $u : \operatorname{Gal}(F_s/F) \to \operatorname{Out}(G_0)(\mathbb{Z})$  whose kernel defines a Galois extension F'/F which is the minimal subextension of  $F_s$  making G an inner form. Our assumption implies that  $\operatorname{Gal}(F'/F)$  is of prime to p degree and so that  $\Gamma = u(\operatorname{Gal}(F_s/F))$  is a finite subgroup of  $\operatorname{Out}(G_0)(\mathbb{Z})$  of order prime to p. We denote by  $\operatorname{Aut}_{\Gamma}(G_0) = \operatorname{Aut}(G_0) \times_{\operatorname{Out}(G_0)} \Gamma$ ; by construction [G] arises from a cohomology class  $\gamma \in H^1(F, \operatorname{Aut}_{\Gamma}(G_0))$ . According to [C-G-R, Thm. 1.2] applied to the ring  $\mathbb{Z}$ , there exists a finite  $\mathbb{Z}$ -subgroup S of  $\operatorname{Aut}_{\Gamma}(G_0)$  such that the map  $H^1(F,S) \to H^1(F,\operatorname{Aut}_{\Gamma}(G_0))$  is onto. Further by the same statement, we can take  $S \subset \operatorname{Aut}_{\Gamma}(G_0, T_0)$  where  $T_0$  is a maximal split  $\mathbb{Z}$ -torus of  $G_0$  fitting in an exact sequence  $1 \to {}_n(T_0) \to S \to \operatorname{Aut}_{\Gamma}(G_0,T_0))/T_0 \to 1$  where n divides the square of the order of the finite group  $\operatorname{Aut}_{\Gamma}(G_0, T_0)/T_0$ . We have a sequence  $1 \to W_0 \to W_0$  $\operatorname{Aut}_{\Gamma}(G_0,T_0))/T_0 \to \Gamma \to 1$  so that  $\operatorname{Aut}_{\Gamma}(G_0,T_0))/T_0$  is finite constant of prime to p order. It implies that  $_n(T_0)$  is a finite diagonalisable Z–group scheme whose base change to T is étale. Altogether  $S_T$  is a finite étale T-subgroup of  $\operatorname{Aut}_{\Gamma}(G_0)_T$  of prime to p order such that  $\gamma$  admits a reduction to  $S_T$ . A fortiori,  $[G] \in H^1(F, \operatorname{Aut}(G_0))$ admits a reduction to  $S_T$ .

The same method works as well for the class  $[(E, G)] \in H^1(F, G_0 \times \operatorname{Aut}(G_0))$  by using the exact sequence  $1 \to G_0 \rtimes_{\mathbb{Z}} G_{0,ad} \to G_0 \rtimes_{\mathbb{Z}} \operatorname{Aut}(G_0) \to \operatorname{Out}(G_0) \to 1$  and provides a finite étale *T*-subgroup of *S'* of  $(G_0 \rtimes_{\mathbb{Z}} \operatorname{Aut}(G_0))_T$  of prime to *p*-order such that the  $G_0 \rtimes_{\mathbb{Z}} \operatorname{Aut}(G_0)$ -torsor over  $\operatorname{Spec}(F)$  arises from a *S'*-torsor.

Proposition 3.3,  $(ii) \Longrightarrow (i)$ , applies and shows that there exists a regular proper model  $\mathfrak{X}$  of X with special fiber  $Y = \mathfrak{X}_k$  such that for every point  $y \in Y$ , then  $E(F_y) \neq \emptyset$ . In other words [E] belongs in  $\operatorname{III}_{\mathfrak{X}}(F,G)$  and a fortiori to  $\operatorname{III}_{patch}(F,G)$ .

(2) If G is a retract rational F-variety, we have that  $\operatorname{III}_{patch}(F, G) = 1$  [H-H-K2, Thm. 9.1] (see [K, Thm. 2.2.4] for the extension to the retract rational case). The first part of the statement enables us to conclude that  $\operatorname{III}_{div}(F, G) = 1$  in this case.  $\Box$ 

**Remarks 3.5.** (a) If G is a torus, the condition is that the minimal splitting extension of T is of prime to p degree. Note that for norm one tori, the main theorem had been established by Sumit Chandra Mishra [M, Thm. 5.1].

(b) The assumptions are not p are not sharp. For example, if G is semisimple of type  $G_2$ , the statement excludes the primes 2 and 3 but excluding 2 is enough. The point is that we can deal with the subgroup  $\mu_2 \times \mu_2 \times \mathbb{Z}/2\mathbb{Z} \subset S$  in the proof.

**Corollary 3.6.** Let G be a semisimple quasi-split F-group which is quasi-split by a Galois prime to p field extension. Assume that p does not divide the order of  $W_0$ . In each case

(i) G is split;

*(ii)* G is simply connected or adjoint;

we have  $\coprod_{div}(F,G) = 1$ .

*Proof.* Such a F-group is a rational K-variety so the result follows from Theorem 3.4.

This applies as well to many other groups in view of the rationality results of Chernousov-Platonov [C-P] and others. This is the case for example for an isotropic k-group of type  $F_4$ .

#### 4. Other Tate-Schafarevich sets

Given an affine algebraic F-group G, we can also study the local-global principle for G-torsors with respect to the following set of completions of F:

(1) Completions associated to the set  $\Omega_F$  of all non-trivial *T*-valuations on *F* (i.e., those whose valuation ring contains *T*).

(2) Completions associated to the set  $\Omega_F^1$  of all non-trivial rank one *T*-valuations on *F*.

(3) Completions associated to the set  $\Omega^1_{F,dvr}$  of all non-trivial rank one discrete T-valuations on F.

We have  $\Omega_{F,div} \subset \Omega^1_{F,dvr} \subset \Omega^1_{F,dvr} \subset \Omega_F$ . This leads to three Tate-Shafarevich sets as for example  $\operatorname{III}^1_{dvr}(F,G) = \operatorname{ker}\left(H^1(F,G) \to \prod_{v \in \Omega_{dvr}} H^1(F_v,G)\right)$  which is denoted simply by  $\operatorname{III}^1(F,G)$  in the references [C-T-P-S, H-H-K2]. We have the following inclusions

 $\mathrm{III}_{\Omega_{F}}^{1}(F,G) \subset \mathrm{III}_{\Omega_{F}^{1}}^{1}(F,G) \subset \mathrm{III}_{patch}^{1}(F,G) \subset \mathrm{III}_{dvr}^{1}(F,G) \subset \mathrm{III}_{div}^{1}(F,G).$ 

According to [H-H-K-P, Thm. 3.4], we have  $\operatorname{III}_{\Omega_F}^1(F,G) = \operatorname{III}_{\Omega_F^1}^1(F,G) = \operatorname{III}_{patch}^1(F,G)$ . The fact that  $\operatorname{III}_{\Omega_F}^1(F,G) = \operatorname{III}_{\Omega_F^1}^1(F,G) = \operatorname{III}_{dvr}^1(F,G)$  is known in two situations requiring the residue field k to be algebraically closed [H-H-K2, Thm. 8.10, cases (i) and (iii)].

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Coming back to the present paper, the of Theorem 3.4 has then the following consequence.

#### **Corollary 4.1.** under the assumptions of Theorem 3.4, we have then

$$\operatorname{III}_{\Omega_F}^1(F,G) = \operatorname{III}_{\Omega_F^1}^1(F,G) = \operatorname{III}_{patch}^1(F,G) = \operatorname{III}_{dvr}^1(F,G) = \operatorname{III}_{div}^1(F,G)$$

and those sets are trivial when G is a retract F-rational variety.

The constant case. This the case when we deal with a reductive T-group scheme G. In this case, in view of [H-H-K2, Thm. 8.10, case (ii)] (see also of [C-T-O-H-H-K-P-S2, Thm. 3.2]), there is no need of any condition on the characteristic exponent p of the residue field k for having  $\coprod_{patch}^{1}(F,G) = \coprod_{dvr}^{1}(F,G)$ . Inspection of the proof shows that only divisorial valuations of F are involved so that  $\coprod_{patch}^{1}(F,G) = \coprod_{div}^{1}(F,G)$ . This applies in particular when  $G_F$  is a retract rational F-variety; since  $\coprod_{patch}^{1}(F,G) = 1$ , we have that  $\coprod_{div}^{1}(F,G) = 1$  in characteristic free. The interest of our result is then mostly for the non-constant case.

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