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Exact controllability to eigensolutions of the heat equation via bilinear controls on two-dimensional domains

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Abstract

The exact controllability of heat type equations in the presence of bilinear controls have been successfully studied in the recent works [1, 3, 14] motivated by the numerous application to engineering, neurobiology, chemistry, and life science. Nevertheless, the result has been only achieved for 1–dimensional domains due to limit of the existing techniques. In this work, we develop a new strategy to ensure the so-called exact controllability to the eigensolutions of heat-type equations via bilinear control on the 2–dimensional domains. The result is implied by the null-controllability of a suitable linearized equation, and the main novelty of the work is the strategy of its proof. First, the null-controllability in a finite dimensional subspace has to be ensured via the solvability of a suitable moment problem. Explicit bounds on the control cost w.r.t. to the dimension of the controlled space are also required. Second, the controllability can be extended to the whole Hilbert space, thanks to the Lebeau-Robbiano-Miller method, when the control cost does not growth too fast w.r.t to the dimension of the finite dimensional subspace. We firstly develop our techniques in the general case when suitable hypotheses on the problem are verified. Afterwards, we apply our procedure to the bilinear heat equation on rectangular domains, and we ensure its exact controllability to the eigensolutions. Finally, we study the controllability issue on the square and we use perturbation theory techniques to deal with the presence of multiple eigenvalues for the spectrum of the Dirichlet Laplacian.

Keywords: bilinear control, heat equation, biorthogonal family

2010 MSC: 93B05, 35Q93, 35K05, 93C20

1 Introduction

The majority of the theoretical control problems studied in the literature and involving evolution equations on bounded domains usually consider boundary or internal controls. There, the dynamics is described by a suitable evolution equation, and it is studied in the presence of one, or more, suitable additive terms. In these models, the controls are external elements of the system that intervene in its evolution and affect it. Nevertheless, in many practical problems from engineering, neurobiology, chemistry and life science, the control is not an external element but rather a modification of the principal parameter of the evolution. In these cases, it is then more appropriate to consider evolution equations in the presence of multiplicative controls. We talk about bilinear control when the spatial part of the multiplicative term is fixed and only the time-dependent intensity changes.

Interesting models involving parabolic dynamics in the presence of multiplicative/bilinear controls are for instance the nuclear chain reactions. In these phenomena, the number of particles of the diffusing material increases by the interaction with the surrounding medium, leading to a diffusion process. An example is the nuclear fission that occurs from the collision of neutrons with uranium nuclei, leading to the formation of new neutrons and energy. The new neutrons interact again with the uranium, triggering the continuation of the process. The evolution of the density of neutrons in a point can be modelled by an equation of the form

$$\partial_t \psi(t) - \alpha \Delta \psi(t) - f(t, x) \psi(t) = 0$$

where $\alpha > 0$ is the diffusion coefficient. In the nuclear plant, the reaction is controlled by modifying the value and the sign of the function f , which then represents the multiplicative control on the problem. In the case

$$f(t, x) = u(t) \mu(x)$$

with $\mu(x)$ fixed and $u(t)$ variable, the multiplicative control is actually bilinear. Notice that the multiplicative control is very suitable to the model since, in the practical situation, the reaction chamber is isolated and the neutrons are neither added nor removed. We refer to the book [22] for other examples of models involving parabolic equations in the presence of multiplicative controls.

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Main result: general framework

In this work, we focus our attention on the following bilinear control problem on a Lipschitz domain Ω of \mathbb{R}^2 :

$$(BHE) \quad \begin{cases} \partial_t \psi(t) - \Delta \psi(t) + \langle v(t), Q \rangle \psi(t) = 0, & t \in (0, T) \\ \psi(0) = \psi_0, \end{cases}$$

where $-\Delta$ is a Dirichlet Laplacian, $Q = (Q_1, \dots, Q_q) : \Omega \rightarrow \mathbb{R}^q$ with $q \in \mathbb{N}^*$ is a L^2 -fixed vector valued function, $v = (v_1, \dots, v_q) \in L^2((0, T), \mathbb{R}^q)$ is the control and then

$$\langle v(t), Q \rangle = v_1(t)Q_1(x, y) + \dots + v_q(t)Q_q(x, y).$$

Let us introduce the ordered eigenvalues $(\lambda_k)_{k \in \mathbb{N}^*}$ of the Laplace operator $-\Delta$ with domain

$$D(-\Delta) := \{\psi \in H_0^1(\Omega, \mathbb{R}), \Delta \psi \in L^2(\Omega, \mathbb{R})\}.$$

In this article, we assume that the domain Ω is such that the spectrum of the Dirichlet Laplacian verifies the following property

$$(1) \quad \exists C > 0, \quad |N_\Lambda - C\Lambda| = O(\sqrt{\Lambda}),$$

where $N_\Lambda := \#\{k \in \mathbb{N}^* : \lambda_k \leq \Lambda\}$ with $\Lambda > 0$ is the counting function of the eigenvalues of $-\Delta$. Notice that the identity (1) implies the existence of $C_1, C_2 > 0$ such that

$$(2) \quad C_1\Lambda - C_2\sqrt{\Lambda} \leq N_\Lambda \leq C_1\Lambda + C_2\sqrt{\Lambda}, \quad \forall k \in \mathbb{N}^*.$$

The Weyl's asymptotic (1) is for instance verified for rectangles, smooth domains where the set of the closed generalised geodesics has zero measure in the cotangent space [21, Corollary 29.3.4], "Domains consisting of a finite number of squares" [16, Theorem 12, pp. 432] or domains satisfying a "non-blocking and non-periodicity conditions" [30, Theorem 1.6.1]. The identity (1) also implies the Weyl's asymptotic

$$(3) \quad \frac{\lambda_k}{k} = O(1).$$

We denote now $(\phi_k)_{k \in \mathbb{N}^*}$ some orthonormalized eigenfunctions corresponding to the eigenvalues $(\lambda_k)_{k \in \mathbb{N}^*}$, which form a Hilbert basis for the space $L^2(\Omega, \mathbb{R})$. We introduced the j -th eigensolutions of $-\Delta$ as the functions

$$\Phi_j(t) = e^{-\lambda_j t} \phi_j, \quad t > 0, \quad \forall j \in \mathbb{N}^*.$$

Assumptions I. Assume the following conditions verified for the problem (BHE).

1. The eigenvalues $(\lambda_k)_{k \in \mathbb{N}^*}$ are simple and there exists $C, p_0 > 0$ such that

$$\lambda_{k+1} - \lambda_k \geq \frac{C}{\lambda_k^{p_0}}, \quad \forall k \in \mathbb{N}^*.$$

2. The vector valued function $Q = (Q_1, \dots, Q_q)$ is such that $q = 2$, the second component $Q_2 = 1$, and, for every $l \in \mathbb{N}^*$, there exist $C > 0$ and $p_1 \geq 3/2$ such that

$$|\langle \phi_k, Q_1 \phi_l \rangle_{L^2}| \geq \frac{C}{\lambda_k^{p_1}}, \quad \forall k \in \mathbb{N}^*.$$

We refer to Remark 3.1 for further comments on the choice of the parameter $3/2$ in Assumptions I. We are finally ready to ensure the first main result of the work with the following theorem. Here, we state the local exact controllability of the (BHE) to the eigensolutions of the Dirichlet Laplacian.

Main Theorem A. *Let Assumptions I be verified. For any time $T > 0$, there exists $\delta > 0$ such that, for any $\psi_0 \in L^2(\Omega, \mathbb{R})$ with*

$$\|\psi_0 - \phi_j\|_{L^2} < \delta,$$

there exists $v_1 \in L^2((0, T), \mathbb{R})$ such that $\psi(T; \psi_0, v)$, the solution of (BHE) with control $v = (v_1, 0)$, verifies

$$\psi(T; \psi_0, v) = \Phi_j(T).$$

Proof. We refer to Section 3.2 for the proof of Main Theorem A. □

Main Theorem A ensures the existence of neighbourhoods of ϕ_j in $L^2(\Omega, \mathbb{R})$ which can be exactly controlled to the target $\Phi_j(T)$. The first main consequences of the theorem is the following corollary.

Corollary 1.1. *Let Assumptions I be verified. For any time $T > 0$, there exists $\delta > 0$ such that, for any $\alpha, \beta \in \mathbb{R}^*$ with the same sign and for any initial state $\psi_0 \in L^2(\Omega, \mathbb{R})$ with*

$$\|\psi_0 - \alpha\phi_j\|_{L^2} < |\alpha|\delta,$$

there exists $v_1 \in L^2((0, T), \mathbb{R})$ and $v_2 \in \mathbb{R}$ such that $\psi(T; \psi_0, v)$, the solution of (BHE) with control $v = (v_1, v_2)$, verifies

$$\psi(T; \psi_0, v) = \beta\phi_j.$$

Proof. We refer to Section 3.2 for the proof of Corollary 1.1. \square

Corollary 1.5 shows that, when we add a constant control, not only do we have some more freedom on the initial data w.r.t. Main Theorem A, but also on the target of the controlled dynamics. A simple application is the case of $\alpha = 1$ and $\beta > 0$, which extends Main Theorem A since we can target any $\beta\phi_j$ instead of $\Phi_j(T)$.

Main result: the rectangle case

Let us now discuss the specific case of $\Omega = (0, a) \times (0, b)$, the rectangle with sides $a, b > 0$. In this framework, the ordered eigenvalues $(\lambda_k)_{k \in \mathbb{N}^*}$ are defined by two sequences of numbers $(l_k)_{k \in \mathbb{N}^*}, (m_k)_{k \in \mathbb{N}^*} \subset \mathbb{N}^*$ such that

$$(4) \quad \lambda_k = \left(\frac{l_k^2}{a^2} + \frac{m_k^2}{b^2} \right) \pi^2.$$

Some corresponding eigenfunctions $(\phi_k)_{k \in \mathbb{N}^*}$ are

$$\phi_k = 2 \sin\left(\frac{l_k}{a} \pi x\right) \sin\left(\frac{m_k}{b} \pi y\right).$$

A suitable choice of a and b allows ensuring the exact controllability of Main Theorem A and Corollary 1.1 under simpler hypotheses than Assumptions I which are the following.

Assumptions II. Assume the following conditions verified for the problem (BHE).

1. The sides of the rectangle $a, b > 0$ are such that a^2/b^2 is an algebraic irrational number.
2. The vector valued function $Q = (Q_1, \dots, Q_q)$ is such that $q = 2$, the second component $Q_2 = 1$, and

$$Q_1(x, y) = Q_1^1(x)Q_1^2(y).$$

In addition, for every $l \in \mathbb{N}^*$, there exist $C > 0$ and $p_1, p_2 \geq 5/2$ such that

$$\left| \left\langle \sin\left(\frac{k}{a} \pi x\right), Q_1^1 \sin\left(\frac{l}{a} \pi x\right) \right\rangle_{L^2(0, a)} \right| \geq \frac{C}{k^{p_1}}, \quad \forall k \in \mathbb{N}^*,$$

$$\left| \left\langle \sin\left(\frac{k}{b} \pi y\right), Q_1^2 \sin\left(\frac{l}{b} \pi y\right) \right\rangle_{L^2(0, b)} \right| \geq \frac{C}{k^{p_2}}, \quad \forall k \in \mathbb{N}^*.$$

Remark 1.2. *Let us consider the specific case of the equation (BHE) on the rectangle $\Omega = (0, 1) \times (0, \sqrt[3]{2})$. Assumptions II are verified when we choose for instance the control potential Q :*

$$Q = \left(\frac{x^2 y^2}{2}, 1 \right).$$

We refer to Remark 3.7 in Section 3.3 for further details on this explicit example. Notice that the same result also is valid with different other polynomial potentials Q as explained in the mentioned remark.

We are ready to state the exact controllability to the eigensolutions of (BHE) in this new framework.

Main Theorem B. *Let Assumptions II be verified. For any time $T > 0$, there exists $\delta > 0$ such that, for any $\psi_0 \in L^2(\Omega, \mathbb{R})$ with*

$$\|\psi_0 - \phi_j\|_{L^2} < \delta,$$

there exists $v_1 \in L^2((0, T), \mathbb{R})$ such that $\psi(T; \psi_0, v)$, the solution of (BHE) with control $v = (v_1, 0)$ verifies

$$\psi(T; \psi_0, v) = \Phi_j(T).$$

Proof. We refer to Section 3.3 for the proof of Main Theorem B. \square

As Corollary 1.1 was a direct consequence of Main Theorem A, we can deduce the following corollary from Main Theorem B.

Corollary 1.3. *Let Assumptions II be verified. For any time $T > 0$, there exists $\delta > 0$ such that, for any $\alpha, \beta \in \mathbb{R}^*$ with the same sign and for any initial state $\psi_0 \in L^2(\Omega, \mathbb{R})$ with*

$$\|\psi_0 - \alpha\phi_j\|_{L^2} < |\alpha|\delta,$$

there exists $v_1 \in L^2((0, T), \mathbb{R})$ and $v_2 \in \mathbb{R}$ such that $\psi(T; \psi_0, v)$, the solution of (BHE) with control $v = (v_1, v_2)$, verifies

$$\psi(T; \psi_0, v) = \beta\phi_j.$$

Proof. We refer to Section 3.3 for the proof of Corollary 1.3. \square

Main result: the square case via perturbation theory

Let us now discuss the specific case of $\Omega = (0, 1) \times (0, 1)$. We would like to prove the local exact controllability for the equation (BHE) in some neighbourhoods of the eigenfunctions of the Laplacian. However, the spectrum of $-\Delta$ on the square presents multiple eigenvalues, introducing a natural ambiguity in the choice of the corresponding eigenfunctions. To avoid such a problem, we consider the operator $-\Delta + \langle v(t), Q \rangle$ with $q = 4$ and $v(t) = (v_1(t), \dots, v_4(t))$ such that $v_1(t) = u \in \mathbb{R}$. We write

$$-\Delta + \langle v(t), Q \rangle = (-\Delta + uQ_1) + \sum_{l=2}^4 v_l(t)Q_l.$$

We consider $u \in \mathbb{R}$ sufficiently small so that uQ_1 plays the role of perturbation of the Dirichlet Laplacian and the spectrum of $-\Delta + uQ_1$ is simple.

- We respectively denote the eigenvalues of $-\Delta + uQ_1$ and a Hilbert basis of $L^2(\Omega)$ made by eigenfunctions:

$$(\lambda_{l,m}^u)_{l,m \in \mathbb{N}^*} \quad \text{and} \quad (\phi_{l,m}^u)_{l,m \in \mathbb{N}^*}.$$

- We introduced the (l, m) -th eigensolutions of $-\Delta + uQ_1$ which are defined as

$$\Phi_{l,m}^u(t) = e^{-\lambda_{l,m}^u t} \phi_{l,m}^u, \quad t > 0, \quad \forall j \in \mathbb{N}^*.$$

Let $(A, D(A))$ be the one-dimensional Dirichlet Laplacian, such that $A = -\partial_x^2$ and $D(A) = H^2 \cap H_0^1((0, 1), \mathbb{R})$. We now introduce a set of assumptions which replace Assumptions I in this new framework.

Assumptions III. Assume the following conditions verified for the problem (BHE).

1. The vector valued function $Q = (Q_1, \dots, Q_q)$ is such that $q = 4$ and the last component $Q_4 = 1$.
2. The potential $Q_1(x, y) = Q_1(x)$ depends only on the variable x , it verifies $\int_0^1 Q_1(x) dx = 0$, and

$$\langle \cos(2k\pi x), Q_1 \rangle_{L^2(0,1)} \neq \langle \cos(2n\pi x), Q_1 \rangle_{L^2(0,1)}, \quad \forall k, n \in \mathbb{N}^*, \quad k \neq n.$$

3. We have $Q_2(x, y) = Q_2^1(x)Q_2^2(y)$ and, for every $l \in \mathbb{N}^*$, there exist $C > 0$ and $p_1, p_2 \geq 5/2$ such that

$$\left| \langle \sin(k\pi x), Q_2^1 \sin(l\pi x) \rangle_{L^2(0,1)} \right| \geq \frac{C}{k^{p_1}}, \quad \forall k \in \mathbb{N}^*,$$

$$\left| \langle \sin(k\pi y), Q_2^2 \sin(l\pi y) \rangle_{L^2(0,1)} \right| \geq \frac{C}{k^{p_2}}, \quad \forall k \in \mathbb{N}^*.$$

4. Let $p_1 > 0$ be the parameter such that the previous point is verified. There holds

$$Q_1, Q_2^1 \in \left\{ f \in H^{p_1-2}((0, 1), \mathbb{R}) : f' \in D(|A|^{\frac{p_1-4}{2}}) \text{ when } p_1 > 4 \right\}.$$

5. The potential $Q_3(x, y) = Q_3(x)$ is such that, for every $l \in \mathbb{N}^*$, there exist $C > 0$ and $p_3 \geq 5/2$ such that

$$\left| \langle \sin(k\pi x), Q_3 \sin(l\pi x) \rangle_{L^2(0,1)} \right| \geq \frac{C}{k^{p_3}}, \quad \forall k \in \mathbb{N}^*.$$

Remark 1.4. An example of control potential Q such that Assumptions III are verified is the following:

$$Q = ((3x^2 - 1), x^2y^2, (3x^2 - 1), 1).$$

We refer to Remark 4.1 in Section 4 for further details on this matter. Notice that the same result also is valid with different other polynomial potentials Q as in Remark 1.2.

We ensure a local exact controllability to the (l, m) -th eigensolution of $-\Delta$ for the equation (BHE) in small time when Assumptions III are verified.

Main Theorem C. Let Q verify Assumption III. For every $l, m \in \mathbb{N}^*$, there exist $R > 0$ and a countable subset V of \mathbb{R} such that, for any $u \in [-R, R] \setminus V$, the following local exact controllability to the eigensolutions is verified. For any time $T > 0$, there exists $\delta > 0$ such that, for any $\psi_0 \in L^2(\Omega, \mathbb{R})$ with

$$\|\psi_0 - \phi_{l,m}^u\|_{L^2} < \delta,$$

there exists $(v_2, v_3) \in L^2((0, T), \mathbb{R}^2)$ such that $\psi(T; \psi_0, v)$, the solution of (BHE) with control $v = (u, v_2, v_3, 0)$ verifies

$$\psi(T; \psi_0, v) = \Phi_{l,m}^0(T).$$

Proof. We refer to Section 4.4 for the proof of Main Theorem C. □

Main Theorem C ensures the existence of neighbourhoods of $L^2(\Omega, \mathbb{R})$ where an exact controllability result for the equation (BHE) is ensured. Notice that even if the initial state has to be chosen in the neighbourhood of the perturbed eigenfunctions $\phi_{l,m}^u$, the target of the dynamics is an unperturbed eigensolution

$$\Phi_{l,m}^0(t) = e^{-\lambda_{l,m}^0 t} \phi_{l,m}^0.$$

In the proof, we at first target $\Phi_{l,m}^u(t)$ and after we prove that we can also reach $\Phi_{l,m}^0(t)$.

Corollary 1.5. Let Q verify Assumptions III. For every $l, m \in \mathbb{N}^*$, there exist $R > 0$ and a countable subset V of \mathbb{R} such that, for any $u \in [-R, R] \setminus V$, the following local exact controllability to the eigensolutions is verified. For any time $T > 0$, there exists $\delta > 0$ such that, for any $\alpha, \beta \in \mathbb{R}^*$ with the same sign and for any initial state $\psi_0 \in L^2(\Omega, \mathbb{R})$ with

$$\|\psi_0 - \alpha \phi_{l,m}^u\|_{L^2} < |\alpha| \delta,$$

there exists $(v_2, v_3, v_4) \in L^2((0, T), \mathbb{R}^3)$ such that $\psi(T; \psi_0, v)$, the solution of (BHE) with control $v = (u, v_2, v_3, v_4)$, verifies

$$\psi(T; \psi_0, v) = \beta \Phi_{l,m}^0.$$

Proof. We refer to Section 4.4 for the proof of Corollary 1.5. □

Notice that Main Theorem C (resp. Corollary 1.5) ensures the controllability with 3 (resp. 4) controls, but it is actually possible to obtain the result with only 2 of them (resp. 3). Indeed, its proof (Section 4) can be obtained with one control less, as explained in Remark 4.10. From this perspective, an example of control potential Q verifying Main Theorem C (resp. Corollary 1.5) is $Q = ((3x^2 - 1), x^2y^2, (3x^2 - 1), 1)$ as explained in Remark 1.4. However, the same result can be also obtained with the control potential

$$Q = ((3x^2 - 1), x^2y^2, 0, 1).$$

Some reference and existing results

It is well known that there is an obstruction to the exact controllability of dynamics via multiplicative bilinear controls as explained by Ball, Marsden, and Slemrod in [7]. Here, the authors proved that the exact controllability of bilinear evolution equations can not be ensured in the whole space where they are defined when specific assumptions on the control are verified. The lack of controllability, not only affects the study of the bilinear heat equation, but also the bilinear Schrödinger equation or the bilinear wave equation.

The negative result presented in [7] is the main reason why the exact controllability for bilinear evolution equations have been addressed in literature with different types of controllability. The exact controllability of bilinear wave or Schrödinger equations was studied, for instance, in suitable subspaces. This idea was introduced by Beauchard and Laurent in [8, 9], who showed a hidden regularizing effect allowing to prove well-posedness and controllability in higher-regularity spaces for which the lack of controllability from [7] does not apply. Nevertheless, this approach can not be applied to the bilinear heat equation as (BHE).

Consider the abstract evolution equation on a Hilbert space X

$$(5) \quad \begin{cases} y'(t) + Ay(t) + u(t)By(t) = 0, & t \in (0, T), \\ y(0) = y_0, \end{cases}$$

when $A : D(A) \subseteq X \rightarrow X$ is a densely defined, self-adjoint linear operator with compact resolvent, B is a bounded linear operator and u is a bilinear control. Exact controllability results of (5) were firstly proved by Alabau-Boussouira, Cannarsa and Urbani in [1, 3]. They considered $(\phi_k)_{k \in \mathbb{N}^*}$ some normalized eigenfunctions of A and $(\lambda_k)_{k \in \mathbb{N}^*}$ the corresponding eigenvalues. They proved the local controllability to the so-called j -th eigensolutions

$$\Phi_j(t) = e^{-\lambda_j t} \phi_j.$$

In detail, they proved that under the gap condition

$$(6) \quad \exists \gamma > 0, \forall k \in \mathbb{N}^*, \quad |\sqrt{\lambda_{k+1} - \lambda_j} - \sqrt{\lambda_k - \lambda_j}| \geq \gamma > 0$$

and the spreading assumption

$$(7) \quad \langle B\phi_j, \phi_j \rangle \neq 0, \quad |\lambda_k - \lambda_j|^q |\langle B\phi_j, \phi_k \rangle| \geq b, \quad \forall k \neq j,$$

with $p, q > 0$, system (5) is exactly controllable to the j -th eigensolution at any time $T > 0$. In other words, they showed the possibility of steering any state sufficiently close to ϕ_j into $\Phi_j(T)$ in a time $T > 0$. A direct application of their works is the controllability of a heat equation as (BHE) in the 1-dimensional case of $\Omega = (0, 1)$. We also refer to the works [2, 15] for similar results on the subject.

These results were later extended in [14] by the second author, together with Cannarsa and Urbani where they studied (BHE) on network-type domains. In this framework, the spectrum of the Laplacian $-\Delta$ has different properties w.r.t. to the interval $(0, 1)$ since it is only possible to show the existence of $N \in \mathbb{N}^*$ such that

$$(8) \quad \sqrt{\lambda_{k+N}} - \sqrt{\lambda_k} \geq \gamma > 0.$$

For some specific structure of the network, it is also verified a weak gap condition of the form:

$$(9) \quad \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq a_k,$$

with $a_k \asymp k^{-p}$ and $p \in \mathbb{N}^*$. In [14], the authors extended the existing techniques of solvability of moment problems to such a framework and proved the local controllability to the eigensolutions for any time $T > 0$.

The central part of the local exact controllability to the eigensolutions is the solvability of a suitable moment problem. Heuristically speaking, one has to prove: for any $(x_k)_{k \in \mathbb{N}^*} \subset \mathbb{R}$ in a specific space, there exists $u \in L^2((0, T), \mathbb{R})$ such that

$$(10) \quad x_k = \int_0^T e^{\lambda_k s} u(s) ds, \quad \forall k \in \mathbb{N}^*.$$

This property was widely studied in literature according to the different behavior of the eigenvalues $(\lambda_k)_{k \in \mathbb{N}^*}$. The solvability when the spectral gap (6) is verified has been well-known for a very long time, and it was ensured for instance in the works [10, 18, 19]. On the other hand, the property under the validity of the weaker spectral hypothesis (8) is more recent, and it was studied [4, 6, 11]. Here, the authors also provided some upper and lower bounds on the norm of the biorthogonal family at any fixed time. The dependence on the time of the upper bound was initially made explicit in the work [14] and the result was generalized in [12]. We refer to this last work for a careful analysis of the modern results and techniques on the solvability of moment problems.

Finally, we want to mention the preprint [5] which appeared contemporary to our paper. In this work, the authors also deal with the solvability of moment problems in higher-dimensional framework.

Our technique: local controllability to eigensolutions for two dimensional domains

Our techniques are inspired by the idea of Alabau-Boussouira, Cannarsa and Urbani in [2, 3]: the local controllability of (BHE) is implied by the null-controllability of suitable linearized systems and by suitable estimates of the norm of the control. The natural approach is to linearize (BHE) w.r.t. an eigenfunction and prove the null-controllability of the linearized evolution via the solvability of a suitable moment problem as (10). However, the asymptotic behaviour of the eigenvalues λ_k is different from the 1-dimensional case. For 2-dimensional domains, we only know the existence of a sequence $(N_k)_{k \in \mathbb{N}^*} \subseteq \mathbb{N}^*$ such that $N_k \rightarrow +\infty$ and

$$(11) \quad \inf_{k \in \mathbb{N}^*} \sqrt{\lambda_{k+N_k}} - \sqrt{\lambda_k} > 0.$$

In this case, the solvability of the moment problem can not be directly studied via the existing techniques in literature which usually require stronger hypotheses on the spectrum as the spectral gap (8).

The main novelty of the work is how we overcome the issue of the solvability of the moment problem in higher dimensional domains and then prove the null-controllability. We use the Miller adaptation of the Lebeau-Robbiano method [25, 26] to show that the null-controllability of the linearized equation in

$$E_\Lambda := \text{span}\{\phi_k, k \in J_\Lambda\} \quad \text{with} \quad \Lambda > 0,$$

might be extended to the whole $L^2(\Omega, \mathbb{R})$. This result holds when suitable bounds w.r.t. Λ on the control norm are verified. The Lebeau-Robbiano-Miller method allows us to solve a suitable finite dimensional moment problems (10) where the spectrum behaves as (8) instead of (11). This idea is inspired by the work [28] which deals with additive control problems. We develop our technique in an abstract setting at first (Section 2) and we apply it later to proving Main Theorem A (Section 3), Main Theorem B (Section 3) and Main Theorem C (Section 4).

- Main Theorem A provides the exact controllability to eigensolutions under the hypothesis that the domain Ω is such that the Weyl's asymptotics (1) and the weak gap (9) are verified. The peculiarity of the proof appears when we study the finite dimensional moment problem to ensure the null-controllability in E_Λ . Here, we need to study suitable bounds for the norm of the control and establish its dependency w.r.t. the parameter Λ . It appears that from the Weyl's asymptotics (1), two-dimensional domains provide suitable growth w.r.t. Λ which allows applying the method exposed above. Notice that it does not seem to be the case for higher-dimensional domains.
- Main Theorem B presents the controllability result in the case of suitable rectangles $\Omega = (0, a) \times (0, b)$. We prove that when a^2/b^2 is an algebraic irrational number, then (9) is guaranteed. Main Theorem B provides an explicit example for our exact controllability result of Main Theorem A.
- Finally, Main Theorem C deals with the case where $\Omega = (0, 1) \times (0, 1)$ is a square. Here, the spectrum of the Dirichlet Laplacian presents multiple eigenvalues and (9) is obviously not fulfilled. This fact represents an obstruction to the solvability of the moment problem and then to the controllability. To circumvent this difficulty, we use an additional control $u \in \mathbb{R}^*$ so that the control term uQ_1 perturbs the Laplacian $-\Delta$.
 1. We choose $u \in \mathbb{R}^*$ so that, not only the spectrum of $-\Delta + uQ_1$ is simple, but also (9) is satisfied. This idea allows us to control neighbourhoods of perturbed eigenfunctions $\phi_{l,m}^u$ to perturbed eigensolutions $\Phi_{l,m}^u(T)$. Notice that our result is verified for u as small as desired.
 2. To prove Main Theorem C and then the reachability of unperturbed eigensolutions, we use an additional control. The main trick here is that the perturbed eigensolutions $\Phi_{l,m}^u(T)$ have separate variables when $Q_1(x, y) = Q_1(x)$ (Assumptions III). Then, we reduce the dynamics (BHE) starting from the perturbed eigensolution to a one-dimensional control problem where classical techniques applies. This fact allows us to finally steer the perturbed eigensolution $\Phi_{l,m}^u(T)$ to the unperturbed one $\Phi_{l,m}^0(T)$.

Our proofs are applied to two-dimensional cases since, in the intermediate step of the proof, the growth of the control cost w.r.t. Λ is suitable in this framework. Higher dimensional domains are not considered here as, in such cases, the Weyl asymptotics (1) degrades the control cost that seems to growth too much, preventing the use of the Lebeau-Robbiano-Miller method. Nevertheless, it might be possible to enhance our technique in order to allow worse estimations on the control cost and then treat at least the three-dimensional framework of the cube. We plan to elucidate this fact in future investigations.

Scheme of the work

The manuscript is composed as follows. In Section 2, we introduce the Lebeau-Robbiano-Miller method in an abstract setting. In Section 3, we use the abstract techniques from Section 2 to prove Main Theorem A and Main Theorem B. Main Theorem C is proved in Section 4 and technical results from this work are proved in Appendix.

Main notations

Let us introduce the main notations adopted in the manuscript.

- We denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in \mathbb{R}^q and $\| \cdot \|$ is the corresponding norm.

- We consider $H^s = H^s(\Omega, \mathbb{R})$ for $s \geq 0$ and $L^p = L^p(\Omega, \mathbb{R})$ for $p \geq 1$ the standard Sobolev and Lebesgue spaces of functions $f : \Omega \rightarrow \mathbb{R}$ endowed with the norms $\|\cdot\|_s$ and $\|\cdot\|_{L^p}$. The space L^2 is also equipped with the scalar product

$$\langle f, g \rangle_{L^2} = \int_{\Omega} f(x)g(x)dx.$$

We call $H_0^1 = H_0^1(\Omega, \mathbb{R})$ the spaces of functions $f \in H^1$ such that $f(\partial\Omega) = 0$.

- Let $(a, b) \in \mathbb{R}$. We call $L^p(a, b) = L^p((a, b), \mathbb{R})$ for $p \geq 1$ and $L^2(a, b)$ is equipped with the norm $\|\cdot\|_{L^2(a,b)}$ corresponding to the scalar product

$$\langle f, g \rangle_{L^2(a,b)} = \int_a^b f(x)g(x)dx.$$

We denote $H_0^1(a, b) = H_0^1((a, b), \mathbb{R})$ the space of functions $f \in H^1(a, b)$ such that $f(a) = f(b) = 0$.

- Let $C^s = C^s(\Omega, \mathbb{R})$ with $s \in \mathbb{N} \cup \{\infty\}$ be the space of s -times continuously differentiable functions $f : \Omega \rightarrow \mathbb{R}$.
- Consider a Banach space X . We denote by $\|\cdot\|_X$ the norm of a Banach space X and $C^s([0, T], X)$ the space of s -times continuously differentiable functions $f : [0, T] \rightarrow X$. When X is also a Hilbert space, $\langle \cdot, \cdot \rangle_X$ represents the corresponding scalar product.

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2 Local controllability to eigensolutions in the abstract setting

Let X be a separable Hilbert space equipped by a scalar product $\langle \cdot, \cdot \rangle_X$ associated to the norm $\|\cdot\|_X$. Let $A : D(A) \subseteq X \rightarrow X$ be a densely defined linear operator such that

- A is self-adjoint;
- there exists $\sigma \geq 0$ such that $\langle Ax, x \rangle \geq -\sigma \|x\|_X^2$ for every $x \in D(A)$;
- there exists $\lambda > -\sigma$ such that the operator $(\lambda I + A)^{-1} : X \rightarrow X$ is compact.

In this framework, the spectrum of A is purely discrete and consists of a sequence of ordered positive real numbers $(\lambda_k)_{k \in \mathbb{N}^*}$. We denote by $(\phi_k)_{k \in \mathbb{N}^*}$ the associated eigenfunctions, forming a Hilbert basis of X . The operator $-A$ generates a strongly continuous semigroup by e^{-tA} . We consider the parabolic problem

$$(12) \quad \begin{cases} \varphi'(t) + A\varphi(t) + v(t)B\varphi(t) = 0, & t > 0, \\ \varphi(0) = \varphi_0, \end{cases}$$

where $v \in L^2((0, T), \mathbb{R})$ and $B : X \rightarrow X$ is a bounded linear operator. The well-posedness of the equation (BHE) in the spirit of Proposition 3.4 is well-known, as proved in [7]: we call by $\varphi(t; \varphi_0, v)$ the solution with initial state φ_0 and control v . We introduce the j -th eigensolution of the operator A as the solution of the problem corresponding to $v = 0$ and $\varphi_0 = \phi_j$:

$$\Phi_j(t) := e^{-\lambda_j t} \phi_j.$$

2.1 Local controllability to eigensolutions and null-controllability of the linearized problem

The aim of this section is to show how to ensure the local exact controllability of (12) to the j -th eigensolutions of the operator A via the null-controllability of a suitable linear problem. This idea was firstly introduced in the abstract setting by Alabau-Boussouira, Cannarsa and Urbani in [3]. Here, we slightly improve their result in order to cover our framework.

Definition 2.1. Let $j \in \mathbb{N}^*$. The problem (12) is locally exactly controllable to the j -th eigensolution of A in time $T > 0$ if there exists $\delta > 0$ such that, for any $\varphi_0 \in X$ with $\|\varphi_0 - \phi_j\|_X < \delta$, there exists a control $v \in L^2((0, T), \mathbb{R})$ such that

$$\varphi(T; \varphi_0, u) = \Phi_j(T).$$

We introduce now the linear problem

$$(13) \quad \begin{cases} \xi'(t) + A\xi(t) + v(t)B\phi_j = 0, & t \in (0, T), \\ \xi(0) = \xi_0 \in X. \end{cases}$$

We denote by $\xi(\cdot; \xi_0, u)$ the (mild) solution of the linear problem (13) corresponding to the initial condition ξ_0 and control v given by

$$\xi(t; \xi_0, v) = e^{-tA}\xi_0 + \int_0^t e^{-(t-s)A}v(s)B\phi_j ds.$$

We now introduce the partial null-controllability definition we shall use in what follows.

Definition 2.2. Let E be a closed subspace of X . We say that the problem (13) is null-controllable in E and in a time $T > 0$, if there exists a constant $K_E(T) > 0$ such that, for any $\xi_0 \in X$, there exists a control $v \in L^2((0, T), \mathbb{R})$ such that

$$\Pi_E \xi(T; \xi_0, v) = 0 \quad \text{and} \quad \|v\|_{L^2(0, T)} \leq K_E(T) \|\xi_0\|_X,$$

where Π_E denotes the orthogonal projection on E in X . The best constant $K_E(T)$ is called the control cost, and it is defined as

$$K_E(T) := \sup_{\|\xi_0\|_X=1} \inf \left\{ \|v\|_{L^2(0, T)} : \Pi_E \xi(T; \xi_0, v) = 0 \right\}.$$

When $E = X$, we just say that the problem (13) is null-controllable and we denote $K_E(T) = K(T)$.

In this framework, we shall use the following local controllability result for the bilinear control problem.

Theorem 2.3. *If the problem (13) is null-controllable in any $T > 0$ and there exist some constants $\nu, T_0 > 0, \gamma \geq 1$ such that*

$$(14) \quad K(\tau) \leq e^{\nu/\tau^\gamma}, \quad \forall 0 < \tau \leq T_0.$$

then, the dynamics (12) is locally controllable to the j -st eigensolution of A in any time $T > 0$.

The proof of the theorem is postponed to the appendix B and it directly follows from the techniques developed in [3]. Here, the authors ensure the same statement when we consider the parameter $\gamma = 1$. However, the proof of Theorem 2.3 does not really differ from such specific case, as explained in the appendix.

2.2 Null-controllability in filtered spaces using the moment method

Let $T > 0$ and $\Lambda > 0$. Recalling that we denote by

$$J_\Lambda := \{k \in \mathbb{N}, \lambda_k \leq \Lambda\}, \quad E_\Lambda := \text{span}\{\phi_k, k \in J_\Lambda\}.$$

We introduce a classical result of null-controllability on finite-dimensional spaces.

Proposition 2.4. *Let $\Lambda > 0$ and $j \in \mathbb{N}^*$. Assume that the operator B satisfies the following identities:*

$$\langle B\phi_j, \phi_k \rangle_X \neq 0, \quad \forall k \in J_\Lambda.$$

If there exists $(\sigma_{\Lambda, j})_{j \in J_\Lambda} \subset L^2((0, T), \mathbb{R})$ such that

$$\int_0^T e^{\lambda_k t} \sigma_{\Lambda, j}(t) dt = \delta_{j, k}, \quad \forall j, k \in J_\Lambda,$$

where $\delta_{j, k}$ denotes the Kronecker symbol, then the problem (13) is null-controllable in E_Λ in any time $T > 0$ with control cost

$$K_{E_\Lambda}(T)^2 \leq (\#J_\Lambda) \frac{\sup_{k \in J_\Lambda} \|\sigma_k\|_{L^2(0, T)}^2}{\inf_{k \in J_\Lambda} |\langle B\phi_j, \phi_k \rangle_X|^2}$$

Proof. By the Duhamel formula, the problem (13) is null-controllable in E_Λ and in a time $T > 0$, if there exists a function $v_\Lambda \in L^2((0, T), \mathbb{R})$ such that, for any $k \in J_\Lambda$, the following identities are verified:

$$\int_0^T e^{t\lambda_k} v_\Lambda(t) dt = d_k, \quad \forall k \in J_\Lambda,$$

where $d_k = -\frac{\langle \xi_0, \phi_k \rangle_X}{\langle B\phi_j, \phi_k \rangle_X}$. It is immediate that the function v_Λ exists for every $T > 0$ and it can be defined by the linear combination

$$v_\Lambda = \sum_{k \in J_\Lambda} d_k \sigma_{\Lambda, k}.$$

The estimation on v_Λ follows by writing

$$\begin{aligned} \|v_\Lambda\|_{L^2(0, T)}^2 &\leq (\#J_\Lambda) \sum_{k \in J_\Lambda} |d_k|^2 \|\sigma_{\Lambda, k}\|_{L^2(0, T)}^2 \leq (\#J_\Lambda) \frac{\sup_{k \in J_\Lambda} \|\sigma_{\Lambda, k}\|_{L^2(0, T)}^2}{\inf_{k \in J_\Lambda} |\langle B\phi_j, \phi_k \rangle_X|^2} \|\Pi_{E_\Lambda} \xi_0\|_X^2 \\ &\leq (\#J_\Lambda) \frac{\sup_{k \in J_\Lambda} \|\sigma_{\Lambda, k}\|_{L^2(0, T)}^2}{\inf_{k \in J_\Lambda} |\langle B\phi_j, \phi_k \rangle_X|^2} \|\xi_0\|_X^2. \end{aligned}$$

□

We recall the following proposition from [14] that gives a construction of biorthogonal families.

Proposition 2.5. [14, Proposition 3.3] *Let $(\nu_k)_{k \in \mathbb{N}^*}$ be a sequence of ordered non-negative real numbers such that $\sum_{k=2}^{+\infty} \nu_k^{-1}$. Assume that there exists $M \in \mathbb{N}^*$, $\gamma > 0$ and an ordered sequence of decreasing positive real numbers $(a_k)_{k \in \mathbb{N}^*}$ verifying, for every $k \in \mathbb{N}^*$,*

$$(15) \quad \begin{cases} \sqrt{\nu_{k+M}} - \sqrt{\nu_k} \geq \gamma, \\ \sqrt{\nu_{k+1}} - \sqrt{\nu_k} \geq a_k. \end{cases}$$

Then, there exists a sequence of functions $(\sigma_k)_{k \in \mathbb{N}^}$ which is biorthogonal to the family of exponentials $(e^{\nu_k t})_{k \in \mathbb{N}^*}$ in $L^2(0, T)$:*

$$\int_0^T \sigma_k(t) e^{\nu_j t} dt = \delta_{k, j}, \quad k, j \in \mathbb{N}^*.$$

Moreover, the biorthogonal family $(\sigma_k)_{k \in \mathbb{N}^}$ satisfies*

$$\|\sigma_k\|_{L^2(0, T)}^2 \leq C \left(1 + \frac{\gamma^2}{a_k(a_k + 2\sqrt{\nu_1})} \right)^{2M} e^{-2\nu_k T + \frac{C}{T\gamma^2} + \frac{C\sqrt{\nu_k}}{\gamma}} B(T, \gamma),$$

where

$$(16) \quad B(T, \gamma) := \begin{cases} \frac{1}{T} + \frac{1}{T^2\gamma^2}, & T \leq \frac{1}{\gamma^2} \\ C\gamma^2 & T > \frac{1}{\gamma^2}. \end{cases}$$

2.3 The Lebeau-Robbiano-Miller method

In this section, we present the adaptation due to Miller [26] of the well-known Lebeau-Robbiano method to prove the observability of the problem (13). First, we need to introduce the definition of partial observability, which is an adaptation of [24, Proposition 7.7] to our case.

Proposition 2.6. *Let E be a closed subspace of X . The null-controllability of the problem (13) in E at time $T > 0$ with control cost $K_E(T)$ is equivalent to the following observability estimate :*

$$\|e^{-TA} \Pi_E \zeta\|_X \leq K_E(T) \|\langle B\phi_j, e^{-tA} \Pi_E \zeta \rangle_X\|_{L^2(0, T)}, \quad \forall \zeta \in X.$$

Proof. Consider the bounded control operator, with $U = \mathbb{R}$.

$$\begin{aligned} M : U &\rightarrow X \\ v &\mapsto vB\phi_j. \end{aligned}$$

Then its adjoint is given by

$$\begin{aligned} M^* : X &\rightarrow U \\ z &\mapsto \langle B\phi_j, z \rangle_X. \end{aligned}$$

The proof follows by a direct application of [24, Proposition 7.7].

□

We are finally ready to show how to use the Lebeau-Robbiano-Miller method to prove the null-controllability of the problem (13) when partial observability inequalities are verified in suitable finite-dimensional subspaces of X and the control cost satisfies specific inequalities.

Theorem 2.7. *Assume that there exists $C_0 > 0$ and $\alpha \in (0, 1)$ such that for any $T \in (0, 1]$ and for any $\Lambda > 0$, the problem (13) is null-controllable in E_Λ at time $T > 0$ with control cost*

$$K_{E_\Lambda}(T) \leq C_0 e^{C_0 \left(\Lambda^\alpha + \frac{1}{T^{1-\alpha}} \right)}.$$

Then, there exists $C_1 > 0$ such that for any $T \in (0, 1)$, the problem (13) is null-controllable with control cost

$$K(T) \leq C_1 e^{\frac{C_1}{T^{1-\alpha}}}.$$

Proof. Let $\tau > 0$ and $0 < T_1 < T_2 < 1$ such that $\tau = T_2 - T_1$. Let $\varepsilon \in (0, 1)$ to be fixed below. We have, for any $\zeta \in X$, and any $\Lambda > 0$,

$$\|e^{-T_2 A} \zeta\|_X^2 = \|e^{-T_2 A} \Pi_{E_\Lambda} \zeta\|_X^2 + \|e^{-T_2 A} (I - \Pi_{E_\Lambda}) \zeta\|_X^2.$$

From Proposition 2.6, it follows

$$\|e^{-T_2 A} \zeta\|_X^2 \leq K_{E_\Lambda}(\varepsilon \tau)^2 \|\langle B \phi_j, e^{-tA} \Pi_{E_\Lambda} \zeta \rangle_X\|_{L^2(T_2 - \varepsilon \tau, T_2)}^2 + e^{-2\Lambda \tau} \|e^{-T_1 A} (I - \Pi_{E_\Lambda}) \zeta\|_X^2.$$

But, using the Cauchy-Schwarz inequality

$$\begin{aligned} \|\langle B \phi_j, e^{-tA} \Pi_{E_\Lambda} \zeta \rangle_X\|_{L^2(T_2 - \varepsilon \tau, T_2)}^2 &\leq 2 \|\langle B \phi_j, e^{-tA} \zeta \rangle_X\|_{L^2(T_1, T_2)}^2 + 2 \|\langle B \phi_j, e^{-tA} (I - \Pi_{E_\Lambda}) \zeta \rangle_X\|_{L^2(T_2 - \varepsilon \tau, T_2)}^2 \\ &\leq 2 \|\langle B \phi_j, e^{-tA} \zeta \rangle_X\|_{L^2(T_1, T_2)}^2 + 2 \|B \phi_j\|_X^2 \|e^{-tA} (I - \Pi_{E_\Lambda}) \zeta\|_{L^2((T_2 - \varepsilon \tau, T_2), X)}^2 \\ &\leq 2 \|\langle B \phi_j, e^{-tA} \zeta \rangle_X\|_{L^2(T_1, T_2)}^2 + 2\varepsilon \tau e^{-2\Lambda(1-\varepsilon)\tau} \|B \phi_j\|_X^2 \|e^{-T_1 A} (I - \Pi_{E_\Lambda}) \zeta\|_X^2. \end{aligned}$$

The above estimates yield the existence of $C > 0$ such that, for any $\tau > 0$, $0 < T_1 < T_2 < 1$ such that $\tau = T_2 - T_1$, $\varepsilon \in (0, 1)$, $\Lambda > 0$, and any $\zeta \in X$,

$$\|e^{-T_2 A} \zeta\|_X^2 \leq C e^{C \left(\Lambda^\alpha + \frac{1}{(\varepsilon \tau)^{1-\alpha}} \right)} \|\langle B \phi_j, e^{-tA} \zeta \rangle_X\|_{L^2(T_1, T_2)}^2 + \left(C e^{C \left(\Lambda^\alpha + \frac{1}{(\varepsilon \tau)^{1-\alpha}} \right)} e^{-2\Lambda(1-\varepsilon)\tau} + e^{-2\Lambda \tau} \right) \|e^{-T_1 A} \zeta\|_X^2.$$

Now, we choose $\Lambda = \frac{1}{(\varepsilon \tau)^{\frac{1}{1-\alpha}}}$ and there exists $\tilde{C} > 0$ independent on ε , τ , and ζ such that

$$\|e^{-T_2 A} \zeta\|_X^2 \leq \tilde{C} e^{\tilde{C} \frac{1}{(\varepsilon \tau)^{\frac{1}{1-\alpha}}}} \|\langle B \phi_j, e^{-tA} \zeta \rangle_X\|_{L^2(T_1, T_2)}^2 + \left(\tilde{C} e^{\tilde{C} \frac{1}{(\varepsilon \tau)^{\frac{1}{1-\alpha}}} e^{-2 \frac{(1-\varepsilon)}{\varepsilon^{1-\alpha} \tau^{1-\alpha}}} + e^{-2 \frac{1}{\varepsilon^{1-\alpha} \tau^{1-\alpha}}} \right) \|e^{-T_1 A} \zeta\|_X^2.$$

Let $f_\varepsilon(\tau) = \tilde{C}^{-1} e^{-\tilde{C} \frac{1}{(\varepsilon \tau)^{\frac{1}{1-\alpha}}}}$. We fix $\varepsilon \in (0, 1)$ sufficiently small so that the following approximate observability holds : there exists $\tilde{C} > 0$ such that, for any $\tau \in (0, 1)$ and $0 < T_1 < T_2$ satisfying $\tau = T_2 - T_1$, we have

$$f_\varepsilon(\tau) \|e^{-T_2 A} \zeta\|_X^2 \leq \|\langle B \phi_j, e^{-tA} \zeta \rangle_X\|_{L^2(T_1, T_2)}^2 + f_\varepsilon\left(\frac{\tau}{2}\right) \|e^{-T_1 A} \zeta\|_X^2.$$

For any $T \in (0, 1)$ and $k \in \mathbb{N}^*$, we choose $T_2 = T/2^k$ and $T_1 = T/2^{k+1}$ with $k \in \mathbb{N}$, and we obtain

$$f_\varepsilon(T/2^k) \|e^{-T/2^k A} \zeta\|_X^2 \leq \|\langle B \phi_j, e^{-tA} \zeta \rangle_X\|_{L^2(T/2^{k+1}, T/2^k)}^2 + f_\varepsilon(T/2^{k+1}) \|e^{-T/2^{k+1} A} \zeta\|_X^2.$$

Summing over $k \in \mathbb{N}$ yields the following observability estimate

$$\|e^{-T A} \zeta\|_X^2 \leq f_\varepsilon^{-1}(T) \|\langle B \phi_j, e^{-tA} \zeta \rangle_X\|_{L^2(0, T)}^2.$$

Finally, it is sufficient to apply Proposition 2.6 to end the proof. \square

3 Controllability with simple spectrum

The aim of this section is to prove the exact controllability to eigensolutions when the spectrum of the Dirichlet Laplacian is simple. In detail, we provide the proofs of Main Theorem A and Main Theorem B, and their corollaries.

Remark 3.1. Before moving on, we want to underline that the choice of $p_1 > 3/2$ in Assumptions I is reasonable for this problem. Indeed, when Ω is smooth, for every $\psi \in H^2(\Omega, \mathbb{R})$, we know

$$\psi \phi_l \in H^2(\Omega, \mathbb{R}) \cap H_0^1(\Omega, \mathbb{R}) = \left\{ \psi \in H^2 : (\lambda_k \langle \phi_k, \psi \rangle_{L^2})_{k \in \mathbb{N}^*} \in \ell^2 \right\},$$

for every $l \in \mathbb{N}^*$. This last property and the Weyl's asymptotics (3) imply the existence of $C > 0$ such that

$$|\langle \phi_k, f \phi_l \rangle_{L^2}| \leq \frac{C}{\lambda_k^{1+1/2}}, \quad \forall k \in \mathbb{N}^*.$$

The same argument leads to the choice of the parameter $5/2$ in Assumptions II and Assumptions III.

3.1 Some spectral properties and well-posedness

Let us study the spectral behaviour of the Laplacian operator on a two-dimensional Ω . Thanks to the Weyl's asymptotics (3), we know that the eigenvalues of $-\Delta$ are such that $\lambda_k/k = O(1)$. This property implies the existence of an increasing sequence $(\tilde{n}_k)_{k \in \mathbb{N}^*} \subset \mathbb{N}^*$ such that

$$(17) \quad \inf_{k \in \mathbb{N}^*} \lambda_{k+\tilde{n}_k} - \lambda_k > 0.$$

Actually, we can ensure a stronger spectral result with the following proposition.

Proposition 3.2. *There exists $\tilde{C} > 0$ such that the sequence*

$$n_k = \lfloor \tilde{C} \sqrt{k} \rfloor + 1, \quad \forall k \in \mathbb{N}^*,$$

verifies the following spectral gap condition

$$(18) \quad \lambda_{k+n_k} - \lambda_k \geq \sqrt{k}, \quad \forall k \in \mathbb{N}^*.$$

Proof. Recalling (2) and (3), there exists $C > 0$ such that $\sqrt{\lambda_k} \leq C\sqrt{k}$ and

$$N_{\lambda_k + \sqrt{k}} - N_{\lambda_k} \leq C_1 \sqrt{k} + C_2 \left(\sqrt{\lambda_k} + \sqrt{\lambda_k + \sqrt{k}} \right) \leq \left(C_1 + C_2 \left(C + \sqrt{C^2 + 1} \right) \right) \sqrt{k}, \quad \forall k \in \mathbb{N}^*.$$

Notice that $N_{\lambda_k + \sqrt{k}} - N_{\lambda_k}$ is the number eigenvalues between λ_k and $\lambda_k + \sqrt{k}$. Hence, if we want to obtain a spectral gap larger or equal to \sqrt{k} , then we can take $n_k = \lfloor (C_1 + C_2(C + \sqrt{C^2 + 1}))\sqrt{k} \rfloor + 1$ for every $k \in \mathbb{N}^*$ to ensure the result. \square

We apply now Proposition 3.2 to ensure the existence of a spectral gap for the square for the eigenvalues of the Dirichlet Laplacian.

Proposition 3.3. *There exists $\tilde{C} > 0$ such that the sequence*

$$n_k = \lfloor \tilde{C} \sqrt{k} \rfloor + 1, \quad \forall k \in \mathbb{N}^*,$$

verifies the following spectral gap condition

$$(19) \quad \inf_{k \in \mathbb{N}^*} \sqrt{\lambda_{k+n_k}} - \sqrt{\lambda_k} > 0.$$

Proof. Notice that thanks to (3), there exists $C > 0$ such that $\sqrt{\lambda_k} > C\sqrt{k}$ and

$$\sqrt{\lambda_{k+n_k}} - \sqrt{\lambda_k} = \frac{\lambda_{k+n_k} - \lambda_k}{\sqrt{\lambda_{k+n_k}} + \sqrt{\lambda_k}} \geq \frac{\lambda_{k+n_k} - \lambda_k}{2C\sqrt{k+n_k}}, \quad \forall k \in \mathbb{N}^*.$$

We consider Proposition 3.2. Thanks to the validity of (18), the result yields since

$$\sqrt{\lambda_{k+n_k}} - \sqrt{\lambda_k} \geq \frac{\sqrt{k}}{2C\sqrt{k + \tilde{C}\sqrt{k}}}, \quad \forall k \in \mathbb{N}^*.$$

\square

Before moving on with the theory leading to Main Theorem A, we provide the well-posedness result of (BHE). It is a classical result that we rephrase in the next proposition (see for instance in [7]).

Proposition 3.4. *Let $T > 0$, $q \in \mathbb{N}^*$, $Q = (Q_1, \dots, Q_q) \in L^2(\Omega, \mathbb{R}^q)$ and $v = (v_1, \dots, v_q) \in L^2((0, T), \mathbb{R}^q)$. For any $\psi_0 \in L^2(\Omega, \mathbb{R})$, there exists a unique mild solution $\psi \in C^0([0, T], L^2(\Omega, \mathbb{R}))$ of (BHE) such that*

$$\psi(t; \psi_0, u) = e^{t\Delta} \psi_0 - \int_0^t e^{(t-s)\Delta} \langle v(s), Q \rangle \psi(s; \psi_0, v) ds, \quad \forall t \in [0, T].$$

Moreover, there exists a constant $C(T) > 0$ such that $\sup_{t \in [0, T]} \|\psi(t; \psi_0, v)\|_{L^2} \leq C(T) \|\psi_0\|_{L^2}$.

3.2 Proof of Main Theorem A and Corollary 1.1

This section aims to prove Main Theorem A by using the theory exposed in Section 2. We prove the local controllability of the equation (BHE) to the eigensolution of $-\Delta$ from the null-controllability of a suitable linear problem. The null-controllability is ensured in two steps as follows.

- We first prove the null-controllability on a finite-dimensional subspace (Section 2.2) and we establish suitable bounds for the control cost by using the estimates from Proposition 2.5.
- Second, the null-controllability is proved on the whole $L^2(\Omega, \mathbb{R})$ thanks to the Lebau-Robbiano-Miller method from Proposition 2.6 and Theorem 2.7 (Section 2.3).

Let us introduce the following linear problem in $L^2(\Omega, \mathbb{R})$

$$(20) \quad \begin{cases} \xi'(t) - \Delta \xi(t) + v_1(t) Q_1 \phi_j = 0, & t \in (0, T), \\ \xi(0) = \xi_0. \end{cases}$$

The (mild) solution of (20) is denoted

$$\xi(t; \xi_0, v) = e^{t\Delta} \xi_0 + \int_0^t e^{(t-s)\Delta} v_1(s) Q_1 \phi_j ds.$$

Let $\Lambda > 0$. In this framework, J_Λ and E_Λ introduced in Section 2.2 become

$$J_\Lambda = \{n \in \mathbb{N}, \lambda_n \leq \Lambda\}, \quad E_\Lambda := \text{span}\{\phi_n, n \in J_\Lambda\}.$$

We want to exploit the theory exposed in Section 2.1 and for this purpose we need to prove the following result.

Proposition 3.5. *Let Q verify Assumptions I and let $\alpha \in (1/2, 1)$. There exists $C > 0$ such that for any $T \in (0, 1]$ and for any $\Lambda > 0$, the problem (20) is null-controllable in E_Λ at time $T > 0$ with control cost*

$$K_{E_\Lambda}(T) \leq C e^{C(\Lambda^\alpha + \frac{1}{T})}.$$

Proof. Assume $\Lambda > \lambda_1$ otherwise the statement of the theorem is empty. In our hypotheses, the operator $-\Delta$ acting on $X = L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ falls into the scope of Section 2. We set $s_\Lambda = \lfloor \tilde{C}\sqrt{\Lambda} \rfloor + 1$ with $\tilde{C} > 0$ defined in Proposition 3.3. We introduce the sequence $(\nu_k)_{k \in \mathbb{N}^*} \subset \mathbb{R}^+$ such that

$$\begin{cases} \nu_k = \lambda_k, & \forall k \in \mathbb{N}^* : \lambda_k \leq \Lambda, \\ \nu_k = \Lambda \exp(2k), & \forall k \in \mathbb{N}^* : \lambda_k > \Lambda. \end{cases}$$

Notice the existence of $C > 0$ such that, when $\nu_{k+s_\Lambda} \leq \Lambda$, there holds

$$\sqrt{\nu_{k+s_\Lambda}} - \sqrt{\nu_k} \geq \sqrt{\nu_{k+n_k}} - \sqrt{\nu_k} \geq C > 0,$$

thanks to Proposition 3.3. In addition, when $\nu_{k+s_\Lambda} \geq \Lambda$, we have

$$\sqrt{\nu_{k+s_\Lambda}} - \sqrt{\nu_k} \geq \sqrt{\Lambda} \exp(k + s_\Lambda) - \sqrt{\Lambda} \exp(k + s_\Lambda - 1) \geq \sqrt{\Lambda}.$$

The two last inequalities imply

$$\inf_{k \in \mathbb{N}^*} \sqrt{\nu_{k+s_\Lambda}} - \sqrt{\nu_k} > 0.$$

Now, the definition of the sequence $(\nu_k)_{k \in \mathbb{N}^*}$ and Assumptions I yield the existence of $C, p_0 > 0$ such that

$$\sqrt{\nu_{k+1}} - \sqrt{\nu_k} \geq \frac{C}{\nu_k^{p_0}}, \quad \forall k \in \mathbb{N}^*.$$

Then, we can use from Proposition 2.5 w.r.t to the sequence $(\nu_k)_{k \in \mathbb{N}^*}$ and show the existence of $\sigma_{\Lambda, k} \in L^2(0, T)$ satisfying

$$\int_0^T e^{\lambda_j^u t} \sigma_{\Lambda, k}(t) dt = \delta_{j, k}, \quad j, k \in J_\Lambda,$$

and verifying the following estimate

$$\|\sigma_{\Lambda, k}\|_{L^2(0, T)}^2 \leq C \left(\nu_k^{p_0} \right)^{s_\Lambda} \exp \left(C \left(\nu_k^{1/2} + \frac{1}{T} \right) \right)$$

with a suitable $C > 0$ independent on $k \in \mathbb{N}^*$, $\Lambda > 0$ and $T > 0$. Now, we observe that

$$\nu_k^{p_0 s \Lambda} \leq \Lambda^{p_0(\tilde{C}\sqrt{\Lambda}+1)} = e^{p_0(\tilde{C}\sqrt{\Lambda}+1)\log(\Lambda)}.$$

For any $\delta \in (0, 1/2)$, there exists $C > 0$ independent on $k \in \mathbb{N}^*$, $\Lambda > 0$ and $T > 0$ such that

$$(21) \quad \|\sigma_{\Lambda, k}\|_{L^2(0, T)}^2 \leq C \exp\left(C\left(\Lambda^{1/2+\delta} + \frac{1}{T}\right)\right).$$

It is then sufficient to apply Proposition 2.4 along with (21) to obtain the sought controllability result. The proof is concluded since, thanks to Assumptions I, the control cost is such that there exists $C_1, C_2 > 0$ such that

$$K(T) \leq \frac{C_1 \exp\left(C_1\left(1 + \Lambda^{1/2+\delta} + \frac{1}{T}\right)\right)}{\inf_{k \in J_\Lambda} |\langle B\phi_j, \phi_k \rangle_{L^2}|} \leq C_2 \exp\left(C_2\left(1 + \Lambda^{1/2+\delta} + \frac{1}{T}\right)\right).$$

In the last inequality, we used $\#J_\Lambda = N_\Lambda = O(\Lambda)$ thanks to the Weyl asymptotic (1). \square

Corollary 3.6. *Let Q verify Assumptions I and let $\gamma > 1$. There exists $C > 0$ such that for any $T \in (0, 1]$ the problem (20) is null-controllable at time $T > 0$ with control cost*

$$K(T) \leq C e^{\frac{C}{T^\gamma}}.$$

Proof. It is sufficient to combine Theorem 2.7 with Proposition 3.5 and notice that when $\alpha = \frac{1}{2} + \delta$, $0 < \delta < \frac{1}{2}$, then $\frac{\alpha}{1-\alpha} = 1 + \frac{\delta}{1-2\delta}$. \square

We are finally ready to prove Main Theorem A and Corollary 1.1.

Proof of Main Theorem A. The exact controllability to the eigensolutions of (BHE) ensured by Main Theorem A is proved by directly applying Corollary 3.6 along with Theorem 2.3. \square

Proof of Corollary 1.1. The property follows from Main Theorem A and the linearity. Indeed, if we control $\frac{1}{\alpha}\psi_0$ to $\Phi_j(T_1)$ in a time T_1 thanks to Main Theorem A with a control v_1 , then

$$\psi(T_1; \psi_0, v_1) = \alpha\Phi_j(T_1).$$

Afterwards, since the function $Q_2 = 1$, for any $T_2 > 0$, it is possible to define a constant control $v_2 = (0, u_2)$ with $u_2 \in \mathbb{R}$ such that $\log(\alpha/\beta) - (T_1 + T_2)\lambda_j - T_2 u_2 = 0$. We consider the control

$$v(\cdot) = v_1(\cdot)\mathbb{1}_{[0, T_1]}(\cdot) + v_2(\cdot - T_1)\mathbb{1}_{[T_1, T_1+T_2]}(\cdot)$$

and we ensure Corollary 1.5 by choosing $T = T_1 + T_2$ since

$$\psi(T; \psi_0, v) = \psi(T_2; \psi(T_1; \psi_0, v_1), v_2) = e^{-T_2(\Delta+u_2)}\alpha\Phi_j(T_1) = \beta e^{\log(\alpha/\beta)} e^{-(T_1+T_2)\lambda_j - T_2 u_2} \phi_j = \beta\phi_j.$$

\square

3.3 Proof of Main Theorem B and Corollary 1.3

The aim of this subsection is to prove the exact controllability to the eigensolutions in the specific case of the rectangle, when Assumptions II are verified. We just show that Assumptions II implies Assumptions I and then, Main Theorem B is an explicit application of Main Theorem A. Before entering into the details of the proof, we provide an example of rectangle Ω and potential Q such that Assumption II are satisfied.

Remark 3.7. *Consider the bilinear heat equation (BHE) on the rectangle $\Omega = (0, 1) \times (0, \sqrt[3]{2})$. Notice that the number $2^{-\frac{2}{3}}$ is algebraic and irrational, which implies the validity of the first point of Assumptions II. Assume now*

$$Q = \left(\frac{x^2 y^2}{2}, 1\right).$$

We have the validity of the second point of Assumptions II since, for every $m \in \mathbb{N}^*$, we have

$$\langle \sin(k\pi x), x^2 \sin(m\pi x) \rangle_{L^2(0,1)} = \left\langle \sin\left(\frac{k}{2^{\frac{1}{3}}}\pi y\right), \frac{y^2}{2} \sin\left(\frac{m}{2^{\frac{1}{3}}}\pi y\right) \right\rangle_{L^2(0, 2^{\frac{1}{3}})} = \frac{4(-1)^{k+m} k m}{(k^2 - m^2)^2 \pi^2} = O\left(\frac{1}{k^3}\right).$$

The same type of results can be easily obtained with other polynomials. At each step, the integration by part allows us to prove that the previous integrals always are asymptotically polynomially decreasing.

We prove now the validity of a suitable spectral gap for the spectrum for the Dirichlet Laplacian in the following lemma when Assumptions II are verified.

Lemma 3.8. *Let us consider $\Omega = (0, a) \times (0, b)$ with a^2/b^2 an algebraic irrational number. Then, for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \frac{C_\epsilon}{\lambda_k^{\frac{3+2\epsilon}{2}}}, \quad \forall k \in \mathbb{N}^*.$$

Proof. First, Roth's Theorem (see [29]) implies that, when z is an algebraic irrational number, we have that, for every $\epsilon > 0$, there exists $C_\epsilon > 0$ small enough such that

$$\left| z - \frac{n}{m} \right| \geq \frac{C}{m^{2+\epsilon}}, \quad \forall m, n \in \mathbb{Z}^*.$$

We recall the definition of the sequences $(l_k)_{k \in \mathbb{N}^*}, (m_k)_{k \in \mathbb{N}^*} \subset \mathbb{N}^*$ provided in (4). Now, a^2/b^2 is an algebraic irrational number and, for each $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\begin{aligned} |\lambda_{k+1} - \lambda_k| &= \left| \left(\frac{l_{k+1}^2 \pi^2}{a^2} + \frac{m_{k+1}^2 \pi^2}{b^2} \right) - \left(\frac{l_k^2 \pi^2}{a^2} + \frac{m_k^2 \pi^2}{b^2} \right) \right| = \pi^2 \left| \frac{l_{k+1}^2 - l_k^2}{a^2} - \frac{m_k^2 - m_{k+1}^2}{b^2} \right| \\ &= \frac{\pi^2 |m_k^2 - m_{k+1}^2|}{a^2} \left| \frac{l_{k+1}^2 - l_k^2}{m_k^2 - m_{k+1}^2} - \frac{a^2}{b^2} \right| \geq \frac{\pi^2 |m_k^2 - m_{k+1}^2| C_\epsilon}{a^2 |m_k^2 - m_{k+1}^2|^{2+\epsilon}} \geq \frac{\pi^2 C_\epsilon}{a^2 \lambda_{k+1}^{1+\epsilon}}. \end{aligned}$$

Finally, thanks to the Weyl asymptotics (3), there exists $C > 0$ such that $\lambda_{k+1} \leq C \lambda_k$ and

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = \frac{\lambda_{k+1} - \lambda_k}{\sqrt{\lambda_{k+1}} + \sqrt{\lambda_k}} \geq \frac{\pi^2 C_\epsilon}{a^2 \lambda_{k+1}^{\frac{3+2\epsilon}{2}}} \geq \frac{\pi^2 C_\epsilon}{a^2 C^{3+2\epsilon} \lambda_k^{\frac{3+2\epsilon}{2}}}, \quad \forall k \in \mathbb{N}^*.$$

□

We are now in position to discuss the proof of Main Theorem B.

Proof of Main Theorem B. Notice that thanks to Lemma 3.8, the first point of Assumptions II implies the first point of Assumptions I. Second, we recall the Weyl asymptotics (3). Hence, we observe that from the definition of the eigenfunctions in the rectangular framework, the second point of Assumptions II implies the second point of Assumptions I. The proof then follows from the same approach leading to Main Theorem A. □

Proof of Corollary 1.3. The result follows from Main Theorem B as in Corollary 1.1 is due to Main Theorem A. □

4 Controllability with multiple spectrum on the square

This section aims to prove Main Theorem C by using the theory exposed in Section 2. First, we recall the validity of Proposition 3.4 and, when Assumptions III are verified, we have the following additional explicit formula for the mild solution of (BHE)

$$\psi(t; \psi_0, u) = e^{-t(-\Delta + uQ_1)} \psi_0 - \sum_{l=2}^4 \int_0^t e^{-(t-s)(-\Delta + uQ_1)} v_l(s) Q_l \psi(s; \psi_0, v) ds, \quad \forall t \in [0, T].$$

Remark 4.1. *We notice that the control potential*

$$Q = ((3x^2 - 1), x^2 y^2, (3x^2 - 1), 1)$$

verifies Assumptions III. Indeed, the first point is obviously verified. After, for every distinct $k, n \in \mathbb{N}^$, we have*

$$\langle \cos(2k\pi x), (3x^2 - 1) \rangle_{L^2} = \frac{3}{2\pi^2 k^2} \neq \frac{3}{2\pi^2 n^2} = \langle \cos(2n\pi x), (3x^2 - 1) \rangle_{L^2},$$

which implies the validity of the second point. The third point yields with $p_1 = p_2 = 3$ since, for every $k \neq m$,

$$\langle \sin(k\pi x), x^2 \sin(m\pi x) \rangle_{L^2} = \frac{4(-1)^{m+k} m k}{\pi^2 (k^2 - m^2)^2} = O\left(\frac{1}{k^3}\right),$$

$$\langle \sin(m\pi x), x^2 \sin(m\pi x) \rangle_{L^2} = \frac{2\pi^2 m^2 - 3}{12\pi^2 m^2} \neq 0.$$

The fourth condition is verified as $(3x^2 - 1), x^2 \in H^1(0, 1)$ and finally the last point follows from

$$\langle \sin(k\pi x), (3x^2 - 1) \sin(m\pi x) \rangle_{L^2} = 3 \langle \sin(k\pi x), x^2 \sin(m\pi x) \rangle_{L^2} - \delta_{k,m}, \quad \forall k, m \in \mathbb{N}^*.$$

4.1 Spectral properties

We consider the operator $-\Delta + uQ_1$ defined on $H^2(\Omega) \cap H_0^1(\Omega)$ with $\Omega = (0, 1) \times (0, 1)$ when Q verifies Assumptions III and in particular when Q_1 is constant w.r.t. the second coordinate, *i.e.*

$$Q_1(x, y) = Q_1(x).$$

We assume $u \in \mathbb{R}$ being sufficiently small so that uQ_1 plays the role of perturbation of the Dirichlet Laplacian. We start by clarifying the notation adopted in the introduction.

- We recall that we respectively denote the eigenvalues of $-\Delta + uQ_1$ and a Hilbert basis of $L^2(\Omega)$ made by eigenfunctions:

$$(\lambda_{l,m}^u)_{l,m \in \mathbb{N}^*} \quad \text{and} \quad (\phi_{l,m}^u)_{k \in \mathbb{N}^*}.$$

- Thanks to Assumptions III, $Q_1(x, y) = Q_1(x)$ is constant w.r.t. the second coordinate and

$$\lambda_{l,m}^u = \mu_l^u + \mu_m^0, \quad \phi_{l,m}^u(x, y) = f_l^u(x)f_m^0(y),$$

where $(\mu_k^u, f_k^u)_{k \in \mathbb{N}^*}$ are the eigenpairs of the operator $A^u = -\partial_x^2 + uQ_1$ defined in

$$(22) \quad D(A^u) := H^2((0, 1), \mathbb{R}) \cap H_0^1((0, 1), \mathbb{R}).$$

- Clearly, when $u = 0$, the operator A^0 is the one-dimensional Dirichlet Laplacian and

$$\mu_k^0 = k^2\pi^2, \quad f_k^0(\cdot) = \sqrt{2} \sin(k\pi\cdot),$$

which implies

$$\lambda_{l,m}^u = \mu_l^u + m^2\pi^2, \quad \phi_{l,m}^u(x, y) = f_l^u(x)\sqrt{2} \sin(m\pi y).$$

- We introduce the eigenpairs $(\lambda_{l,m}^0, \phi_{l,m}^0)_{l,m \in \mathbb{N}^*}$ of the operator $-\Delta$ without ambiguity as follows

$$\lambda_{l,m}^0 = l^2\pi^2 + m^2\pi^2, \quad \phi_{l,m}^u(x, y) = 2 \sin(l\pi x) \sin(m\pi y).$$

- To distinguish the ordering of the eigenvalues, we denote the ordered eigenvalues of $-\Delta + uQ_1$:

$$\lambda_k^u$$

and we notice that there exists $(l_k)_{k \in \mathbb{N}^*}, (m_k)_{k \in \mathbb{N}^*} \subseteq \mathbb{N}^*$ such that

$$\lambda_k^u = \lambda_{l_k, m_k}^u.$$

- The asymptotic Weyl's law yields $\lim_{k \rightarrow +\infty} \frac{\lambda_k^0}{k} = O(1)$. Such asymptotics can also be extended to the perturbed eigenvalues so that $\lim_{k \rightarrow +\infty} \frac{\lambda_k^u}{k} = O(1)$ (see Proposition 4.3)

4.2 Perturbation theory

We now some spectral properties of the operator $-\Delta + uQ_1$ when $u \in \mathbb{R}$ is sufficiently small and Q_1 is a perturbation of the Dirichlet Laplacian.

Proposition 4.2. *Let Q verify Assumptions III. For every $u > 0$, we have*

$$(23) \quad \mu_k^u = k^2\pi^2 - r_k^u,$$

where

$$(24) \quad r_k^u = u \int_0^1 \langle (f_k^u)^2, Q_1(x) \rangle_{L^2} dt = u \int_0^1 \cos(2k\pi x) Q_1(x) dx + O(|u|)O\left(\frac{1}{k}\right).$$

In addition, the following map is real analytic

$$u \in \mathbb{R} \mapsto (\mu_k^u)_{k \in \mathbb{N}^*} \in \left\{ (a_k)_{k \in \mathbb{N}^*} \subset \mathbb{R} : a_k = k^2\pi^2 + b_k, (b_k)_{k \in \mathbb{N}^*} \in \ell^2 \right\}.$$

Finally, each map $u \in \mathbb{R} \mapsto f_k^u = \sqrt{2} \sin(k\pi\cdot) + O(|u|)O\left(\frac{1}{k}\right) \in L^2((0, 1), \mathbb{R})$ is analytic.

Proof of 4.2. The result follows from [27]. Indeed, the characterization of the eigenvalues μ_k^u follows from [27, Theorem 2.4 and Theorem 3.1] (and their proofs; see [27, relation (3)] for the first identity in (24)). The formula of the corresponding eigenfunctions and the analyticity are due to [27, Theorem 2.3 & Theorem 2.4]. \square

Proposition 4.3. *Let Q verify Assumptions III. For $u > 0$ sufficiently small, there exist $C_1, C_2 > 0$ such that*

$$C_1 \mu_k^0 \leq \mu_k^u \leq C_2 \mu_k^0, \quad \forall k \in \mathbb{N}^*.$$

Moreover, we have

$$\min\{C_1, 1\} \lambda_k^0 \leq \lambda_k^u \leq \max\{C_2, 1\} \lambda_k^0, \quad \forall k \in \mathbb{N}^*.$$

Proof. The inequalities are a direct consequence of (23) and Proposition 4.2. \square

Now, we show that it is possible to perturb the spectrum of the Dirichlet Laplacian in order to have simple eigenvalues verifying the weak spectral gap introduced in the first point Assumptions I. We also recall that thanks to Proposition 3.3, for a suitable increasing sequence composed by numbers $n_j = O(\sqrt{j})$, we have

$$\inf_{j \in \mathbb{N}^*} \sqrt{\lambda_{j+n_j}^0} - \sqrt{\lambda_j^0} > 0.$$

We want to prove that such a property is also verified for the perturbed eigenvalues when u is sufficiently small.

Proposition 4.4. *Let Q verify Assumptions III. For every $\epsilon > 0$, there exists a countable subset V of \mathbb{R} and $R > 0$ such that, for any $u \in [-R, R] \setminus V$, the following properties are verified.*

1. *There exist $C, p > 0$ such that*

$$\begin{aligned} \left| \sqrt{\lambda_{j+1}^u} - \sqrt{\lambda_j^u} \right| &\geq \frac{C}{j^p}, \quad j \in \mathbb{N}^*. \\ \inf_{j \in \mathbb{N}^*} \left| \sqrt{\lambda_{j+n_j}^u} - \sqrt{\lambda_j^u} \right| &> 0. \end{aligned}$$

2. *We have*

$$\|\phi_{k,j}^u - \phi_{k,j}^0\|_{L^2} < \epsilon.$$

Proof. We refer to Appendix A for the proof of Proposition 4.4. \square

Proposition 4.5. *Let Q satisfy Assumptions III and $j \in \mathbb{N}^*$. There exist $R, C > 0$ such that, for any $u \in [-R, R]$, we have*

$$|\langle f_k^u, Q_2^1 f_j^u \rangle_{L^2}| \geq \frac{C}{k^{p_1}}, \quad \forall k \in \mathbb{N}^*.$$

Proof. We refer to Appendix A for the proof of Proposition 4.5. \square

4.3 Proof of Main Theorem C: first step

We are finally ready to use the techniques leading to Main Theorem A to prove Main Theorem C. The first step is the local controllability of equation (BHE) to the eigensolution of $-\Delta + uQ_1$ in a time T_1 which is

$$\Phi_{l,m}^u(T_1) := e^{-\lambda_{l,m}^u T_1} \phi_{l,m}^u = e^{-(\mu_l^u + m^2 \pi^2) T_1} f_l^u(x) \sqrt{2} \sin(m\pi y)$$

where (f_i^u, μ_i^u) is the i -th eigenpair of the operator $A^u = -\partial_x^2 + uQ_1$ with domain $D(A^u) = H^2 \cap H_0^1((0, 1), \mathbb{R})$. The result is proved as in Main Theorem A when u is small and Assumptions III are verified. In details, the result is ensured by proving the null-controllability of the following linear problem in $L^2(\Omega, \mathbb{R})$

$$(25) \quad \begin{cases} \xi'(t) + (-\Delta + uQ_1)\xi(t) + \sum_{k=2}^4 v_k(t) Q_k \phi_{l,m}^u = 0, & t \in (0, T), \\ \xi(0) = \xi_0. \end{cases}$$

The (mild) solution of (13) is denoted

$$\xi(t; \xi_0, v) = e^{-t(-\Delta + uQ_1)} \xi_0 + \sum_{k=2}^4 \int_0^t e^{-(t-s)(-\Delta + uQ_1)} v_k(s) Q_k \phi_{l,m}^u ds.$$

We remind the existence of two sequences $(l_n)_{n \in \mathbb{N}^*}, (m_n)_{n \in \mathbb{N}^*} \subset \mathbb{N}^*$ such that each eigenvalue

$$\lambda_n^u = \lambda_{l_n, m_n}^u \quad \text{corresponds to the eigenfunction} \quad \phi_{l_n, m_n}^u.$$

Let $\Lambda > 0$, J_Λ and E_Λ (introduced in Section 2.2) such that

$$J_\Lambda = \{n \in \mathbb{N}, \lambda_n^u \leq \Lambda\}, \quad E_\Lambda := \text{span}\{\phi_{l_n, m_n}^u, n \in J_\Lambda\}.$$

We want to exploit the theory exposed in Section 2.1 and for this purpose we need to prove the following result.

Proposition 4.6. *Let Q verify Assumptions III and let $\alpha > 1/2$. For every $l, m \in \mathbb{N}^*$, there exist $R > 0$ small and a countable subset V of \mathbb{R} such that for any $u \in [-R, R] \setminus V$, there exists $C > 0$ such that for any $T \in (0, 1]$ and for any $\Lambda > 0$, the problem (25) is null-controllable in E_Λ at time $T > 0$ with control cost*

$$K_{E_\Lambda}(T) \leq Ce^{C(\Lambda^\alpha + \frac{1}{T})}.$$

Proof. The result is ensured as Proposition 3.5 by considering the validity of Proposition 4.3, Proposition 4.4 and Proposition 4.5. \square

Corollary 4.7. *Let Q verify Assumptions III and let $\gamma > 1$. For every $l, m \in \mathbb{N}^*$, there exist $R > 0$ small and a countable subset V of \mathbb{R} such that for any $u \in [-R, R] \setminus V$, there exists $C > 0$ such that for any $T \in (0, 1]$ the problem (25) is null-controllable at time $T > 0$ with control cost*

$$K(T) \leq Ce^{\frac{C}{T^\gamma}}.$$

Proof. As in the proof of Corollary 3.6, we combine Theorem 2.7 with Proposition 4.6 when $\alpha = \frac{1}{2} + \delta$, $0 < \delta < \frac{1}{2}$, and then $\frac{\alpha}{1-\alpha} = 1 + \frac{\delta}{1-2\delta}$. \square

We are finally ready to prove the local controllability of (BHE) to the eigensolutions of $(-\Delta + uQ_1)$.

Proposition 4.8. *Let Q verify Assumptions III. For every $l, m \in \mathbb{N}^*$, there exist $R > 0$ small and a countable subset V of \mathbb{R} such that, for any $u \in [-R, R] \setminus V$, the following result is verified. For any $T > 0$, there exists $\delta > 0$ such that, for any $\psi_0 \in L^2(\Omega, \mathbb{R})$ with*

$$\|\psi_0 - \phi_{l,m}^u\|_{L^2} < \delta,$$

there exists $v_2 \in L^2((0, T), \mathbb{R})$ such that $\psi(T; \psi_0, v)$, the solution of (BHE) with control $v = (u, v_2, 0, 0)$, verifies

$$\psi(T; \psi_0, v) = \Phi_{l,m}^u(T).$$

Proof. The result is direct consequence of Propositions 2.3 and Corollary 4.7. \square

4.4 Proof of Main Theorem C: second step

The second step of the proof of Main Theorem C is the following. Consider the problem (BHE) with control $v = (0, 0, v_3, 0)$ and initial state $\psi_0(x, y) = \psi_0^1(x)\psi_0^2(y)$. Thanks to the separation of variables, the solution ψ of (BHE) can be decomposed as

$$\psi(t, x, y) = \psi^1(t, x)\psi^2(t, y),$$

where ψ^1 and ψ^2 are respectively the solutions of the uncoupled equations

$$(26) \quad \begin{cases} \partial_t \psi^1(t, x) - \partial_x^2 \psi^1(t, x) + v_3(t)Q_3(x)\psi^1(t, x) = 0, & t \in (0, T), \quad x \in (0, 1), \\ \psi^1(t, 0) = \psi^1(t, 1) = 0, & t \in (0, T), \\ \psi^1(0) = \psi_0^1, \end{cases}$$

$$(27) \quad \begin{cases} \partial_t \psi^2(t, y) - \partial_y^2 \psi^2(t, y) = 0, & t \in (0, T), \quad y \in (0, 1), \\ \psi^2(t, 0) = \psi^2(t, 1) = 0, & t \in (0, T), \\ \psi^2(0) = \psi_0^2. \end{cases}$$

Notice that (26) is just a one-dimensional heat equation in the presence of a bilinear control, while the second is a free dynamics. Before proceeding with the proof of Main Theorem C, we need to recall the following local controllability result for the problem (26) ensured in [3] (see [3, Section 5.1]).

Proposition 4.9 ([3]). *Let $Q_3(x)$ be such that, for fixed $j \in \mathbb{N}^*$, there exist $C > 0$ and $p \geq 5/2$ such that*

$$\left| \langle \sin(k\pi x), Q_3 \sin(j\pi x) \rangle_{L^2(0,1)} \right| \geq \frac{C}{k^p}, \quad \forall k \in \mathbb{N}^*.$$

The problem (26) is locally controllable to the j -th eigensolution of the one-dimensional Dirichlet Laplacian $-\partial_x^2$ in any time $T > 0$. In other words, there exists $\delta > 0$ such that, for any $\psi_0^1 \in L^2((0, 1), \mathbb{R})$ with

$$\|\psi_0^1 - \sqrt{2} \sin(j\pi y)\|_{L^2} < \delta,$$

there exists a control $v_3 \in L^2((0, T), \mathbb{R})$ such that $\psi^1(T; \psi_0^1, v_3)$, the solution of (26), verifies

$$\psi^1(T; \psi_0^1, v_3) = e^{-j^2 \pi^2 T} \sqrt{2} \sin(j\pi y).$$

We finally have all the ingredients to conclude the proof of Main Theorem C.

Proof of Main Theorem C. We prove the statement when $l, m = 1$ but the general case can be proved equivalently. Let B be the neighbourhood of $\sqrt{2}\sin(\pi x)$ ensured by Proposition 4.9 and V the countable set from Proposition 4.8. For every $t > 0$, we know

$$\begin{aligned} \mu_1^u &\xrightarrow{|u| \rightarrow 0} \pi^2, \\ e^{-(\mu_1^u - \pi^2)t} f_1^u(x) &\xrightarrow[L^2]{|u| \rightarrow 0} \sqrt{2}\sin(\pi x), \end{aligned}$$

thanks to Proposition 4.2 and Proposition 4.4. Let $T_1 = T_2 = \frac{T}{2}$. We consider $R > 0$ sufficiently small so that Proposition 4.8 is satisfied and at the same time $e^{-(\mu_1^u - \pi^2)\frac{T}{2}} f_1^u(x)$ belongs to B for $u \in [-R, R] \setminus V$. Proposition 4.8 yields the existence of δ such that, for any $\psi_i \in L^2(\Omega, \mathbb{R})$ with

$$\|\psi_i - \phi_{1,1}^u\|_{L^2} < \delta,$$

there exists a control $v_2 \in L^2((0, T_1), \mathbb{R})$ such that, for $w_1 = (u, v_2, 0, 0)$, we have

$$\psi(T_1; \psi_i, w_1) = \Phi_{1,1}^u(T_1)$$

Now, we observe that

$$\Phi_{1,1}^u(T_1) = e^{-\lambda_1^u T_1} f_1^u(x) \sqrt{2}\sin(\pi y) = e^{-\pi^2 T_1} e^{-(\mu_1^u - \pi^2) T_1} f_1^u(x) e^{-\pi^2 T_1} \sqrt{2}\sin(\pi y).$$

The local controllability of (26) from Proposition 4.9 yields the existence of $v_3 \in L^2((0, T_2), \mathbb{R})$ such that

$$\psi^1(T_2; e^{-(\mu_1^u - \pi^2) T_1} f_1^u(x), v_3) = e^{-\pi^2 T_2} \sqrt{2}\sin(\pi x)$$

and the linearity yields

$$\begin{aligned} \psi^1(T_2; e^{-\mu_1^u T_1} f_1^u(x), v_3) &= e^{-\pi^2 T_1} \psi^1(T_2; e^{-(\mu_1^u - \pi^2) T_1} f_1^u(x), v_3) \\ &= e^{-\pi^2 T_1} e^{-\pi^2 T_2} \sqrt{2}\sin(\pi x) = e^{-\pi^2 (T_1 + T_2)} \sqrt{2}\sin(\pi x). \end{aligned}$$

At the same time, $\psi^2(T_2; e^{-\pi^2 T_1} \sqrt{2}\sin(\pi y))$, the solution of (27) with initial state $e^{-\pi^2 T_1} \sqrt{2}\sin(\pi y)$, satisfies

$$\psi^1(T_2; e^{-\pi^2 T_1} \sqrt{2}\sin(\pi y)) = e^{-\pi^2 T_2} e^{-\pi^2 T_1} \sqrt{2}\sin(\pi y) = e^{-\pi^2 (T_1 + T_2)} \sqrt{2}\sin(\pi y).$$

Hence, $\psi(T_2; \Phi_{1,1}^u, w_2)$, the solution of (BHE) with initial state $\Phi_{1,1}^u = e^{-\mu_1^u T_1} f_1^u(x) \sqrt{2}\sin(\pi y)$ and control $w_2 = (0, 0, v_3, 0)$, verifies

$$\psi(T_2; \Phi_{1,1}^u, w_2) = \psi^1(T_2; e^{-\mu_1^u T_1} f_1^u(x), v_3) \psi^2(T_2; e^{-\pi^2 T_1} \sqrt{2}\sin(\pi y)) = 2e^{-\pi^2 (T_1 + T_2)} \sin(\pi x) \sin(\pi y).$$

Finally, the proof is ensured with

$$w = w_1(\cdot) \mathbb{1}_{[0, T/2]} + w_2(\cdot - T/2) \mathbb{1}_{[T/2, T]} = \left(u(\cdot) \mathbb{1}_{[0, T/2]}, v_2(\cdot) \mathbb{1}_{[0, T/2]}, v_3(\cdot - T/2) \mathbb{1}_{[T/2, T]}, 0 \right),$$

which verifies $\psi(T; \psi_i, w) = \psi(T_2; \psi(T_1; \psi_i, w_1), w_2) = \psi(T_2; \Phi_{1,1}^u, w_2) = 2e^{-\pi^2 T} \sin(\pi x) \sin(\pi y)$. \square

Remark 4.10. Notice that Main Theorem C ensures the exact controllability with 3 controls, but it is actually possible to obtain the result with only 2 of them. Indeed, its proof shows that u_1 vanishes in the second part of the dynamics, while u_3 vanishes in the first one. Hence, the result can be proved with 2 controls when Q_1 also plays the role of the potential Q_3 , i.e. Q_1 satisfies the fifth point of Assumptions III. An example of control potential Q verifying Assumptions III is $Q = ((3x^2 - 1), x^2 y^2, (3x^2 - 1), 1)$ (see Remark 4.1), nevertheless the same controllability can be ensured also with the potential $Q = ((3x^2 - 1), x^2 y^2, 0, 1)$.

Proof of Corollary 1.5. The property follows from Main Theorem C exactly as Corollary 1.1 from Main Theorem A. \square

A Some perturbation theory proofs

This appendix introduces all the technical proofs of the perturbation theory results stated in Section 4.2. We recall the operator $A^u = -\partial_x^2 + uQ_1$ defined in $D(A^u) := H^2((0, 1), \mathbb{R}) \cap H_0^1((0, 1), \mathbb{R})$. For every $0 \leq s \leq \frac{2p-1}{2}$, we introduce the spaces $D(|A^u|^{\frac{s}{2}})$ equipped with the norm $\| |A^u|^{\frac{s}{2}} \cdot \|_{L^2(0,1)}$. Notice that $D(|A^u|^{\frac{s}{2}}) \equiv D(|A^0|^{\frac{s}{2}})$, however, we need to ensure the equivalence of the two corresponding norms.

Proposition A.1. *Let Q verify Assumptions III and $0 \leq s \leq p_1 - 2$. There exist $R, C_1, C_2 > 0$ such that, for every $u \in [-R, R]$, we have*

$$C_1 \left\| |A^0|^{\frac{s}{2}} \cdot \right\|_{L^2(0,1)} \leq \left\| |A^u|^{\frac{s}{2}} \cdot \right\|_{L^2(0,1)} \leq C_2 \left\| |A^0|^{\frac{s}{2}} \cdot \right\|_{L^2(0,1)}.$$

Proof. We start by proving the existence of $C > 0$ such that $\left\| |-\partial_x^2 + uQ_1|^{\frac{s}{2}} \cdot \right\|_{L^2(0,1)} \leq C \left\| |-\partial_x^2|^{\frac{s}{2}} \cdot \right\|_{L^2(0,1)}$. When $s = 0$, the statement is trivial. If $s = 2$, then

$$\left\| |-\partial_x^2 + uQ_1| \cdot \right\|_{L^2(0,1)} \leq \left\| |-\partial_x^2| \cdot \right\|_{L^2(0,1)} + \|u\| \|Q_1\|_{H^1(0,1)} \left\| \cdot \right\|_{L^2(0,1)} \leq C \left\| |-\partial_x^2| \cdot \right\|_{L^2(0,1)}.$$

In general, as soon as $s \in 2\mathbb{N}^*$, we have the existence of $C_1, C_2 > 0$ such that

$$\left\| |-\partial_x^2 + uQ_1|^{\frac{s}{2}} \cdot \right\|_{L^2(0,1)} \leq C_1 \left(1 + |u|^{\frac{s}{2}}\right) \left(1 + \|Q_1\|_{H^{s-1}(0,1)}^{\frac{s}{2}}\right) \left\| |-\partial_x^2|^{\frac{s}{2}} \cdot \right\|_{L^2(0,1)} \leq C_2 \left\| |-\partial_x^2|^{\frac{s}{2}} \cdot \right\|_{L^2(0,1)}.$$

Notice that in the last relation, we used the regularity hypothesis on Q_1 ensured by Assumptions III. Indeed, the fourth point of Assumptions III yields that $Q_1 \in H^s((0, 1), \mathbb{R})$ and the operator $\psi \in D(|-\partial_x^2|^{\frac{s}{2}}) \mapsto Q_1\psi \in D(|-\partial_x^2|^{\frac{s}{2}})$ since $Q_1' \in D(|-\partial_x^2|^{\frac{s-2}{2}})$. Classical interpolation arguments (see for instance the proof of [13, Lemma 1]) imply the validity of the relation also for $s \notin 2\mathbb{N}^*$ and there exists $C_3 > 0$ such that

$$\left\| |-\partial_x^2 + uQ_1|^{\frac{s}{2}} \cdot \right\|_{L^2(0,1)} \leq C_3 \left\| |-\partial_x^2|^{\frac{s}{2}} \cdot \right\|_{L^2(0,1)}.$$

The opposite inequality follows by using the same strategy with the decomposition $-\partial_x^2 = (-\partial_x^2 + uQ_1) - uQ_1$. \square

We are finally ready to provide the proof of Proposition 4.4.

Proof of Proposition 4.4. Point 1. We consider $(r_k^u)_{k \in \mathbb{N}^*}$ such that $\mu_k^u = k^2\pi^2 + r_k^u$ introduced by Proposition 4.2. The map

$$u \in \mathbb{R} \mapsto r_l^u - r_n^u = u \int_0^1 \langle (f_l^{tu})^2 - (f_n^{tu})^2, Q_1(x) \rangle_{L^2} dt = \int_0^u \langle (f_l^t)^2 - (f_n^t)^2, Q_1(x) \rangle_{L^2} dt \in \mathbb{R}$$

is an analytic function for every $l, n \in \mathbb{N}^*$ and its derivative is w.r.t. u is

$$g_{l,n}^u = \langle (f_l^u)^2 - (f_n^u)^2, Q_1(x) \rangle_{L^2}.$$

Since $(f_l^0)^2 - (f_n^0)^2 = \sin^2(l\pi x) - \sin^2(n\pi x) = \frac{1}{2}(\cos(2n\pi x) - \cos(2l\pi x))$, the first point of Assumption III yields

$$g_{l,n}^0 \neq 0, \quad \forall l, n \in \mathbb{N}^*, l \neq n.$$

Now, we notice that $g_{l,n}^u$ is also the derivative of the analytic map $u \in \mathbb{R} \mapsto \lambda_l^u - \lambda_n^u$ which is then non-constant for every $l, n \in \mathbb{N}^*$. Its zeros form a discrete subset $V_{l,n}$ of \mathbb{R} and $V := \bigcup_{l,n \in \mathbb{N}^*} V_{l,n}$ is countable. Thus, for every $u \notin V$, the spectrum is simple. Thanks to (24), we know $r_l^u = O(|u|)O(1/k)$ and, for any $u \in [-R, R]$ with sufficiently small R , we have $|r_l^u| < \frac{\pi^2}{4}$ for every $l \in \mathbb{N}^*$. This last identity and Proposition 3.2 implies that

$$\lambda_{j+n_j}^u - \lambda_j^u = O(\sqrt{j}), \quad \forall j \in \mathbb{N}^*.$$

The same approach adopted in the proof of Proposition 3.3 leads to the property

$$\inf_{j \in \mathbb{N}^*} \sqrt{\lambda_{j+n_j}^u} - \sqrt{\lambda_j^u} > 0.$$

Now, we refer to the theory developed in [20] where the authors show that the distance between the eigenvalues of different self-adjoint Laplacian operators is at most polynomially decreasing when it is different from 0 (see for instance [20, Section 4]). Such results and the simplicity of the perturbed spectrum show the existence of

$p \in \mathbb{N}^*$, depending only on the perturbation uQ_1 , such that $|r_l^u - r_{l+1}^u| = O(1/l^p)$ for every $l \in \mathbb{N}^*$. Proposition 4.3 and (3) yield

$$|\lambda_{l+1}^u - \lambda_l^u| = |r_{l+1}^u - r_l^u| = O\left(\frac{1}{l^p}\right) = O\left(\frac{1}{(\sqrt{\lambda_{l+1}^u})^p}\right).$$

The first point of the proposition follows, since $|\sqrt{\lambda_{l+1}^u} - \sqrt{\lambda_l^u}| = O\left(\frac{\lambda_{l+1}^u - \lambda_l^u}{\sqrt{\lambda_l^u}}\right)$.

Point 2. Notice now that $\phi_j^u(x, y) = f_j^u(x)\sqrt{2}\sin(\pi y)$. The second statement follows from the analyticity of the map $u \in \mathbb{R} \mapsto f_j^u \in L^2((0, 1), \mathbb{R})$ ensured by Proposition 4.2. \square

Finally, we want to prove Proposition 4.5, but some intermediate results are required. Notice that the following part of the appendix rephrases the theory presented in [17, Appendix B] but we develop it anyway for the sake of completeness.

Lemma A.2. *Let Q satisfy Assumptions III and Π_k^\perp be the projector onto the orthogonal space of f_k^0 . There exists $R > 0$ such that, for every $u \in [-R, R]$, the operator*

$$(-\partial_x^2 + u\Pi_k^\perp Q_1 - \mu_k^u)\Pi_k^\perp$$

is invertible with bounded inverse from $\Pi_k^\perp D(A^0)$ to $\Pi_k^\perp L^2((0, 1), \mathbb{R})$ for every $k \in \mathbb{N}^$. In addition, there exists a family of uniformly bounded operators O_k in $L(\Pi_k^\perp L^2((0, 1), \mathbb{R}), \Pi_k^\perp L^2((0, 1), \mathbb{R}))$ such that*

$$((-\partial_x^2 + u_0\Pi_k^\perp Q_1 - \mu_k^u)\Pi_k^\perp)^{-1} := ((-\partial_x^2 - \mu_k^u)\Pi_k^\perp)^{-1}O_k.$$

The operators O_k are also uniformly bounded in $L(\Pi_k^\perp D(|A^0|^{\frac{p_1-2}{2}}), \Pi_k^\perp D(|A^0|^{\frac{p_1-2}{2}}))$ with p_1 defined in Assumptions III. Finally, there exists $C > 0$ such that

$$\|((-\partial_x^2 + u_0\Pi_k^\perp Q_1 - \mu_k^u)\Pi_k^\perp)^{-1}\|_{L(\Pi_k^\perp L^2)} \leq \frac{C}{k}, \quad \forall k \in \mathbb{N}^*.$$

Proof. Let us consider $R > 0$ sufficiently small so that $|\mu_k^u - \mu_k^0| < \pi^2$ for every $k \in \mathbb{N}^*$ thanks to the analyticity of the spectrum stated by Proposition 4.2. The operator $-\partial_x^2 - \mu_k^u$ is invertible with bounded inverse when it acts on $\Pi_k^\perp L^2((0, 1), \mathbb{R})$ and u is sufficiently small. If $R < \pi^2/\|Q_1\|_{L(L^2)}$, then

$$\| -u\Pi_k^\perp Q_1((-\partial_x^2 - \mu_k^u)\Pi_k^\perp)^{-1} \|_{L(\Pi_k^\perp L^2)} \leq |u|\|Q_1\|_{L(L^2)} \frac{1}{\min\{|\mu_{k+1}^0 - \mu_k^u|, |\mu_k^u - \mu_{k-1}^0|\}} \leq \frac{|u|\|Q_1\|_{L(L^2)}}{\pi^2} < 1.$$

Then, the corresponding Neumann series converges and the linear bounded operator

$$O_k := \sum_{n=0}^{\infty} (-u\Pi_k^\perp Q_1((-\partial_x^2 - \mu_k^u)\Pi_k^\perp)^{-1})^n = (Id + u\Pi_k^\perp Q_1((-\partial_x^2 - \mu_k^u)\Pi_k^\perp)^{-1})^{-1}$$

is well-defined. This property implies that $(-\partial_x^2 + u_0\Pi_k^\perp Q_1 - \mu_k^u)\Pi_k^\perp$ is invertible, since

$$((-\partial_x^2 + u_0\Pi_k^\perp Q_1 - \mu_k^u)\Pi_k^\perp)^{-1} = ((-\partial_x^2 - \mu_k^u)\Pi_k^\perp)^{-1}O_k.$$

Notice that the multiplication operator w.r.t. $\Pi_k^\perp Q_1$, not only belongs to the space

$$L(\Pi_k^\perp L^2((0, 1), \mathbb{R}), \Pi_k^\perp L^2((0, 1), \mathbb{R})), \quad \text{but also to} \quad L(\Pi_k^\perp D(|A^0|^{\frac{p_1-2}{2}}), \Pi_k^\perp D(|A^0|^{\frac{p_1-2}{2}})).$$

This property is due to the regularity hypothesis on Q_1 stated by the fourth point of Assumptions III and, then, the family of operators $(O_k)_{k \in \mathbb{N}^*}$ is uniformly bounded in such spaces. Finally, there exists $C > 0$ such that

$$\begin{aligned} \|((-\partial_x^2 + u_0\Pi_k^\perp Q_1 - \mu_k^u)\Pi_k^\perp)^{-1}\|_{L(\Pi_k^\perp L^2)} &\leq \|((-\partial_x^2 - \mu_k^u)\Pi_k^\perp)^{-1}\|_{L(\Pi_k^\perp L^2)} \|O_k\|_{L(L^2)} \\ &\leq \frac{\|O_k\|_{L(L^2)}}{\min\{|\mu_{k+1}^0 - \mu_k^u|, |\mu_k^u - \mu_{k-1}^0|\}} \leq \frac{C}{k}. \end{aligned}$$

\square

Lemma A.3. *Let $(B_l)_{l \in \mathbb{N}^*}$ be a family of uniformly bounded operators in $L(D(|A^0|^{\frac{s}{2}}), D(|A^0|^{\frac{s}{2}}))$ with $s > 0$. Let $j \in \mathbb{N}^*$. There exists $C > 0$ such that*

$$|\langle f_k^0, B_k f_j^0 \rangle_{L^2}| \leq \frac{C}{k^s}, \quad |\langle f_k^0, B_j f_j^0 \rangle_{L^2}| \leq \frac{C}{k^s}, \quad \forall k \in \mathbb{N}^*.$$

Proof. Let us prove the result when $j = 1$. The general case equivalently follows. The two inequalities follow from the existence of $C > 0$ such that, for every $k \in \mathbb{N}^*$,

$$\sup_{l \in \mathbb{N}^*} |k^s \langle f_k^0, B_l f_1^0 \rangle_{L^2}|^2 \leq \sup_{l \in \mathbb{N}^*} \sum_{m \in \mathbb{N}^*} |m^s \langle f_m^0, B_l f_1^0 \rangle_{L^2}|^2 \leq \sup_{l \in \mathbb{N}^*} \|B_l f_1^0\|_{D(|A^0|^{\frac{s}{2}})}^2 \leq C.$$

The last relation implies $\sup_{l \in \mathbb{N}^*} |\langle f_k^0, B_l f_1^0 \rangle_{L^2}| \leq \frac{C}{k^s}$ for every $k \in \mathbb{N}^*$ which yields the result. \square

For every $u \in \mathbb{R}$, we decompose the perturbed eigenfunction of the operator A^u as follows

$$f_k^u = a_k f_k^0 + \eta_k$$

defined with $\Pi_k^\perp f_k^u = \eta_k$ and a_k some normalizing constant for every $k \in \mathbb{N}^*$.

Lemma A.4. *Let Q satisfy Assumptions III. There exists $R > 0$ such that, for every $u \in [-R, R]$, we have*

$$\eta_k = -a_k ((-\partial_x^2 + u \Pi_k^\perp Q_1 - \mu_k^u) \Pi_k^\perp)^{-1} u \Pi_k^\perp Q_1 f_k^0$$

and there exists $C > 0$ such that

$$\|\eta_k\|_{L^2} \leq \frac{C}{k} |u|.$$

Finally, the coefficients a_k are such that $\sup_{k \in \mathbb{N}^*} |a_k| \xrightarrow{u \rightarrow 0} 1$.

Proof. By definition, we know that $\mu_k^u f_k^u = (-\partial_x^2 + u Q_1)(a_k f_k^0 + \eta_k)$ and

$$a_k \mu_k^u f_k^0 + \mu_k^u \eta_k = -a_k \partial_x^2 f_k^0 - \partial_x^2 \eta_k + a_k u Q_1 f_k^0 + u Q_1 \eta_k.$$

By projecting onto the orthogonal space of f_k^0 , we obtain

$$\mu_k^u \eta_k = -\partial_x^2 \eta_k + a_k u \Pi_k^\perp Q_1 f_k^0 + u \Pi_k^\perp Q_1 \eta_k.$$

Now, Lemma A.2 ensures that $(-\partial_x^2 + u \Pi_k^\perp Q_1 - \mu_k^u) \Pi_k^\perp$ is invertible with bounded inverse and

$$(28) \quad \eta_k = -u a_k ((-\partial_x^2 + u \Pi_k^\perp Q_1 - \mu_k^u) \Pi_k^\perp)^{-1} \Pi_k^\perp Q_1 f_k^0.$$

Clearly, the coefficients a_k are uniformly bounded and the existence of there exists $C > 0$ such that

$$\|\eta_k\|_{L^2} \leq \frac{C}{k} |u|$$

follows from Proposition A.2. Finally, this property ensures that $\sup_{k \in \mathbb{N}^*} |a_k| \xrightarrow{u \rightarrow 0} 1$. \square

We are finally ready to prove Proposition 4.5.

Proof of Proposition 4.5. Let us prove the result when $j = 1$. The general case is proved equivalently. Let us start by using the third point of Assumptions III in the following decomposition:

$$(29) \quad \begin{aligned} |\langle f_k^u, Q_2^1 f_1^u \rangle_{L^2}| &= |\langle a_k f_k^0 + \eta_k, Q_2^1 (a_1 f_1^0 + \eta_1) \rangle_{L^2}| \\ &\geq C \frac{a_k a_1}{k^{p_1}} - |a_k \langle f_k^0, Q_2^1 \eta_1 \rangle_{L^2} + a_1 \langle \eta_k, Q_2^1 f_1^0 \rangle_{L^2} + \langle \eta_k, Q_2^1 \eta_1 \rangle_{L^2}|. \end{aligned}$$

We want to prove

$$|a_k \langle f_k^0, Q_2^1 \eta_1 \rangle_{L^2} + a_1 \langle \eta_k, Q_2^1 f_1^0 \rangle_{L^2} + \langle \eta_k, Q_2^1 \eta_1 \rangle_{L^2}| = O\left(\frac{1}{k^{p_1}}\right) O(u).$$

To this purpose, we start by studying

$$\langle f_m^0, Q_2^1 \eta_l \rangle_{L^2}$$

for every $l \neq m$. We use Lemma A.3 and Lemma A.4 as follows

$$(30) \quad \begin{aligned} \langle f_m^0, Q_2^1 \eta_l \rangle_{L^2} &= -\langle f_m^0, Q_2^1 a_l ((-\partial_x^2 + u \Pi_l^\perp Q_1 - \mu_l^u) \Pi_l^\perp)^{-1} u \Pi_l^\perp Q_1 f_l^0 \rangle_{L^2} \\ &= -a_l u \langle \Pi_l^\perp Q_2^1 f_m^0, ((-\partial_x^2 - \mu_l^u) \Pi_l^\perp)^{-1} O_l \Pi_l^\perp Q_1 f_l^0 \rangle_{L^2} \\ &= -a_l u \langle ((-\partial_x^2 - \mu_l^u) \Pi_l^\perp)^{-1} \Pi_l^\perp Q_2^1 f_m^0, O_l \Pi_l^\perp Q_1 f_l^0 \rangle_{L^2}. \end{aligned}$$

Now, we notice that

$$\begin{aligned}
((-\partial_x^2 - \mu_l^u)\Pi_l^\perp)^{-1}\Pi_l^\perp Q_2^1 f_m^0 &= \Pi_l^\perp Q_2^1 ((-\partial_x^2 - \mu_l^u)\Pi_l^\perp)^{-1} f_m^0 - [\Pi_l^\perp Q_2^1, ((-\partial_x^2 - \mu_l^u)\Pi_l^\perp)^{-1}] f_m^0 \\
&= \Pi_l^\perp Q_2^1 ((-\partial_x^2 - \mu_l^u)\Pi_l^\perp)^{-1} f_m^0 \\
(31) \quad &+ ((-\partial_x^2 - \mu_l^u)\Pi_l^\perp)^{-1} [\Pi_l^\perp Q_2^1, -\partial_x^2] ((-\partial_x^2 - \mu_l^u)\Pi_l^\perp)^{-1} f_m^0 \\
&= \frac{1}{\mu_m^0 - \mu_l^u} \left(\Pi_l^\perp Q_2^1 + ((-\partial_x^2 - \mu_l^u)\Pi_l^\perp)^{-1} \Pi_l^\perp [Q_2^1, -\partial_x^2] \right) f_m^0.
\end{aligned}$$

Let us denote M_l the uniformly bounded operators in $L(D(|A^0|^{\frac{p_1-2}{2}}), D(|A^0|^{\frac{p_1-2}{2}}))$ such that

$$M_l := \left(\Pi_l^\perp Q_2^1 + ((-\partial_x^2 - \mu_l^u)\Pi_l^\perp)^{-1} \Pi_l^\perp [Q_2^1, -\partial_x^2] \right)$$

By using together (30) and (31), we obtain

$$\langle f_m^0, Q_2^1 \eta_l \rangle_{L^2} = -\frac{a_l u}{\mu_m^0 - \mu_l^u} \langle T_l f_m^0, f_l^0 \rangle_{L^2}$$

where $T_l := (O_l \Pi_l^\perp Q_1)^* M_l$ are uniformly bounded operators in $L(D(|A^0|^{\frac{p_1-2}{2}}), D(|A^0|^{\frac{p_1-2}{2}}))$, and then

$$\begin{aligned}
(32) \quad \langle f_k^0, Q_2^1 \eta_1 \rangle_{L^2} &= -\frac{a_1 u}{\mu_k^0 - \mu_1^u} \langle T_1 f_k^0, f_1^0 \rangle_{L^2}, \\
\langle \eta_k, Q_2^1 f_1^0 \rangle_{L^2} &= \langle Q_2^1 \eta_k, f_1^0 \rangle_{L^2} = -\frac{a_k u}{\mu_1^0 - \mu_k^u} \langle T_k^* f_k^0, f_1^0 \rangle_{L^2}.
\end{aligned}$$

Now, we can prove with the same strategy the existence of suitable uniformly bounded operators \tilde{T}_k in $L(D(|A^0|^{\frac{p_1-2}{2}}), D(|A^0|^{\frac{p_1-2}{2}}))$ such that the following identity is verified

$$(33) \quad \langle \eta_k, Q_2^1 \eta_1 \rangle_{L^2} = -\frac{a_k a_1 u^2}{\mu_1^0 - \mu_k^u} \langle \tilde{T}_k f_k^0, f_1^0 \rangle_{L^2}.$$

We recall that, thanks to Proposition 4.3, we have $\mu_k^u \sim \mu_k^0 = \pi^2 k^2$. We finally use together (32) and (33) in (29). Thanks to Assumptions I, there exists $C_1 > 0$ such that

$$|\langle f_k^u, Q_2^1 f_1^u \rangle_{L^2}| \geq \frac{C}{k^{p_1}} - \frac{C_1 |u|}{k^2} \left| \langle T_1 f_k^0, f_1^0 \rangle_{L^2} + \langle T_k^* f_k^0, f_1^0 \rangle_{L^2} + u \langle \tilde{T}_k f_k^0, f_1^0 \rangle_{L^2} \right|$$

Finally, Lemma A.4 allows us to consider $R > 0$ sufficiently small so that $\sup_{l \in \mathbb{N}^*} |a_l|$ is close to 1 and Lemma A.3 yields the existence of $C_2, C_3 > 0$ such that

$$|\langle f_k^u, Q_2^1 f_1^u \rangle_{L^2}| \geq \frac{C a_k a_1}{k^{p_1}} - C_2 \frac{|u|}{k^2} \frac{1}{k^{p_1-2}} \geq \frac{C_3}{k^{p_1}}.$$

□

B Proof of the local bilinear controllability

The aim of this appendix is to ensure Theorem 2.3. However, before proceeding with the proof of the theorem, we need some intermediate results. We also recall the notation introduced in Section 2.

B.1 A preliminary lemma

Let us first introduce some notation. We set $T_f = \min(T, \frac{\pi^2}{6}, \frac{\pi^2}{6} T_0)$, and $T_1 = \frac{6}{\pi^2} T_f \leq 1$. We then introduce a partition of the time interval $[0, T]$, given by $\tau_0 = 0$ and $\tau_n := \sum_{k=1}^n T_k$, $n \in \mathbb{N}^*$, where $T_k := k^{-2} T_1$, $k \in \mathbb{N}$. Note that $\lim_{n \rightarrow +\infty} \tau_n = T_f$. Let $n \in \mathbb{N}$ and $v_n \in L^2(\tau_{n-1}, \tau_n)$. We denote by $\eta(\cdot; \eta_{n-1}, \tau_{n-1}, v_n)$ the unique solution of

$$\begin{aligned}
(34) \quad &\left\{ \begin{aligned} \eta'(t) + A\eta(t) + v_n(t)B(\eta(t) + \phi_j) &= 0, \quad t \in (\tau_{n-1}, \tau_n), \\ \eta(\tau_{n-1}) &= \eta_{n-1}, \end{aligned} \right.
\end{aligned}$$

with $\eta_n = \eta(\tau_n; \eta_{n-1}, \tau_{n-1}, v_n)$. We shall also denote by $\xi(\cdot; \eta_{n-1}, \tau_{n-1}, v_n)$ the unique solution of

$$\begin{aligned}
(35) \quad &\left\{ \begin{aligned} \xi'(t) + A\xi(t) + v_n(t)B\phi_j &= 0, \quad t \in (\tau_{n-1}, \tau_n), \\ \xi(\tau_{n-1}) &= \eta_{n-1}. \end{aligned} \right.
\end{aligned}$$

We may remark that $\zeta := \eta(\cdot; \eta_0, 0, v_1) - \xi(\cdot; \eta_0, 0, v_1)$ satisfies

$$(36) \quad \begin{cases} \zeta'(t) + A\zeta(t) + v_n(t)B\eta(t) = 0, & t \in (\tau_{n-1}, \tau_n), \\ \zeta(0) = 0. \end{cases}$$

Moreover, if as v_n is a null-control for the linear system (35), then $\zeta(\tau_n) = \eta(\tau_n; \eta_{n-1}, \tau_{n-1}, v_n)$.

Finally, we set $\Gamma_0 > 0$ sufficiently large so that the following inequality is satisfied

$$(37) \quad K(T) + N(T) \leq e^{\frac{\Gamma_0}{T^\gamma}},$$

where $N(T)^2 := \|B\|^2 K(T)^2 e^{(4\sigma + \|B\| + 1)T + 2\|B\|\sqrt{T}} (1 + \|B\|K(T)^2)$. We prove the following lemma.

Lemma B.1. *There exists $\alpha > 1$ such that for any $\eta_0 \in X$ satisfying $\|\eta_0\|_X \leq e^{-\alpha \frac{\Gamma_0}{T_1^\gamma}}$, for any $n \in \mathbb{N}^*$, there exists $v_n \in L^2(\tau_{n-1}, \tau_n)$ such that*

1. $\|v_n\|_{L^2(\tau_{n-1}, \tau_n)} \leq K(T_n) \|\eta_{n-1}\|_X$
2. $\xi(\tau_n; \eta_{n-1}, \tau_{n-1}, v_n) = 0$
3. $\|\eta(\tau_n; \eta_{n-1}, \tau_{n-1}, v_n)\|_X \leq \prod_{j=1}^n N(T_j) 2^{n-j} \|\eta_0\|_X^{2^n}$.

Proof. The two first items comes from the null controllability of the linear system, on the time interval (τ_{n-1}, τ_n) of length T_n , $n \in \mathbb{N}^*$. Let us prove the third item by induction. Assume first $n = 1$. We moreover have by hypothesis

$$K(T_1) \|\eta_0\| \leq K(T_1) e^{-\alpha \frac{\Gamma_0}{T_1^\gamma}} \leq e^{(1-\alpha) \frac{\Gamma_0}{T_1^\gamma}} \leq 1,$$

assuming $\alpha > 1$. We then may apply [3, Proposition 2.4] for system (36) to obtain

$$\|\eta(\tau_1; \eta_0, 0, v_1)\|_X = \|\zeta(T_1)\|_X \leq N(T_1) \|\eta_0\|_X^2.$$

Now assume that the three items holds for any $k \in \{1, \dots, n-1\}$. Then, by the null-controllability of the linear system, there exists $v_n \in L^2(\tau_{n-1}, \tau_n)$ such that $\xi(\tau_n; \eta_{n-1}, \tau_{n-1}, v_n) = 0$, with estimate

$$\|v_n\|_{L^2(\tau_{n-1}, \tau_n)} \leq K(T_n) \|\eta(\tau_{n-1}; \eta_{n-2}, \tau_{n-2}, v_{n-1})\|_X.$$

Moreover, by induction hypothesis,

$$\begin{aligned} K(T_n) \|\eta_{n-1}\| &\leq e^{\frac{\Gamma_0}{T_n^\gamma}} \|\eta_{n-1}\| \leq e^{\frac{\Gamma_0}{T_n^\gamma}} \prod_{k=1}^{n-1} N(T_k) 2^{n-1-k} \|\eta_0\|_X^{2^{n-1}} \\ &\leq \exp\left(\frac{\Gamma_0}{T_1^\gamma} \sum_{j=k}^n 2^{n-1-k} k^{2\gamma}\right) \|\eta_0\|_X^{2^{n-1}}. \end{aligned}$$

Yet assuming $\|\eta_0\|_X \leq e^{-\alpha \frac{2\Gamma_0}{T_1^\gamma}}$, we deduce

$$K(T_n) \|\eta_{n-1}\| \leq \exp\left(\frac{\Gamma_0 2^n}{T_1^\gamma} \left(\sum_{k=1}^n 2^{-(1+k)} k^{2\gamma} - \frac{\alpha}{2}\right)\right) \leq 1,$$

for $\alpha \geq \sum_{k=1}^{+\infty} 2^{-(1+k)} k^{2\gamma}$. As a result, applying [3, Proposition 2.4] for system (36), we obtain

$$\|\eta(\tau_n; \eta_{n-1}, \tau_{n-1}, v_n)\|_X \leq N(T_n) \|\eta_{n-1}\|_X^2.$$

Using the induction hypothesis, we deduce

$$\|\eta(\tau_n; \eta_{n-1}, \tau_{n-1}, v_n)\|_X \leq \prod_{k=1}^n N(T_k) 2^{n-k} \|\eta_0\|_X^{2^n},$$

and this ends the proof. \square

B.2 Proof of Theorem 2.3

We are now ready to proceed to the proof of Theorem 2.3. Recalling (12):

$$(38) \quad \begin{cases} \varphi'(t) + A\varphi(t) + v(t)B\varphi(t) = 0, & t > 0, \\ \varphi(0) = \varphi_0 \in X, \end{cases}$$

for some control $v \in L^2(0, T_f)$. We first assume that $j = 1$ and $\lambda_1 = 0$. Note that in this case, $\phi_1 = \Phi_1$. We take $v(t) := \sum_{k=0}^{+\infty} \mathbb{1}_{[\tau_{k-1}, \tau_k)}(t)v_k(t)$, where $v_k \in L^2(\tau_{k-1}, \tau_k)$ is given by Lemma B.1 with $\eta_0 = \varphi_0$. Hence, the difference $\eta := \varphi - \phi_1$ between the solution and the target satisfies (34). Note that for any $n \in \mathbb{N}$, $t \in [\tau_{n-1}, \tau_n]$, $\eta(t) = \eta_n(t)$. From Lemma B.1, we deduce for any $n \in \mathbb{N}$,

$$(39) \quad \|\varphi(\tau_n) - \phi_1\|_X \leq \prod_{k=1}^n N(T_k)^{2^{n-k}} \|\varphi_0 - \phi_1\|_X^{2^n}.$$

Yet, by (37), for any $n \in \mathbb{N}^*$

$$N(T_j)^{2^{n-j}} \leq \exp\left(\frac{\Gamma_0 2^n}{T_1^\gamma} \sum_{k=1}^n 2^{-k}\right) \leq \exp\left(\frac{\pi^2 \Gamma_0 2^n}{6T_1^\gamma}\right).$$

Assuming $\|\varphi_0 - \phi_1\|_X \leq \exp\left(-\alpha \frac{\gamma_0}{T_1^\gamma}\right)$, we deduce from (39),

$$\|\varphi(\tau_n) - \phi_1\|_X \leq \exp\left(\left(1 - \frac{6\alpha}{\pi^2}\right) \frac{\pi^2 \Gamma_0 2^n}{6T_1^\gamma}\right).$$

by choosing $\alpha > \frac{6}{\pi^2}$, and letting n to infinity, we obtain $\varphi(T_f) = 0$. This ends the proof in the case $\lambda_1 = 0$.

Now assume $j = 1$ and $\lambda_1 \neq 0$. Then, $\Phi_1(t) = e^{-t\lambda_1} \phi_1$. Let us define $z(t) := e^{t\lambda_1} \varphi(t)$ and it satisfies

$$(40) \quad \begin{cases} z'(t) + A_1 z(t) + v(t)Bz(t) = 0, & t > 0, \\ z(0) = \varphi_0 \in X, \end{cases}$$

where $A_1 = A - \lambda_1 I$. Its first eigenpair is given by $(\tilde{\lambda}_1, \tilde{\phi}_1) = (0, \phi_1)$. We now see immediately (see [2, Lemma 4.1], [3, Proof of Theorem 1.2]) that the operator A_1 satisfies all the requested properties to use the above arguments to prove that there exists $T_0 > 0$ such that for any $T \in (0, T_0]$, there exists $R_T > 0$ and $v \in L^2(0, T)$ such that $z(T) = \phi_1$. This implies

$$0 = \|z(T) - \phi_1\|_X = \|e^{-T\lambda_1} z(T) - e^{-T\lambda_1} \phi_1\|_X = \|\varphi(T) - \Phi_1(T)\|_X.$$

Finally, when $j > 1$, we can proceed as above by considering $z(t) := e^{t\lambda_j} \varphi(t)$ and $A_j = A - \lambda_j I$. The same arguments previously adopted end the proof.

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