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► **To cite this version:**

Thierry Denoeux. Combination of Dependent Gaussian Random Fuzzy Numbers. Eighth International Conference on Belief Functions (BELIEF 2024), Sep 2024, Belfast, United Kingdom. hal-04621423

HAL Id: hal-04621423

<https://hal.science/hal-04621423>

Submitted on 24 Jun 2024

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Combination of Dependent Gaussian Random Fuzzy Numbers

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Abstract. Gaussian random fuzzy numbers are random fuzzy sets generalizing Gaussian random variables and possibility distributions. They define belief functions on the real line that can be conveniently combined by the product-intersection rule under the independence assumption. In this paper, we provide formulas for the combination of an arbitrary number of Gaussian random fuzzy numbers whose dependence is described by a correlation matrix.

Keywords: Evidence theory, Dempster-Shafer theory, belief functions, random fuzzy sets, information fusion.

1 Introduction

In recent papers [3][5][7], I introduced a theory of epistemic random fuzzy sets as an extension of both the Dempster-Shafer theory of evidence [9] and possibility theory [10]. In this new theoretical framework, uncertain and/or fuzzy pieces of evidence are represented by random fuzzy sets inducing belief functions, and independent items of evidence are combined by the product-intersection rule generalizing both Dempster’s rule of combination and the normalized product-intersection operator of possibility theory.

Gaussian random fuzzy numbers (GRFNs) introduced in [5] are an important practical model making it possible to represent evidence about continuous real variables. A GRFN is characterized by its mean, its variance and its imprecision. Gaussian random variables and Gaussian fuzzy numbers are recovered, respectively, in the special cases of infinite precision and zero variance. As shown in [5], GRFNs define a parametric family of belief functions on the real line, closed under the product-intersection rule.

As Dempster’s rule, the product-intersection rule introduced in [3][5] assumes independence of the combined pieces of evidence. Formulas for the combination of independent GRFNs are given in [5]. However, the independence assumption is sometimes too restrictive in applications. For instance, opinions from different experts, or predictions based on correlated features or overlapping datasets (using, e.g., the ENNreg model [4]) often cannot be treated as independent. The

combination of dependent GRFNs is, thus, an important problem; it is addressed in this paper.

The rest of this paper is organized as follows. Necessary notions about epistemic random fuzzy sets are first recalled in Section 2. Formulas for the combination of n GRFNs with an arbitrary covariance matrix are then derived in Section 3. Finally, examples are presented in Section 4 and Section 5 concludes the paper.

2 Background

General definitions and results about random fuzzy sets will first be recalled in Sections 2.1. GRFNs will then be addressed in Section 2.2.

2.1 Epistemic Random Fuzzy Sets

Epistemic Random Fuzzy Set (ERFS) theory is based on two main components: the representation of evidence by random fuzzy sets (inducing belief and plausibility functions), and a combination mechanism: the product-intersection rule for pooling independent evidence.

Let $(\Omega, \Sigma_\Omega, P)$ denote a probability space, (Θ, Σ_Θ) a measurable space, and \tilde{X} a mapping from Ω to the set $[0, 1]^\Theta$ of fuzzy subsets of Θ . For any $\alpha \in [0, 1]$, let ${}^\alpha\tilde{X}$ be the mapping from Ω to 2^Θ that maps each $\omega \in \Omega$ to the (weak) α -cut of $\tilde{X}(\omega)$. If, for any $\alpha \in [0, 1]$, ${}^\alpha\tilde{X}$ is $\Sigma_\Omega - \Sigma_\Theta$ strongly measurable [8], the tuple $(\Omega, \Sigma_\Omega, P, \Theta, \Sigma_\Theta, \tilde{X})$ is said to be a *random fuzzy set* (RFS) [2].

In ERFS theory, a RFS represents a piece of evidence, which may be unreliable, vague (fuzzy), or both. The set Ω is seen as a *set of interpretations* of a piece of evidence about a variable θ taking values in Θ . If interpretation $\omega \in \Omega$ holds, we only know that θ is constrained by the possibility distribution defined by fuzzy set $\tilde{X}(\omega)$. Standard Dempster-Shafer theory only considers the case of unambiguous evidence, in which every image $\tilde{X}(\omega)$ is crisp; mapping \tilde{X} is then a random set. In contrast, possibility theory only imposes a flexible constraint on the unknown quantity, without considering that this constraint may be itself uncertain. By considering both vagueness and uncertainty, ERFS is, thus, more flexible, allowing for faithful representation of many different kinds of evidence.

To any RFS, we can associate a belief function representing one's beliefs based on the available evidence. For technical reasons, we assume hereafter any RFS \tilde{X} to verify the following normalization conditions: (1) For all $\omega \in \Omega$, $\tilde{X}(\omega)$ is either the empty set, or a normal fuzzy set, and (2) the image $\tilde{X}(\omega)$ is almost surely nonempty, i.e., $P(\{\omega \in \Omega : \tilde{X}(\omega) = \emptyset\}) = 0$. For any $\omega \in \Omega$, a conditional possibility measure $\Pi_{\tilde{X}(\omega)}$ and a dual conditional necessity measure $N_{\tilde{X}(\omega)}$ on Θ can be defined as follows: for any $B \subseteq \Theta$, $\Pi_{\tilde{X}(\omega)}(B) = \sup_{\theta \in B} \tilde{X}(\omega)(\theta)$, and $N_{\tilde{X}(\omega)}(B) = 1 - \Pi_{\tilde{X}(\omega)}(B^c)$ if $\tilde{X}(\omega) \neq \emptyset$ and $N_{\tilde{X}(\omega)}(B) = 0$ otherwise. For any $B \in \Sigma_\Theta$, let $Bel_{\tilde{X}}(B)$ and $Pl_{\tilde{X}}(B)$ denote, respectively, the *expected necessity*

and the *expected possibility* of B wrt P . The corresponding mappings $Bel_{\tilde{X}} : \Sigma_{\Theta} \rightarrow [0, 1]$ and $Pl_{\tilde{X}} : \Sigma_{\Theta} \rightarrow [0, 1]$, are, respectively, belief and plausibility functions [2, 11].

Product-intersection rule. Given two normal fuzzy subsets \tilde{F} and \tilde{G} of Θ , their normalized product intersection is defined as

$$(\tilde{F} \odot \tilde{G})(\theta) = \begin{cases} \frac{\tilde{F}(\theta)\tilde{G}(\theta)}{\text{hgt}(\tilde{F} \cdot \tilde{G})} & \text{if } \text{hgt}(\tilde{F} \cdot \tilde{G}) > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

where $\text{hgt}(\tilde{F} \cdot \tilde{G}) = \sup_{\theta \in \Theta} \tilde{F}(\theta)\tilde{G}(\theta)$ is the height of the product intersection of \tilde{F} and \tilde{G} . This operation is associative; as shown in [6], it is the only normalized intersection operator having this property. The normalized product intersection can be extended to RFSs as follows. Let $(\Omega_i, \Sigma_i, P_i, \Theta, \Sigma_{\Theta}, \tilde{X}_i)$, $i = 1, 2$, be two RFSs representing independent pieces of evidence. Their product intersection is defined as the RFS $(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2, \tilde{P}_{12}, \Theta, \Sigma_{\Theta}, \tilde{X}_1 \oplus \tilde{X}_2)$, where $\tilde{X}_1 \oplus \tilde{X}_2$ is the mapping from $\Omega_1 \times \Omega_2$ to $[0, 1]^{\Theta}$ defined as $(\tilde{X}_1 \oplus \tilde{X}_2)(\omega_1, \omega_2) = \tilde{X}_1(\omega_1) \odot \tilde{X}_2(\omega_2)$, $\Sigma_1 \otimes \Sigma_2$ is the tensor product of Σ_1 and Σ_2 , and \tilde{P}_{12} is the probability measure on $(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2)$ obtained by conditioning the product measure $P_1 \times P_2$ by the fuzzy set of consistent pairs (ω_1, ω_2) , defined as $\tilde{F}(\omega_1, \omega_2) = \text{hgt}(\tilde{X}_1(\omega_1) \cdot \tilde{X}_2(\omega_2))$, i.e.,

$$\forall A \in \Sigma_1 \otimes \Sigma_2, \tilde{P}_{12}(A) = \frac{\int_{\Omega_1} \int_{\Omega_2} A(\omega_1, \omega_2) \tilde{F}(\omega_1, \omega_2) dP_2(\omega_2) dP_1(\omega_1)}{\int_{\Omega_1} \int_{\Omega_2} \tilde{F}(\omega_1, \omega_2) dP_2(\omega_2) dP_1(\omega_1)}, \quad (2)$$

where $A(\cdot, \cdot)$ denotes the indicator function of A . The *degree of conflict* between the two pieces of evidence is defined as one minus the denominator in the right-hand side of (2). The product intersection of RFSs is commutative and associative. It extends both Dempster's rule for combining random sets, and the normalized product intersection (1) for combining possibility distributions.

2.2 Gaussian Random Fuzzy Numbers

Gaussian Fuzzy Numbers (GFNs) play the same role in quantitative possibility theory as Gaussian random variables (GRVs) in probability theory. They are defined as fuzzy subsets of \mathbb{R} with membership function $x \mapsto \exp(-\frac{h}{2}(x-m)^2)$, where $m \in \mathbb{R}$ is the *mode* and $h \in [0, +\infty]$ is the *precision*. A GFN with mode m and precision h will be denoted by $\text{GFN}(m, h)$. GFNs are easily combined by the normalized product-intersection operator, as the following property holds: $\text{GFN}(m_1, h_1) \odot \text{GFN}(m_2, h_2) = \text{GFN}(m_{12}, h_1 + h_2)$, with $m_{12} = (h_1 m_1 + h_2 m_2) / (h_1 + h_2)$.

Let us now consider a GRV $M : \Omega \rightarrow \mathbb{R}$ with mean μ and variance σ^2 . The mapping $\tilde{X} : \Omega \rightarrow [0, 1]^{\mathbb{R}}$ such that $\tilde{X}(\omega) = \text{GFN}(M(\omega), h)$ defines a

random fuzzy set called a *Gaussian random fuzzy number* (GRFN) with mean μ , variance σ^2 and precision h . A GRFN can, thus, be seen as a GFN whose mode is uncertain and described by a Gaussian probability distribution. It is defined by a location parameter μ , and two parameters h and σ^2 corresponding, respectively, to possibilistic and probabilistic uncertainty. A GRV or a GFN is recovered when, respectively, $h = +\infty$ or $\sigma^2 = 0$. Formulas for the contour function $pl_{\tilde{X}} = Pl_{\tilde{X}}(\{x\})$ as well as for the lower and upper cumulative distribution functions (cdfs) $Bel_{\tilde{X}}((-\infty, x])$ and $Pl_{\tilde{X}}((-\infty, x])$ are given in [5].

As shown in [5], the family of GRFNs is closed under the product-intersection combination operation \oplus . Let $M_1 \sim N(\mu_1, \sigma_1^2)$ and $M_2 \sim N(\mu_2, \sigma_2^2)$ be two independent GRVs, and let $\tilde{X}_1 = \text{GFN}(M_1, h_1)$ and $\tilde{X}_2 = \text{GFN}(M_2, h_2)$ be corresponding GRFNs. To combine \tilde{X}_1 and \tilde{X}_2 by the product-intersection rule, we proceed as follows [5]:

1. We condition the joint probability distribution of (M_1, M_2) by the fuzzy subset \tilde{F} of \mathbb{R} defined by $\tilde{F}(m_1, m_2) = \text{hgt}(\text{GFN}(m_1, h_1) \cdot \text{GFN}(m_2, h_2)) = \exp(-0.5\bar{h}(m_1 - m_2)^2)$, where $\bar{h} = h_1 h_2 / (h_1 + h_2)$. This conditional distribution is normal with mean $\tilde{\boldsymbol{\mu}}$ and covariance matrix $\tilde{\boldsymbol{\Sigma}}$, whose expressions are given in [5].
2. The combined random fuzzy set $\tilde{X} = \tilde{X}_1 \oplus \tilde{X}_2$ is $\text{GFN}(M_c, h_1 + h_2)$, where $M_c \sim N(\mu_c, \sigma_c^2)$ with $\mu_c = \mathbf{h}^{*T} \tilde{\boldsymbol{\mu}}$ and $\sigma_c^2 = \mathbf{h}^{*T} \tilde{\boldsymbol{\Sigma}} \mathbf{h}^*$, where

$$\mathbf{h}^* = (h_1, h_2)^T / (h_1 + h_2)$$

is the vector of normalized precisions.

3 Extension to n Dependent GRFNs

The formulas for the product intersection of two GRFNs mentioned in Section 2.2 were established under the assumption that the underlying GRVs are independent. In this section, we generalize these formulas to the combination of n *dependent* GRFNs $\text{GFN}(M_i, h_i)$, $i = 1, \dots, n$, where (M_1, \dots, M_n) has a multidimensional normal distribution with an arbitrary covariance matrix. After preliminaries exposed in Section 3.1, we prove our main result in Section 3.2.

3.1 Preliminaries

Let us first recall that a Gaussian fuzzy vector (GFV) with mode $\mathbf{m} \in \mathbb{R}^n$ and symmetric, positive semidefinite (PSD) precision matrix $\mathbf{H} \in \mathbb{R}^{n \times n}$ as a fuzzy subset of \mathbb{R}^n with membership function

$$\mathbf{x} \mapsto \exp(-0.5(\mathbf{x} - \mathbf{m})^T \mathbf{H}(\mathbf{x} - \mathbf{m})).$$

It is denoted as $\text{GFV}(\mathbf{m}, \mathbf{H})$. The results derived in Section 3.2 are based on the following propositions, which are direct consequences of results about the prod-

uct of univariate and multivariate normal densities proved in [1]. Propositions 1 and 2 generalize¹, respectively, Propositions 3 and 11 in [5].

Proposition 1 *Let $GFN(m_i, h_i)$, $i = 1, \dots, n$, be n GFNs.*

1. *The height of their product intersection is*

$$\tilde{F}(m_1, \dots, m_n) = \exp \left[-\frac{1}{2} \left(\sum_{i=1}^n h_i m_i^2 - \frac{(\sum_{i=1}^n h_i m_i)^2}{\sum_{i=1}^n h_i} \right) \right]. \quad (3)$$

2. *Their normalized product intersection is a GFN with precision $h = \sum_{i=1}^n h_i$ and mode $m = (1/h) \sum_{i=1}^n h_i m_i$.*

Proposition 2 *Let $GFV(\mathbf{m}_1, \mathbf{H}_1)$ and $GFV(\mathbf{m}_2, \mathbf{H}_2)$ be two GRVs. Assuming that \mathbf{H}_1 is positive definite and \mathbf{H}_2 is PSD,*

1. *The height of their product intersection is*

$$\exp \left(-\frac{1}{2} (\mathbf{m}_1 - \mathbf{m}_2)^T \mathbf{H}_2 [\mathbf{I} + \mathbf{H}_1^{-1} \mathbf{H}_2]^{-1} (\mathbf{m}_1 - \mathbf{m}_2) \right), \quad (4)$$

where \mathbf{I} is the $n \times n$ identity matrix;

2. *Their normalized product intersection is a GFV with precision matrix $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$ and mode $\mathbf{m} = \mathbf{H}^{-1} (\mathbf{H}_1 \mathbf{m}_1 + \mathbf{H}_2 \mathbf{m}_2)$.*

3.2 Main Result

Let us consider n GRFNs $GFN(M_i, h_i)$, $i = 1, \dots, n$, where $\mathbf{M} = (M_1, \dots, M_n)$ has a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$. We first assume this distribution to be non-degenerate, but this assumption will be relaxed later. To combine these n GRFNs by the product intersection rule, we first need to condition the distribution of \mathbf{M} by the fuzzy subset \tilde{F} of consistent tuples m_1, \dots, m_n given by (3). We first remark that (3) can be written in vector form as $\tilde{F}(\mathbf{m}) = \exp [-\frac{1}{2} \mathbf{m}^T \mathbf{A} \mathbf{m}]$, where $\mathbf{m} = (m_1, \dots, m_n)^T$ and \mathbf{A} is the symmetric and PSD matrix

$$\mathbf{A} = \text{diag}(\mathbf{h}) - \frac{\mathbf{h} \mathbf{h}^T}{\mathbf{1}^T \mathbf{h}},$$

where $\mathbf{h} = (h_1, \dots, h_n)^T$ and $\mathbf{1} = (1, \dots, 1)^T$. The conditional density of \mathbf{M} given fuzzy event \tilde{F} is

$$f(\mathbf{m} | \tilde{F}) = \frac{f(\mathbf{m}) \tilde{F}(\mathbf{m})}{\int f(\mathbf{m}) \tilde{F}(\mathbf{m}) d\mathbf{m}}. \quad (5)$$

¹ The rigorous proof of Proposition 2 cannot be given here for lack of space; it will be given in an extended version of this paper.

From Proposition 2, $f(\mathbf{m}|\tilde{F}) \propto \exp\left(-\frac{1}{2}(\mathbf{m} - \tilde{\boldsymbol{\mu}})^T \tilde{\boldsymbol{\Sigma}}^{-1}(\mathbf{m} - \tilde{\boldsymbol{\mu}})\right)$ with

$$\tilde{\boldsymbol{\Sigma}}^{-1} = \boldsymbol{\Sigma}^{-1} + \mathbf{A} \quad \text{and} \quad \tilde{\boldsymbol{\mu}} = \tilde{\boldsymbol{\Sigma}}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \mathbf{A}\mathbf{0}) = \tilde{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}.$$

Writing $\tilde{\boldsymbol{\Sigma}}^{-1} = \boldsymbol{\Sigma}^{-1}(\mathbf{I} + \boldsymbol{\Sigma}\mathbf{A})$, we get

$$\tilde{\boldsymbol{\Sigma}} = (\mathbf{I} + \boldsymbol{\Sigma}\mathbf{A})^{-1}\boldsymbol{\Sigma} \quad \text{and} \quad \tilde{\boldsymbol{\mu}} = (\mathbf{I} + \boldsymbol{\Sigma}\mathbf{A})^{-1}\boldsymbol{\mu}. \quad (6)$$

Remark 1 *We have established (6) under the assumption that the distribution of \mathbf{M} has a density (i.e., $\boldsymbol{\Sigma}$ is nonsingular). However, it can be shown that $\mathbf{I} + \boldsymbol{\Sigma}\mathbf{A}$ is nonsingular. As, when singular, $\boldsymbol{\Sigma}$ is the limit of a sequence of nonsingular covariance matrices, (6) remains true by continuity, even when the distribution of \mathbf{M} is degenerate.*

Combination. From Proposition 1, the product intersection of the n GRFNs $\text{GFN}(M_i, h_i)$, $i = 1, \dots, n$ GRFN is the GRFN $\text{GFN}(M_c, \sum_{i=1}^n h_i)$ with

$$M_c = \frac{\sum_{i=1}^n h_i M_i}{\sum_{i=1}^n h_i} = \mathbf{h}^{*T} \mathbf{M} \sim N(\mu_c, \sigma_c^2), \quad (7)$$

where $\mathbf{h}^* = (h_1, \dots, h_n) / \sum_{i=1}^n h_i$, $\mu_c = \mathbf{h}^{*T} \tilde{\boldsymbol{\mu}}$ and $\sigma_c^2 = \mathbf{h}^{*T} \tilde{\boldsymbol{\Sigma}} \mathbf{h}^*$.

Degree of conflict. The degree of conflict between the n GRFNs is defined as one minus the denominator on the right-hand side of (5). From (4) in Proposition 2, and noticing that $|\tilde{\boldsymbol{\Sigma}}|/|\boldsymbol{\Sigma}| = |\tilde{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1}| = |\mathbf{I} + \boldsymbol{\Sigma}\mathbf{A}|^{-1}$, we get

$$\kappa = 1 - |\mathbf{I} + \boldsymbol{\Sigma}\mathbf{A}|^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{\mu}^T \mathbf{A}[\mathbf{I} + \boldsymbol{\Sigma}\mathbf{A}]^{-1}\boldsymbol{\mu}\right). \quad (8)$$

Remark 2 *The equations given in [5] for the product intersection of two independent GRFNs and their degree of conflict can be recovered, respectively, from (7) and (8) when $n = 2$ and $\boldsymbol{\Sigma}$ is diagonal.*

4 Numerical Example

Let us consider two GRFNs $\tilde{X}_1 = \text{GFN}(M_1, h_1)$ and $\tilde{X}_2 = \text{GFN}(M_2, h_2)$ with $M_1 \sim N(1, 4)$, $M_2 \sim N(3, 1)$, $h_1 = 2$, and $h_2 = 1$. Their contour functions as well as their lower and upper cdfs are shown in Figure 1a. Figures 2a and 2b show ellipses with 95% coverage probability for the unconditional distribution of random vector (M_1, M_2) and its conditional distribution given \tilde{F} for, respectively, $\rho = 0.9$ and $\rho = -0.9$. The combined GRFNs for three values of the correlation coefficient $\rho \in \{-1, 0, 1\}$ are shown in Figure 1b. It is clear that the assumed correlation coefficient strongly influences the result of the combination. Figures 3a and 3b show, respectively, the mean and standard deviation of the combined GRFN as functions of ρ . The standard deviation appears to be particularly sensitive to the value of ρ .

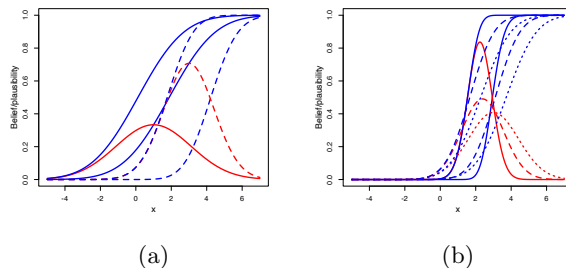


Fig. 1: Two GRFNs (a) and their combination (b) assuming $\rho = -1$ (solid lines), $\rho = 0$ (broken lines) and $\rho = 1$ (dotted lines). Each GRFN is represented by its contour function (red curve) and by its lower and upper cdfs (blue curves).

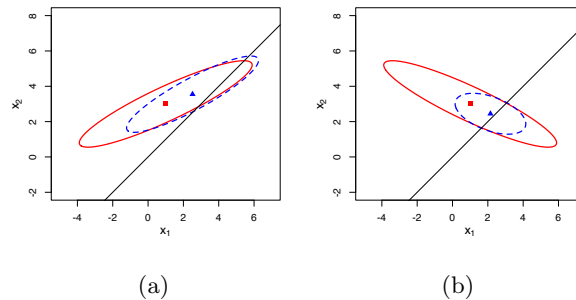


Fig. 2: 95% coverage probability ellipses for the unconditional distribution of random vector (M_1, M_2) (solid red curve) and its conditional distribution given \tilde{F} (dashed blue curve), for $\rho = 0.9$ (a) and $\rho = -0.9$ (b).

Figure 3c shows the degree of conflict as a function of ρ . It reaches a minimum value of 0.629 for $\rho = 0.625$. In real applications where the sources cannot be assumed to be independent but the precise value of the correlation is unknown, the value corresponding to the minimum conflict between the sources could be chosen. Figures 4a and 4b show, respectively, the minimum degree of conflict as a function of the distance $\Delta = |\mu_1 - \mu_2|$ between means and the corresponding correlation coefficient $\hat{\rho}$ with, as before, $\sigma_1 = 2$, $\sigma_2 = 1$, $h_1 = 2$, and $h_2 = 1$. Figure 4a also displays the conflict for the product-intersection rule ($\rho = 0$) and the complete positive dependence rule ($\rho = 1$). Interestingly, we have $\hat{\rho} = 1$ when the conflict is low and $\hat{\rho} = -1$ when the conflict is high, while $\hat{\rho}$ takes values between -1 and 1 for intermediate distances.

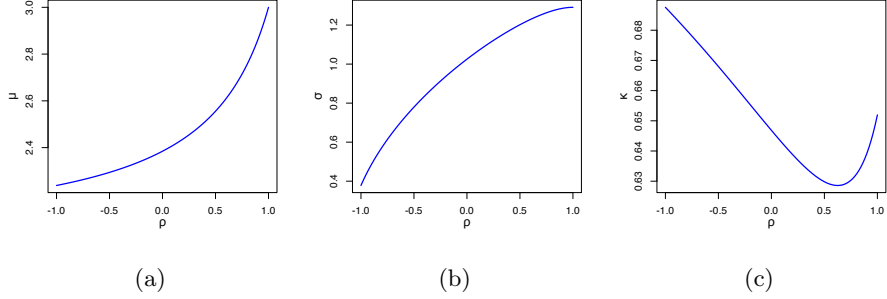


Fig. 3: Mean (a), standard deviation (b) and degree of conflict (c) as functions of ρ for the combination of the two GRFNs shown in Figure 1a.

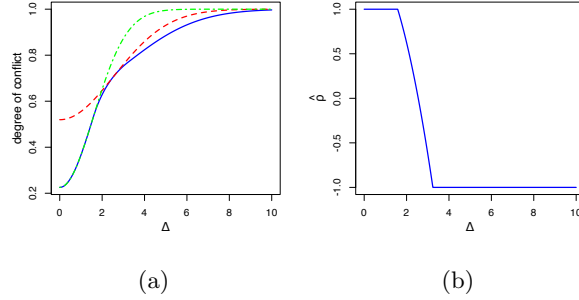


Fig. 4: (a): Minimum degree of conflict (solid blue curve), degree of conflict for $\rho = 0$ (dashed red curve) and $\rho = 1$ (green dashed-dotted curve) vs. $\Delta = |\mu_1 - \mu_2|$. (b): Minimum-conflict correlation coefficient $\hat{\rho}$ as a function of Δ .

5 Conclusions

We provided formulas for the combination of $n \geq 2$ GRFNs with an arbitrary correlation matrix. The results can be directly applied to the combination of transformed GRFNs and mixtures of GRFNs, as defined in [7]. In practice, the correlation matrix will rarely be known and it will need to be estimated. In the case where n sources provide GRFNs and no ground truth is available, the correlation coefficients minimizing the conflict can be determined. This approach was exemplified in Section 4 in the case of two sources; it can be easily extended to n sources using a suitable representation of the correlation matrix. In the supervised case such as, e.g., a regression task where n ENNreg models [4] provide predictive GRFNs and ground truth is available, a loss function such as the generalized negative log-likelihood introduced in [4] can be minimized. Another

problem, not addressed in this paper, is the consideration of the reliability or “relevance” of information sources in the combination. These and other research directions will be explored in future work.

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