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► **To cite this version:**

Bertrand Burgardt, Anh Le Van. A Fully Regularized Direct Boundary Formulation For Three-dimensional Elastoplastic Problems. WIT Transactions on Modelling and Simulation, 1998, 21, pp.1-9. 10.2495/BE980101 . hal-04621253v2

**HAL Id: hal-04621253**

**<https://hal.science/hal-04621253v2>**

Submitted on 22 Aug 2024

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# A Fully Regularized Direct Boundary Formulation for Three-dimensional Elastoplastic Problems

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## Abstract

The present work is devoted to a fully regularized boundary formulation for three-dimensional elastoplastic problems. The proposed algorithm is based on an integral representation of the strain field in which no integrals have to be understood in the principal value sense. Following the current approach used in Finite Element procedures, an initial strain formulation and an implicit integration scheme are considered. The elastoplastic relationships take into account an isotropic and linear kinematic hardening material satisfying the von Mises criterion.

## 1 Introduction

It is well established that the boundary formulation can be applied to various problems presenting geometrical or material nonlinearities (Mukherjee & Chandra [7]). Whereas algorithms for solving non-linear equations are intensively developed within the domain finite element method, strangely enough very few papers were devoted to 3D elastoplastic problems with somewhat sophisticated hardening laws, especially those involving the concept of the tangent operator. More often than not the algorithms are explicit as proposed by Telles [11] or Banerjee [2]. Recently many works have investigated implicit procedures for integral equations on account of their unconditional stability (see Telles & Carrer [13]).

In the present paper, an integral representation for strains is proposed. As the basic component of the numerical implementation, this expression is characterized by being fully regularized, even at irregular boundary points. Next it is shown how to incorporate the radial return algorithm into the boundary element formulation in order to numerically solve the elastoplastic problems. This approach contains some extensions of the recent work of Bonnet & Mukherjee [5].

## 2 Boundary Integral Formulation

Let  $\Omega$  be a three-dimensional open bounded domain subjected to a distribution of initial plastic strain  $\varepsilon^p(x)$  the boundary  $S$  of which is a piecewise Liapunov manifold. The integral representation of the displacement field  $u(x)$  is now well established (see e.g. Bonnet [4] or Balas, Sladek & Sladek [1]) and is given by :

$$u_k(x) = \int_S \tilde{t}_i(y) U_i^k(x, y) dS(y) - \int_S T_i^k(x, y) \tilde{u}_i(y) dS(y) + \int_{\Omega} \Sigma_{ij}^k(x, y) \varepsilon_{ij}^p(y) dV(y) \quad (1)$$

where  $U_i^k(x, y)$  denotes the Kelvin solution in  $R^3$  (for the definition of  $T_i^k(x, y)$  and  $\Sigma_{ij}^k(x, y)$  see Bonnet [4]) and in the Direct Boundary Element formulation,  $\tilde{u}_i(y)$  represents the trace of  $u_i(x)$  on  $S$  and  $\tilde{t}_i(y)$  the stress vector on  $S$ .

Differentiation of (1) with respect to  $x$  yields the representation formula for the displacement gradient of internal points. A regularized formulation of the displacement gradient is given in Bonnet [4], Bonnet & Mukherjee [5]. Up to now boundary strains (or boundary stresses) are computed from the boundary stress, displacement and plastic strain fields (see Telles [11], Telles & Brebbia [12]). Nevertheless recent works (see Guiggiani [6]) underscore the poor accuracy and the limitations of such a procedure in elasticity and plainly show the advantage to use integral representation for the boundary strain. Bui & Bonnet [3] gave an integral representation of the displacement gradient field at smooth points on  $S$  and Sladek, Sladek & Levan [10] extended this approach to bodies subjected to initial strain. The

present section gives an integral representation of the boundary displacement gradients valid for a manifold  $S$  having edge and corner singularities.

Let us define the following quantity.

$$\begin{aligned} \frac{\widetilde{\partial u}}{\partial n_i}(y) = & \frac{1}{G} \left( \delta_{ik} - \frac{n_i(y)n_k(y)}{2(1-\nu)} \right) \tilde{t}_k^{elas}(y) - \\ & - \frac{\nu}{1-\nu} D_k \tilde{u}_k(y) n_i(y) - n_k(y) D_i \tilde{u}_k(y) \end{aligned} \quad (2)$$

where  $G$  and  $\nu$  denotes the shear modulus and the Poisson ratio respectively.  $D_j(\cdot)$  represents the surface gradient operator on  $S$  and  $\tilde{t}_j^{elas}(y)$  is the elastic traction vector given by the formula :

$$\tilde{t}_k^{elas}(y) = \tilde{t}_k(y) + (C_{klmn} \varepsilon_{mn}^p(y)) n_l(y) \quad (3)$$

where  $C$  denotes the elastic fourth-order tensor.

Let us now introduce the "reconstitued" displacement gradient on  $S$ .

$$\widetilde{\nabla} u_{ij}(y) \equiv \tilde{h}_{ij}(y) = D_j \tilde{u}_i(y) + \frac{\widetilde{\partial u}}{\partial n_i}(y) n_j(y) \quad (4)$$

It can be proved that the gradient of a displacement field satisfying Navier-Cauchy's equations in  $\Omega$  admits of the following integral representation for  $x \in \Omega$ .

$$\begin{aligned} u_{k,l}(x) = & \int_S D_{lj} U_i^k(x, y) \tilde{\sigma}_{ij}(y) - \int_S T_i^k(x, y) \tilde{h}_{il}(y) dS(y) \\ & - \int_{\Omega} \Sigma_{ij,l}^k(x, y) [\varepsilon_{ij}^p(y) - \varepsilon_{ij}^p(x)] dV(y) \\ & + \int_S \Sigma_{ij}^k(x, y) n_l(y) [\varepsilon_{ij}^p(y) - \varepsilon_{ij}^p(x)] dS(y) \end{aligned} \quad (5)$$

where  $D_{lb}(\cdot)$  can be identified with the surface curl operator and is defined by the relation  $D_{lb}(\cdot)(y) = n_l(y) D_b(\cdot)(y) - n_b(y) D_l(\cdot)(y)$ .  $\tilde{\sigma}_{ij}(y)$  represents the "reconstitued" boundary stress.

$$\tilde{\sigma}_{ij}(y) = C_{ijab} (\tilde{h}_{ab}(y) - \varepsilon_{ab}^p(y)) \quad (6)$$

Assuming that  $\tilde{h}_{ij}(y) \in C^{0,\alpha}(S)$  and  $\varepsilon_{ij}^p(y) \in C^{0,\alpha}(\Omega)$ , we obtain the following expression for the integral representation of the boundary displacement gradient.

$$\begin{aligned} \forall z \in S, u_{k,l}(z) = & \int_S U_i^k(z, y) D_{jl} \tilde{\sigma}_{ij}(y) dS(y) + \tilde{h}_{kl}(z) \\ & - \int_S T_i^k(z, y) [\tilde{h}_{il}(y) - \tilde{h}_{il}(z)] dS(y) \\ & + \int_\Omega \Sigma_{ij}^k(z, y) \varepsilon_{ij,l}^p(y) dV(y) \end{aligned} \quad (7)$$

The above expression facilitates the numerical implementation as will be shown latter. It is noteworthy that the computation of the displacement gradient at either internal or boundary points are based on the same structure since the integral kernels arising in relations (7) and (5) are identical. All the integrals are regular or weakly singular so that the numerical computation can be performed with standard Gaussian quadrature rules.

### 3 Radial Return Algorithm (RRA)

For the domain finite element purposes, use is constantly made of the so-called radial return algorithm (RRA) developed by Simo & Taylor [9] for rate-independent plasticity. The main advantages of the RRA are the unconditional stability and the obtention of the consistent tangent operator. Bonnet & Mukherjee [5] were the firsts to adapt this algorithm to boundary element method.

Let  $d^p$  be the plastic strain rate,  $\bar{\varepsilon}^p$  be the cumulated plastic strain and  $\alpha$  the back-stress (the center of the yield surface). The hardening rule is characterized by a function  $\kappa(\bar{\varepsilon}^p)$ , the evolution of  $\alpha$  is defined by the following relation and the yield condition is given by

$$f(\sigma, \alpha, \kappa) \equiv \|s - \alpha\| - \sqrt{\frac{2}{3}} \kappa(\bar{\varepsilon}^p) = 0 \quad (8)$$

$$\dot{\alpha} = \frac{2}{3} H'(\bar{\varepsilon}^p) d^p \quad (9)$$

where  $s$  is the deviatoric part of  $\sigma$  and  $H'(\bar{e}^p)$  the plastic modulus. Use will be made of the RRA, given in Simo & Taylor [9] for rate-independent plasticity and briefly summarized below. Let  $\Delta\varepsilon_{n+1}$  be a given strain increment then a trial deviatoric stress is defined as

$$s_{n+1}^E = s_n + 2G\Delta e_{n+1} \quad (10)$$

If  $f(s_{n+1}^E, \alpha_n, \kappa_n) \leq 0$ , the trial deviatoric stress is elastic and  $\sigma_{n+1}$  is given by :

$$\sigma_{n+1} = K(I \otimes I) : \Delta\varepsilon_{n+1} + 2G\Delta e_{n+1} + \sigma_n \quad (11)$$

where  $K$  is the bulk modulus and  $I$  the second order unit tensor. If  $f(s_{n+1}^E, \alpha_n, \kappa_n) > 0$ , the trial deviatoric stress lies outside of the elastic region enclosed by the yield surface and  $\sigma_{n+1}$  is obtained by the following equations which express the projection of  $s_{n+1}^E$  onto the yield surface :

$$\sigma_{n+1} = K(I \otimes I) : \varepsilon_{n+1} + s_{n+1} \quad \bar{e}_{n+1}^p = \bar{e}_n^p + \sqrt{\frac{2}{3}}\Delta\gamma_{n+1} \quad (12)$$

$$s_{n+1} = \alpha_{n+1} + \sqrt{\frac{2}{3}}\kappa_{n+1}\hat{n} \quad \hat{n} = \frac{s_{n+1}^E - \alpha_n}{\|s_{n+1}^E - \alpha_n\|} \quad (13)$$

$$\alpha_{n+1} = \alpha_n + \sqrt{\frac{2}{3}}\Delta H_{n+1}\hat{n} \quad (14)$$

and  $\Delta\gamma_{n+1}$  solves the nonlinear consistency equation

$$\|s_{n+1}^E - \alpha_n\| - \sqrt{\frac{2}{3}}\kappa_{n+1} - 2G\Delta\gamma_{n+1} - \sqrt{\frac{2}{3}}\Delta H_{n+1} = 0 \quad (15)$$

in which  $\Delta H_{n+1} = H(\bar{e}_{n+1}^p) - H(\bar{e}_n^p)$ .

## 4 Numerical implementation

In order to solve the system of integral equations presented in Section 2, the boundary  $S$  and the part of the domain  $\Omega$  which is likely to yield have to be discretized. Let us assume that  $(\tilde{u}_n, \tilde{t}_n, \varepsilon_n^p)$  are the solution of the evolutive problem at the time-like parameter value  $t_n$ . One then has to find the fields  $(\tilde{u}_{n+1}, \tilde{t}_{n+1}, \varepsilon_{n+1}^p)$  defined at  $t_{n+1}$ , consistent with the yield criterion, the equilibrium equation and the boundary conditions for an given load increment  $(\Delta\tilde{u}_{n+1}, \Delta\tilde{t}_{n+1})$ . Let

us denote  $\Delta(\cdot)_{n+1} = (\cdot)_{n+1} - (\cdot)_n$ .

The enforcement of the boundary conditions at  $t_{n+1}$  reads after discretization :

$$[H]\{\Delta u_{n+1}\} - [G]\{\Delta t_{n+1}\} = [Q]\{C : \Delta \varepsilon_{n+1}^p\} \quad (16)$$

Let us denote by  $\{\Delta y_{n+1}\}$  and  $\{\Delta f_{n+1}\}$  the vectors collecting respectively the unknown and known boundary variables. The above formula can be rearranged as follows :

$$[A]\{\Delta y_{n+1}\} = \{\Delta f_{n+1}\} + [Q]\{C : \Delta \varepsilon_{n+1}^p\} \quad (17)$$

As for the strain field in  $\bar{\Omega}$ , it can be obtained using Relation (5) for internal points or Relation (7) for boundary points (see Bonnet & Mukherjee [5]). For internal points all the integrals arising in the present formulation are regular if  $\varepsilon_{ij}^p(y) \in C^{0,\alpha}(\Omega)$  which makes it possible to use a classical continuous interpolation for the plastic strains.

Let us take for each finite boundary element and internal cell the following interpolation :

$$\tilde{u}(y(\xi)) = N_e(\xi).u^e \quad \tilde{t}(y(\xi)) = N_e(\xi).t^e \quad (18)$$

$$\varepsilon^p(y(\xi)) = N_e(\xi).(\varepsilon^p)^e \quad \tilde{h}(y(\xi)) = N_e(\xi).h^e \quad (19)$$

When obtaining Relation (7), one has assumed that  $\tilde{h}_{ij}(y) \in C^{0,\alpha}(S)$ . In fact, differentiation of Relation (18) yields only piecewise continuous interpolation of  $\tilde{h}_{ij}$ . In order to overcome this difficulty, Polch, Cruse & Huang [8] interpolated  $D_{lb}(\tilde{u}_k)(y)$  and  $\tilde{u}_k(y)$  independently. This procedure is extended here to the entire boundary displacement gradient. Eventually, the variational formulation consisting of minimizing the error between both interpolations gives rise to :

$$[M]\{h\} = [B_u]\{u\} + [B_t]\{t\} + [B_{\varepsilon^p}]\{\varepsilon^p\} \quad (20)$$

where  $[M]$  is an invertible symmetric matrix.

Discretizing Relations (5) and (7), and denoting by  $\{\Delta \varepsilon_{n+1}^p\}$  the strain increment vector, one obtains :

$$\{\Delta \varepsilon_{n+1}^p\} = [D]\{\Delta h_{n+1}\} + [Q']\{C : \Delta \varepsilon_{n+1}^p\} \quad (21)$$

Using the relation (20) and substituting  $\Delta y_n$  in the previous equation via (17), one finally has :

$$\{\Delta\varepsilon_{n+1}\} = \{\Delta\varepsilon_{n+1}^{elas}\} + [S]\{C : \Delta\varepsilon_{n+1}^p\} \quad (22)$$

where  $\{\Delta\varepsilon_{n+1}^{elas}\}$  is the elastic strain increment which results from the increment  $(\Delta\tilde{u}_{n+1}, \Delta\tilde{t}_{n+1})$ . Incorporating the partition elastic strain-plastic strain  $\{C : \Delta\varepsilon_{n+1}^p\} = \{C : \Delta\varepsilon_{n+1}\} - \{\Delta\sigma_{n+1}\}$  into the previous relation, one obtains the following nonlinear equation for  $\Delta\varepsilon_{n+1}$ .

$$\begin{aligned} \{G(\Delta\varepsilon_{n+1})\} &\equiv [S]\{\sigma_{n+1}(\Delta\varepsilon_{n+1}) + \sigma_n - C : \Delta\varepsilon_{n+1}\} + \\ &\quad \{\Delta\varepsilon_n\} - \{\Delta\varepsilon_{n+1}^{elas}\} \\ &= \{0\} \end{aligned} \quad (23)$$

The solution of Equation (23) is carried out using the Newton-Raphson scheme and the consistent tangent operator (CTO)  $C_{n+1}$ . The additive correction  $\delta\varepsilon_{n+1}^i = \Delta\varepsilon_{n+1}^{i+1} - \Delta\varepsilon_{n+1}^i$  to  $\Delta\varepsilon_{n+1}^i$  solves the following linear equation.

$$([S].[C - C_{n+1}^i] - [I]).\{\delta\varepsilon_{n+1}^i\} = \{G(\Delta\varepsilon_n^i)\} \quad (24)$$

where  $C_{n+1}^i = \frac{\partial\sigma_{n+1}}{\partial\Delta\varepsilon_{n+1}}(\Delta\varepsilon_{n+1}^i)$ . The expression of  $C_{n+1}$  associated with the RRA is given by Simo & Taylor [9] :

$$C_{n+1} = K.I \otimes I + 2G\beta(\mathcal{I} - \frac{1}{3}I \otimes I) - 2G\gamma\hat{n} \otimes \hat{n} \quad (25)$$

where

$$\beta = \sqrt{\frac{2}{3}} \frac{\kappa(\bar{e}_{n+1}^p) + \Delta H_n}{\|s_{n+1}^E - \alpha_n\|} \quad \gamma = \frac{1}{1 + \frac{[\kappa' + H']_{n+1}}{3G}} - (1 - \beta) \quad (26)$$

and  $\mathcal{I}$  denotes the fourth-order symmetric unit tensor.

## 5 Conclusion

In the present paper, an integral representation for strains has been proposed. Involved as the basic component in the numerical implementation, this expression is characterized by being fully regularized, even at irregular boundary points. It has also been shown how to incorporate the radial return algorithm into the boundary element formulation. Further investigations are under progress in order to numerically validate the proposed formulation and the results will be presented in next papers.



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