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Rogue waves in the NLS hierarchy of order 6

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Abstract

Rational solutions to the sixth equation of the NLS hierarchy are given. We construct explicit expressions of these solutions for the first orders depending on multi-parameters. Patterns of these solutions in the (x, t) plane according the different values of the parameters are studied.

Key Words : NLS hierarchy, rational solutions.

PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

We consider the sixth equation of the NLS hierarchy of order 6 (*NLS6*) which can be written as

$$\begin{aligned} u_t + u_{7x} + 14|u|^2 u_{5x} + 42\bar{u}u_x u_{4x} + 28u\bar{u}_x u_{4x} + 14uu_x \bar{u}_{4x} + 70|u|^4 u_{3x} \\ + 98|u_x|^2 u_{3x} + 70\bar{u}u_{2x} u_{3x} + 42u\bar{u}_{2x} u_{3x} + 28u_x^2 \bar{u}_{3x} + 28uu_{2x} \bar{u}_{3x} + 280\bar{u}|u|^2 u_x u_{2x} \\ + 140|u|^2 u\bar{u}_x u_{2x} + 140|u|^2 uu_x \bar{u}_{2x} + 70\bar{u}u_{2x}^2 + 112u_x |u_{2x}|^2 + 70\bar{u}^2 u_x^3 + 280|u|^2 |u_x|^2 u_x \\ + 70u^2 |u_x|^2 \bar{u}_x + 140|u|^6 u_x \end{aligned} \quad (1)$$

with as usual the subscripts meaning partial derivatives and \bar{u} the complex conjugate of u .

The NLS hierarchy includes different classical equations. The first one is the NLS equation [1]; the second one is the mKdV equation [2]; the third one is the LPD equation [3].

Many works has been done for these first three equations of the NLS hierarchy; for example, we can mention the following works, for the NLS equation [4], the mKdV equation [5], the LPD equation [3, 6, 7]. However, very few studies

have been carried out for the following orders of this hierarchy. Here we explicitly construct solutions of the order equation six of this hierarchy.

We construct explicitly rational solutions for the first orders and study the patterns of the modulus of the solutions in the (x, t) plane.

2 Rational solutions of order 1 to the NLS6 equation

Theorem 2.1 *The function $v(x, t)$ defined by*

$$v(x, t) = \frac{-3 + 4x^2 - 1120xt + 78400t^2}{1 + 4x^2 - 1120xt + 78400t^2} \quad (2)$$

is a solution to the (NLS6) equation (1).

Proof: We have to replace the expression of the solution given by (2) and verify that (1) is satisfied.

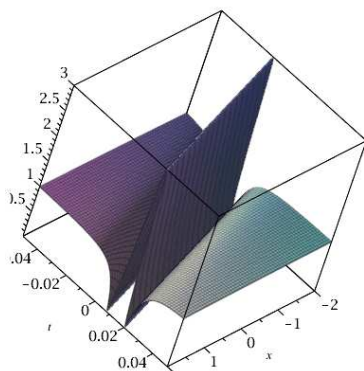


Figure 1. Solution of order 1 to (NLS6).

We get a smooth solution of the equation (1).

3 Rational solutions of order 2 of the NLS6 equation depending on 2 real parameters

Theorem 3.1 *The function $v(x, t)$ defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)} \quad (3)$$

with

$$\begin{aligned}
n(x, t) = & -(-64x^6 + 53760tx^5 + 2304b_1x^5 + 144x^4 - 34560b_1^2x^4 + 768ia_1x^4 - \\
& 18816000t^2x^4 - 1612800b_1tx^4 - 768a_1^2x^4 + 430080ta_1^2x^3 + 19353600b_1^2tx^3 + \\
& 18432b_1a_1^2x^3 + 276480b_1^3x^3 - 188160tx^3 - 430080ia_1tx^3 - 18432ia_1x^3b_1 - \\
& 4992b_1x^3 + 3512320000t^3x^3 + 451584000b_1t^2x^3 - 368793600000t^4x^2 - 165888b_1^2a_1^2x^2 + \\
& 6144ia_1^3x^2 + 90316800ia_1t^2x^2 + 4032000b_1tx^2 - 7741440b_1ta_1^2x^2 + 5760a_1^2x^2 + \\
& 58752b_1^2x^2 - 4064256000b_1^2t^2x^2 + 7741440ia_1b_1tx^2 - 90316800t^2a_1^2x^2 + 62092800t^2x^2 - \\
& 1244160b_1^4x^2 + 180x^2 - 63221760000b_1t^3x^2 - 3072a_1^4x^2 - 116121600b_1^3tx^2 - \\
& 1152ia_1x^2 + 165888ia_1b_1^2x^2 - 322560ta_1^2x + 4425523200000b_1t^4x + 1083801600b_1t^2a_1^2x + \\
& 379330560000b_1^2t^3x - 835430400b_1t^2x + 348364800b_1^4tx + 16257024000b_1^3t^2x + \\
& 46448640b_1^2ta_1^2x - 4608ia_1xb_1 - 1083801600ia_1xb_1t^2 - 7902720000t^3x + 2985984b_1^5x - \\
& 28062720b_1^2tx + 663552b_1^3a_1^2x - 50688b_1a_1^2x - 1720320ia_1^3tx + 8429568000t^3a_1^2x - \\
& 46448640ia_1xb_1^2t - 73728ia_1^3xb_1 + 36864b_1a_1^4x - 663552ia_1xb_1^3 - 5616b_1x - \\
& 967680ia_1tx - 290304b_1^3x + 20652441600000t^5x + 860160ta_1^4x - 8429568000ia_1t^3x - \\
& 292320tx - 720ia_1 + 1536ia_1^3 - 50577408000b_1t^3a_1^2 - 5160960b_1ta_1^4 + 995328ia_1b_1^4 + \\
& 10321920ia_1^3b_1t - 645120b_1ta_1^2 - 45 + 8386560ia_1b_1t + 350353920000t^4 + 63866880b_1^3t + \\
& 12288ia_1^5 - 92897280b_1^3ta_1^2 + 295034880000ia_1t^4 + 96768b_1^2a_1^2 - 67737600t^2a_1^2 + \\
& 51631104000b_1t^3 + 120422400ia_1^3t^2 - 7761600t^2 + 947520b_1t + 221184ia_1^3b_1^2 - \\
& 481890304000000t^6 - 3251404800b_1^2t^2a_1^2 - 132765696000000b_1^2t^4 + 2777241600b_1t^2 - \\
& 24385536000b_1^4t^2 - 758661120000b_1^3t^3 - 995328b_1^4a_1^2 - 110592b_1^2a_1^4 - 123914649600000b_1t^5 - \\
& 295034880000t^4a_1^2 - 60211200t^2a_1^4 + 69120ia_1b_1^2 + 158054400ia_1t^2 + 92897280ia_1b_1^3t + \\
& 3251404800ia_1b_1^2t^2 + 50577408000ia_1b_1t^3 - 418037760b_1^5t + 8448a_1^4 + 518400b_1^4 - \\
& 2985984b_1^6 - 4096a_1^6 + 1872a_1^2 + 18000b_1^2)e^{2ia_1}
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & 64x^6 - 2304b_1x^5 - 53760tx^5 + 34560b_1^2x^4 + 1612800b_1tx^4 + 48x^4 + \\
& 18816000t^2x^4 + 768a_1^2x^4 - 430080ta_1^2x^3 + 384b_1x^3 - 19353600b_1^2tx^3 - 18432b_1a_1^2x^3 - \\
& 3512320000t^3x^3 + 80640tx^3 - 276480b_1^3x^3 - 451584000b_1t^2x^3 - 2096640b_1tx^2 + \\
& 368793600000t^4x^2 - 1152a_1^2x^2 + 108x^2 + 165888b_1^2a_1^2x^2 + 7741440b_1ta_1^2x^2 + \\
& 4064256000b_1^2t^2x^2 + 116121600b_1^3tx^2 + 63221760000b_1t^3x^2 + 90316800t^2a_1^2x^2 - \\
& 39513600t^2x^2 + 1244160b_1^4x^2 - 17280b_1^2x^2 + 3072a_1^4x^2 - 1083801600b_1t^2a_1^2x + \\
& 16450560b_1^2tx - 967680ta_1^2x - 110880tx - 379330560000b_1^2t^3x + 5795328000t^3x + \\
& 124416b_1^3x - 348364800b_1^4tx - 16257024000b_1^3t^2x - 4608b_1a_1^2x - 4425523200000b_1t^4x - \\
& 2448b_1x - 46448640b_1^2ta_1^2x - 663552b_1^3a_1^2x - 2985984b_1^5x - 36864b_1a_1^4x - \\
& 860160ta_1^4x + 564480000b_1t^2x - 20652441600000t^5x - 8429568000t^3a_1^2x + \\
& 5160960b_1ta_1^4 - 40642560b_1^3t + 69120b_1^2a_1^2 - 1964390400b_1^2t^2 - 38986752000b_1t^3 + \\
& 58564800t^2 + 158054400t^2a_1^2 + 8386560b_1ta_1^2 - 276595200000t^4 + 13276569600000b_1^2t^4 + \\
& 418037760b_1^5t + 9 + 758661120000b_1^3t^3 + 995328b_1^4a_1^2 + 110592b_1^2a_1^4 + 24385536000b_1^4t^2 + \\
& 295034880000t^4a_1^2 + 60211200t^2a_1^4 + 2116800b_1t + 123914649600000b_1t^5 + \\
& 481890304000000t^6 + 92897280b_1^3ta_1^2 + 3251404800b_1^2t^2a_1^2 + 50577408000b_1t^3a_1^2 + \\
& 6912a_1^4 - 269568b_1^4 + 2985984b_1^6 + 4096a_1^6 + 1584a_1^2 + 20016b_1^2
\end{aligned}$$

is a solution to the (NLS6) equation (1).

Proof: It is sufficient to replace the expression of the solution given by (3), and check that the relation (1) is verified.

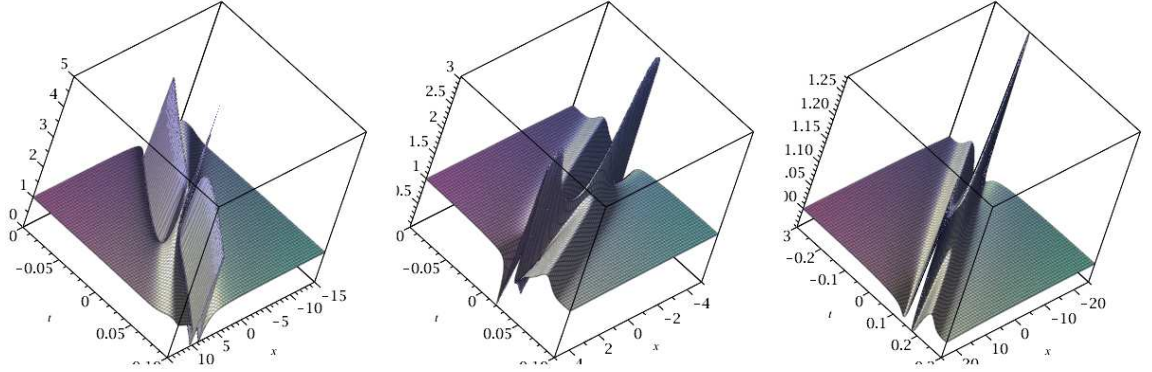


Figure 2. Solution of order 2 to the equation (1); to the left $a_1 = 0, b_1 = 0$; in the center $a_1 = 1, b_1 = 0$; to the right $a_1 = 3, b_1 = 0$.

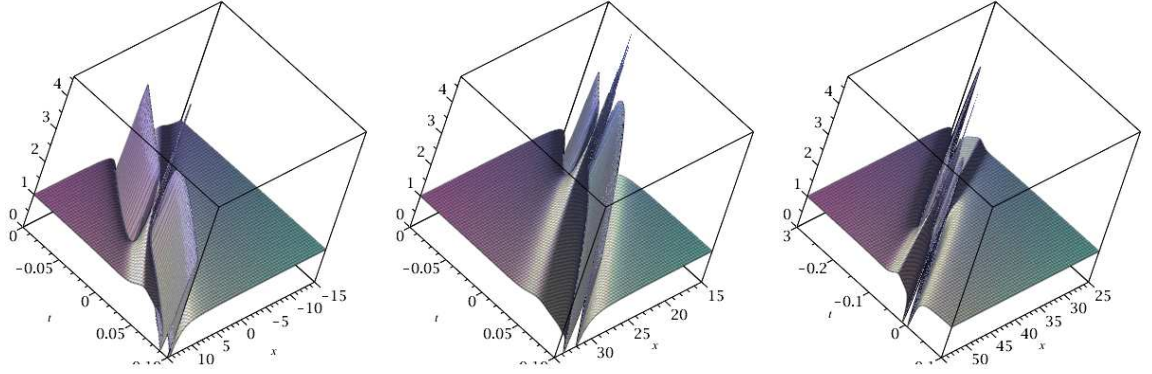


Figure 3. Solution of order 2 to the equation (1); to the left $a_1 = 0, b_1 = 1$; in the center $a_1 = 0, b_1 = 4$; to the right $a_1 = 0, b_1 = 10$.

4 Rational solutions of order 3 of the NLS6 equation

The solution depending on 4 real parameters being too long, we only present in the appendix. Here we give the solution without parameters.

Theorem 4.1 *The function $v(x, t)$ defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)} \quad (4)$$

with

$$n(x, t) = 4096 x^{12} - 6881280 t x^{11} - 18432 x^{10} + 5298585600 t^2 x^{10} - 2472673280000 t^3 x^9 + 60211200 t x^9 + 778892083200000 t^4 x^8 - 59609088000 t^2 x^8 - 57600 x^8 - 174471826636800000 t^5 x^7 -$$

$$\begin{aligned}
& 18063360 tx^7 + 30346444800000 t^3 x^7 + 28497065017344000000 t^6 x^6 + 92665036800 t^2 x^6 - \\
& 172800 x^6 - 9417513369600000 t^4 x^6 - 3419647802081280000000 t^7 x^5 - 218695680 tx^5 + \\
& 1915224824217600000 t^5 x^5 - 61552705536000 t^3 x^5 + 226800 x^4 - 630637056000 t^2 x^4 + \\
& 29921918268211200000000 t^8 x^4 - 262302530273280000000 t^6 x^4 + 19127111270400000 t^4 x^4 - \\
& 1094688000 tx^3 - 18618082477998080000000000 t^9 x^3 + 24092973151027200000000 t^7 x^3 + \\
& 284118589440000 t^3 x^3 - 3315956023296000000 t^5 x^3 - 38170137600000000 t^4 x^2 + \\
& 781959464075919360000000000 t^{10} x^2 + 1052489894400 t^2 x^2 - 142809155371008000000000 t^8 x^2 + \\
& 113400 x^2 + 330931419807744000000 t^6 x^2 - 703080000 tx + 49507173861949440000000000 t^9 x - \\
& 214611005952000 t^3 x + 1379170050048000000 t^5 x - 17856001875640320000000 t^7 x - \\
& 19904422721932492800000000000 t^{11} x - 14175 + 30758094323712000000 t^6 \\
& + 23221826508921241600000000000 t^{12} + 404965962016358400000000 t^8 \\
& - 764187658074193920000000000 t^{10} - 632512540800 t^2 + 63315462551040000 t^4
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & 4096 x^{12} - 6881280 tx^{11} + 6144 x^{10} + 5298585600 t^2 x^{10} - 2472673280000 t^3 x^9 + \\
& 25804800 tx^9 + 778892083200000 t^4 x^8 - 37933056000 t^2 x^8 + 34560 x^8 - 174471826636800000 t^5 x^7 - \\
& 121282560 tx^7 + 22254059520000 t^3 x^7 + 28497065017344000000 t^6 x^6 + 143242444800 t^2 x^6 + \\
& 149760 x^6 - 7434878976000000 t^4 x^6 - 3419647802081280000000 t^7 x^5 - 56125440 tx^5 + \\
& 1582142246092800000 t^5 x^5 - 75714379776000 t^3 x^5 + 54000 x^4 - 188987904000 t^2 x^4 + \\
& 299219182682112000000000 t^8 x^4 - 223442896158720000000 t^6 x^4 + 21605404262400000 t^4 x^4 + \\
& 395539200 tx^3 - 18618082477998080000000000 t^9 x^3 + 20984202421862400000000 t^7 x^3 - \\
& 12770795520000 t^3 x^3 - 3593524838400000000 t^5 x^3 + 28265816678400000 t^4 x^2 + \\
& 781959464075919360000000000 t^{10} x^2 + 121775270400 t^2 x^2 - 1264881090428928000000000 t^8 x^2 + \\
& 48600 x^2 + 350361236865024000000 t^6 x^2 - 162388800 tx + 4442951500431360000000000 t^9 x + \\
& 324417719808000 t^3 x - 5029695627264000000 t^5 x - 18633194557931520000000 t^7 x - \\
& 19904422721932492800000000000 t^{11} x + 2025 + 259682974580736000000 t^6 \\
& + 23221826508921241600000000000 t^{12} + 418566833956454400000000 t^8 \\
& - 693100434067292160000000000 t^{10} + 432678153600 t^2 + 2070646986240000 t^4
\end{aligned}$$

is a solution to the (NLS6) equation (1).

Proof: We have to check that the relation (1) is verified when we replace the expression of the solution given by (5).

In the following, we give the patterns of the modules of the solutions according to different values of the parameters.

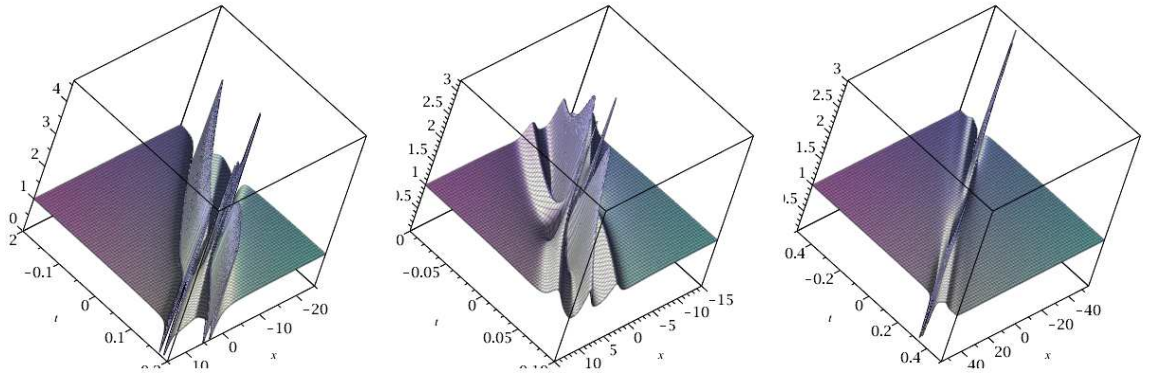


Figure 4. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 0$; in the center $a_1 = 1, b_1 = 0, a_2 = 0, b_2 = 0$; to the right $a_1 = 5, b_1 = 0, a_2 = 0, b_2 = 0$.

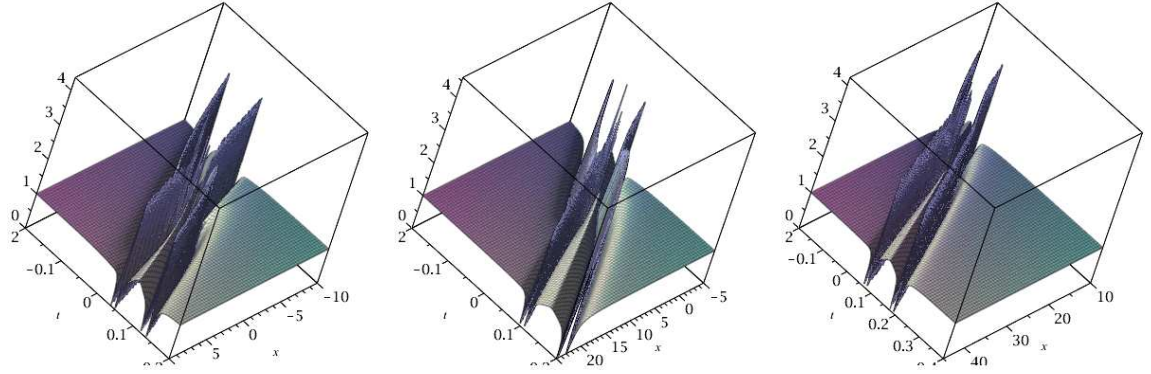


Figure 5. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, 1, a_2 = 0, b_2 = 0$; in the center $a_1 = 0, b_1 = 1, a_2 = 0, b_2 = 0$; to the right $a_1 = 0, b_1 = 5, a_2 = 0, b_2 = 0$.

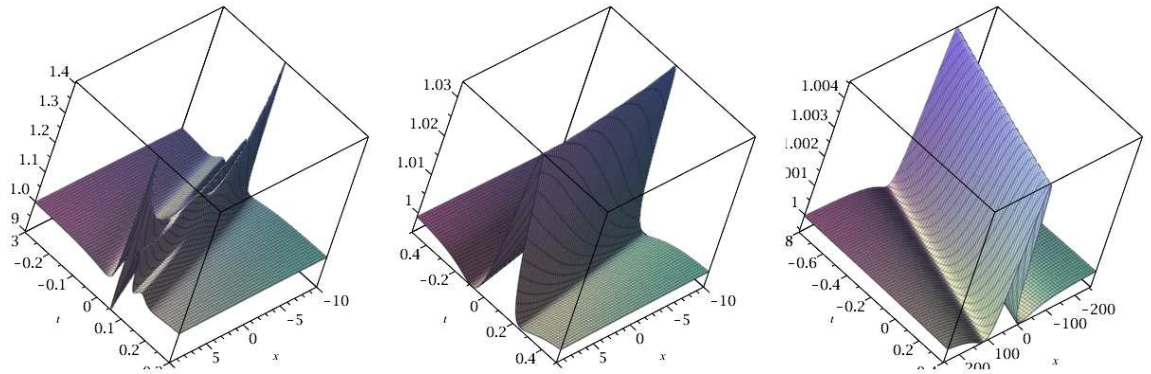


Figure 6. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, a_2 = 0, 5, b_2 = 0$; in the center $a_1 = 0, b_1 = 0, a_2 = 1, b_2 = 0$; to the right $a_1 = 0, b_1 = 5, a_2 = 3, b_2 = 0$.

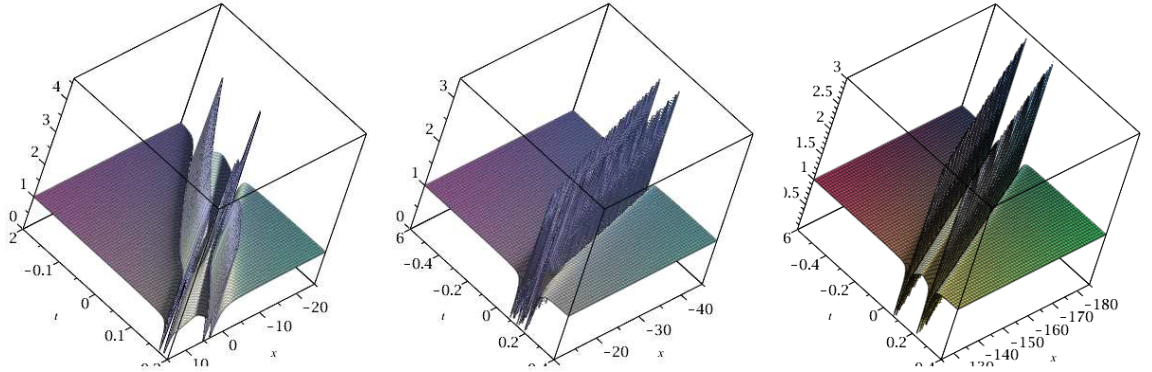


Figure 7. Solution of order 3 to (1); to the left $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 0, 5$; in the center $a_1 = 0, b_1 = 0, a_2 = 0, b_2 = 1$; to the right $a_1 = 0, b_1 = 5, a_2 = 0, b_2 = 5$.

5 Conclusion

Rational solutions to the (NLS6) equation have been constructed for the first orders. In all these N-order solutions we get quotient of a polynomial of degree $N(N + 1)$ in x and t for the numerator by a polynomial of degree $N(N + 1)$ in x and t for the denominator.

Unlike other equations belonging to this NLS hierarchy, for example, the NLS equation [37], the mKdV equation [59], or the Lakshmanan Porsezian Daniel equation [62], we do not recover the structure of triangles with peaks which appear in function of the different values of the parameters.

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Appendix

Solution of order 3 to the (NLS6) equation depending on 4 real parameters :
The function $v(x, t)$ defined by

$$v(x, t) = \left(1 - 24 \frac{n(x, t)}{d(x, t)}\right) e^{i(2a_1 - 6a_2 + 20t)} \quad (5)$$

with

$$\begin{aligned} n(x, t) = & 675 - 86400(8b_1 - 80b_2 + 560t)(32b_2 - 448t) + 15(1 + (4a_1 - 24a_2)^2)(2x - \\ & 12b_1 + 60b_2 - 280t)^8 + (210 - 60(4a_1 - 24a_2)^2 + 50(4a_1 - 24a_2)^4 + 7680(4a_1 - \\ & 24a_2)a_2)(2x - 12b_1 + 60b_2 - 280t)^6 + (-720(4a_1 - 24a_2)^2(8b_1 - 80b_2 + 560t) - \\ & 5760b_1 - 11520b_2 + 564480t)(2x - 12b_1 + 60b_2 - 280t)^5 - 172800(4a_1 - 24a_2)^3 a_2 + \\ & (450(4a_1 - 24a_2)^2 - 150(4a_1 - 24a_2)^4 + 70(4a_1 - 24a_2)^6 + 19200(4a_1 - \\ & 24a_2)^3 a_2 - 450 + 5400(8b_1 - 80b_2 + 560t)^2 - 460800a_2^2 + 57600(4a_1 - 24a_2)a_2)(2x - \\ & 12b_1 + 60b_2 - 280t)^4 + (-2400(4a_1 - 24a_2)^4(8b_1 - 80b_2 + 560t) + 460800(4a_1 - \\ & 24a_2)(8b_1 - 80b_2 + 560t)a_2 + 57600b_1 - 806400b_2 + 7257600t - 7200(4a_1 - \\ & 24a_2)^2(16b_1 - 128b_2 + 672t))(2x - 12b_1 + 60b_2 - 280t)^3 + (6750(4a_1 - 24a_2)^4 + \\ & 420(4a_1 - 24a_2)^6 + 45(4a_1 - 24a_2)^8 - 2700(4a_1 - 24a_2)^2(5 + 4(8b_1 - 80b_2 + \\ & 560t)^2 - 3072a_2^2) - 675 - 10800(8b_1 - 80b_2 + 560t)^2 - 2764800a_2^2 + 691200(4a_1 - \\ & 24a_2)a_2 - 230400(4a_1 - 24a_2)^3 a_2)(2x - 12b_1 + 60b_2 - 280t)^2 + (-1680(4a_1 - \\ & 24a_2)^6(8b_1 - 80b_2 + 560t) - 460800(4a_1 - 24a_2)^3(8b_1 - 80b_2 + 560t)a_2 - \\ & 10800(4a_1 - 24a_2)^2(8b_1 - 272b_2 + 3248t) + 86400b_1 - 1209600b_2 + 10886400t + \\ & 43200(8b_1 - 80b_2 + 560t)^3 + 11059200(8b_1 - 80b_2 + 560t)a_2^2 + 3600(4a_1 - \\ & 24a_2)^4(8b_1 + 80b_2 - 1680t) - 86400(4a_1 - 24a_2)(16(8b_1 - 80b_2 + 560t)a_2 + \\ & 16a_2(32b_2 - 448t)))(2x - 12b_1 + 60b_2 - 280t) + 450(4a_1 - 24a_2)^4(-17 + \\ & 28(8b_1 - 80b_2 + 560t)^2 + 3072a_2^2) + 10800(4a_1 - 24a_2)(-16a_2 + 64(8b_1 - 80b_2 + \\ & 560t)^2 a_2 + 16384a_2^3) + 675(4a_1 - 24a_2)^2(-3 + 16(8b_1 - 80b_2 + 560t)^2 + 4096a_2^2 - \\ & 128(8b_1 - 80b_2 + 560t)(32b_2 - 448t)) + i(-345600(8b_1 - 80b_2 + 560t)^2 a_2 + (4a_1 - \\ & 24a_2)(2x - 12b_1 + 60b_2 - 280t)^{10} + (-60a_1 + 840a_2 + 5(4a_1 - 24a_2)^3)(2x - \\ & 12b_1 + 60b_2 - 280t)^8 + (-600a_1 - 240a_2 - 140(4a_1 - 24a_2)^3 + 10(4a_1 - 24a_2)^5 + \\ & 3840(4a_1 - 24a_2)^2 a_2)(2x - 12b_1 + 60b_2 - 280t)^6 + (-240(4a_1 - 24a_2)^3(8b_1 - \\ & 80b_2 + 560t) - 23040(8b_1 - 80b_2 + 560t)a_2 + 720(4a_1 - 24a_2)(8b_1 - 176b_2 + \\ & 1904t))(2x - 12b_1 + 60b_2 - 280t)^5 + 417600(4a_1 - 24a_2)^4 a_2 + (-450(4a_1 - \\ & 24a_2)^3 - 210(4a_1 - 24a_2)^5 + 10(4a_1 - 24a_2)^7 + 4800(4a_1 - 24a_2)^4 a_2 + 450(4a_1 - \\ & 24a_2)(-3 + 12(8b_1 - 80b_2 + 560t)^2 - 1024a_2^2) - 14400a_2 + 28800(4a_1 - \\ & 24a_2)^2 a_2)(2x - 12b_1 + 60b_2 - 280t)^4 + (-480(4a_1 - 24a_2)^5(8b_1 - 80b_2 + 560t) + \\ & 230400(4a_1 - 24a_2)^2(8b_1 - 80b_2 + 560t)a_2 + 7200(4a_1 - 24a_2)(8b_1 - 48b_2 + \\ & 112t) - 2400(4a_1 - 24a_2)^3(16b_1 - 128b_2 + 672t) - 230400(8b_1 - 80b_2 + 560t)a_2 - \\ & 230400a_2(32b_2 - 448t))(2x - 12b_1 + 60b_2 - 280t)^3 + (1710(4a_1 - 24a_2)^5 - \\ & 60(4a_1 - 24a_2)^7 + 5(4a_1 - 24a_2)^9 - 900(4a_1 - 24a_2)^3(7 + 4(8b_1 - 80b_2 + \\ & 560t)^2 - 3072a_2^2) + 675(4a_1 - 24a_2)(7 + 16(8b_1 - 80b_2 + 560t)^2 + 4096a_2^2) - \\ & 345600a_2 - 345600(8b_1 - 80b_2 + 560t)^2 a_2 - 88473600a_2^3 - 115200(4a_1 - \\ & 24a_2)^4 a_2)(2x - 12b_1 + 60b_2 - 280t)^2 + (-240(4a_1 - 24a_2)^7(8b_1 - 80b_2 + \\ & 560t) - 115200(4a_1 - 24a_2)^4(8b_1 - 80b_2 + 560t)a_2 + 10800(4a_1 - 24a_2)(24b_1 - \\ & 400b_2 + 3920t + 4(8b_1 - 80b_2 + 560t)^3 + 1024(8b_1 - 80b_2 + 560t)a_2^2) + \\ & 3600(4a_1 - 24a_2)^3(24b_1 - 176b_2 + 784t) + 720(4a_1 - 24a_2)^5(56b_1 - 400b_2 + \\ & 1680t) + 345600(8b_1 - 80b_2 + 560t)a_2 + 691200a_2(32b_2 - 448t) - 43200(4a_1 - \end{aligned}$$

$$\begin{aligned}
& 24 a_2)^2(16(8 b_1 - 80 b_2 + 560 t)a_2 + 16 a_2(32 b_2 - 448 t))(2 x - 12 b_1 + 60 b_2 - \\
& 280 t) + 90(4 a_1 - 24 a_2)^5(-107 + 28(8 b_1 - 80 b_2 + 560 t)^2 + 3072 a_2^2) + 5400(4 a_1 - \\
& 24 a_2)^2(176 a_2 + 64(8 b_1 - 80 b_2 + 560 t)^2 a_2 + 16384 a_2^3) - 225(4 a_1 - 24 a_2)^3(11 + \\
& 80(8 b_1 - 80 b_2 + 560 t)^2 + 20480 a_2^2 + 128(8 b_1 - 80 b_2 + 560 t)(32 b_2 - 448 t)) - \\
& 675(4 a_1 - 24 a_2)(-7 + 56(8 b_1 - 80 b_2 + 560 t)^2 + 22528 a_2^2 - 128(8 b_1 - 80 b_2 + \\
& 560 t)(32 b_2 - 448 t) - 128(32 b_2 - 448 t)^2) - 1440(4 a_1 - 24 a_2)^8 a_2 - 9600(4 a_1 - \\
& 24 a_2)^6 a_2 - 2764800(8 b_1 - 80 b_2 + 560 t)a_2(32 b_2 - 448 t) - 870(4 a_1 - 24 a_2)^7 + \\
& 25(4 a_1 - 24 a_2)^9 + (4 a_1 - 24 a_2)^{11} - 151200 a_2 + 265420800 a_2^3) - 11520(4 a_1 - \\
& 24 a_2)^7 a_2 - 195840(4 a_1 - 24 a_2)^5 a_2 + 86400(32 b_2 - 448 t)^2 + (2 x - 12 b_1 + 60 b_2 - \\
& 280 t)^{10} + 27000(8 b_1 - 80 b_2 + 560 t)^2 + 2190(4 a_1 - 24 a_2)^6 + 495(4 a_1 - 24 a_2)^8 + \\
& 11(4 a_1 - 24 a_2)^{10} + 23500800 a_2^2
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & 2024 - 777600(8 b_1 - 80 b_2 + 560 t)(32 b_2 - 448 t) + 120(8 b_1 - 80 b_2 + \\
& 560 t)(2 x - 12 b_1 + 60 b_2 - 280 t)^9 + 1440(32 b_2 - 448 t)(2 x - 12 b_1 + 60 b_2 - \\
& 280 t)^7 - 115200(4 a_1 - 24 a_2)^7 a_2 - 518400(4 a_1 - 24 a_2)^5 a_2 + 265420800(8 b_1 - \\
& 80 b_2 + 560 t)^2 a_2^2 + (-120(4 a_1 - 24 a_2)^2 + 5760(4 a_1 - 24 a_2)a_2 + 120)(2 x - 12 b_1 + \\
& 60 b_2 - 280 t)^8 + (480(4 a_1 - 24 a_2)^2 - 240(4 a_1 - 24 a_2)^4 + 15360(4 a_1 - 24 a_2)^3 a_2 + \\
& 2320 + 2160(8 b_1 - 80 b_2 + 560 t)^2 + 1290240 a_2^2 - 92160(4 a_1 - 24 a_2)a_2)(2 x - \\
& 12 b_1 + 60 b_2 - 280 t)^6 + (-720(4 a_1 - 24 a_2)^4(8 b_1 - 80 b_2 + 560 t) - 276480(4 a_1 - \\
& 24 a_2)(8 b_1 - 80 b_2 + 560 t)a_2 + 4320(4 a_1 - 24 a_2)^2(8 b_1 - 176 b_2 + 1904 t) - 51840 b_1 + \\
& 103680 b_2 + 2177280 t)(2 x - 12 b_1 + 60 b_2 - 280 t)^5 + (-1440(4 a_1 - 24 a_2)^4 + \\
& 11520(4 a_1 - 24 a_2)^5 a_2 + 240(4 a_1 - 24 a_2)^2(56 + 135(8 b_1 - 80 b_2 + 560 t)^2 - \\
& 11520 a_2^2) + 518400(4 a_1 - 24 a_2)a_2 + 345600(4 a_1 - 24 a_2)^3 a_2 + 3360 + 32400(8 b_1 - \\
& 80 b_2 + 560 t)^2 - 13824000 a_2^2 + 86400(8 b_1 - 80 b_2 + 560 t)(32 b_2 - 448 t))(2 x - \\
& 12 b_1 + 60 b_2 - 280 t)^4 + (-960(4 a_1 - 24 a_2)^6(8 b_1 - 80 b_2 + 560 t) + 921600(4 a_1 - \\
& 24 a_2)^3(8 b_1 - 80 b_2 + 560 t)a_2 - 43200(4 a_1 - 24 a_2)^2(24 b_1 - 272 b_2 + 2128 t) - \\
& 7200(4 a_1 - 24 a_2)^4(48 b_1 - 448 b_2 + 2912 t) + 345600 b_1 - 5529600 b_2 + 53222400 t - \\
& 86400(8 b_1 - 80 b_2 + 560 t)^3 - 22118400(8 b_1 - 80 b_2 + 560 t)a_2^2 + 172800(4 a_1 - \\
& 24 a_2)(16(8 b_1 - 80 b_2 + 560 t)a_2 - 16 a_2(32 b_2 - 448 t))(2 x - 12 b_1 + 60 b_2 - 280 t)^3 + \\
& (13440(4 a_1 - 24 a_2)^6 + 240(4 a_1 - 24 a_2)^8 - 240(4 a_1 - 24 a_2)^4(-326 + 45(8 b_1 - \\
& 80 b_2 + 560 t)^2 - 34560 a_2^2) + 480(4 a_1 - 24 a_2)^2(-76 + 135(8 b_1 - 80 b_2 + 560 t)^2 + \\
& 311040 a_2^2) - 4147200(4 a_1 - 24 a_2)^3 a_2 - 414720(4 a_1 - 24 a_2)^5 a_2 - 64800(4 a_1 - \\
& 24 a_2)(-96 a_2 + 64(8 b_1 - 80 b_2 + 560 t)^2 a_2 + 16384 a_2^3) + 12144 - 97200(8 b_1 - \\
& 80 b_2 + 560 t)^2 + 8294400 a_2^2 + 518400(32 b_2 - 448 t)^2)(2 x - 12 b_1 + 60 b_2 - 280 t)^2 + \\
& (-360(4 a_1 - 24 a_2)^8(8 b_1 - 80 b_2 + 560 t) - 276480(4 a_1 - 24 a_2)^5(8 b_1 - 80 b_2 + \\
& 560 t)a_2 - 1440(4 a_1 - 24 a_2)^6(8 b_1 - 240 b_2 + 2800 t) + 32400(4 a_1 - 24 a_2)^4(8 b_1 + \\
& 112 b_2 - 2128 t) + 64800(4 a_1 - 24 a_2)^2(-40 b_1 + 752 b_2 - 7728 t + 4(8 b_1 - 80 b_2 + \\
& 560 t)^3 + 1024(8 b_1 - 80 b_2 + 560 t)a_2^2) - 777600(4 a_1 - 24 a_2)(16(8 b_1 - 80 b_2 + \\
& 560 t)a_2 + 32 a_2(32 b_2 - 448 t)) - 172800(4 a_1 - 24 a_2)^3(48(8 b_1 - 80 b_2 + 560 t)a_2 + \\
& 16 a_2(32 b_2 - 448 t)) - 648000 b_1 + 8553600 b_2 - 74390400 t + 259200(8 b_1 - 80 b_2 + \\
& 560 t)^3 + 331776000(8 b_1 - 80 b_2 + 560 t)a_2^2 - 1036800(8 b_1 - 80 b_2 + 560 t)^2(32 b_2 - \\
& 448 t) + 265420800 a_2^2(32 b_2 - 448 t))(2 x - 12 b_1 + 60 b_2 - 280 t) + 80(4 a_1 - \\
& 24 a_2)^6(191 + 63(8 b_1 - 80 b_2 + 560 t)^2 + 6912 a_2^2) + 21600(4 a_1 - 24 a_2)^3(-368 a_2 + \\
& 64(8 b_1 - 80 b_2 + 560 t)^2 a_2 + 16384 a_2^3) + 240(4 a_1 - 24 a_2)^4(599 + 135(8 b_1 - \\
& 80 b_2 + 560 t)^2 - 57600 a_2^2 - 360(8 b_1 - 80 b_2 + 560 t)(32 b_2 - 448 t)) - 16200(4 a_1 - \\
& 24 a_2)(496 a_2 + 1280(8 b_1 - 80 b_2 + 560 t)^2 a_2 + 65536 a_2^3 + 2048(8 b_1 - 80 b_2 +
\end{aligned}$$

$560 t) a_2 (32 b_2 - 448 t) + 24 (4 a_1 - 24 a_2)^2 (3881 + 12150 (8 b_1 - 80 b_2 + 560 t)^2 + 7257600 a_2^2 + 21600 (8 b_1 - 80 b_2 + 560 t) (32 b_2 - 448 t) + 21600 (32 b_2 - 448 t)^2) - 1920 (4 a_1 - 24 a_2)^9 a_2 + 518400 (32 b_2 - 448 t)^2 + 356400 (8 b_1 - 80 b_2 + 560 t)^2 + 518400 (8 b_1 - 80 b_2 + 560 t)^4 + 3720 (4 a_1 - 24 a_2)^8 + 120 (4 a_1 - 24 a_2)^{10} + (1 + (2 x - 12 b_1 + 60 b_2 - 280 t)^2 + (4 a_1 - 24 a_2)^2)^6 + 33973862400 a_2^4 + 223948800 a_2^2$
is a solution to the *(NLS6)* equation (1).