Rogue waves in the NLS hierarchy of order 6

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Abstract

Rational solutions to the sixth equation of the NLS hierarchy are given. We construct explicit expressions of these solutions for the first orders depending on multi-parameters. Patterns of these solutions in the (x, t) plane according the different values of the parameters are studied.

Key Words : NLS hierarchy, rational solutions.

PACS numbers :
33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

We consider the sixth equation of the NLS hierarchy of order 6 ($NLS6$) which can be written as

$$\begin{aligned}
&u_t + u_{7x} + 14|u^2|u_{5x} + 42\overline{u}u_x u_{4x} + 28u\overline{u}_x u_{4x} + 14uu_x\overline{u}_{4x} + 70|u|^4 u_{3x} \\
&+98|u_x|^2 u_{3x} + 70\overline{u}u_{2x}u_{3x} + 42u\overline{u}_{2x}u_{3x} + 28u_x^2\overline{u}_{3x} + 28uu_{2x}\overline{u}_{3x} + 280\overline{u}|u|^2 u_x u_{2x} \\
&+140|u|^2 u\overline{u}_x u_{2x} + 140|u|^2 uu_x\overline{u}_{2x} + 70\overline{u}u_{2x}^2 + 112u_x|u_{2x}|^2 + 70\overline{u}^2 u_x^3 + 280|u|^2|u_x|^2 u_x \\
&+70u^2|u_x|^2\overline{u}_x + 140|u|^6 u_x
\end{aligned} \tag{1}$$

with as usual the subscripts meaning partail derivatives and $\overline{u}$ the complex conjugate of u.
The NLS hierarchy includes different classical equations. The first one is the NLS equation [1]; the second one is the mKdV equation [2]; the third one is the LPD equation [3].

Many works has been done for these first three equations of the NLS hierarchy; for example, we can mention the following works, for the NLS equation [4], the mKdV equation [5], the LPD equation [3, 6, 7]. However, very few studies

have been carried out for the following orders of this hierarchy. Here we explicitly construct solutions of the order equation six of this hierarchy.
We construct explicitly rational solutions for the first orders and study the patterns of the modulus of the solutions in the (x,t) plane.

2 Rational solutions of order 1 to the NLS6 equation

Theorem 2.1 *The function $v(x,t)$ defined by*

$$v(x,t) = \frac{-3 + 4\,x^2 - 1120\,xt + 78400\,t^2}{1 + 4\,x^2 - 1120\,xt + 78400\,t^2} \tag{2}$$

is a solution to the $(NLS6)$ *equation* (1).

Proof: We have to replace the expression of the solution given by (2) and verify that (1) is satisfied.

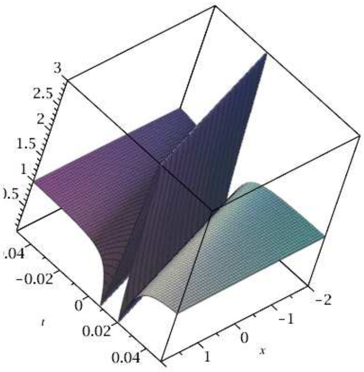


Figure 1. Solution of order 1 to $(NLS6)$.

We get a smooth solution of the equation (1).

3 Rational solutions of order 2 of the NLS6 equation depending on 2 real parameters

Theorem 3.1 *The function $v(x,t)$ defined by*

$$v(x,t) = \frac{n(x,t)}{d(x,t)} \tag{3}$$

with

$n(x,t) = -(-64\,x^6 + 53760\,tx^5 + 2304\,b_1x^5 + 144\,x^4 - 34560\,{b_1}^2x^4 + 768\,ia_1x^4 - 18816000\,t^2x^4 - 1612800\,b_1tx^4 - 768\,{a_1}^2x^4 + 430080\,t{a_1}^2x^3 + 19353600\,{b_1}^2tx^3 + 18432\,b_1{a_1}^2x^3 + 276480\,{b_1}^3x^3 - 188160\,tx^3 - 430080\,ia_1tx^3 - 18432\,ia_1x^3b_1 - 4992\,b_1x^3 + 3512320000\,t^3x^3 + 451584000\,b_1t^2x^3 - 368793600000\,t^4x^2 - 165888\,{b_1}^2{a_1}^2x^2 + 6144\,i{a_1}^3x^2 + 90316800\,ia_1t^2x^2 + 4032000\,b_1tx^2 - 7741440\,b_1t{a_1}^2x^2 + 5760\,{a_1}^2x^2 + 58752\,{b_1}^2x^2 - 4064256000\,{b_1}^2t^2x^2 + 7741440\,ia_1b_1tx^2 - 90316800\,t^2{a_1}^2x^2 + 62092800\,t^2x^2 - 1244160\,{b_1}^4x^2 + 180\,x^2 - 63221760000\,b_1t^3x^2 - 3072\,{a_1}^4x^2 - 116121600\,{b_1}^3tx^2 - 1152\,ia_1x^2 + 165888\,ia_1{b_1}^2x^2 - 322560\,t{a_1}^2x + 4425523200000\,b_1t^4x + 1083801600\,b_1t^2{a_1}^2x + 379330560000\,{b_1}^2t^3x - 835430400\,b_1t^2x + 348364800\,{b_1}^4tx + 16257024000\,{b_1}^3t^2x + 46448640\,{b_1}^2t{a_1}^2x - 4608\,ia_1xb_1 - 1083801600\,ia_1xb_1t^2 - 7902720000\,t^3x + 2985984\,{b_1}^5x - 28062720\,{b_1}^2tx + 663552\,{b_1}^3{a_1}^2x - 50688\,b_1{a_1}^2x - 1720320\,i{a_1}^3tx + 8429568000\,t^3{a_1}^2x - 46448640\,ia_1x{b_1}^2t - 73728\,i{a_1}^3xb_1 + 36864\,b_1{a_1}^4x - 663552\,ia_1x{b_1}^3 - 5616\,b_1x - 967680\,ia_1tx - 290304\,{b_1}^3x + 20652441600000\,t^5x + 860160\,t{a_1}^4x - 8429568000\,ia_1t^3x - 292320\,tx - 720\,ia_1 + 1536\,i{a_1}^3 - 50577408000\,b_1t^3{a_1}^2 - 5160960\,b_1t{a_1}^4 + 995328\,ia_1{b_1}^4 + 10321920\,i{a_1}^3b_1t - 645120\,b_1t{a_1}^2 - 45 + 8386560\,ia_1b_1t + 350353920000\,t^4 + 63866880\,{b_1}^3t + 12288\,i{a_1}^5 - 92897280\,{b_1}^3t{a_1}^2 + 295034880000\,ia_1t^4 + 96768\,{b_1}^2{a_1}^2 - 67737600\,t^2{a_1}^2 + 51631104000\,b_1t^3 + 120422400\,i{a_1}^3t^2 - 7761600\,t^2 + 947520\,b_1t + 221184\,i{a_1}^3{b_1}^2 - 481890304000000\,t^6 - 3251404800\,{b_1}^2t^2{a_1}^2 - 13276569600000\,{b_1}^2t^4 + 2777241600\,{b_1}^2t^2 - 24385536000\,{b_1}^4t^2 - 758661120000\,{b_1}^3t^3 - 995328\,{b_1}^4{a_1}^2 - 110592\,{b_1}^2{a_1}^4 - 123914649600000\,b_1t^5 - 295034880000\,t^4{a_1}^2 - 60211200\,t^2{a_1}^4 + 69120\,ia_1{b_1}^2 + 158054400\,ia_1t^2 + 92897280\,ia_1{b_1}^3t + 3251404800\,ia_1{b_1}^2t^2 + 50577408000\,ia_1b_1t^3 - 418037760\,{b_1}^5t + 8448\,{a_1}^4 + 518400\,{b_1}^4 - 2985984\,{b_1}^6 - 4096\,{a_1}^6 + 1872\,{a_1}^2 + 18000\,{b_1}^2)\mathrm{e}^{2\,ia_1}$

and

$d(x,t) = 64\,x^6 - 2304\,b_1x^5 - 53760\,tx^5 + 34560\,{b_1}^2x^4 + 1612800\,b_1tx^4 + 48\,x^4 + 18816000\,t^2x^4 + 768\,{a_1}^2x^4 - 430080\,t{a_1}^2x^3 + 384\,b_1x^3 - 19353600\,{b_1}^2tx^3 - 18432\,b_1{a_1}^2x^3 - 3512320000\,t^3x^3 + 80640\,tx^3 - 276480\,{b_1}^3x^3 - 451584000\,b_1t^2x^3 - 2096640\,b_1tx^2 + 368793600000\,t^4x^2 - 1152\,{a_1}^2x^2 + 108\,x^2 + 165888\,{b_1}^2{a_1}^2x^2 + 7741440\,b_1t{a_1}^2x^2 + 4064256000\,{b_1}^2t^2x^2 + 116121600\,{b_1}^3tx^2 + 63221760000\,b_1t^3x^2 + 90316800\,t^2{a_1}^2x^2 - 39513600\,t^2x^2 + 1244160\,{b_1}^4x^2 - 17280\,{b_1}^2x^2 + 3072\,{a_1}^4x^2 - 1083801600\,b_1t^2{a_1}^2x + 16450560\,{b_1}^2tx - 967680\,t{a_1}^2x - 110880\,tx - 379330560000\,{b_1}^2t^3x + 5795328000\,t^3x + 124416\,{b_1}^3x - 348364800\,{b_1}^4tx - 16257024000\,{b_1}^3t^2x - 4608\,b_1{a_1}^2x - 4425523200000\,b_1t^4x - 2448\,b_1x - 46448640\,{b_1}^2t{a_1}^2x - 663552\,{b_1}^3{a_1}^2x - 2985984\,{b_1}^5x - 36864\,b_1{a_1}^4x - 860160\,t{a_1}^4x + 564480000\,b_1t^2x - 20652441600000\,t^5x - 8429568000\,t^3{a_1}^2x + 5160960\,b_1t{a_1}^4 - 40642560\,{b_1}^3t + 69120\,{b_1}^2{a_1}^2 - 1964390400\,{b_1}^2t^2 - 38986752000\,b_1t^3 + 58564800\,t^2 + 158054400\,t^2{a_1}^2 + 8386560\,b_1t{a_1}^2 - 276595200000\,t^4 + 13276569600000\,{b_1}^2t^4 + 418037760\,{b_1}^5t + 9 + 758661120000\,{b_1}^3t^3 + 995328\,{b_1}^4{a_1}^2 + 110592\,{b_1}^2{a_1}^4 + 24385536000\,{b_1}^4t^2 + 295034880000\,t^4{a_1}^2 + 60211200\,t^2{a_1}^4 + 2116800\,b_1t + 123914649600000\,b_1t^5 + 481890304000000\,t^6 + 92897280\,{b_1}^3t{a_1}^2 + 3251404800\,{b_1}^2t^2{a_1}^2 + 50577408000\,b_1t^3{a_1}^2 + 6912\,{a_1}^4 - 269568\,{b_1}^4 + 2985984\,{b_1}^6 + 4096\,{a_1}^6 + 1584\,{a_1}^2 + 20016\,{b_1}^2$

is a solution to the (NLS6) equation (1).

Proof: It is sufficient to replace the expression of the solution given by (3), and check that the relation (1) is verified.

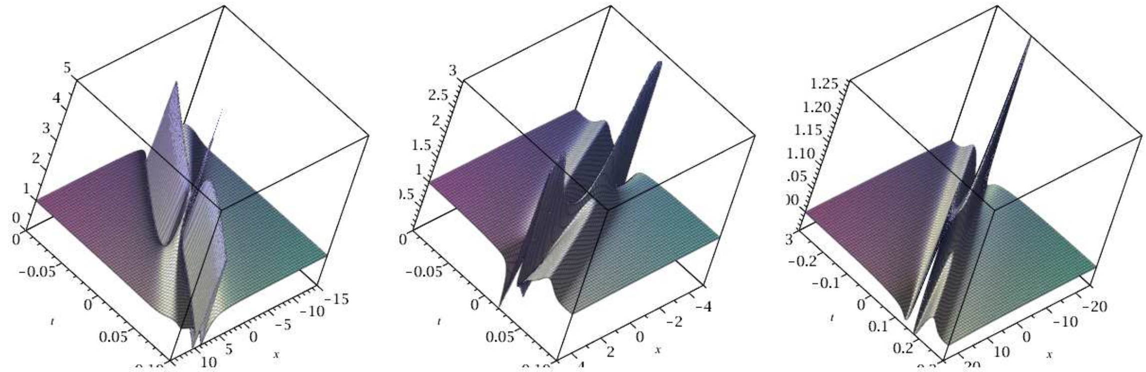


Figure 2. Solution of order 2 to the equation (1); to the left $a_1 = 0$, $b_1 = 0$; in the center $a_1 = 1$, $b_1 = 0$; to the right $a_1 = 3$, $b_1 = 0$.

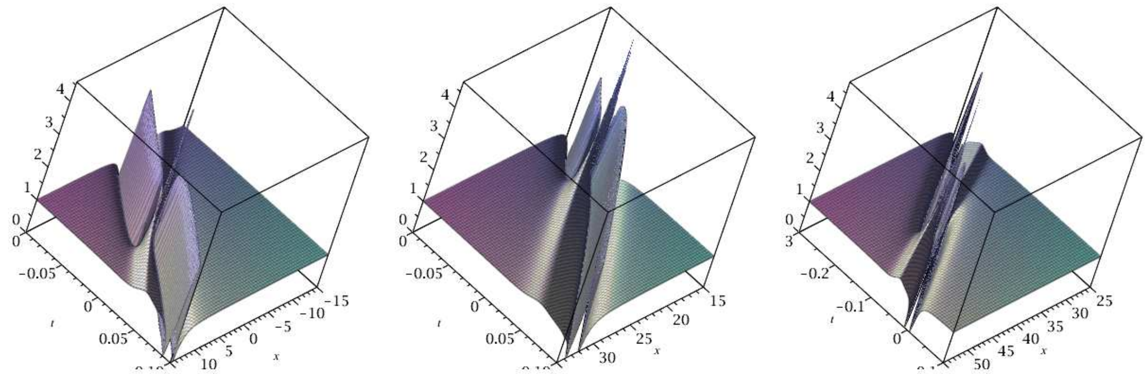


Figure 3. Solution of order 2 to the equation (1); to the left $a_1 = 0$, $b_1 = 1$; in the center $a_1 = 0$, $b_1 = 4$; to the right $a_1 = 0$, $b_1 = 10$.

4 Rational solutions of order 3 of the NLS6 equation

The solution depending on 4 real parameters being too long, we only present in the appendix. Here we give the solution without parameters.

Theorem 4.1 *The function $v(x,t)$ defined by*

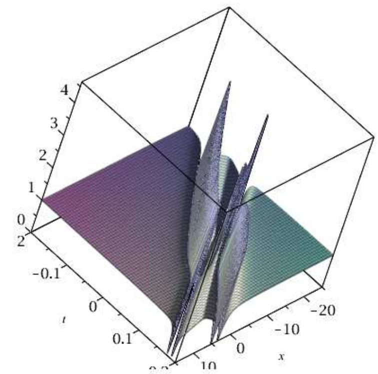
$$v(x,t) = \frac{n(x,t)}{d(x,t)} \tag{4}$$

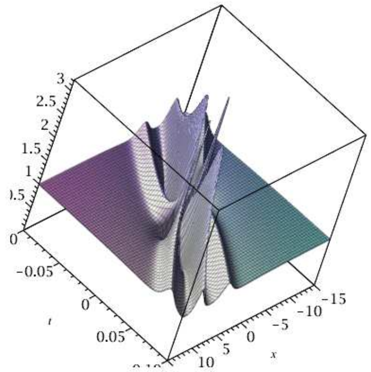
with
$n(x,t) = 4096\,x^{12} - 6881280\,tx^{11} - 18432\,x^{10} + 5298585600\,t^2x^{10} - 2472673280000\,t^3x^9 + 60211200\,tx^9 + 778892083200000\,t^4x^8 - 59609088000\,t^2x^8 - 57600\,x^8 - 174471826636800000\,t^5x^7 -$

$18063360\,tx^7+30346444800000\,t^3x^7+28497065017344000000\,t^6x^6+92665036800\,t^2x^6-172800\,x^6-9417513369600000\,t^4x^6-34196478020812800000000\,t^7x^5-218695680\,tx^5+1915224824217600000\,t^5x^5-61552705536000\,t^3x^5+226800\,x^4-630637056000\,t^2x^4+29921918268211200000000\,t^8x^4-262302530273280000000\,t^6x^4+19127111270400000\,t^4x^4-1094688000\,tx^3-186180824779980800000000000\,t^9x^3+240929731510272000000000\,t^7x^3+284118589440000\,t^3x^3-3315956023296000000\,t^5x^3-381701376000000000\,t^4x^2+781959464075919360000000000\,t^{10}x^2+1052489894400\,t^2x^2-142809155371008000000000\,t^8x^2+113400\,x^2+330931419807744000000\,t^6x^2-703080000\,tx+495071738619494400000000000\,t^9x-214611005952000\,t^3x+137917005004800000\,t^5x-178560018756403200000000\,t^7x-1990442272193249280000000000\,t^{11}x-14175+30758094323712000000\,t^6$
$+2322182650892124160000000000000\,t^{12}+4049659620163584000000000\,t^8$
$-764187658074193920000000000\,t^{10}-632512540800\,t^2+63315462551040000\,t^4$
and
$d(x,t)=4096\,x^{12}-6881280\,tx^{11}+6144\,x^{10}+5298585600\,t^2x^{10}-2472673280000\,t^3x^9+25804800\,tx^9+778892083200000\,t^4x^8-37933056000\,t^2x^8+34560\,x^8-174471826636800000\,t^5x^7-121282560\,tx^7+22254059520000\,t^3x^7+28497065017344000000\,t^6x^6+143242444800\,t^2x^6+149760\,x^6-7434878976000000\,t^4x^6-34196478020812800000000\,t^7x^5-56125440\,tx^5+1582142246092800000\,t^5x^5-75714379776000\,t^3x^5+54000\,x^4-188987904000\,t^2x^4+29921918268211200000000\,t^8x^4-223442896158720000000\,t^6x^4+21605404262400000\,t^4x^4+395539200\,tx^3-186180824779980800000000000\,t^9x^3+209842024218624000000000\,t^7x^3-12770795520000\,t^3x^3-3593524838400000000\,t^5x^3+282658166784000000\,t^4x^2+781959464075919360000000000\,t^{10}x^2+121775270400\,t^2x^2-126488109042892800000000\,t^8x^2+48600\,x^2+350361236865024000000\,t^6x^2-162388800\,tx+444295150043136000000000000\,t^9x+324417719808000\,t^3x-5029695627264000000\,t^5x-186331945579315200000000\,t^7x-1990442272193249280000000000\,t^{11}x+2025+25968297458073600000\,t^6$
$+2322182650892124160000000000000\,t^{12}+4185668339564544000000000\,t^8$
$-693100434067292160000000000\,t^{10}+432678153600\,t^2+2070646986240000\,t^4$
is a solution to the $(NLS6)$ *equation* (1).

Proof: We have to check that the relation (1) is verified when we replace the expression of the solution given by (5).

In the following, we give the patterns of the modules of the solutions according to different values of the parameters.





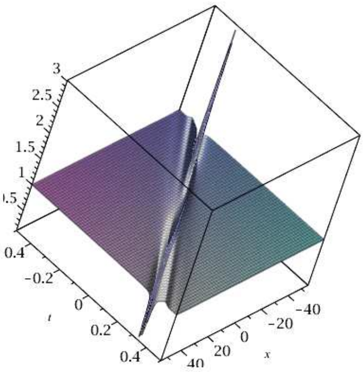


Figure 4. Solution of order 3 to (1); to the left $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$; in the center $a_1 = 1$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$; to the right $a_1 = 5$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$.

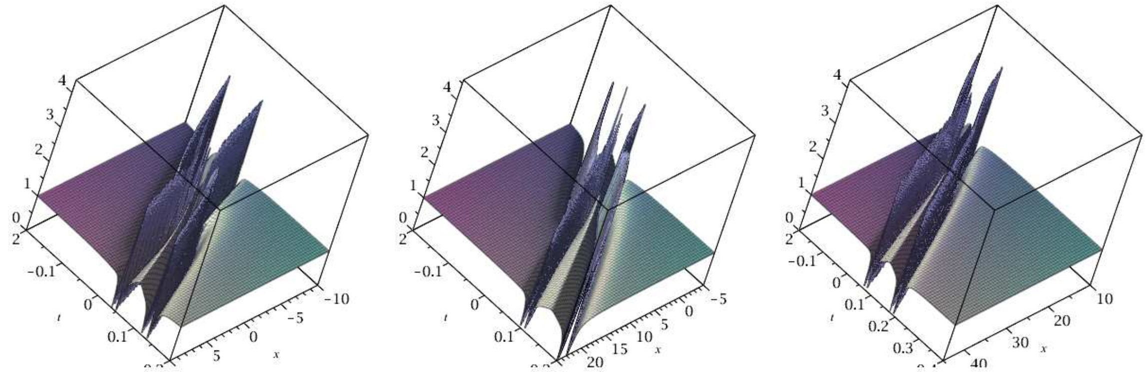


Figure 5. Solution of order 3 to (1); to the left $a_1 = 0$, $b_1 = 0,1$, $a_2 = 0$, $b_2 = 0$; in the center $a_1 = 0$, $b_1 = 1$, $a_2 = 0$, $b_2 = 0$; to the right $a_1 = 0$, $b_1 = 5$, $a_2 = 0$, $b_2 = 0$.

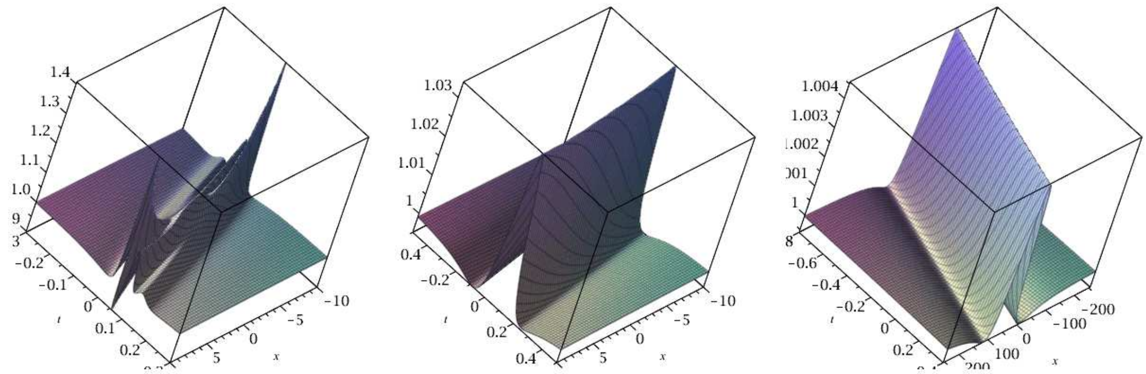


Figure 6. Solution of order 3 to (1); to the left $a_1 = 0$, $b_1 = 0$, $a_2 = 0,5$, $b_2 = 0$; in the center $a_1 = 0$, $b_1 = 0$, $a_2 = 1$, $b_2 = 0$; to the right $a_1 = 0$, $b_1 = 5$, $a_2 = 3$, $b_2 = 0$.

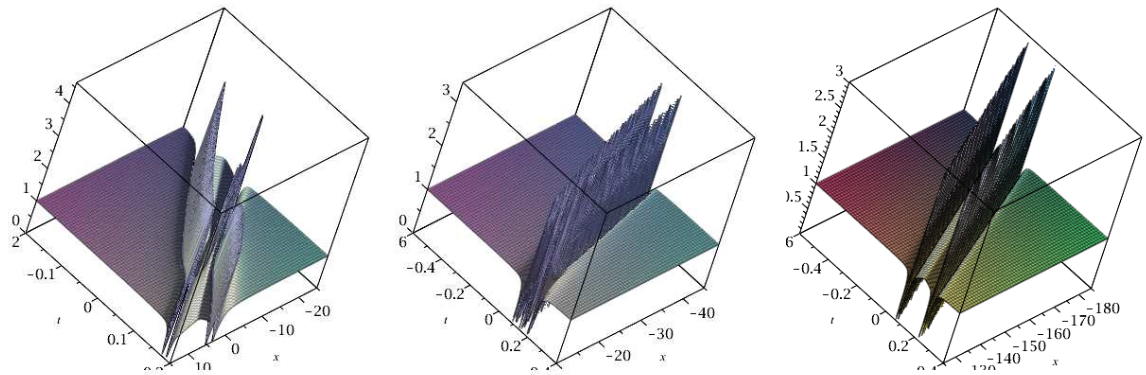


Figure 7. Solution of order 3 to (1); to the left $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0,5$; in the center $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, $b_2 = 1$; to the right $a_1 = 0$, $b_1 = 5$, $a_2 = 0$, $b_2 = 5$.

5 Conclusion

Rational solutions to the $(NLS6)$ equation have been constructed for the first orders. In all these N-order solutions we get quotient of a polynomial of degree $N(N+1)$ in x and t for the numerator by a polynomial of degree $N(N+1)$ in x and t for the denominator.
Unlike other equations belonging to this NLS hierarchy, for example, the NLS equation [37], the mKdV equation [59], or the Lakshmanan Porsezian Daniel equation [62], we do not recover the structure of triangles with peaks which appear in function of the different values of the parameters.

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Appendix

Solution of order 3 to the (NLS6) equation depending on 4 real parameters :
The function $v(x,t)$ defined by

$$v(x,t) = \left(1 - 24\frac{n(x,t)}{d(x,t)}\right) \mathrm{e}^{i(2\,a_1 - 6\,a_2 + 20\,t)} \tag{5}$$

with

$n(x,t) = 675 - 86400\,(8\,b_1 - 80\,b_2 + 560\,t)(32\,b_2 - 448\,t) + 15\,(1 + (4\,a_1 - 24\,a_2)^2)(2\,x -$
$12\,b_1 + 60\,b_2 - 280\,t)^8 + (210 - 60\,(4\,a_1 - 24\,a_2)^2 + 50\,(4\,a_1 - 24\,a_2)^4 + 7680\,(4\,a_1 -$
$24\,a_2)a_2)(2\,x - 12\,b_1 + 60\,b_2 - 280\,t)^6 + (-720\,(4\,a_1 - 24\,a_2)^2(8\,b_1 - 80\,b_2 + 560\,t) -$
$5760\,b_1 - 11520\,b_2 + 564480\,t)(2\,x - 12\,b_1 + 60\,b_2 - 280\,t)^5 - 172800\,(4\,a_1 - 24\,a_2)^3 a_2 +$
$(450\,(4\,a_1 - 24\,a_2)^2 - 150\,(4\,a_1 - 24\,a_2)^4 + 70\,(4\,a_1 - 24\,a_2)^6 + 19200\,(4\,a_1 -$
$24\,a_2)^3 a_2 - 450 + 5400\,(8\,b_1 - 80\,b_2 + 560\,t)^2 - 460800\,{a_2}^2 + 57600\,(4\,a_1 - 24\,a_2)a_2)(2\,x -$
$12\,b_1 + 60\,b_2 - 280\,t)^4 + (-2400\,(4\,a_1 - 24\,a_2)^4(8\,b_1 - 80\,b_2 + 560\,t) + 460800\,(4\,a_1 -$
$24\,a_2)(8\,b_1 - 80\,b_2 + 560\,t)a_2 + 57600\,b_1 - 806400\,b_2 + 7257600\,t - 7200\,(4\,a_1 -$
$24\,a_2)^2(16\,b_1 - 128\,b_2 + 672\,t))(2\,x - 12\,b_1 + 60\,b_2 - 280\,t)^3 + (6750\,(4\,a_1 - 24\,a_2)^4 +$
$420\,(4\,a_1 - 24\,a_2)^6 + 45\,(4\,a_1 - 24\,a_2)^8 - 2700\,(4\,a_1 - 24\,a_2)^2(5 + 4\,(8\,b_1 - 80\,b_2 +$
$560\,t)^2 - 3072\,{a_2}^2) - 675 - 10800\,(8\,b_1 - 80\,b_2 + 560\,t)^2 - 2764800\,{a_2}^2 + 691200\,(4\,a_1 -$
$24\,a_2)a_2 - 230400\,(4\,a_1 - 24\,a_2)^3 a_2)(2\,x - 12\,b_1 + 60\,b_2 - 280\,t)^2 + (-1680\,(4\,a_1 -$
$24\,a_2)^6(8\,b_1 - 80\,b_2 + 560\,t) - 460800\,(4\,a_1 - 24\,a_2)^3(8\,b_1 - 80\,b_2 + 560\,t)a_2 -$
$10800\,(4\,a_1 - 24\,a_2)^2(8\,b_1 - 272\,b_2 + 3248\,t) + 86400\,b_1 - 1209600\,b_2 + 10886400\,t +$
$43200\,(8\,b_1 - 80\,b_2 + 560\,t)^3 + 11059200\,(8\,b_1 - 80\,b_2 + 560\,t){a_2}^2 + 3600\,(4\,a_1 -$
$24\,a_2)^4(8\,b_1 + 80\,b_2 - 1680\,t) - 86400\,(4\,a_1 - 24\,a_2)(16\,(8\,b_1 - 80\,b_2 + 560\,t)a_2 +$
$16\,a_2(32\,b_2 - 448\,t)))(2\,x - 12\,b_1 + 60\,b_2 - 280\,t) + 450\,(4\,a_1 - 24\,a_2)^4(-17 +$
$28\,(8\,b_1 - 80\,b_2 + 560\,t)^2 + 3072\,{a_2}^2) + 10800\,(4\,a_1 - 24\,a_2)(-16\,a_2 + 64\,(8\,b_1 - 80\,b_2 +$
$560\,t)^2 a_2 + 16384\,{a_2}^3) + 675\,(4\,a_1 - 24\,a_2)^2(-3 + 16\,(8\,b_1 - 80\,b_2 + 560\,t)^2 + 4096\,{a_2}^2 -$
$128\,(8\,b_1 - 80\,b_2 + 560\,t)(32\,b_2 - 448\,t)) + i(-345600\,(8\,b_1 - 80\,b_2 + 560\,t)^2 a_2 + (4\,a_1 -$
$24\,a_2)(2\,x - 12\,b_1 + 60\,b_2 - 280\,t)^{10} + (-60\,a_1 + 840\,a_2 + 5\,(4\,a_1 - 24\,a_2)^3)(2\,x -$
$12\,b_1 + 60\,b_2 - 280\,t)^8 + (-600\,a_1 - 240\,a_2 - 140\,(4\,a_1 - 24\,a_2)^3 + 10\,(4\,a_1 - 24\,a_2)^5 +$
$3840\,(4\,a_1 - 24\,a_2)^2 a_2)(2\,x - 12\,b_1 + 60\,b_2 - 280\,t)^6 + (-240\,(4\,a_1 - 24\,a_2)^3(8\,b_1 -$
$80\,b_2 + 560\,t) - 23040\,(8\,b_1 - 80\,b_2 + 560\,t)a_2 + 720\,(4\,a_1 - 24\,a_2)(8\,b_1 - 176\,b_2 +$
$1904\,t))(2\,x - 12\,b_1 + 60\,b_2 - 280\,t)^5 + 417600\,(4\,a_1 - 24\,a_2)^4 a_2 + (-450\,(4\,a_1 -$
$24\,a_2)^3 - 210\,(4\,a_1 - 24\,a_2)^5 + 10\,(4\,a_1 - 24\,a_2)^7 + 4800\,(4\,a_1 - 24\,a_2)^4 a_2 + 450\,(4\,a_1 -$
$24\,a_2)(-3 + 12\,(8\,b_1 - 80\,b_2 + 560\,t)^2 - 1024\,{a_2}^2) - 14400\,a_2 + 28800\,(4\,a_1 -$
$24\,a_2)^2 a_2)(2\,x - 12\,b_1 + 60\,b_2 - 280\,t)^4 + (-480\,(4\,a_1 - 24\,a_2)^5(8\,b_1 - 80\,b_2 + 560\,t) +$
$230400\,(4\,a_1 - 24\,a_2)^2(8\,b_1 - 80\,b_2 + 560\,t)a_2 + 7200\,(4\,a_1 - 24\,a_2)(8\,b_1 - 48\,b_2 +$
$112\,t) - 2400\,(4\,a_1 - 24\,a_2)^3(16\,b_1 - 128\,b_2 + 672\,t) - 230400\,(8\,b_1 - 80\,b_2 + 560\,t)a_2 -$
$230400\,a_2(32\,b_2 - 448\,t))(2\,x - 12\,b_1 + 60\,b_2 - 280\,t)^3 + (1710\,(4\,a_1 - 24\,a_2)^5 -$
$60\,(4\,a_1 - 24\,a_2)^7 + 5\,(4\,a_1 - 24\,a_2)^9 - 900\,(4\,a_1 - 24\,a_2)^3(7 + 4\,(8\,b_1 - 80\,b_2 +$
$560\,t)^2 - 3072\,{a_2}^2) + 675\,(4\,a_1 - 24\,a_2)(7 + 16\,(8\,b_1 - 80\,b_2 + 560\,t)^2 + 4096\,{a_2}^2) -$
$345600\,a_2 - 345600\,(8\,b_1 - 80\,b_2 + 560\,t)^2 a_2 - 88473600\,{a_2}^3 - 115200\,(4\,a_1 -$
$24\,a_2)^4 a_2)(2\,x - 12\,b_1 + 60\,b_2 - 280\,t)^2 + (-240\,(4\,a_1 - 24\,a_2)^7(8\,b_1 - 80\,b_2 +$
$560\,t) - 115200\,(4\,a_1 - 24\,a_2)^4(8\,b_1 - 80\,b_2 + 560\,t)a_2 + 10800\,(4\,a_1 - 24\,a_2)(24\,b_1 -$
$400\,b_2 + 3920\,t + 4\,(8\,b_1 - 80\,b_2 + 560\,t)^3 + 1024\,(8\,b_1 - 80\,b_2 + 560\,t){a_2}^2) +$
$3600\,(4\,a_1 - 24\,a_2)^3(24\,b_1 - 176\,b_2 + 784\,t) + 720\,(4\,a_1 - 24\,a_2)^5(56\,b_1 - 400\,b_2 +$
$1680\,t) + 345600\,(8\,b_1 - 80\,b_2 + 560\,t)a_2 + 691200\,a_2(32\,b_2 - 448\,t) - 43200\,(4\,a_1 -$

$24\,a_2)^2(16\,(8\,b_1-80\,b_2+560\,t)a_2+16\,a_2(32\,b_2-448\,t)))(2\,x-12\,b_1+60\,b_2-280\,t)+90\,(4\,a_1-24\,a_2)^5(-107+28\,(8\,b_1-80\,b_2+560\,t)^2+3072\,{a_2}^2)+5400\,(4\,a_1-24\,a_2)^2(176\,a_2+64\,(8\,b_1-80\,b_2+560\,t)^2a_2+16384\,{a_2}^3)-225\,(4\,a_1-24\,a_2)^3(11+80\,(8\,b_1-80\,b_2+560\,t)^2+20480\,{a_2}^2+128\,(8\,b_1-80\,b_2+560\,t)(32\,b_2-448\,t))-675\,(4\,a_1-24\,a_2)(-7+56\,(8\,b_1-80\,b_2+560\,t)^2+22528\,{a_2}^2-128\,(8\,b_1-80\,b_2+560\,t)(32\,b_2-448\,t)-128\,(32\,b_2-448\,t)^2)-1440\,(4\,a_1-24\,a_2)^8a_2-9600\,(4\,a_1-24\,a_2)^6a_2-2764800\,(8\,b_1-80\,b_2+560\,t)a_2(32\,b_2-448\,t)-870\,(4\,a_1-24\,a_2)^7+25\,(4\,a_1-24\,a_2)^9+(4\,a_1-24\,a_2)^{11}-151200\,a_2+265420800\,{a_2}^3)-11520\,(4\,a_1-24\,a_2)^7a_2-195840\,(4\,a_1-24\,a_2)^5a_2+86400\,(32\,b_2-448\,t)^2+(2\,x-12\,b_1+60\,b_2-280\,t)^{10}+27000\,(8\,b_1-80\,b_2+560\,t)^2+2190\,(4\,a_1-24\,a_2)^6+495\,(4\,a_1-24\,a_2)^8+11\,(4\,a_1-24\,a_2)^{10}+23500800\,{a_2}^2$

and

$d(x,t)=2024-777600\,(8\,b_1-80\,b_2+560\,t)(32\,b_2-448\,t)+120\,(8\,b_1-80\,b_2+560\,t)(2\,x-12\,b_1+60\,b_2-280\,t)^9+1440\,(32\,b_2-448\,t)(2\,x-12\,b_1+60\,b_2-280\,t)^7-115200\,(4\,a_1-24\,a_2)^7a_2-518400\,(4\,a_1-24\,a_2)^5a_2+265420800\,(8\,b_1-80\,b_2+560\,t)^2{a_2}^2+(-120\,(4\,a_1-24\,a_2)^2+5760\,(4\,a_1-24\,a_2)a_2+120)(2\,x-12\,b_1+60\,b_2-280\,t)^8+(480\,(4\,a_1-24\,a_2)^2-240\,(4\,a_1-24\,a_2)^4+15360\,(4\,a_1-24\,a_2)^3a_2+2320+2160\,(8\,b_1-80\,b_2+560\,t)^2+1290240\,{a_2}^2-92160\,(4\,a_1-24\,a_2)a_2)(2\,x-12\,b_1+60\,b_2-280\,t)^6+(-720\,(4\,a_1-24\,a_2)^4(8\,b_1-80\,b_2+560\,t)-276480\,(4\,a_1-24\,a_2)(8\,b_1-80\,b_2+560\,t)a_2+4320\,(4\,a_1-24\,a_2)^2(8\,b_1-176\,b_2+1904\,t)-51840\,b_1+103680\,b_2+2177280\,t)(2\,x-12\,b_1+60\,b_2-280\,t)^5+(-1440\,(4\,a_1-24\,a_2)^4+11520\,(4\,a_1-24\,a_2)^5a_2+240\,(4\,a_1-24\,a_2)^2(56+135\,(8\,b_1-80\,b_2+560\,t)^2-11520\,{a_2}^2)+518400\,(4\,a_1-24\,a_2)a_2+345600\,(4\,a_1-24\,a_2)^3a_2+3360+32400\,(8\,b_1-80\,b_2+560\,t)^2-13824000\,{a_2}^2+86400\,(8\,b_1-80\,b_2+560\,t)(32\,b_2-448\,t))(2\,x-12\,b_1+60\,b_2-280\,t)^4+(-960\,(4\,a_1-24\,a_2)^6(8\,b_1-80\,b_2+560\,t)+921600\,(4\,a_1-24\,a_2)^3(8\,b_1-80\,b_2+560\,t)a_2-43200\,(4\,a_1-24\,a_2)^2(24\,b_1-272\,b_2+2128\,t)-7200\,(4\,a_1-24\,a_2)^4(48\,b_1-448\,b_2+2912\,t)+345600\,b_1-5529600\,b_2+53222400\,t-86400\,(8\,b_1-80\,b_2+560\,t)^3-22118400\,(8\,b_1-80\,b_2+560\,t){a_2}^2+172800\,(4\,a_1-24\,a_2)(16\,(8\,b_1-80\,b_2+560\,t)a_2-16\,a_2(32\,b_2-448\,t)))(2\,x-12\,b_1+60\,b_2-280\,t)^3+(13440\,(4\,a_1-24\,a_2)^6+240\,(4\,a_1-24\,a_2)^8-240\,(4\,a_1-24\,a_2)^4(-326+45\,(8\,b_1-80\,b_2+560\,t)^2-34560\,{a_2}^2)+480\,(4\,a_1-24\,a_2)^2(-76+135\,(8\,b_1-80\,b_2+560\,t)^2+311040\,{a_2}^2)-4147200\,(4\,a_1-24\,a_2)^3a_2-414720\,(4\,a_1-24\,a_2)^5a_2-64800\,(4\,a_1-24\,a_2)(-96\,a_2+64\,(8\,b_1-80\,b_2+560\,t)^2a_2+16384\,{a_2}^3)+12144-97200\,(8\,b_1-80\,b_2+560\,t)^2+8294400\,{a_2}^2+518400\,(32\,b_2-448\,t)^2)(2\,x-12\,b_1+60\,b_2-280\,t)^2+(-360\,(4\,a_1-24\,a_2)^8(8\,b_1-80\,b_2+560\,t)-276480\,(4\,a_1-24\,a_2)^5(8\,b_1-80\,b_2+560\,t)a_2-1440\,(4\,a_1-24\,a_2)^6(8\,b_1-240\,b_2+2800\,t)+32400\,(4\,a_1-24\,a_2)^4(8\,b_1+112\,b_2-2128\,t)+64800\,(4\,a_1-24\,a_2)^2(-40\,b_1+752\,b_2-7728\,t+4\,(8\,b_1-80\,b_2+560\,t)^3+1024\,(8\,b_1-80\,b_2+560\,t){a_2}^2)-777600\,(4\,a_1-24\,a_2)(16\,(8\,b_1-80\,b_2+560\,t)a_2+32\,a_2(32\,b_2-448\,t))-172800\,(4\,a_1-24\,a_2)^3(48\,(8\,b_1-80\,b_2+560\,t)a_2+16\,a_2(32\,b_2-448\,t))-648000\,b_1+8553600\,b_2-74390400\,t+259200\,(8\,b_1-80\,b_2+560\,t)^3+331776000\,(8\,b_1-80\,b_2+560\,t){a_2}^2-1036800\,(8\,b_1-80\,b_2+560\,t)^2(32\,b_2-448\,t)+265420800\,{a_2}^2(32\,b_2-448\,t))(2\,x-12\,b_1+60\,b_2-280\,t)+80\,(4\,a_1-24\,a_2)^6(191+63\,(8\,b_1-80\,b_2+560\,t)^2+6912\,{a_2}^2)+21600\,(4\,a_1-24\,a_2)^3(-368\,a_2+64\,(8\,b_1-80\,b_2+560\,t)^2a_2+16384\,{a_2}^3)+240\,(4\,a_1-24\,a_2)^4(599+135\,(8\,b_1-80\,b_2+560\,t)^2-57600\,{a_2}^2-360\,(8\,b_1-80\,b_2+560\,t)(32\,b_2-448\,t))-16200\,(4\,a_1-24\,a_2)(496\,a_2+1280\,(8\,b_1-80\,b_2+560\,t)^2a_2+65536\,{a_2}^3+2048\,(8\,b_1-80\,b_2+$

$560\,t)a_2(32\,b_2-448\,t))+24\,(4\,a_1-24\,a_2)^2(3881+12150\,(8\,b_1-80\,b_2+560\,t)^2+7257600\,{a_2}^2+21600\,(8\,b_1-80\,b_2+560\,t)(32\,b_2-448\,t)+21600\,(32\,b_2-448\,t)^2)-1920\,(4\,a_1-24\,a_2)^9a_2+518400\,(32\,b_2-448\,t)^2+356400\,(8\,b_1-80\,b_2+560\,t)^2+518400\,(8\,b_1-80\,b_2+560\,t)^4+3720\,(4\,a_1-24\,a_2)^8+120\,(4\,a_1-24\,a_2)^{10}+(1+(2\,x-12\,b_1+60\,b_2-280\,t)^2+(4\,a_1-24\,a_2)^2)^6+33973862400\,{a_2}^4+223948800\,{a_2}^2$
is a solution to the $(NLS6)$ equation (1).