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LOOP TORSORS AND ABHYANKAR'S LEMMA

PHILIPPE GILLE

ABSTRACT. We define the notion of loop torsors under certain group schemes defined over the localization of a regular henselian ring A at a strict normal crossing divisor D . We provide a Galois cohomological criterion for classifying those torsors. We revisit also the theory of loop torsors on Laurent polynomial rings.

Keywords: Reductive group schemes, normal crossing divisor, parabolic subgroups.

MSC 2000: 14L15, 20G15, 20G35.

1. INTRODUCTION

The first result of quadratic form theory is Gauss diagonalization, a regular quadratic form q of dimension n over a field k of characteristic $\neq 2$ is isometric to a diagonal quadratic form $a_1x_1^2 + \cdots + a_nx_n^2$. Let O_n be the orthogonal group attached to the diagonal quadratic form $q_0 = x_1^2 + \cdots + x_n^2$. Serre's viewpoint is that the Galois cohomological set $H^1(k, O_n)$ classifies isometry classes of rank n [52, §III.1.2]. The diagonal embedding $\mu_2^n \subset O_n$, induces a map

$$(k^\times/k^{\times 2})^2 \rightarrow H^1(k, \mu_2)^n \rightarrow H^1(k, O_n)$$

of Galois cohomology sets where we used Kummer theory to identify $H^1(k, \mu_2)$ with $k^\times/k^{\times 2}$. It maps an n -uple (a_1, \dots, a_n) of invertible scalars to the isometry class of the quadratic form $a_1x_1^2 + \cdots + a_nx_n^2$. Gauss diagonalization rephrases by saying that any O_n -torsor admits a reduction to the k -subgroup μ_2^n . This is a remarkable fact and several other important algebraic constructions are based on finite subgroups of algebraic groups: construction of cyclic central simple algebras, Cayley-Dickson doubling process for composition algebras, Tits first construction of Albert algebras, ... This kind of torsors plays also an important role in the theory of essential dimension [45], it relates to ramification issues [12]. In [10], there is a general result for reducing torsors to an uniform finite subgroup, it had been extended to semilocal rings in [11].

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The torsors reducing to a finite étale subgroup are important special cases of loop torsors over a base ring (or a scheme). Those have been defined firstly for a connected variety X over the field k (with separable closure k_s) equipped with a base point $x_0 \in X(k)$. We denote by $x : \text{Spec}(k_s) \rightarrow X_{k_s}$ the point given by (x_0, id_{k_s}) .

For a smooth algebraic k -group G , the loop classes consist in the image of the map

$$H^1(\pi_1(X, x), G(k_s)) \rightarrow H^1(\pi_1(X, x), G(X_{k_s})) \hookrightarrow H^1(X, G).$$

Loop torsors are then constructed with a quite special kind of Galois cocycles (called loop cocycles) In the reference [27], we investigated a theory of loop torsors over the ring of Laurent polynomials $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ over a field k of characteristic zero. The main application is the study of forms of toroidal Lie algebras [8] and conjugacy of Cartan subalgebras in extended affine Lie algebras [9].

Given a linear algebraic k -group G , the loop G -torsors over R_n arise then with Galois cocycles which do not involve any denominator. This fact is important and permits some extra functoriality. Using Bruhat-Tits' theory, this permitted to relate the study of those torsors to that of reductive algebraic groups over the field of iterated Laurent series $F_n = k((t_1)) \dots ((t_n))$. One important result is the acyclicity fact $H_{loop}^1(R_n, G) \xrightarrow{\sim} H^1(F_n, G)$ providing a nice dictionary between loop R_n -torsors under G and F_n -torsors under G [27, Thm. 8.1].

The first part of the paper is to extend a bunch of the above results to an arbitrary base field and also to allow certain useful locally algebraic groups which are not algebraic (e.g. automorphism groups of reductive groups). This requires some preliminary work in the legacy of SGA3 involving ind-quasi-affine schemes (Gabber). Furthermore we have to limit ourselves to a smaller class of loop torsors called tame torsors for extending the results. We warn the reader that *tame loop torsors* are called simply *loop torsors* in the note [22]. Beyond the tame case, loop torsors are of course of interest, see the Remark 3.3, but the main issue in this paper is the tame case.

The next issue is to start a similar approach with the localization A_D of a regular henselian ring A at a strict normal crossing divisor D and to relate with algebraic groups defined over a natural field associated to A and D , namely the completion K_v of the fraction field K with respect to the valuation arising from the blow-up of $\text{Spec}(A)$ at its maximal ideal. It can be seen as a thickening of the previous setting and this explains why we have to consider the case of Laurent polynomials first. A nice example is the local ring $A = \mathbb{Z}_p[[x]]$ for a prime p where $D = \text{div}(x) + \text{div}(p)$; we have $K = \mathbb{Q}_p((x))$ and K_v has residue field $\mathbb{F}_p((y))$.

Another way to relate the two settings (say the Laurent setting and the Abhyankar setting) is the following example. We can take $A = k[[t_1, \dots, t_n]]$ with divisor $t_1 t_2 \dots t_n = 0$. In this case $A_D = k[[t_1, \dots, t_n]][\frac{1}{t_1}, \dots, \frac{1}{t_n}]$ contains the ring R_n and we have $K_v \cong k(\frac{t_1}{t_n}, \dots, \frac{t_{n-1}}{t_n})(t_n)$. The two rings R_n and A_D are close in the sense they share the same tame covers.

Returning to the case of general A_D , the main result is the injectivity of the base change map $H_{loop}^1(A_D, \tilde{G}) \rightarrow H^1(K_v, G)$ for a smooth A -group scheme \tilde{G} which is the extension of a twisted constant S -group scheme by a reductive A -group scheme G (Thm. 6.9). Furthermore this base change map controls reducibility and isotropy issues for the relevant twisted group schemes. The applications concern local-global principles for torsors and homogeneous spaces à la Harbater-Hartmann-Krashen, see [24, 25].

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2. PRELIMINARIES

Let S be a base scheme. We recall that a *geometric point* \bar{s} is a map $\bar{s} : \text{Spec}(F) \rightarrow S$ where F is an algebraic closed field. We define the notion of *quasi-geometric point* as a map $s : \text{Spec}(E) \rightarrow S$ where E is an algebraic closed field.

2.1. Grothendieck’s fundamental group. We assume that S is connected. If S is noetherian and is equipped with a geometric point \bar{s} , Grothendieck defined the fundamental group $\pi_1(S, \bar{s})$ [49]; the main result is that the category of finite étale covers of S is equivalent to that of finite $\pi_1(S, \bar{s})$ -sets. This has been extended to an arbitrary connected scheme in the Stack Project [53, Tag 0BQ8]. In Fu’s book, it was noticed that one can deal also with a quasi-geometric point $s : \text{Spec}(k) \rightarrow S$ in the noetherian case and that the category of finite étale covers of S is also equivalent to the category of finite $\pi_1(S, s)$ -sets [18, Thm. 3.2.12]; this was extended to an arbitrary connected scheme [51, Thm. 2.3.27]. Let us explain why it provides the same theory by considering the geometric point $\bar{s} : \text{Spec}(\bar{k}) \rightarrow \text{Spec}(k) \xrightarrow{s} S$ where \bar{k} denotes an algebraic closure of the separably closed field k . According to the proposition 2.3.35 of the last reference, we have a natural morphism $id_* : \pi_1(S, \bar{s}) \rightarrow \pi_1(S, s)$ which is actually an isomorphism since it induces an equivalence of categories on finite Galois sets. If k is a field and k_s is a separable closure, we can take as base point $s : \text{Spec}(k_s) \rightarrow \text{Spec}(k)$. In this case, there is a canonical isomorphism $\text{Gal}(k_s/k) \xrightarrow{\sim} \pi_1(\text{Spec}(k), s)$, see [18, Prop. 3.2.14] which avoids then to deal with an algebraic closure of k . This provides some freedom which is used for example in the recent paper [47].

If S is assumed furthermore quasi-compact and quasi-separated, then (S, s) admits a universal cover (S^{sc}, s^{sc}) in the sense of [56] which is connected and simply connected (*ibid*, prop. 3.4).

2.2. Ind-quasi-affine schemes. We use fpqc covers and topology in the sense of Kleiman, see [55, §2.3.2] or alternatively [53, Tag 03NW]. The fpqc topology is the default topology for dealing with sheaf torsors under a group scheme.

A scheme X is ind-quasi-affine if every quasi-compact open of X is quasi-affine; a morphism of schemes $f : X \rightarrow S$ is ind-quasi-affine if $f^{-1}(V)$ is ind-quasi-affine for each affine open V in S [53, 0AP5]. This notion is stable by base change and local for the fpqc topology [53, Tags 0AP7, 0AP8].

Example 2.1. Field case. We assume that $S = \text{Spec}(k)$ for a field k . We remind the reader that the assignment $X \rightarrow X(k_s)$ provides an equivalence of categories between the category of étale k -schemes and the category of Galois sets [14, I.4.6.4]. Furthermore the correspondence exchanges monomorphisms with injections so that monomorphisms in the category of étale k -schemes are clopen immersions. The correspondence exchanges epimorphisms with surjections so that so that epimorphisms in the category of étale k -schemes are the surjective étale morphisms.

The above correspondence induces furthermore an equivalence of categories between Galois modules and commutative étale k -groups. For example we can deal with the k -group μ_{l^∞} of l^∞ -roots of unity for any prime l invertible in k .

2.3. Constant schemes. A constant S -scheme is a S -scheme isomorphic to $M_S = \bigsqcup_{m \in M} S$ for a set M [50, I.8]. For each S -scheme T we have

$$M_S(T) = \left\{ \text{locally constant functions } T_{\text{top}} \rightarrow M \right\}$$

denoted by $C^0(T, M)$. On the other hand we have

$$\text{Hom}_{S\text{-sch}}(M_S, T) = \text{Hom}_{\text{Sets}}(M, \text{Hom}_S(S, T)).$$

The functor $M \mapsto M_S$ is faithful if S is not empty and is fully faithful if S is connected. Given two sets M and N , we have then by taking $T = N_S$

$$\text{Hom}_{S\text{-sch}}(M_S, N_S) = \text{Hom}_{\text{Sets}}(M, C^0(S, N))$$

whence a morphism $C^0(S, \text{Hom}_{\text{Sets}}(M, N)) \rightarrow \text{Hom}_{S\text{-sch}}(M_S, N_S)$ which maps a function f to $m \mapsto f_m$ with $f_m(s) = f(s)(m)$. Furthermore we have

$$\left(\text{Hom}_{\text{Sets}}(M, N)_S \right)(T) = \text{Hom}_{S\text{-sch}}(M_S, N_S)(T)$$

if T is locally connected or if M is finite. We obtained then a monomorphism of S -functors

$$\text{Hom}_{\text{Sets}}(M, N)_S \rightarrow \underline{\text{Hom}}_{\text{sch}}(M_S, N_S)$$

which is an isomorphism when M is finite.

Lemma 2.2. *An S -morphism $f : M_S \rightarrow N_S$ is étale.*

Proof. We are allowed to localize on the source N_S [53, Tag 02GJ, (3)] so that the statement reduces to the case of $M_S = S$ (since each $S_m \rightarrow M_S$ is an S -morphism). An S -morphism $f : S \rightarrow N_S$ is given by a locally constant function $h : S \rightarrow N$ whose image is denoted by I . It provides a partition $S = \sqcup_{i \in I} S_i$ in clopen subschemes such that f reads $\sqcup_{m \in I} f_i$ where $f_i : S_i \rightarrow N_S$ is the composite $S_i \rightarrow S \cong S \xrightarrow{i \text{ piece}} N_S$. Again we can reduce to each S_i and f_i is then an open immersion and a fortiori étale. \square

2.4. Twisted constant schemes. A *twisted constant* S -scheme X is an S -scheme which is locally isomorphic to a constant scheme with respect to the fpqc topology, that is, there exists a fpqc cover $(S_i)_{i \in I}$ such that each $X \times_S S_i$ is a constant S_i -scheme M_{i, S_i} [50, X.5.1]. Such a cover is called a splitting cover. The twisted constant S -scheme X is said *quasi-isotrivial* (resp. *isotrivial*) if X admits an étale (resp. finite étale) splitting cover (that is, we can take the $(S_i)_{i \in I}$ such that $\sqcup_{i \in I} S_i$ is an étale cover, resp. a finite étale cover).

If $s \in S$, $X_{\bar{s}} \xrightarrow{\sim} M_{\bar{k}(s)}$ for a set M ; the isomorphism class of M is called the *type* at s of X . The function *type* is locally constant on S .

We denote by Twc_S the full subcategory of Sch_S whose objects are twisted constant S -schemes. We have the analogous notion for S -group schemes (which are actually those of the SGA3 reference).

Lemma 2.3. *Let X be a twisted constant S -scheme.*

- (1) X is ind-quasi-affine over S .
- (2) The morphism $X \rightarrow S$ is a separated étale morphism and satisfies the valuative criterion of properness.

Proof. Both statements are local for the fpqc topology according to [53, Tag 0AP8], [15, 2.2.7.1.(i), 4.17.7.4] and a result of M. Lara on the valuative criterion of properness [36, lemma 2.40]. We can assume that X is constant by fpqc localization, that is, $X = M_S$ for a set M and furthermore that $S = \text{Spec}(A)$ is affine.

(1) We are given a quasi-compact open subset $U \subset X = M_S$ and want to show that it is quasi-affine. We have $U = \bigsqcup_{m \in M} U_m$ where U_m is a quasi-compact open subset of $\text{Spec}(A)$. There there exists a finite subset $M_0 \subset M$ such that $U = \bigsqcup_{m \in M_0} U_i$. Since each U_i is quasi-affine, it follows that U is quasi-affine.

(2) It is clear that M_S is separated étale over S . For each S -valuation ring A of fraction field K , we have $M_S(A) = M = M_S(K)$ so that M_S satisfies the valuative criterion of properness. \square

Lemma 2.4. (1) *The morphisms in the category Twc_S are étale.*

- (2) *The monomorphisms in the category Twc_S are the clopen immersions.*
- (3) *The epimorphisms in the category Twc_S are the surjective étale morphisms.*

Of course the analogous statements hold in the category of group schemes.

Proof. All statements are local with respect to the fpqc topology so boil down to the constant case already handled in §2.3. \square

Lemma 2.5. *Let $f : X \rightarrow Y$ be an S -morphism between twisted constant S -schemes.*

(1) *If f is a clopen immersion and Y is constant then there exists a partition $S = \sqcup_{i \in I} S_i$ such that each X_{S_i} is constant.*

(2) *If f is a clopen immersion and Y is quasi-isotrivial then X is quasi-isotrivial.*

Proof. (1) We assume that $Y = N_S$ for a set S . Lemma 2.4.(1) shows that $X \rightarrow N_S = Y$ is a clopen immersion so that $N_S = X \sqcup X'$ where $X = \sqcup_{n \in N} X_n$, $X' = \sqcup_{n \in N} X'_n$ with $S = X_n \sqcup X'_n$ for each $n \in N$. It provides a partition $S = \sqcup_{i \in I} S_i$ such that $X_{S_i} = \sqcup_{n \in N_i} S$ for $N_i \subset N$.

(2) We apply the previous reasoning after base change by a splitting étale morphism $S' \rightarrow S$ of Y . \square

Let $(S_i)_{i \in I} \rightarrow S$ be a fpqc cover. According to Gabber [53, Tag 0APK], any $(S_i)_{i \in I}/S$ -data descent of ind-quasi-affine schemes $(X_i \rightarrow S_i)$ is effective. Let us state a few applications of that.

Lemma 2.6. *The category Twc_S satisfies fpqc descent.*

Proof. Let $(S_i)_{i \in I} \rightarrow S$ be a fpqc cover and consider a data descent twisted constant schemes $(X_i \rightarrow S_i)$. Since each X_i is ind-quasi-affine over S_i (Lemma 2.3.(1)), Gabber's result provides an S -scheme X together with isomorphisms $X \times_S S_i \cong X_i$. It follows that X is twisted constant over S so we are done. \square

Gabber's result yields also the following generalization of the case of affine group schemes.

Lemma 2.7. *Let G be an ind-quasi-affine S -group scheme.*

(1) *Sheaf G -torsors are representable by ind-quasi-affine S -schemes.*

(2) *If \mathcal{E} is a sheaf G -torsor, then the inner twist ${}^{\mathcal{E}}G$ is representable by an ind-quasi-affine S -group scheme.*

Lemma 2.8. *Let G be a twisted constant S -group scheme.*

(1) *The G -torsors (for the fpqc topology) are representable by twisted constant S -schemes. Furthermore they are quasi-isotrivial.*

(2) *If \mathcal{E} is a sheaf G -torsor, then the inner twist ${}^{\mathcal{E}}G$ is representable by a twisted constant S -scheme.*

(3) *Let H be a twisted constant S -group scheme and let $f : H \rightarrow G$ be a monomorphism. Then f is a clopen immersion and the fpqc quotient G/H is representable by a twisted constant S -scheme. Furthermore if G is isotrivial (resp. quasi-isotrivial) then the twisted constant S -scheme G/H is isotrivial (resp. quasi-isotrivial).*

(4) If H is normal in G , the fpqc quotient G/H is representable by a twisted constant S -group scheme. Furthermore if G is isotrivial (resp. quasi-isotrivial) then G/H is isotrivial (resp. quasi-isotrivial).

Proof. (1) Let \mathcal{E} be a sheaf J -torsor, it is representable by an S -scheme E in view of Lemma 2.7.(1) Furthermore E is a twisted constant S -scheme according to Lemma 2.3. Since E is étale and $E \times_S J \xrightarrow{\sim} E \times_S E$, the G -torsor E is quasi-isotrivial.

(2) Lemma 2.7.(2) insures representability of ${}^{\mathcal{E}}G$ and this S -scheme is twisted constant according to 2.6.

(3) In view of Lemma 2.4.(1), H is a clopen immersion. Once again the question is local for the fpqc topology. We can assume that $G = \Gamma_S$ and $H = \Theta_S$ are constant. Since $\mathrm{Hom}_{S\text{-gr}}(H, G) = \left(\mathrm{Hom}_{\mathrm{gr}}(\Theta, \Gamma) \right)(S)$, there exists a partition $S = \sqcup_i S_i$ such that $f_{S_i} = (f_i)_{S_i}$ where each $f_i : \Theta \rightarrow \Gamma$ is an abstract group homomorphism. Since f is a monomorphism, so is each f_i . The fpqc quotient G/H is representable by $\sqcup_i (\Gamma/f_i(\Theta))_{S_i}$ so is a twisted constant S -scheme.

We assume now that G is isotrivial, that is, there exists a finite étale cover $S' = \sqcup_{i \in I} S_i$ of S such that $G_{S_i} \xrightarrow{\sim} (\Gamma_i)_{S_i}$ where the Γ_i 's are abstract groups. Without loss of generality we can assume that $G_{S'} \xrightarrow{\sim} (\Gamma)_{S'}$. The locally isotrivial variant holds as well by Zariski localization.

(4) We let to reader to mimick the preceding reasoning for the case H normal in G . □

We discuss variations of [50, X.7.0.3].

Proposition 2.9. *Assume that S is locally noetherian, connected and normal. Let K be the fraction field of X , let K_s be the separable closure of K and let $\eta_s : \mathrm{Spec}(K_s) \rightarrow S$ be the associated point. Let $\pi_1(S, \eta_s)$ be the Grothendieck fundamental group of S .*

(1) *An object X of Twc_S is isomorphic to a disjoint union $\bigsqcup_{i \in I} X_i$ where each X_i is a finite étale connected cover of S . Furthermore we have $X(S) = X(K)$.*

(2) *Assume that S is noetherian. Then the category of continuous $\pi_1(S, \eta_s)$ -sets and the category Twc_S are equivalent.*

Proof. (1) We are given an object X of Twc_S . Since it is étale the irreducible components $(X_i)_{i \in I}$ are its connected components and are open according to [50, before X.5.14]. Furthermore denoting by η_i the generic point of X_i , it is above the generic point η of S and $X_{i,K} = \{\eta_i\}$ so that $K_i = \kappa(X_i)$ is a finite separable extension of K . In view of Lemma 2.3.(2), X satisfies the valuative criterion of properness, and do the closed subschemes X_i 's of X . Summarizing each X_i is étale separated over S and satisfies the valuative criterion of properness. Let \tilde{X}_i be the normalization of S with respect to the finite field extension K_i/K [29, 12.42], it is finite over X_i (*ibid*, 12.50). Since X_i is normal [39, I.3.17.(b)], we obtain then a birational morphism $p_i : X_i \rightarrow \tilde{X}_i$ by means of the universal property of the normalization (*ibid*, 12.44.(iii)). We observe

that p_i is separated in view of [29, 9.13.(2)]. Since X_i (and \tilde{X}_i) satisfies the valuative criterion of properness we have

$$\begin{array}{ccccc} \mathrm{Hom}_S(\tilde{X}_i, X_i) & = & X_i(\tilde{R}_i) & = & X_i(K_i) \\ & & \downarrow & & \downarrow = \\ \mathrm{Hom}_S(\tilde{X}_i, \tilde{X}_i) & = & \tilde{X}_i(\tilde{R}_i) & = & \tilde{X}_i(K_i). \end{array}$$

This provides a section $s_i : \tilde{X}_i \rightarrow X_i$ of p_i . Since $p_i \circ s_i$ is generically id_{X_i} , we have $p_i \circ s_i = id_{X_i}$ and similarly we have $s_i \circ p_i = id_{\tilde{X}_i}$. Each p_i is an isomorphism and we have then a decomposition $X = \bigsqcup_{i \in I} X_i$ where the S -schemes X_i 's are finite étale and connected.

Next we want to show that $X(S) = X(K)$. Since S is dense in $\mathrm{Spec}(K)$ and X is separated, the map $X(S) \rightarrow X(K)$ is injective. Let K_i be the function field of S_i . Since $\mathrm{Spec}(K_i) = S_{i,K}$, we have $S(K) = \bigsqcup_{i \in I} X_i(K) = \bigsqcup_{i \in I} \mathrm{Hom}_{K\text{-alg}}(K_i, K)$. It follows that $I_0 \cong S(K)$ where I_0 stands for the indices i such that $K = K_i$ or equivalently $S = S_i$. Thus $X(S) \rightarrow X(K)$ is onto and bijective.

(2) The assumptions imply that S is qcqs [53, Tags 01OV, 01OY]; let S^{sc} be the universal cover of S as defined in [56, prop. 3.4]. Since a constant S -scheme M_S is ind-quasi-affine (Lemma 2.3.(1)), we can twist it by a continuous morphism $f : \pi_1(S, \eta_s) \rightarrow \mathrm{Aut}(M)$. This defines a functor from the category of continuous $\pi_1(S, \eta)$ -sets to the category Twc_S , $(M, f) \mapsto M^f$. Extended to S^{sc} , this functor is nothing but $M \mapsto M_{S^{sc}}$ which is an equivalence of categories between the category of sets and that of constant S^{sc} -schemes. By considering the action of $\Pi_1(S, \eta_s)$ on both sides and taking invariants, it follows that the functor $(M, f) \mapsto M^f$ is fully faithful. The essential surjectivity follows from (1). \square

Remark 2.10. In the proof of (1), if X is quasi-isotrivial, each X_i is finite in view of [50, X.5.13]. So the new case is when X is not quasi-isotrivial. Another way to prove the statement is to use Bhatt-Scholze's theory [1, Thm. 1.10].

Corollary 2.11. *Assume that S is connected, normal and noetherian. Let $X = \bigsqcup_{i \in I} S_i$ be a twisted constant S -scheme where each S_i is a finite étale cover of S . Then the following are equivalent:*

- (i) *the S_i 's admit a common splitting finite Galois cover;*
- (ii) *X is isotrivial;*
- (iii) *X is quasi-isotrivial.*

Proof. Again we can deal with the universal cover S^{sc} of S as defined in [56, prop. 3.4]. The implications (i) \implies (ii) \implies (iii) are obvious. Let us establish (iii) \implies (i). We assume that X is quasi-isotrivial. Next we consider the continuous surjective map $q : \mathrm{Gal}(K_s/K) \rightarrow \pi_1(S, \eta_s)$ and denote by H_i the preimage of G_i for each $i \in I$. We denote by $G = \bigcap_{i \in I} G_i$ and by $H = q^{-1}(G) = \bigcap_{i \in I} H_i$. We have

$X_K \cong \bigsqcup_{i \in I} \text{Spec}(K_s^{H_i})$. Since X_K is quasi-isotrivial, Example 2.1 provides a normal open subgroup \tilde{H} of $\text{Gal}(K_s/K)$ such that $K_s^{\tilde{H}}$ is the minimal splitting field of X_K . More precisely \tilde{H} is the largest normal open subgroup of H such that $\tilde{H} \subset H_i$ for all i . We put $\tilde{G} = q(\tilde{H})$, it is an open normal subgroup of $\pi_1(S, \eta_s)$ such that $\tilde{G} \subset G_i$ for each i . We put $\tilde{S} = S^{sc}/G$, this is a finite Galois cover of S which splits each S_i . \square

2.5. Extensions of twisted constant group schemes. If $G \rightarrow S$ is an ind-quasi-affine, fpqc descent [53, Tag 0APK] implies that sheaf fpqc G -torsors are representable by ind-quasi-affine S -schemes. Similarly if E is a fpqc sheaf $\text{Aut}(G)$ -torsor, then the inner twist ${}^E G$ of G by E is representable by an ind-quasi-affine S -group scheme. In the spirit of [50, VI_BV9.2], we have the following fact.

Lemma 2.12. *Let $u : G' \rightarrow G$ be a monomorphism of S -group schemes and assume that the fpqc quotient G/G' is representable by an S -scheme X . Then G' is ind-quasi-affine over S if and only if the quotient morphism $q : G \rightarrow X$ is ind-quasi-affine.*

Proof. If q is ind-quasi-affine so is $G' \rightarrow S$ by base change. We assume that $G' \rightarrow S$ is quasi-affine. Let $(X_i)_{i \in I}$ be an fpqc cover of X such that $q^{-1}(X_i) \cong X_i \times_S G'$. Since $q^{-1}(X_i)$ is ind-quasi-affine over X_i for each i , fpqc descent enables us to conclude that q is ind-quasi-affine [53, Tag 0AP8]. \square

We deal with an S -group scheme \tilde{G} fitting in an exact sequence of the shape

$$(2.1) \quad 1 \rightarrow G \rightarrow \tilde{G} \rightarrow J \rightarrow 1$$

where G is affine and J is twisted S -constant. Combining Lemmas 2.3 and 2.12 yields that \tilde{G} is ind-quasi-affine so that Lemma 2.7 applies to \tilde{G} .

Lemma 2.13. *We assume that S is connected, normal, locally noetherian. We denote by K the function field of S .*

- (1) *The map $H^1(S, J) \rightarrow H^1(K, J)$ is injective.*
- (2) *J -torsors are isotrivial.*

Proof. (1) The standard torsion argument reduces to establish the triviality of the kernel of $H^1(S, J) \rightarrow H^1(K, J)$. We have seen that J -torsors are representable by twisted constant S -schemes. Let $[E]$ be an element of the kernel of $H^1(S, J) \rightarrow H^1(K, J)$. According to Proposition 2.9, we have $E(S) = E(K)$. It follows that $E(S) \neq \emptyset$ so that $[E] = 1$.

(2) Let E be a J -torsor. Corollary 2.9 states that $E \cong \bigsqcup_{i \in I} S_i$ where the S_i 's are connected finite étale covers of S . In particular $E(S_i) \neq \emptyset$ for each $i \in I$ so that E is isotrivial. \square

Remark 2.14. With Grothendieck's method of [50, X.5], one can prove that statement provided the twisted S -group scheme J is quasi-isotrivial.

Example 2.15. Let us illustrate the statement for a field k with the k -group μ_{l^∞} of l^∞ root of unity attached to an invertible prime l . Given $a \in k^\times$, we consider the Galois set $\varinjlim_n \{x_n \in k_s^\times \mid x_n^{l^n} = a a^{l^n}\}$ where the transition maps are $x_n \mapsto x_n^l$, it defines a twisted constant k -scheme X which is a μ_{l^∞} -torsor. Putting $b = \sqrt[l]{a} \in k_s$, the elements $b_n = b^{l^{n-1}}$ defines a point of $X(k)$ so that this μ_{l^∞} -torsor is isotrivial.

Lemma 2.16. *Assume that S is connected, locally noetherian and normal. In the sequence (2.1), assume that G is reductive. Then the \tilde{G} -torsors over S are semi-locally isotrivial (and a fortiori quasi-isotrivial).*

Proof. It is known for J by Lemma 2.13.(2) and for G by [50, XXIV.4.16]. The dévissage from these two cases is similar with the argument of the proof of [50, XXIV.4.24]. \square

From now on we assume that G is reductive. The S -group \tilde{G} acts on its normal S -subgroup G and we consider the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & G & \longrightarrow & \tilde{G} & \longrightarrow & J \longrightarrow 1 \\ & & \downarrow \text{Int} & & \downarrow \text{Int} & & \downarrow h \\ 1 & \longrightarrow & G_{ad} = G/C(G) & \longrightarrow & \text{Aut}(G) & \longrightarrow & \text{Out}(G) \longrightarrow 1. \end{array}$$

where the bottom exact sequence is [50, XXIV.1.1]. Note that $\text{Out}(G)$ is a twisted constant S -group scheme (but not necessarily quasi-isotrivial, this is the case however if G is quasi-isotrivial).

2.6. Normalizers, I. Let P be an S -parabolic subgroup of G equipped with a Levi S -subgroup L . The normalizer $N_{\tilde{G}}(P, L)$ is representable by a separated smooth S -group scheme which is S -closed in \tilde{G} [21, Lemme 3.4.54]. We claim that the sequence $1 \rightarrow G \rightarrow \tilde{G} \rightarrow J \rightarrow 1$ induces an exact sequence of S -group schemes

$$(2.2) \quad 1 \rightarrow L \rightarrow N_{\tilde{G}}(P, L) \rightarrow J_{P,L} \rightarrow 1$$

where $J_{P,L}$ is an S -subgroup scheme of J . Since $L = N_G(P, L)$ is smooth (and a fortiori flat over S), we know that $N_{\tilde{G}}(P, L)/L$ is representable by an A -group scheme $J_{P,L}$ which is locally of finite presentation [50, XVI.2.3]. Furthermore the homomorphism $J_{P,L} \rightarrow J$ is a monomorphism. According to [50, VI_B.9.2.(xii)], $J_{P,L}$ is S -smooth, so is S -étale since J is étale. We shall say more on $J_{P,L}$ in Lemma 2.18. To pursue we deal with the following special case.

Lemma 2.17. (1) *The S -functor $\text{Aut}(G, P, L)$ is representable by a smooth S -group scheme which fits in an exact sequence of smooth S -group schemes*

$$1 \rightarrow L/C(G) \rightarrow \text{Aut}(G, P, L) \rightarrow \text{Out}(G, P, L) \rightarrow 1.$$

(2) *The S -group scheme $\text{Out}(G, P, L)$ is twisted constant and is a clopen S -subgroup of $\text{Out}(G)$.*

(3) The fppf quotient $\text{Out}(G)/\text{Out}(G, P, L)$ is representable by a finite étale S -scheme.

Proof. (1) This is the special case of the above fact when taking $\text{Aut}(G, P, L)$ for \tilde{G} .
 (2) The statement is local for the fpqc topology so that we can assume that G is split and that P is a standard parabolic subgroup. If G is adjoint, then $\text{Out}(G, P, L)$ and $\text{Out}(G)$ are finite constant [21, lemme 5.1.2] so that the statement is obvious. We consider the exact sequence $1 \rightarrow C(G) \rightarrow G \rightarrow G_{ad} \rightarrow 1$ and the natural map $\text{Aut}(G) \rightarrow \text{Aut}(G_{ad})$. The correspondence [21, Lemma 3.2.1.(2)] shows that $\text{Aut}(G, P, L) = \text{Aut}(G) \times_{\text{Aut}(G_{ad})} \text{Aut}(G_{ad}, P_{ad}, L_{ad})$. We obtain a commutative diagram

$$\begin{array}{ccc} \text{Aut}(G, P, L) & \xrightarrow{\sim} & \text{Aut}(G) \times_{\text{Aut}(G_{ad})} \text{Aut}(G_{ad}, P_{ad}, L_{ad}) \\ \downarrow & & \downarrow \\ \text{Out}(G, P, L) & \xrightarrow{r} & \text{Out}(G) \times_{\text{Out}(G_{ad})} \text{Out}(G_{ad}, P_{ad}, L_{ad}) \end{array}$$

Claim 2.6.1. *The bottom horizontal map r is an isomorphism.*

Since $\text{Out}(G, P, L) \rightarrow \text{Out}(G)$ is a monomorphism so is r . It is then enough to prove that the right vertical map is an epimorphism of flat sheaves. Let $u \in \left(\text{Out}(G) \times_{\text{Out}(G_{ad})} \text{Out}(G_{ad}, P_{ad}, L_{ad}) \right)(T)$ for an S -scheme T . Up to localize for the flat topology, u is represented by elements $a \in \text{Aut}(G)(T)$ and $b \in \text{Aut}(G_{ad}, P_{ad}, L_{ad})(T)$ having same image in $\text{Out}(G_{ad})(T)$. It means that there exists $y \in G_{ad}(T)$ such that $a \text{Int}(y) = b \in \text{Aut}(G_{ad})$. The pair $(a \text{Int}(y), b)$ defines an element of $\left(\text{Aut}(G) \times_{\text{Aut}(G_{ad})} \text{Aut}(G_{ad}, P_{ad}, L_{ad}) \right)(T)$ mapping to u . The claim is then established.

It follows that $\text{Out}(G, P, L) \rightarrow \text{Out}(G)$ is a clopen immersion. Also since the category of twisted S -group schemes is stable by cartesian product, we obtain that $\text{Out}(G, P, L)$ is a twisted constant S -group scheme.

(3) We can continue with the same reductions. We have seen that $\text{Out}(G_{ad})/\text{Out}(G_{ad}, P_{ad}, L_{ad})$ is finite S -étale. According to §2.4, the fppf quotient $\text{Out}(G)/\text{Out}(G, P, L)$ is representable by a twisted constant S -scheme and so is $\text{Out}(G_{ad})/\text{Out}(G_{ad}, P_{ad}, L_{ad})$. The map $\text{Out}(G)/\text{Out}(G, P, L) \rightarrow \text{Out}(G_{ad})/\text{Out}(G_{ad}, P_{ad}, L_{ad})$ is a monomorphism so is a clopen immersion according to Lemma 2.4.(1). Thus $\text{Out}(G)/\text{Out}(G, P, L)$ is finite S -étale. \square

Lemma 2.18. (1) *The map $J_{P,L} \rightarrow J$ is a clopen immersion and $J_{P,L}$ is a twisted constant S -group scheme. If J is furthermore quasi-isotrivial, so is $J_{P,L}$.*

(2) *The fppf quotient $J/J_{P,L}$ is representable by a finite étale S -scheme.*

Proof. (1) The S -functor $\text{Aut}(G, P, L)$ is representable by a smooth S -scheme [21, prop. 3.4.3] and we have $N_{\tilde{G}}(P, L) \xrightarrow{\sim} \tilde{G} \times_{\text{Aut}(G)} \text{Aut}(G, P, L)$. We obtain then the

commutative diagram

$$\begin{array}{ccc} N_{\tilde{G}}(P, L) & \xrightarrow{\sim} & \tilde{G} \times_{\text{Aut}(G)} \text{Aut}(G, P, L) \\ \downarrow & & \downarrow \\ J_{P,L} & \xrightarrow{r} & J \times_{\text{Out}(G)} \text{Out}(G, P, L). \end{array}$$

Claim 2.6.2. *The bottom map r is an isomorphism.*

Since $J_{P,L} \rightarrow J$ is a monomorphism so is r . It is then enough to prove that the right vertical map is an epimorphism of flat sheaves. For that we deal with an S -scheme T and an element $u \in \left(J \times_{\text{Out}(G)} \text{Out}(G, P, L) \right)(T)$. Up to localize for the flat topology we may assume that there exists $\tilde{g} \in \tilde{G}(T)$ and $a \in \text{Aut}(G, P, L)(T)$ such that \tilde{g} and a have same image in $\text{Out}(G)(T)$. It means (again up to localize) that there exists $y \in G(T)$ such that $\text{int}(\tilde{g}y) = a$. It follows that $\tilde{g}y$ normalizes (P, L) so that $(\tilde{g}y, a)$ defines an element of $\left(\tilde{G} \times_{\text{Aut}(G)} \text{Aut}(G, P, L) \right)(T)$ which maps to u . The claim is then established.

According to Lemma 2.17.(2), the map $\text{Out}(G, P, L) \rightarrow \text{Out}(G, P)$ is a clopen immersion between twisted constant S -group schemes. Claim 2.6.2 implies that $J_{P,L} \rightarrow J$ is a clopen immersion and also that $J_{P,L}$ is a twisted constant S -group. Finally it is quasi-isotrivial since J is (Lemma 2.5.(2)).

(2) According to §2.4, the fppf quotient $J/J_{P,L}$ is representable by a twisted constant S -scheme. On the other hand we know that $\text{Out}(G)/\text{Out}(G, P, L)$ is representable by finite S -étale scheme. The map $J/J_{P,L} \rightarrow \text{Out}(G)/\text{Out}(G, P, L)$ is a monomorphism so is a clopen immersion according to Lemma 2.4.(1). Thus $J/J_{P,L}$ is finite S -étale. \square

The conclusion is that the sequence (2.2) has the same shape than the initial sequence $1 \rightarrow G \rightarrow \tilde{G} \rightarrow J \rightarrow 1$.

2.7. Normalizers, II. Now let P be an S -parabolic subgroup of G . According to [21, prop. 3.4.3], the fppf sheaf $N_{\tilde{G}}(P)$ is representable by a smooth S -scheme which is closed in \tilde{G} . Furthermore the quotient $\tilde{G}/N_{\tilde{G}}(P)$ is representable by a smooth S -scheme. In the same manner as in the previous section, we can construct an exact sequence of smooth S -group schemes

$$1 \rightarrow P \rightarrow N_{\tilde{G}}(P) \rightarrow J_P \rightarrow 1$$

such that J_P is a twisted constant S -group and that J/J_P is representable by a finite étale S -scheme. A complement is the following (which extends [50, XXII.5.8.5]).

Lemma 2.19. *The scheme $\tilde{G}/N_{\tilde{G}}(P)$ is a projective S -scheme.*

Proof. We establish first that the S -scheme $\tilde{G}/N_{\tilde{G}}(P)$ is proper. In view of [15, 2.2.7.1.(vii)], the statement is local with respect to the fpqc topology; since J/J_P is finite étale over S we can assume that $J/J_P = S \sqcup S \cdots \sqcup S$ (d times) and that there exist $\tilde{g}_1, \dots, \tilde{g}_d \in \tilde{G}(S)$ mapping to the pieces of J/J_P . In this case we have $\tilde{G}/N_{\tilde{G}}(P) \xrightarrow{\sim} G/P \sqcup \cdots \sqcup G/P$ (d times) which is proper over $S \sqcup \cdots \sqcup S$ so that $\tilde{G}/N_{\tilde{G}}(P)$ is proper over S .

The assignment $\tilde{g} \rightarrow [\tilde{g}P]$ defines a monomorphism $h : \tilde{G}/N_{\tilde{G}}(P) \rightarrow \text{Par}(G)$. Since $\tilde{G}/N_{\tilde{G}}(P)$ is S -proper, the morphism h is proper [53, Tag 01W6, (2)] so is a closed immersion [15, 3.8.11.5]. Since $\text{Par}(G)$ is S -projective, we conclude that the S -scheme $\tilde{G}/N_{\tilde{G}}(P)$ is projective. \square

2.8. Reductive group schemes and reducibility. Let H be a reductive S -group scheme. We denote by $\text{Par}(H)$ the total scheme of parabolic subgroups of H [50, XXVI.3]. It decomposes as $\text{Par}(H) = \text{Par}^+(H) \sqcup S$, where S corresponds to the fact that H itself is a parabolic subgroup scheme. The scheme $\text{Par}^+(H)$ is called the total scheme of proper parabolic subgroups and is also projective over S .

We say that H is *reducible* if it admits a proper parabolic subgroup P such that P contains a Levi subgroup L . The opposite notion is *irreducible*.

If S is affine, the notion of reducibility for H is equivalent to the existence of a proper parabolic subgroup P [50, XXVI.2.3], so there is no ambiguity with the terminology of [26].

We say that H over S is *isotropic* if H admits a subgroup isomorphic to $\mathbb{G}_{m,S}$. The opposite notion is *anisotropic*. According to [21, Thm. 7.3.1.(ii)], if S is connected, H is isotropic if and only if H is reducible or the radical torus $\text{rad}(H)$ is isotropic.

By extension, if an S -group scheme M acts on H , we say that the action is *reducible* if it normalizes a couple (P, L) where P is a proper parabolic subgroup of H and L a Levi subgroup of P . The action is otherwise called *irreducible*.

We say that the action of M on H is *isotropic* if it centralizes a S -subgroup \mathbb{G}_m of H . Otherwise the action is *anisotropic*. We shall use several times the following statement which was implicitly used in [27, §2.4].

Corollary 2.20. [23, cor. B.2] *Assume that S is connected. The following are equivalent:*

- (i) *The action of M on H is isotropic;*
- (ii) *The action of M on $\text{rad}(H)$ is isotropic or the action of M on H is reducible.*

A useful complement is the following.

Proposition 2.21. [23, prop. B.3] *Assume that $S = \text{Spec}(R)$ is affine and connected, that M is a flat affine R -group scheme whose geometric fibers are linearly reductive (e.g. M is of multiplicative type). If M normalizes an R -parabolic subgroup P of H , there exists $\lambda : \mathbb{G}_m \rightarrow H$ which is M -invariant such that $P = P_H(\lambda)$. In particular $L = C_H(\lambda)$ is a Levi subgroup of P which is normalized by M .*

3. LOOP TORSORS ON VARIETIES

We come back here on variant of results of [13, 27]. We extend them to the characteristic free case and to a wider class of k -group schemes.

3.1. Various fundamental groups. Let k be a field. We denote by $p \geq 1$ the characteristic exponent of k and by k_s/k be an absolute Galois closure. We denote by $\Gamma_k = \text{Gal}(k_s/k)$ the absolute Galois group. Let X be a geometrically connected k -scheme of finite type. We choose a k_s -point x , this is a quasi geometric point of X so that we can deal with the fundamental group $\pi_1(X, x)$ as explained in §2.1. We remind the following important sequence to the reader.

Proposition 3.1.1. [18, Prop. 3.3.7] *We have an exact sequence of profinite groups*

$$(3.1) \quad 1 \rightarrow \pi_1(X_{k_s}, x) \rightarrow \pi_1(X, x) \rightarrow \text{Gal}(k_s/k) \rightarrow 1.$$

By using geometric base points, this is equivalent to the homotopy sequence of Grothendieck [49, théorème IX.6.1].

For a profinite group \mathcal{G} , we denote by $\mathcal{G}^{(p')}$ its maximal prime to p quotient it is the quotient of \mathcal{G} by the closure of the subgroup $\mathcal{G}^{[p]}$ generated by the pro- p -Sylow subgroups of \mathcal{G} . We can define then $\pi_1(X, x)^{(p')}$. Since $\pi_1(X_{k_s}, x)^{[p]}$ is normal and closed in $\pi_1(X, x)$, we can define the quotient $\pi_1(X, x)^{(p'-geo)} = \pi_1(X, x)/\pi_1(X_{k_s}, x)^{[p]}$. Altogether we have a commutative exact diagram of exact sequences of profinite groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_{k_s}, x) & \longrightarrow & \pi_1(X, x) & \longrightarrow & \text{Gal}(k_s/k) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 1 & \longrightarrow & \pi_1(X_{k_s}, x)^{(p')} & \longrightarrow & \pi_1(X, x)^{(p'-geo)} & \longrightarrow & \text{Gal}(k_s/k) \longrightarrow 1 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(X_{k_s}, x)^{(p')} & \longrightarrow & \pi_1(X, x)^{(p')} & \longrightarrow & \text{Gal}(k_s/k)^{(p')} \longrightarrow 1 \end{array}$$

By Galois theory we have associated the cover $X^{sc, p'-geo}$ (resp. $X^{sc, p'}$) of X named the universal geometrical p' -over (resp. universal p' -cover). In particular k_s admits the maximal p' -Galois subextension $k^{(p')}$.

On the other hand, if X is smooth, Kerz and Schmidt showed that the four notions of tamely ramified covers of X coincide [35, Thm. 4.4]. As explained in the introduction of [32], tameness of a covering should be thought of as “at most tamely ramified along the boundary of compactifications over the base”. We use mostly here the so-called *divisor-tameness* definition. We say then that a finite connected étale cover $Y \rightarrow X$ is *divisor-tame* if for every normal compactification X^c of X and every point $x \in X^c \setminus X$ of codimension 1, the discrete rank one valuation v_x on $k(X)$ associated

with x is tamely ramified in the finite, separable field extension $k(Y)/k(X)$ ¹. Let (X^{tsc}, x^{tsc}) be the *simply connected tame cover* of (X, x) . We have the fundamental exact sequence of profinite groups

$$(3.2) \quad 1 \rightarrow \pi_1^t(X_{k_s}, x) \rightarrow \pi_1^t(X, x) \rightarrow \text{Gal}(k_s/k) \rightarrow 1.$$

Since p' -Galois covers of X_{k_s} are tame, $\pi_1^t(X_{k_s}, x)$ maps onto $\pi_1(X_{k_s}, x)^{(p')}$ so that we have a commutative exact diagram of exact sequences of profinite groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_{k_s}, x) & \longrightarrow & \pi_1(X, x) & \longrightarrow & \text{Gal}(k_s/k) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 1 & \longrightarrow & \pi_1^t(X_{k_s}, x) & \longrightarrow & \pi_1^t(X, x) & \longrightarrow & \text{Gal}(k_s/k) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(X_{k_s}, x)^{(p')} & \longrightarrow & \pi_1(X, x)^{(p'-geo)} & \longrightarrow & \text{Gal}(k_s/k) \longrightarrow 1. \end{array}$$

Remark 3.1. A basic tamely ramified cover is of the shape Y_l where $Y \rightarrow X$ is a geometrically connected M -torsor for some finite quotient M of $\pi_1^t(X, x)$ and l/k a finite Galois extension such that M_l is constant. In this case we have $\text{Gal}(Y_l/X) = M(l) \rtimes \text{Gal}(l/k)$. We observe that every continuous finite quotient of $\pi_1^t(X, x)$ factorizes through such a cover.

3.2. Loop torsors. Let G be a locally algebraic group defined over k . We get then a map

$$H^1(\pi_1(X, x), G(k_s)) \rightarrow H^1(\pi_1(X, x), G(X^{sc})) \hookrightarrow H^1(X, G)$$

whose image is denoted by $H_{loop}^1(X, G)$. We say that a sheaf G -torsor over X is *loop* if its class belongs to $H_{loop}^1(X, G)$.

We have variants called tame loop torsors (resp. p' -loop torsors) when we consider the group $\pi_1^t(X, x)$ (resp. $\pi_1(X, x)^{(p'-geo)}$). This gives rise to the subsets

$$H_{loop}^{1, p'-geo}(X, G) \subseteq H_{loop}^{1, tame}(X, G) \subseteq H_{loop}^1(X, G).$$

Finally the image of the map

$$H^1(\pi_1(X, x)^{(p')}, G(k^{(p')})) \rightarrow H^1(\pi_1(X, x)^{(p')}, G(X^{sc, p'})) \hookrightarrow H^1(X, G)$$

is called the class of p' -loop torsors.

From now on we assume that $x \in X(k_s)$ is a k -point. The sequence (3.1) comes with a splitting $s_x : \text{Gal}(k_s/k) \rightarrow \pi_1(X, x)$. We have then a decomposition

$$\pi_1(X, x) = \pi_1(X_{k_s}, x) \rtimes \text{Gal}(k_s/k).$$

¹Denoting by K_x the completion of $k(X)$ for the valuation v_x , it means that in the decomposition $k(Y) \otimes_{k(X)} K_x = L_1 \times \cdots \times L_r$, each finite separable field extension L_i/K is tamely ramified, that is, the residue field extension is separable and the ramification indices are coprime with the characteristic exponent of the residue field.

It follows that the profinite group $\pi_1(X, x)$ is equipped with a structure of affine algebraic k -group. We are given a loop cocycle $\eta \in Z^1(\pi_1(X, x), \mathbf{G}(k_s))$. Its restriction $\eta|_{\Gamma_k}$ is called the *arithmetic part* of η and its denoted by η^{ar} : it is an element of $Z^1(\Gamma_k, \mathbf{G}(k_s))$. Next we consider the restriction of η to $\pi_1(X_{k_s}, x)$ that we denote by η^{geo} and called the *geometric part* of η . By taking into account the Galois action, the map $\eta^{geo} : \pi_1(X_{k_s}, x) \rightarrow {}_{\eta^{ar}}G$ is a homomorphism of algebraic k -groups.

Lemma 3.2. (1) *The map $\eta \rightarrow (\eta^{ar}, \eta^{geo})$ provides a bijection between $Z^1(\pi_1(X, x), G(k_s))$ and the couples (z, ϕ) where $z \in Z^1(\Gamma_k, G(k_s))$ and $\phi : \pi_1(X, x) \rightarrow {}_zG$ is a k -group homomorphism.*

(2) *We have an exact sequence of pointed sets*

$$(3.3) \quad 1 \rightarrow \text{Hom}_{k-gr}(\pi_1(X_{k_s}, x), G)/G(k) \rightarrow H^1(\pi_1(X, x), G(k_s)) \xrightarrow{\text{Res}_x} H^1(\Gamma_k, G(k_s)) \rightarrow 1.$$

Furthermore the first map is injective.

(3) *We have a decomposition*

$$H^1(\pi_1(X, x), G(k_s)) = \bigsqcup_{[z] \in H^1(k, G)} \text{Hom}_{k-gr}(\pi_1(X_{k_s}, x), {}_zG)/{}_zG(k).$$

Proof. (1) This is similar with [27, lemma 3.7].

(2) Part (1) defines a map

$$\text{Hom}_{k-gr}(\pi_1(X_{k_s}, x), G) \rightarrow Z^1(\pi_1(X, x), G(k_s)) \rightarrow H^1(\pi_1(X, x), G(k_s))$$

which is $G(k)$ -invariant. It applies a k -homomorphism $\phi : \pi_1(X_{k_s}, x) \rightarrow G$ to the loop cocycle $\tilde{\phi} : \pi_1(X, x) \rightarrow G$ defined by $\tilde{\phi}(\tau, \sigma) = \phi(\sigma)$ for $(\tau, \sigma) \in \pi_1(X, x) \rtimes \Gamma_k$. The first part shows that the only thing to do is to establish injectivity for the map $\text{Hom}_{k-gr}(\pi_1(X_{k_s}, x), G)/G(k) \rightarrow H^1(\pi_1(X, x), G(k_s))$. We are given two k -homomorphisms $\phi, \psi : \pi_1(X, x), G(k_s))$ having same image in $H^1(\pi_1(X, x), G(k_s))$. This means that there exists $g \in G(k_s)$ such that $\tilde{\psi}(\tau, \sigma) = g^{-1} \tilde{\phi}(\tau, \sigma) \sigma g$ for all $(\tau, \sigma) \in \pi_1(X, x) \rtimes \Gamma_k$ (observe that τ acts trivially on $G(k_s)$). Since $1 = \tilde{\psi}(1, \sigma) = g^{-1} \tilde{\phi}(1, \sigma) \sigma g = g^{-1} \sigma g$ we obtain that $g \in G(k)$. Thus ϕ and ψ are $G(k)$ -conjugated.

(3) By considering the fibers of the surjective map $\text{Res}_x : H^1(\pi_1(X, x), G(k_s)) \rightarrow H^1(k, G)$, we have a decomposition

$$H^1(\pi_1(X, x), G(k_s)) = \bigsqcup_{[z] \in H^1(k, G)} \text{Res}_x^{-1}([z]).$$

Assertion (2) provides a bijection $\text{Hom}_{k-gr}(\pi_1(X_{k_s}, x), G)/G(k) \xrightarrow{\sim} \text{Res}_x^{-1}([1])$ and the usual twisting argument provides a bijection $\text{Hom}_{k-gr}(\pi_1(X_{k_s}, x), {}_zG)/{}_zG(k) \xrightarrow{\sim} \text{Res}_x^{-1}([z])$ for each $[z] \in H^1(k, G)$. The above decomposition provides then the desired decomposition. \square

We have of course analogous statements for the other kinds of loop torsors.

Remark 3.3. Loop torsors which are not tame are called wild. An important example is the following. Assuming that $p > 1$, we consider the Artin-Schreier cover $f : \mathbb{G}_{a,k} \rightarrow \mathbb{G}_{a,k}$, $x \mapsto x^p - x$. This is a cyclic Galois cover of order p which is widely ramified at infinity. If $g \in G(k)$ is an element of order p , it defines a loop cocycle for G and this cover. This construction suggested by Serre is a key ingredient to the study of unipotent elements of semisimple groups, see [20].

3.3. Extensions of reductive groups. We are mostly interested in k -groups of the next shape. Let $1 \rightarrow G \rightarrow \tilde{G} \rightarrow J \rightarrow 1$ be an exact sequence of locally algebraic groups with $G = (\tilde{G})^0$ reductive and J is twisted constant. It follows that J is étale, that \tilde{G} is smooth and that J is the group of connected components of \tilde{G} [14, II.5.1.8]. We have seen that the \tilde{G} -sheaf torsors are representable and also that the inner twist of \tilde{G} by such a torsor is representable by a locally algebraic group which is of the same shape than \tilde{G} (see §2.5). We observe that $J(k_s) = J(X^{sc})$ so that

$$(3.4) \quad H^1(\pi_1(X, x), J(k_s)) = H^1(\pi_1(X, x), J(X^{sc})) = \ker(H^1(X, J) \rightarrow H^1(X^{sc}, J))$$

in view of [21, cor. 2.9.2]. In other words, this kernel consists of loop torsors.

Lemma 3.4. *Let ϕ, ϕ' be two loop cocycles with value in $\tilde{G}(k_s)$ having same image in $H^1(X, J)$. Then there exists $\tilde{g} \in \tilde{G}(k_s)$ such that ϕ and $\sigma \mapsto \tilde{g}^{-1} \phi' \sigma(\tilde{g})$ have same image in $Z^1(\pi_1(X, x), J(k_s))$.*

Proof. According to the fact (3.4), the loop cocycles ϕ, ϕ' have same image in $Z^1(\pi_1(X, x), J(k_s))$. Since $\tilde{G}(k_s)$ maps onto $J(k_s)$, it follows that there exists $\tilde{g} \in \tilde{G}(k_s)$ such that ϕ and $\sigma \mapsto \tilde{g}^{-1} \phi' \sigma(\tilde{g})$ have same image in $Z^1(\pi_1(X, x), J(k_s))$. \square

We say that a loop cocycle $\phi : \pi_1(X, x) \rightarrow \tilde{G}(k_s)$ is *reducible* if the k -homomorphism $\phi^{geo} : \pi_1(X_{k_s}, x) \rightarrow {}_{\phi^{ar}}\tilde{G}$ is reducible, that is, normalizes a pair (P, L) where P is a proper parabolic k -subgroup of ${}_{\phi^{ar}}\tilde{G}$ and L a Levi subgroup of P .

Lemma 3.5. *Let $\phi \in Z^1(\pi_1(X, x), G(k_s))$ be a purely geometric loop cocycle. Let (P, L) be a pair normalized by ϕ^{geo} where P is a k -parabolic subgroup of G and L is a Levi k -subgroup of P . We assume that (P, L) is minimal for this property (with respect to the inclusion). Then the loop cocycle ϕ takes value in $N_{\tilde{G}}(P, L)(k_s)$ and it is irreducible seen as loop cocycle for $N_{\tilde{G}}(P, L)$.*

Proof. We put $\tilde{L} = N_{\tilde{G}}(P, L)$. The assumption implies that the geometric loop cocycle ϕ takes value in $\tilde{L}(k_s)$. We assume that the

ψ of ϕ in $Z^1(\pi_1(X, x), \tilde{L}(k_s))$ is reducible, that is, there exists a pair (Q, M) normalized by $\psi^{geo} = \phi^{geo}$ such that Q is proper k -parabolic subgroup of $L = (\tilde{L})^0$ and M a

Levi subgroup of Q . We have a Levi decomposition $P = U \rtimes L$ and remind the reader that $P' = U \rtimes Q$ is a k -parabolic subgroup of G satisfying $P' \subsetneq P$ [3, prop. 4.4.c]. Also MM is a Levi subgroup of P' normalized by ϕ^{geo} contradicting the minimality of (P, L) . \square

We say that a loop cocycle $\phi : \pi_1(X, x) \rightarrow \tilde{G}(k_s)$ is *isotropic* if the k -homomorphism $\phi^{geo} : \pi_1(X_{k_s}, x) \rightarrow {}_{\phi^{ar}}\tilde{G}$ is isotropic, that is, centralizes a non trivial k -split subtorus of ${}_{\phi^{ar}}G$. We record now the formal following consequence of Corollary 2.20.

Corollary 3.6. *The following are equivalent:*

- (i) ϕ is isotropic;
- (ii) ϕ is reducible or the torus $({}_{\phi^{ar}}C^{\phi^{geo}})^0$ is isotropic.

4. LOOP TORSORS ON LAURENT POLYNOMIALS

4.1. Basic tame étale covers. Let $n \geq 1$ be an integer. We denote by $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ the ring of Laurent polynomial and by K_n its fraction field. For $m \geq 1$ prime to the characteristic exponent of k , we put $R_{n,m} = k[t_1^{\pm \frac{1}{m}}, \dots, t_n^{\pm \frac{1}{m}}]$.

We take $1 \in \mathbb{G}_m^n(k)$ as base point. An example of tame cover of R_n is $R_{n,m} \otimes_k l = l[t_1^{\pm \frac{1}{m}}, \dots, t_n^{\pm \frac{1}{m}}]$ where l is a finite Galois field extension of k containing a primitive m -root of unity. Covers of this shape are called nice basic tame covers.

Lemma 4.1. (1) *The above basic tame cover is Galois and we have $\text{Gal}(R_{n,m} \otimes_k l / R_n) = \mu_m(l)^n \rtimes \text{Gal}(l/k)$.*

(2) *The inductive limit of the nice basic tame covers is the universal tame cover of R_n and $\pi_1(R_n, 1) = \mathbb{Z}'(1)^n \rtimes \text{Gal}(k_s/k)$.*

Proof. (1) The R_n -algebra $R_{n,m} \otimes_k l$ is connected, étale and free of rank $m^n[l : k]$ over R_n . The finite group $\Gamma = \mu_m(l)^n \rtimes \text{Gal}(l/k)$ acts on $R_{n,m} \otimes_k l$ by

$$((\zeta_1, \dots, \zeta_n)\sigma) \cdot (t_i^{1/m} \otimes x) = \zeta_i t_i^{1/m} \sigma(x).$$

According to [54, prop. 5.3.7], the $(R_{n,m} \otimes_k l)^\Gamma$ -algebra $R_{n,m} \otimes_k l$ is Galois of group Γ . Since $R_n = (R_{n,m} \otimes_k l)^\Gamma$, we obtained the wished statement.

(2) We consider the smooth compactification $X^c = (\mathbb{P}_k^1)^n$ of $X = (\mathbb{G}_m)^n$ and its boundary is a normal crossing divisor. In this case an étale connected cover $Y \rightarrow X$ is tamely ramified if the discrete valuations of $k(X)$ defined by the prime divisors of $X^c \setminus X$ ramify tamely in the extension $k(Y)/k(X)$ [35, Thm. 4.4]. Without loss of generality we can assume that k is separably closed and we have to prove that the map $\pi_1^t(R_n, 1) \rightarrow \mathbb{Z}'(1)^n$ (arising from the basic covers) is an isomorphism. The compactification \mathbb{P}_k^1 of $\mathbb{G}_{m,k}$ is obviously good in the sense that it is the complement a normal crossing divisor. According to Orgogozo's theorem [43, Thm. 5.1], we have a decomposition $\pi_1^t(R_n, 1) = \pi_1(R_1, 1)^n$. We are reduced then to the case of \mathbb{G}_m and to show that any connected Galois tame cover is a Kummer cover. Let $f : Y \rightarrow \mathbb{G}_m$

be a tamely ramified connected Galois cover of group G . Then the field extension $k(Y)/k(t)$ is tamely ramified at 0 and ∞ so that $k(Y) \otimes_{k(t)} k((t)) \cong k((t^{\frac{1}{a}}))^{G/G_0}$ and $k(Y) \otimes_{k(t)} k((t^{-1})) \cong k((t^{-\frac{1}{b}}))^{G/G_\infty}$ where G_0 (resp. G_∞) is the inertia group at 0 (resp. ∞) and a (resp. b) the ramification index at 0 (resp. ∞). By assumption a and b are prime to the characteristic exponent of k . We put $n = g.c.m.(a, b)$ and consider the Kummer cover $h_n : \mathbb{G}'_m \rightarrow \mathbb{G}_m, u \mapsto t^n$. Then $Y \times_{\mathbb{G}_m} \mathbb{G}'_m$ is a finite G -cover of \mathbb{G}'_m which is unramified at 0 and ∞ . Since the projective line is simply connected, it follows that $Y \times_{\mathbb{G}_m} \mathbb{G}'_m = \mathbb{G}'_m \times_k G_k$. It follows that f is dominated by h_n , so that f is a Kummer cover as well. \square

Remarks 4.2. (a) In the case $p > 1$, Katz-Gabber's correspondence provides many more étale covers of \mathbb{G}_m [34, thm. 1.4.1].

(b) Since $\pi^{p'-geo}(R_n, 1)$ is a quotient of $\pi^t(R_n, 1)$, Lemma 4.1.(2) shows that the map $\pi^t(R_n, 1) \rightarrow \pi^{p'-geo}(R_n, 1)$ is an isomorphism.

(c) We observe that for the tame basic covers have trivial Picard groups and so have the other tame covers by using inflation-restriction sequences. This is not true anymore for arbitrary finite étale covers if $p \geq 2$ even in the case k algebraically closed. According to [40, Thm. 4 .6], there are examples of connected Galois covers X (for suitable alternating groups) of the affine line \mathbb{A}_k^1 which are smooth affine curve whose smooth completion has positive genus. It follows that X has an infinite Picard group and so has the cover $X \times_{\mathbb{A}_k^1} \mathbb{G}_m$ of \mathbb{G}_m .

4.2. Fixed point statement. The following is a mild generalization of [27, Thm. 7.1] so that we let the reader to check that its proof can be readily adapted.

Theorem 4.3. *Let G be a k -group scheme locally of finite presentation acting on a projective k -scheme Z . Let ϕ be a tame-loop cocycle for G . Then $Y = (\phi_{ar} Z)^{\phi^{geo}}$ is a projective k -scheme and the following are equivalent:*

- (i) $Y(k) \neq \emptyset$;
- (ii) $Y(R_n) \neq \emptyset$;
- (iii) $(\phi Z)(R_n) \neq \emptyset$;
- (iv) $(\phi Z)(K_n) \neq \emptyset$;
- (v) $(\phi Z)(F_n) \neq \emptyset$.

Remarks 4.4. (a) Note that the projectivity of Z is used to insure that the twisted fppf R_n -sheaf ϕZ by Galois descent is representable according to [5, §6.2].

(b) If Z is proper smooth over k , the statement is still true if the various twists are understood in the category of flat sheaves.

Let $1 \rightarrow G \rightarrow \tilde{G} \rightarrow J \rightarrow 1$ be an exact sequence of k -groups as in §3.3.

Proposition 4.5. *Let $\phi : \pi^t(R_n, 1) \rightarrow \tilde{G}(k_s)$ be a tame-loop cocycle. Then the following are equivalent:*

- (i) ϕ is reducible;
- (ii) the R_n -reductive group scheme ${}_\phi G$ is reducible;
- (iii) $({}_\phi G)_{K_n}$ is reducible;
- (iv) $({}_\phi G)_{F_n}$ is reducible.

Proof. We apply Theorem 4.3 to the total variety of parabolic subgroups $Z = \text{Par}^+(G)$, see §2.8. It is projective and is equipped with a natural action of \tilde{G} . We have an isomorphism of R_n -schemes ${}_\phi \text{Par}^+(G) = \text{Par}^+({}_\phi G)$ so that the assertions (iii), (iv) and (v) of Theorem 4.3 corresponds respectively to the assertions (ii), (iii) and (iv) of Proposition 4.5; again reductibility in those cases is equivalent to the existence of a proper parabolic subgroup. It remains to deal with (i). We have $Y = ({}_{\phi^{ar}} \text{Par}^+(G))^{\phi^{geo}} = (\text{Par}^+({}_{\phi^{ar}} G))^{\phi^{geo}}$ so that $Y(k)$ is the set of proper parabolic k -subgroups of ${}_{\phi^{ar}} G$ which are normalized by ϕ^{geo} . According to Proposition 2.21, P admits a Levi subgroup L which is normalized by ϕ^{geo} . Thus ϕ is reducible. \square

4.3. Tame Galois cohomology. We define the tame Galois cohomology set by

$$H_{tame}^1(F_n, \tilde{G}) = \bigsqcup H^1(F_{n,m} \otimes_k l / F_n, \tilde{G}).$$

where m runs over the positive integers which are prime to the characteristic exponent of k and l runs over the (finite) Galois extensions of k .

4.4. Acyclicity. We extend the injectivity part of [27, Thm. 8.1] beyond the affine case and in characteristic free. Let $1 \rightarrow G \rightarrow \tilde{G} \rightarrow J \rightarrow 1$ be a sequence of smooth k -groups such that $G = (\tilde{G})^0$ is reductive and J is a twisted constant k -group scheme. The main result of this section is the following.

Theorem 4.6. *The map $H_{loop, tame}^1(R_n, \tilde{G}) \rightarrow H_{tame}^1(F_n, \tilde{G})$ is injective.*

We start by dealing with the following special case.

Lemma 4.7. *The map $H_{loop, tame}^1(R_n, J) \rightarrow H_{tame}^1(F_n, J)$ is bijective.*

Proof. For each nice basic tame cover $R_{n,m} \otimes_k l$ of R_n , we have isomorphisms

$$\begin{array}{ccc} H^1(\mu_m^n(l) \rtimes \text{Gal}(l/k), J(l)) & \xrightarrow{\sim} & H^1(R_{n,m} \otimes_k l / R_n, J) \\ \downarrow \wr & & \downarrow \\ H^1(\mu_m^n(l) \rtimes \text{Gal}(l/k), J(F_{n,m} \otimes_k l)) & \xrightarrow{\sim} & H^1(F_{n,m} \otimes_k l / F_n, J) \end{array}$$

so that the map $H^1(R_{n,m} \otimes_k l / R_n, J) \rightarrow H^1(F_{n,m} \otimes_k l / F_n, J)$ is an isomorphism. By taking the inductive limit on those covers [38] we get the wished statement. \square

The proof of Theorem 4.6 goes by steps and uses crucially the notion of reductibility and of isotropicity. By using Theorem 7.1 of the appendix, we extend verbatim [27, lemma 7.12].

Lemma 4.8. *If $[\phi], [\phi'] \in H^1(\pi_1^t(R_n, 1), \tilde{G}(k_s))$ have same image in $H^1(F_n, \tilde{G})$, then $[\phi^{ar}] = [\phi'^{ar}] \in H^1(k, \tilde{G})$. \square*

Proposition 4.9. *Let $\phi : \pi^t(R_n, 1) \rightarrow \tilde{G}(k_s)$ be a tame-loop cocycle. Then the following are equivalent:*

- (i) ϕ is isotropic;
- (ii) the R_n -reductive group scheme ${}_\phi G$ is isotropic;
- (iii) $({}_\phi G)_{K_n}$ is isotropic;
- (iv) $({}_\phi G)_{F_n}$ is isotropic.

Proof. The implications (i) \implies (ii) \implies (iii) \implies (iv) are obvious. Let us show the implication (iv) \implies (i) by induction on n , the case $n = 0$ being obvious. We reason by sake of contradiction and assume that ϕ is anisotropic. We can deal with a nice basic tame cover $R_{n,m} \otimes_k l$ of R_n such that G_l is split. We put $\Gamma = \mu_m(l)^n \rtimes \text{Gal}(l/k)$.

Without loss of generality we can assume that $\phi^{ar} = 1$, i.e. $C_G(\phi^{geo})^0$ is anisotropic. According to Corollary 3.6, ϕ is then irreducible and the torus $C_G(\phi^{geo})^0$ is anisotropic.

We want to establish that $({}_\phi G)_{F_n}$ is anisotropic. According to the Bruhat-Tits-Rousseau's theorem [7, 5.1.27] applied to the field $F_n = F_{n-1}((t_n))$, this rephrases to show that the extended Bruhat-Tits building $\mathcal{B}_e(({}_\phi G)_{F_n})$ consists in one point. According to the tamely ramified descent theorem [46, prop. 5.1.1] (see also [44]), we have

$$\mathcal{B}_e(({}_\phi G)_{F_n}) = \mathcal{B}_e(G_{F_{n,m} \otimes_k l})^{\Gamma_\phi}$$

where the fixed points are taken with respect to the twisted Galois action relatively to ϕ . The right handside contains the center c_e which is fixed so that we have to prove that $\{c_e\} = \mathcal{B}_e(G_{F_{n,m} \otimes_k l})^{\Gamma_\phi}$. In other words we have to prove that $\{c\} = \mathcal{B}(G_{F_{n,m} \otimes_k l})^{\Gamma_\phi}$ and that $0 = E^{\mu_m(l)^n \text{Gal}(l/k)}$ where $E = \widehat{C}_l \otimes_{\mathbb{Z}} \mathbb{R}$.

The toral part. We have

$$0 = (\widehat{C}^0)^{\Gamma_\phi} = \text{Hom}_{F_n}(\mathbb{G}_m, {}_\phi C_{F_n})$$

so that $E^{\Gamma_\phi} = (\widehat{C}^0)_{F_n} \otimes_{\mathbb{Z}} \mathbb{R} = 0$.

The semisimple part. According to Proposition 4.5, $({}_\phi G)_{F_n}$ is irreducible so that $\mathcal{B}({}_\phi G_{F_n}) \xrightarrow{\sim} \mathcal{B}(G_{F_{n,m} \otimes_k l})^{\Gamma_\phi}$ consists in the point c . \square

Remark 4.10. The former proof was not correct [27, cor. 7.2.(3)] since we implicitly used the new Corollary 3.6.

Proposition 4.11. [27, Thm. 7.9] *Let ϕ, ϕ' be purely geometrical tame-loop cocycles given by $\phi^{geo}, \phi'_{geo} : \mu_m^n \rightarrow \tilde{G}$. Assume that ϕ is anisotropic. Then the following are equivalent:*

- (i) ϕ^{geo} and ϕ'_{geo} are $\tilde{G}(k)$ -conjugated;
- (ii) $[\phi] = [\phi'] \in H^1(R_n, \tilde{G})$;
- (iii) $[\phi] = [\phi'] \in H^1(K_n, \tilde{G})$;
- (iv) $[\phi] = [\phi'] \in H^1(F_n, \tilde{G})$.

Proof. We prove the statement by induction on $n \geq 0$, the case $n = 0$ being obvious. The implications (i) \implies (ii) \implies (iii) \implies (iv) are obvious. Let us prove the implication (iv) \implies (i).

We work at finite level with a basic tame cover $R_{n,m} \otimes_k l$ of R_n such that G_l is split. Our assumption is that there exists $\tilde{g} \in \tilde{G}(F_{n,m} \otimes_k l)$ such that

$$(4.1) \quad \phi(\sigma) = \tilde{g}^{-1} \phi'(\sigma) \sigma(\tilde{g}).$$

for all $\sigma \in \Gamma = \text{Gal}(R_{n,m} \otimes_k l / R_n) = \mu_m(l)^n \rtimes \text{Gal}(l/k)$. The key step is the following.

Claim 4.4.1. $\tilde{g} \in \tilde{G}(F_{n-1,m} \otimes_k k[[t_n^{\frac{1}{m}}]])$.

We consider the extended Bruhat-Tits building $\mathcal{B}_{e,n} = \mathcal{B}_e(G_{F_{n,m,l}})$. It comes with an action of $\tilde{G}(F_{n,m,l}) \rtimes \text{Gal}(F_{n,m} \otimes_k l / F_n)$ and we denote by c the hyperspecial point (which is sometimes called the center of the building) which is the unique point fixed by $(DG)^{sc}(F_{n-1,m} \otimes_k l[[t_n]])$ [6, 9.1.19.(c)]. According to Lemma 8.2, $\tilde{G}(F_{n-1,m} \otimes_k l[[t_n^{\frac{1}{m}}]])$ is the stabilizer of c for the standard action of $\tilde{G}(F_{n,m} \otimes_k l)$ on $\mathcal{B}_{e,n}$ so that we have to prove that $\tilde{g}.c = c$. Denoting by \star the twisted action of Γ on $\mathcal{B}_{e,n}$, we have that

$$(4.2) \quad (\mathcal{B}_{e,n})^{\Gamma_\phi} = \{c\}.$$

Indeed ${}_\phi G_{F_n}$ is anisotropic according to the Bruhat-Tits-Rousseau's theorem [7, 5.1.27]. Since c belongs to $(\mathcal{B}_{e,n})^{\Gamma_\phi}$, it follows that $\mathcal{B}_{e,n}^{\Gamma_\phi} = \{c\}$. For each $\sigma \in \Gamma$, we have

$$\begin{aligned} \sigma \star (\tilde{g}.c) &= \phi(\sigma) \sigma(\tilde{g}).\sigma(c) \\ &= \phi(\sigma) \sigma(\tilde{g}).c \quad [c \text{ is invariant under } \tilde{\Gamma}_{m,n}] \\ &= \tilde{g}.\phi'(\sigma) c \quad [\text{relation 4.1}] \\ &= \tilde{g}.c \quad [\phi(\gamma) \in \tilde{G}(k_s)]. \end{aligned}$$

Thus $\tilde{g}.c = c$ so that $\tilde{g} \in \tilde{G}(F_{n-1,m} \otimes_k l[[t_n^{\frac{1}{m}}]])$. Since ϕ and ϕ' are purely geometrical, the equation (4.1) implies that \tilde{g} is $\text{Gal}(l/k)$ -invariant. Thus $\tilde{g} \in \tilde{G}(F_{n-1,m} \otimes_k k[[t_n^{\frac{1}{m}}]])$ which establishes the Claim.

We can then specialize the relation (4.1) with respect to the Γ -equivariant map $\tilde{G}(F_{n-1,m} \otimes_k k[[t_n^{\frac{1}{n}}]]) \rightarrow \tilde{G}(F_{n-1,m})$ and obtain

$$(4.3) \quad \phi(\sigma) = \tilde{g}_n^{-1} \phi'(\sigma) \sigma(\tilde{g}_n) \quad (\sigma \in \Gamma)$$

with $\tilde{g}_n \in \tilde{G}(F_{n-1,m})$. We consider the transporter

$$X = \{x \in \tilde{G} \mid \phi_n^{geo} = x^{-1} \phi_n'^{geo} x\}$$

where ϕ_n^{geo} (resp. $\phi_n'^{geo}$) stands for the restriction of ϕ^{geo} (resp. ϕ'^{geo}) to the last factor μ_n . Equation 4.3 tells us that $X(F_{n-1,m}) \neq \emptyset$ so that X is not empty. It follows that X is a $\tilde{G}^{\phi_n^{geo}}$ -torsor and Theorem 7.1 yields that $X(k) \neq \emptyset$. Without loss of generality we can then assume that $\phi_n^{geo} = \phi_n'^{geo}$.

We put $\tilde{H} = \tilde{G}^{\phi_n^{geo}}$ and $H = (G^{\phi_n^{geo}})^0$. Then H is reductive and $H = (\tilde{H})^0$. The restriction ψ (resp. ϕ') of ϕ to $\mu_n(l)^{r-1} \rtimes \text{Gal}(l/k)$ of ϕ (resp. ϕ') take values in \tilde{H} , we see them as tame-loop cocycles for \tilde{H} .

Claim 4.4.2. *ψ is anisotropic as tame-loop cocycle with value in \tilde{H} .*

The F_n -group ${}_{\psi}H \times_{F_n}$ is a subgroup of ${}_{\phi}G \cong_{\phi} G$ which is anisotropic according to Proposition 4.9. A fortiori ${}_{\psi}H$ is F_{n-1} -anisotropic so that the same statement shows that ψ is anisotropic as tame-loop cocycle with value in \tilde{H} . The Claim is established.

The equation (4.3) applied to the element $(1, \dots, 1, \zeta_n)$ shows that $\tilde{g}_n \in \tilde{H}(F_{n-1,m})$ so that $[\psi] = [\psi'] \in H^1(F_{n-1}, \tilde{H})$, the induction hypothesis shows that $\psi(\sigma) = \tilde{h} \psi'(\sigma) \sigma(\tilde{h})$ for all $\sigma \in \mu_n(l)^{r-1} \rtimes \text{Gal}(l/k)$. Thus $\psi(\sigma) = \tilde{h} \psi'(\sigma) \sigma(\tilde{h})$ for all $\sigma \in \mu_n(l)^r \rtimes \text{Gal}(l/k)$. \square

Corollary 4.12. [27, Thm. 7.8] *Let ϕ, ϕ' be tame-loop cocycles with values in $\tilde{G}(k_s)$. Assume that ϕ is anisotropic. Then the following are equivalent:*

- (i) *There exists $\tilde{g} \in \tilde{G}(k_s)$ such that $\phi'(\sigma) = \tilde{g}^{-1} \phi(\sigma) \sigma(\tilde{g})$ for all $\sigma \in \pi_1^t(R_n, 1)$.*
- (ii) $[\phi] = [\phi'] \in H^1(R_n, \tilde{G})$;
- (iii) $[\phi] = [\phi'] \in H^1(K_n, \tilde{G})$;
- (iv) $[\phi] = [\phi'] \in H^1(F_n, \tilde{G})$.

Furthermore under these assumptions we have $[\phi^{ar}] = [\phi'^{ar}] \in H^1(k, \tilde{G})$.

Proof. Once again the implications (i) \implies (ii) \implies (iii) \implies (iv) are obvious. We prove now (iv) \implies (i). Lemma 4.8 shows that $[\phi^{ar}] = [\phi'^{ar}] \in H^1(k, \tilde{G})$ which is the last fact of the statement. Without loss of generality we can assume that $\phi^{ar} = \phi'^{ar}$. Up to twist \tilde{G} by ϕ^{ar} , the usual torsion argument boils down to the case $\phi^{ar} = 1$. This case is handled by Proposition 4.11, so we are done. \square

We proceed now to the proof of Theorem 4.6.

Proof. We deal with the injectivity of the map $H_{loop}^1(R_n, \tilde{G}) \rightarrow H^1(F_n, \tilde{G})$, that is to show that the fiber at the class $[\phi]$ of any loop cocycle ϕ consists in one element. If ϕ is an anisotropic tame-loop cocycle, Corollary 4.12 shows that the fiber at $[\phi]$ of $H_{loop}^1(R_n, \tilde{G}) \rightarrow H^1(F_n, \tilde{G})$ is $\{[\phi]\}$.

A first generalization is the irreducible case. We assume then that the tame-loop cocycle ϕ is irreducible. One again we can assume that $\phi^{ar} = 1$. Let ϕ' be another tame-loop cocycle such that $[\phi'] = [\phi] \in H^1(F_n, \tilde{G})$. According to Lemma 4.7 the map $H^1(R_n^{tsc}/R_n, J) \rightarrow H_{tame}^1(F_n, J)$ is bijective. It follows that ϕ and ϕ' have same image in $H^1(R_n, J)$. Lemma 3.4 permits to assume without lost of generality that ϕ and ϕ' have same image in $Z^1(\pi_1^t(R_n), J(k_s))$. We denote by J_1 the image of $\phi^{geo} : \widehat{\mathbb{Z}}'(1)^n \rightarrow J$, this is a finite smooth algebraic k -group of multiplicative type. We put $\tilde{G}_1 = G_1 \times_J J_1$, by construction ϕ and ϕ' have value in $\tilde{G}_1(k_s)$. To avoid any confusion we denote them by ϕ_1 and ϕ'_1 . We consider the commutative diagram of pointed sets

$$\begin{array}{ccccc} (J/J_1)(k_s)^{\pi_1^t(R_n, 1)_{\phi_1}} & \longrightarrow & H^1(\pi_1^t(R_n, 1), \phi_1 G_1(k_s)) & \longrightarrow & H^1(\pi_1^t(R_n, 1), \phi_1 \tilde{G}(k_s)) \\ \downarrow = & & \downarrow & & \downarrow \\ (\phi_1(J/J_1))(F_n) & \longrightarrow & H^1(F_n, \phi_1 G_1) & \longrightarrow & H^1(F_n, \phi_1 \tilde{G}). \end{array}$$

The second one is associated to the exact sequence $1 \rightarrow \phi_1(\tilde{G}_1) \rightarrow \phi_1(\tilde{G}) \rightarrow \phi_1(J/J_1) \rightarrow 1$ of F_n -spaces and the first one is associated to the exact sequence of $\pi_1^t(R_n, 1)$ -sets $1 \rightarrow \phi_1(\tilde{G}_1(k_s)) \rightarrow \phi_1(\tilde{G}(k_s)) \rightarrow \phi_1(J/J_1)(k_s) \rightarrow 1$. By diagram chase involving the torsion bijection $H^1(\pi_1^t(R_n, 1), \phi_1 \tilde{G}_1(k_s)) \xrightarrow{\sim} H^1(\pi_1^t(R_n, 1), \tilde{G}_1(k_s))$, we see that we can arrange ϕ'_1 in order that ϕ'_1 has same image than ϕ_1 in $H^1(F_n, \tilde{G}_1)$. We can work then with then \tilde{G}_1 which is generated by G and the image of ϕ_1^{geo} .

Since the k -torus C is central in G , the k -subgroup $C^{\phi^{geo}}$ is central in \tilde{G}_1 . We denote by C_0 the maximal split k -subtorus of $C^{\phi^{geo}}$ and consider the central exact sequence of algebraic k -groups

$$1 \rightarrow C_0 \rightarrow \tilde{G}_1 \rightarrow \tilde{G}_1/C_0 \rightarrow 1.$$

The gain is that the image of ϕ_1 in $Z^1(\pi_1^t(R_n, 1), (\tilde{G}_1/C_0)(k_s))$ is an anisotropic tame-loop cocycle by applying the criterion of Corollary 3.6. Since $H^1(R_n, C_0) = H^1(F_n, C_0) = 1$, we obtain the following commutative diagram

$$\begin{array}{ccc} H^1(R_n, \tilde{G}_1) & \hookrightarrow & H^1(R_n, \tilde{G}_1/C_0) \\ \downarrow & & \downarrow \\ H^1(F_n, \tilde{G}_1) & \hookrightarrow & H^1(F_n, \tilde{G}_1/C_0) \end{array}$$

where the horizontal maps are injections [28, III.3.4.5.(iv)]. Corollary 4.12 shows that $[\phi_1]$ and $[\phi'_1]$ have same image in $H^1(R_n, \tilde{G}_1/C_0)$. The diagram shows that $[\phi_1] = [\phi'_1] \in H^1(R_n, \tilde{G}_1)$. By pushing in $H^1(R_n, \tilde{G})$ we get that $[\phi] = [\phi'] \in H^1(R_n, \tilde{G})$ as desired.

We deal now with the general case. The above reduction (with \tilde{G}_1) permits to assume that J is finite étale so that \tilde{G} is affine and also that ϕ is purely geometric. Let (P, L) be a pair normalized by ϕ^{geo} where P is k -parabolic subgroup of G , L is a k -Levi subgroup of P and which is minimal for this property (with respect to the inclusion). Then the tame-loop cocycle ϕ takes value in $N_{\tilde{G}}(P, L)(k_s)$. We have an exact sequence (2.2)

$$1 \rightarrow L \rightarrow N_{\tilde{G}}(P, L) \rightarrow J_P \rightarrow 1$$

of smooth affine k -groups where $J_P \subset J$ is a clopen k -subgroup and J_P is a twisted constant (Lemma 2.18). We denote by ψ the image of ϕ in $Z^1(\pi_1^t(R_n, \cdot), N_{\tilde{G}}(P, L)(k_s))$. Lemma 3.5.(2) states that ψ is irreducible.

We deal now with the tame-loop cocycle ϕ' having same image in $H^1(F_n, \tilde{G})$ as ϕ . Lemma 4.8 implies that ϕ' is purely geometrical. We consider the projective k -variety $X = \tilde{G}/N_{\tilde{G}}(P)$ (Lemma 2.19). Theorem 4.3 shows that $(\phi X)(F_n) \neq \emptyset$ so that $(\phi' X)(F_n) \neq \emptyset$. The same result shows that $X^{\phi^{geo}}(k)$ are not empty. We pick $x' \in X^{\phi^{geo}}(k)$ and choose $x \in X^{\phi^{geo}}(k)$ such that $G_x = P$. observe and choose $\tilde{g} \in \tilde{G}(k_s)$ such that $x' = \tilde{g}.x$. For $\sigma \in \text{Gal}(k_s/k)$, we have that $x' = \sigma(\tilde{g}).x$ so that $\sigma \rightarrow n_\sigma = \tilde{g}^{-1}\sigma(\tilde{g})$ is a 1-cocycle with value in $\tilde{G}_x(k_s) = N_{\tilde{G}}(P)(k_s)$. For $\sigma \in \text{Gal}(k_s/k)$, we have $\phi'(\sigma).x' = x'$ so that $\phi'(\sigma).\tilde{g}.x = \tilde{g}.x$. It follows that $\tilde{g}^{-1}\phi'(\sigma)\tilde{g}.x = x$. Since $n_\sigma x = x$ it follows that $\phi''(\sigma) = \tilde{g}^{-1}\phi'(\sigma)$ fixes x . Up to replace ϕ' by ϕ'' we can then assume that ϕ' takes value in $N_{\tilde{G}}(P)(k_s)$. Proposition 2.21 tells us that P admits a Levi subgroup L' normalized by ϕ'^{geo} . Then $L' = {}^g L$ for some $g \in P(k)$. Up to replace ϕ' by $g^{-1}\phi'$, we can then assume that ϕ' has value in $N_{\tilde{G}}(P, L)$.

We note that (P, L) is minimal for this property (otherwise it will not be minimal for ϕ). The above argument tells us that the image ψ' of ϕ' in $Z^1(\pi_1^t(R_n, \cdot), N_{\tilde{G}}(P, L)(k_s))$ is irreducible. We consider now the commutative diagram

$$\begin{array}{ccc} H^1(R_n, \tilde{G}) & \longrightarrow & H^1(F_n, \tilde{G}) \\ \uparrow & & \parallel \\ H^1(R_n, N_{\tilde{G}}(P, L))_{irr} & \longrightarrow & H^1(F_n, N_{\tilde{G}}(P, L))_{irr} \end{array} .$$

The second horizontal map is well-defined in view of Proposition 4.5. The right vertical map is injective (*ibid*, lemme 4.2.1.(2)). We have seen that ϕ, ϕ' define tame-loop elements of $H^1(R_n, N_{\tilde{G}}(P, L))_{irr}$ which give then the same image in $H^1(F_n, N_{\tilde{G}}(P))$. Taking into account the already handled irreducible case, diagram chasing enables

us to conclude that $[\phi] = \phi' \in H^1(R_n, N_{\tilde{G}}(P, L))$. Thus $[\phi] = [\phi'] \in H^1(R_n, \tilde{G})$ as desired. \square

Remark 4.13. The reduction involving \tilde{G}_1 is unfortunately missing in the original proof, i.e. [27, Thm. 8.1].

5. TAME FUNDAMENTAL GROUP À LA GROTHENDIECK-MURRE

5.1. Abhyankar's lemma. Let $X = \text{Spec}(A)$ be a regular local scheme (not assumed henselian at this stage). Let k be the residue field of A and $p \geq 1$ be its characteristic exponent. We put $\hat{\mathbb{Z}}' = \prod_{l \neq p} \mathbb{Z}_l$. Let K be the fraction field of A , and let K_s be a separable closure of K . It determines a base point $\xi : \text{Spec}(K) \rightarrow X$ so that we can deal with the Grothendieck fundamental group $\Pi_1(X, \xi)$ [49].

We assume that A is of dimension $r \geq 1$. Let (f_1, \dots, f_r) be a regular sequence of A and consider the divisor $D = \sum D_i = \sum \text{div}(f_i)$, it has strict normal crossings. We put $U = X \setminus D = \text{Spec}(A_D)$.

We recall that a finite étale cover $V \rightarrow U$ is *tamely ramified* with respect to D if the associated étale K -algebra $L = L_1 \times \dots \times L_a$ is tamely ramified at the D'_i 's, that is, for each i , there exists j_i such that for the Galois closure \tilde{L}_{j_i}/K of L_{j_i}/K , the inertia group associated to v_{D_i} has order prime to p [49, XIII.2.0].

Grothendieck and Murre defined the tame (*modéré* in French) fundamental group $\Pi_1^D(U, \xi)$ with respect to $U \subset X$ as defined in [49, XIII.2.1.3] and [30, §2]. This is a profinite quotient of $\Pi_1(U, \xi)$ whose quotients by open subgroups provides finite connected tame covers of U .

Let $V \rightarrow U$ be a finite étale tame cover. In this case Abhyankar's lemma states that there exists a flat Kummer cover $X' = \text{Spec}(A') \rightarrow X$ where

$$A' = A[T_1, \dots, T_r]/(T_1^{n_1} - f_1, \dots, T_r^{n_r} - f_r)$$

and the n_i 's are coprime to p such that $V' = V \times_X X' \rightarrow X'$ extends uniquely to a finite étale cover $Y' \rightarrow X'$ [49, XIII.5.2].

Lemma 5.1. *Let $V \rightarrow U$ be a finite étale cover which is tame. Then $\text{Pic}(V) = 0$.*

Proof. We use the same notation as above. We know that X' is regular [49, XIII.5.1] so a fortiori locally factorial. It follows that the restriction maps $\text{Pic}(X') \rightarrow \text{Pic}(V') \rightarrow \text{Pic}(V)$ are surjective [15, 21.6.11]. Since A' is finite over the local ring A , it is semilocal so that $\text{Pic}(A') = \text{Pic}(X') = 0$. Thus $\text{Pic}(V) = 0$ as desired. \square

From now on we assume that A is henselian. According to [15, 18.5.10], the finite A -ring A' is a finite product of henselian local rings. We observe that $A' \otimes_A k = k[T_1, \dots, T_r]/(T_1^{n_1}, \dots, T_r^{n_r})$ is a local Artinian algebra so that A' is connected. It follows that A' is a henselian local ring. Its maximal ideal is $\mathfrak{m}' = \mathfrak{m} \otimes_A A' + \langle T_1, \dots, T_r \rangle$ so that $A'/\mathfrak{m}' = k$. Since there is an equivalence of categories between finite étale covers of A (resp. A') and étale k -algebras [15, 18.5.15], the base change from A to

A' provides an equivalence of categories between the category of finite étale covers of A and that of A' .

It follows that the above finite étale cover $Y' \rightarrow X'$ descends uniquely to a finite étale cover $\tilde{f} : \tilde{Y} \rightarrow X$. From now on, we assume that V is furthermore connected, it implies that

$$H^0(V, \mathcal{O}_V) = B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r)$$

where B is a finite connected étale cover of A . It follows that $V \rightarrow U$ is a quotient of a Galois cover of the shape

$$B_n = \left(B[T_1, \dots, T_r]/(T_1^n - f_1, \dots, T_r^n - f_r) \right) \otimes_A A_D$$

where B is Galois cover of A containing a primitive n -th root of unity. We have

$$\text{Gal}(B_n/A_D) = \left(\prod_{i=1}^r \mu_n(B) \right) \rtimes \text{Gal}(B/A).$$

Passing to the limit we obtain an isomorphism

$$\pi_1^t(U, \xi) \cong \left(\prod_{i=1}^r \widehat{\mathbb{Z}}'(1) \right) \rtimes \pi_1(X, \xi).$$

We denote by $f : U^{tsc} \rightarrow U$ the profinite étale cover associated to the quotient $\pi_1^t(U, \xi)$ of $\pi_1(U, \xi)$. According to [30, thm. 2.4.2], it is the universal tamely ramified cover of U . It is a localization of the inductive limit \tilde{B}' of the B'_n . On the other hand we consider the inductive limit \tilde{B} of the B 's and observe that \tilde{B}' is a \tilde{B} -ring.

Remark 5.1.1. The A -algebra B_n is isomorphic to $B[T_1^{\pm 1}, \dots, T_r^{\pm 1}]/(T_1^n - f_1, \dots, T_r^n - f_r)$ which is the form given in [22, §2.10].

5.2. Blow-up. We follow a blowing-up construction arising from [15, lemma 15.1.1.6]. We denote by \widehat{X} the blow-up of $X = \text{Spec}(A)$ at its closed point, this is a regular scheme [37, §8.1, Thm. 1.19] and the exceptional divisor $E \subset \widehat{X}$ is a Cartier divisor isomorphic to \mathbb{P}_k^{r-1} . We denote by $R = \mathcal{O}_{\widehat{X}, \eta}$ the local ring at the generic point η of E . The ring R is a DVR of fraction field K and of residue field $F = k(E) = k(t_1, \dots, t_{r-1})$ where t_i is the image of $\frac{t_i}{f_r} \in R$ by the specialization map. We denote by $v : K^\times \rightarrow \mathbb{Z}$ the discrete valuation associated to R . We put $\underline{A}_D = k[t_1^{\pm 1}, \dots, t_{r-1}^{\pm 1}]$.

We deal now with a Galois extension B_n of A_D as above. Since B is a connected finite étale cover of A , B is regular and local; it is furthermore henselian [15, 18.5.10]. We denote by L the fraction field of B and by L_n that of B_n . We have $[L_n : L] = n^r$ and want to extend the valuation v to L and to L_n .

We denote by $l = B/\mathfrak{m}_B$ the residue field of B , this is a finite Galois field extension of k . Also (T_1, \dots, T_r) is a system of parameters for B . We denote by $w : L^\times \rightarrow \mathbb{Z}$ the discrete valuation associated to the exceptional divisor of the blow-up of $\text{Spec}(B)$ at

its closed point. Then w extends v and L_w/K_v is an unramified extension of degree $[L : K]$ and of residual extension $F_l = l(t_1, \dots, t_{r-1})/k(t_1, \dots, t_{r-1})$.

On the other hand we denote by $w_n : L_n^\times \rightarrow \mathbb{Z}$ the discrete valuation associated to the exceptional divisor of the blow-up of $\text{Spec}(B_n)$ at its closed point. We put $l_n = B_n/\mathfrak{m}_{B_n}$, we have $l = l_n$. The valuation $\frac{w_n}{n}$ on L_n extends w and its residual extension is $F_{l,n} = l(t_1^{1/n}, \dots, t_{r-1}^{1/n})/k(t_1, \dots, t_{r-1})$ so that $[F_{l,n} : F_l] = n^{r-1}$. Furthermore the ramification index e_n of L_n/L is $\geq n$. Since $n^r \leq e_n [F_{l,n} : F_l] \leq [L_n : K] = n^r$ (where the last inequality is [4, §VI.3, prop. 2]) it follows that $e_n = n$. The same statement shows that the map $L_w \otimes_L L_n \rightarrow L_{w_n}$ is an isomorphism. To summarize L_{w_n}/L_w is tamely ramified of ramification index n and of degree n^r . Altogether we have $L_{w_n} = L_w \otimes_K L_n$ so that L_{w_n} is Galois over K_v of group $\prod_{i=1}^r \mu_n(B) \rtimes \text{Gal}(B/A) = \prod_{i=1}^r \mu_n(l) \rtimes \text{Gal}(l/k)$.

We denote by Δ the diagonal embedding $\mu_n(l) \subset \prod_{i=1}^r \mu_n(l)$. We put $L_{w_n}^\Delta = L_n^{\Delta(\mu_n(B))}$. Since t_r is a uniformizing parameter of K_v and since $\Delta(\zeta) \cdot t_r = \zeta \cdot t_r$ for each $\zeta \in \mu_n(B)$, it follows that $(L_{w_n})^\Delta$ is the maximal unramified extension of L_{w_n}/K_v . As in (??), we have a factorization

$$(5.1) \quad \begin{array}{ccc} O_{w_n} & \longrightarrow & l(t_1^{\frac{1}{n}}, \dots, t_{r-1}^{\frac{1}{n}}) \\ \parallel & & \parallel \\ B_n \cap O_{w_n} & \longrightarrow & l[t_1^{\pm \frac{1}{n}}, \dots, t_{r-1}^{\pm \frac{1}{n}}]. \end{array}$$

We put $\underline{B}_n = l[t_1^{\pm \frac{1}{n}}, \dots, t_{r-1}^{\pm \frac{1}{n}}]$. An important point is the equivariance of the above diagram for the action of $\text{Gal}(B_n/A_D)$. In particular it provides an exact sequence

$$(5.2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mu_n(B) & \xrightarrow{\Delta} & \mu_n(B)^r \rtimes \text{Gal}(B/A) & \longrightarrow & \left(\mu_n(B)^r / \mu_n(B) \right) \rtimes \text{Gal}(B/A) \longrightarrow 1 \\ & & \uparrow = & & \uparrow \wr & & \uparrow \wr \\ 1 & \longrightarrow & \mu_n(B) & \longrightarrow & \text{Gal}(B_n/A_D) & \longrightarrow & \text{Gal}(\underline{B}_n/A_D) \longrightarrow 1. \end{array}$$

5.3. Tame loop cocycles and tame loop torsors. Let G be an X -group scheme locally of finite presentation. A tame-loop cocycle is an element of $Z^1(\pi_1^t(U), G(\tilde{B}))$ and it defines a Galois cocycle in $Z^1(\pi_1^t(U), G(U^{tsc}))$. We denote by $Z_{loop}^1(\pi_1^t(U), G(U^{tsc}))$ the image of the map $Z^1(\pi_1^t(U), G(\tilde{B})) \rightarrow Z^1(\pi_1^t(U), G(U^{tsc}))$ and by $H_{loop}^1(\pi_1^t(U), G)$ the image of the map

$$Z^1(\pi_1^t(U), G(\tilde{B})) \rightarrow H^1(\pi_1^t(U), G(U^{tsc})) \rightarrow H^1(U, G).$$

We say that a G -torsor E over U (resp. an fppf sheaf G -torsor) is a tame-loop torsor if its class belongs to $H_{tame}^1(U, G) \subset H^1(U, G)$.

A given class $\gamma \in H_{tame}^1(U, G)$ is represented by a 1-cocycle $\phi : \text{Gal}(B_n/A_D) \rightarrow G(B)$ for some cover B_n/A as above. Its restriction $\phi^{ar} : \text{Gal}(B'_n/A) \rightarrow G(B'_n)$ to the subgroup $\text{Gal}(B'_n/A)$ of $\text{Gal}(B_n/A_D)$ is called the ‘‘arithmetic part’’ and the other restriction $\phi^{geo} : \prod_{i=1}^r \mu_n(B) \rightarrow \mathfrak{G}(B)$ is called the geometric part. We observe in the sequel that ϕ^{geo} is a B -group homomorphism.

Indeed for $\sigma \in \text{Gal}(B/A)$ and $\tau \in \prod_{i=1}^r \mu_n(B)$ the computation of [27, page 16] shows that $\phi^{geo}(\sigma\tau\sigma^{-1}) = \phi^{ar}(\sigma) \sigma\phi(\tau) \phi^{ar}(\sigma)^{-1}$ so that ϕ^{geo} descends to a homomorphism of A -group schemes $\phi^{geo} : \mu_n^r \rightarrow \phi^{ar}G$. Altogether this provides a parameterization of tame-loop cocycles.

Lemma 5.2. (1) For B_n/A_D as above, the map $\phi \mapsto (\phi^{ar}, \phi^{geo})$ provides a bijection between $Z_{tame}^1(\text{Gal}(B_n/A_D), G(B))$ and the couples (z, η) where $z \in Z^1(\text{Gal}(B/A), G(B))$ and $\eta : \prod_{i=1}^r \mu_n \rightarrow {}_zG$ is an A -group homomorphism.

(2) The map $\phi \mapsto (\phi^{ar}, \phi^{geo})$ provides a bijection between $Z_{tame}^1(\pi^1(U, \xi)^t, G(\tilde{B}))$ and the couples (z, η) where $z \in Z^1(\pi^1(X, \xi), G(\tilde{B}))$ and $\eta : \prod_{i=1}^r \hat{\mathbb{Z}}' \rightarrow {}_zG$ is an A -group homomorphism.

(3) We have an exact sequence of pointed sets

$$1 \rightarrow \text{Hom}_{A-gr}(\mu_n^r, G)/G(A) \rightarrow H^1(\text{Gal}(B_n/A), G(B)) \xrightarrow{Res} H^1(\text{Gal}(B/A), G(B)) \rightarrow 1$$

and the first map is injective.

(4) We have a decomposition

$$H^1(\text{Gal}(B_n/A), G(B)) = \bigsqcup_{[z] \in H^1(A, G)} \text{Hom}_{A-gr}(\mu_n^r, {}_zG)/{}_zG(A).$$

Proof. This is similar with Lemma 3.2. □

We examine more closely the case of a finite étale X -group scheme \mathfrak{F} of constant degree d .

Lemma 5.3. (1) $\mathfrak{F}(\tilde{B}) = \mathfrak{F}(X^{tsc}) = \mathfrak{F}(U^{tsc})$.

(2) We assume that d is prime to p . We have $H_{tame}^1(U, \mathfrak{F}) = H^1(U, \mathfrak{F})$.

(3) We assume that d is prime to p . Let $f : \mathfrak{F} \rightarrow \mathfrak{H}$ be a homomorphism of A -group schemes (locally of finite type). Then $f_*\left(H^1(U, \mathfrak{F})\right) \subset H_{tame}^1(U, \mathfrak{H})$.

Proof. (1) We are given a cover B_n/A_D as above such that $\mathfrak{F}_{B_n} \cong \Gamma_{B_n}$ is finite constant. Since B and B_n are connected, the map $\mathfrak{F}(B) \rightarrow \mathfrak{F}(B_n)$ reads as the

identity $\Gamma \cong \mathfrak{F}(B) \rightarrow \mathfrak{F}(B_n) \cong \Gamma$ so is bijective. By passing to the limit we get $\mathfrak{F}(\tilde{B}) = \mathfrak{F}(U^{tsc})$.

(2) Let \mathfrak{E} be a \mathfrak{F} -torsor over U . This is a finite étale U -scheme. Since U is noetherian and connected, we have a decomposition $\mathfrak{E} = V_1 \sqcup \cdots \sqcup V_l$ where each V_i is a connected finite étale U -scheme of constant degree d_i . We have $d_1 + \cdots + d_l = d$ so that we can assume that d_1 is prime to p . We have then $\mathfrak{E}(V_1) \neq \emptyset$.

It follows that $f_1 : V_1 \rightarrow U$ is a finite étale cover so that there exists a factorization $U^{tsc} \rightarrow V_1 \xrightarrow{h} U$ of f so that $\mathfrak{E}(U^{tsc}) \neq \emptyset$. Therefore $[\mathfrak{E}]$ arises from $H^1(\pi_1^t(U, \xi), \mathfrak{F}(U^{tsc})) \subset H^1(U, \mathfrak{F})$. It follows that $H^1(\pi_1^t(U, \xi), \mathfrak{F}(U^{tsc})) \xrightarrow{\sim} H^1(U, \mathfrak{F})$. We use now (1) and obtain the desired bijection $H^1(\pi_1^t(U, \xi), \mathfrak{F}(B)) \xrightarrow{\sim} H^1(U, \mathfrak{F})$.

(3) This follows readily from (2). \square

5.4. Specializing purely geometric tame loop cocycles. Let $\phi : \text{Gal}(B_n/A) \rightarrow G(B)$ be a purely geometrical loop cocycle and consider the underlying A -homomorphism $\phi^{geo} : (\mu_{n,A})^r \rightarrow G$ and its restriction to the diagonal μ_n . We consider the A -group scheme $G_{\Delta_\phi} = G^{\phi^{geo}(\mu_n)} / \phi^{geo}(\mu_n)$ which is a central quotient of the fixed point locus of the diagonal μ_n . Then ϕ^{geo} induces a homomorphism $(\mu_{n,A})^r / \mu_{n,A} \rightarrow G_{\Delta_\phi}$. By taking the closed fiber we get a homomorphism $\mu_{n,k}^r / \mu_{n,k} \rightarrow G_{\Delta_\phi,k}$ whence a purely geometrical tame loop cocycle $\underline{\phi} : \text{Gal}(\underline{B}_n / \underline{A}_D) \rightarrow G_{\Delta_\phi}(l)$. We call it the specialization of ϕ .

Remark 5.4. This notion is quite ad-hoc since G_{Δ_ϕ} depends highly of ϕ .

Lemma 5.5. *The monomorphism $G^{\phi^{geo}} / \phi^{geo}(\mu_n) \rightarrow G_{\Delta_\phi} = G^{\phi^{geo}(\mu_n)} / \phi^{geo}(\mu_n)$ induces an open immersion of A -group schemes $G^{\phi^{geo}} / \phi^{geo}(\mu_n) \rightarrow (G_{\Delta_\phi})^{\phi^{geo}}$.*

Proof. To show that the induced map $h : G^{\phi^{geo}} / \phi^{geo}(\mu_n) \rightarrow (G_{\Delta_\phi})^{\phi^{geo}}$ is an open immersion, we can reason fiberwise according to [15, 17.9.5]. We are reduced to the case of an algebraically closed A -field E and since the two E -groups are smooth, it is enough to show that the (injective) map

$$(5.3) \quad \text{Lie}(h)(E) : \text{Lie}\left(G^{\phi^{geo}} / \phi^{geo}(\mu_n)\right)(E) \rightarrow \text{Lie}\left((G_{\Delta_\phi})^{\phi^{geo}}\right)(E)$$

is an isomorphism. Since μ_n is étale, we have $\text{Lie}\left(G^{\phi^{geo}}\right) \xrightarrow{\sim} \text{Lie}\left(G^{\phi^{geo}} / \phi^{geo}(\mu_n)\right)$. On the other hand, we have

$$\text{Lie}\left((G_{\Delta_\phi})^{\phi^{geo}}\right)(E) \xrightarrow{\sim} \left(\text{Lie}\left((G_{\Delta_\phi})^{\phi^{geo}}\right)(E)\right)^{\phi^{geo}}$$

according to [13, A.8.10.(1)]. Similarly we have $\text{Lie}\left(G^{\phi^{geo}(\mu_n)}\right) \xrightarrow{\sim} \text{Lie}(G_{\Delta_\phi})$ so altogether we are then reduced to show (for showing the bijectivity of (5.3)) that the

map

$$\mathrm{Lie}\left(G^{\phi^{geo}}\right)(E) \rightarrow \left(\mathrm{Lie}\left(G^{\phi^{geo}(\mu_n)}\right)(E)\right)^{\phi^{geo}}$$

is an isomorphism. This last fact follows again from [13, A.8.10.(1)] applied to the smooth E -group scheme $G^{\phi^{geo}(\mu_n)}$. \square

5.5. Twisting by tame-loop torsors. We assume that the A -group scheme G acts on an A -scheme Z . Let $\phi : \left(\prod_{i=1}^r \mu_n\right)(B) \rtimes \mathrm{Gal}(B/A) \rightarrow G(B)$ be a tame-loop cocycle. It gives rise to an A -action of μ_n^r on ${}_{\phi^{ar}}Z$. We denote by $({}_{\phi^{ar}}Z)^{\phi^{geo}}$ the fixed point locus for this action, it is representable by a closed A -subscheme of ${}_{\phi^{ar}}Z$ [13, A.8.10.(1)]. We have a closed embedding $({}_{\phi^{ar}}Z)^{\phi^{geo}} \times_X U \subset {}_{\phi}Z$ of U -schemes.

Assume that furthermore that ϕ is purely geometrical. Then ϕ induces an action of μ_n^r/μ_n on $Z_{\delta_{\phi}} := Z^{\phi^{geo}(\mu_n)}$. We have also an action of $G_{\Delta_{\phi}}$ on $Z_{\delta_{\phi}}$. By taking the closed fiber we obtain an action of $G_{\Delta_{\phi,k}}$ on $Z_{\delta_{\phi,k}}$ so we can twist $Z_{\delta_{\phi,k}}$ by $\underline{\phi}$ for obtaining the \underline{A}_D -scheme $\underline{\phi}(Z_{\delta_{\phi,k}} \times_k \underline{A}_D)$.

6. FIXED POINTS METHOD

We slightly refine [22, Thm. 3.1].

Theorem 6.1. *Let $X = \mathrm{Spec}(A)$ be a henselian regular local scheme as above. We denote by $v : K^{\times} \rightarrow \mathbb{Z}$ the discrete valuation associated to the exceptional divisor E of the blow-up of X at its closed point.*

Let G be an A -group scheme locally of finite presentation acting on a projective smooth A -scheme Z . Let ϕ be a tame-loop cocycle for G . Then $Y = ({}_{\phi^{ar}}Z)^{\phi^{geo}}$ is a smooth projective A -scheme and the following are equivalent:

- (i) $Y(A) \neq \emptyset$;
- (ii) $Y(A_D) \neq \emptyset$;
- (iii) $({}_{\phi}Z)(A_D) \neq \emptyset$;
- (iv) $({}_{\phi}Z)(K_v) \neq \emptyset$;
- (i') $Y(k) \neq \emptyset$;
- (ii') $Y(\underline{A}_D) \neq \emptyset$.

If furthermore ϕ is purely geometrical this is also equivalent to the two next assertions

- (iii') $(\underline{\phi}Z_{\delta_{\phi,k}})(\underline{A}_D) \neq \emptyset$;
- (iv') $(\underline{\phi}Z_{\delta_{\phi,k}})(k(t_1, \dots, t_{r-1})) \neq \emptyset$.

Once again the projectivity of Z is used to insure that the twisted fppf A_D -sheaf ${}_{\phi}Z$ by Galois descent is representable according to [5, §6.2] as well as other twists. The generalization to the proper smooth case is similar with Remark 4.4.

Proof. The smoothness of Y follows of [13, A.8.10.(2)] so that the henselian lemma provides the equivalence $(i) \iff (i')$. For the other implications we can assume without loss of generality that ϕ is purely geometrical. We claim that we have the following implications

$$\begin{array}{ccccccc} (i) & \implies & (ii) & \implies & (iii) & \implies & (iv) \\ \updownarrow & & & & & & \\ (i') & \iff & (ii') & \iff & (iii') & \iff & (iv'). \end{array}$$

The first horizontal line is obvious and the second one is Theorem 4.3 applied to the action on G_{Δ_ϕ} on Z_{δ_ϕ} by noticing that $Y_k = (Z_{\delta_\phi, k})^{\phi^{geo}}$. It is then enough to prove the implication $(iv) \implies (iv')$.

Let $\phi : \text{Gal}(B_n/A_D) \rightarrow G(B)$ be a tame-loop 1-cocycle for some Galois cover B_n/A_D as above for some n prime to p . We assume that $(\phi Z)(K_v) \neq \emptyset$. By definition we have

$$(\phi Z)(K_v) = \{z \in Z(L_{w_n}) \mid \phi(\sigma) \cdot \sigma(z) = z \ \forall \sigma \in \text{Gal}(L_n/K)\}$$

and our assumption is that this set is non-empty. Let O_{w_n} be the valuation ring of $Z(L_{w_n})$. Since Z is projective over X , we have a specialization map $Z(L_{w_n}) = Z(\mathcal{O}_{w_n}) \rightarrow Z_k(l(t_1^{\frac{1}{n}}, \dots, t_{r-1}^{\frac{1}{n}}))$. We get that the set

$$\left\{ z \in Z_k(k(t_1^{\frac{1}{n}}, \dots, t_{r-1}^{\frac{1}{n}})) \mid \phi(\sigma) \cdot \sigma(z) = z \ \forall \sigma \in \text{Gal}(L_{w_n}/K_v) \cong \mu_n(l)^r \rtimes \text{Gal}(l/k) \right\}$$

is not empty. This set is nothing but $(\phi Z_{\delta_\phi, k})(k(t_1, \dots, t_{r-1}))$ so we win. \square

6.1. Irreducibility and anisotropy. Let G be a reductive A -group scheme and consider an exact sequence $1 \rightarrow G \rightarrow \tilde{G} \rightarrow J \rightarrow 1$ where J is a twisted constant A -group scheme.

Lemma 6.2. *Let ϕ be a purely geometric tame-loop cocycle for \tilde{G} . Let (P, L) be a pair normalized by ϕ^{geo} where P is an A -parabolic subgroup of G and L is a Levi A -subgroup of P . We assume that (P, L) is minimal for this property (with respect to the inclusion). Then the tame-loop cocycle ϕ takes value in $N_{\tilde{G}}(P, L)(\tilde{B})$ and it is irreducible seen as tame-loop cocycle for $N_{\tilde{G}}(P, L)$.*

Proof. We put $\tilde{L} = N_{\tilde{G}}(P, L)$. The assumption implies that the tame-loop cocycle ϕ takes value in $\tilde{L}(k_s)$. We assume that the image of ϕ in $Z^1(\pi^t(X, x), \tilde{L}(k_s))$ is reducible, that is, there exists a pair (Q, M) normalized by ϕ^{geo} such that Q is a proper k -parabolic subgroup of $L = (\tilde{L})^0$ and M a Levi subgroup of Q . We have a Levi decomposition $P = U \rtimes L$ and remind to the reader that $P' = U \rtimes Q$ is a k -parabolic subgroup of G satisfying $P' \subsetneq P$ [3, prop. 4.4.c]. Also M is a Levi subgroup of P' normalized by ϕ^{geo} contradicting the minimality of (P, L) . \square

We observe that $J(\tilde{B}) = J(X^{tsc})$ so that
 (6.1)

$$H^1(\pi_1^t(X, x), J(\tilde{B})) = H^1(\pi_1^t(X, x), J(X^{tsc})) = \ker(H^1(X, J) \rightarrow H^1(X^{tsc}, J))$$

in view of [21, cor. 2.9.2].

Lemma 6.3. *Let ϕ, ϕ' be two tame-loop cocycles with value in $\tilde{G}(\tilde{B})$ having same image in $H^1(A_D, J)$. Then there exists $\tilde{g} \in \tilde{G}(\tilde{B})$ such that ϕ and $\sigma \mapsto \tilde{g}^{-1} \phi' \sigma(\tilde{g})$ have same image in $Z^1(\pi_1^t(X, x), J(\tilde{B}))$.*

Proof. According to the fact (6.1), the tame-loop cocycles ϕ, ϕ' have same image in $Z^1(\pi_1^t(X, x), J(\tilde{B}))$. Since $\tilde{G} \rightarrow J$ is smooth, the Hensel lemma shows that $\tilde{G}(B) \rightarrow J(l)$ for each finite connected étale cover B of A with residue field k . By taking the limit in the Galois tower, we obtain that $\tilde{G}(\tilde{B})$ maps onto $J(\tilde{B})$. It follows that there exists $\tilde{g} \in \tilde{G}(\tilde{B})$ such that ϕ and $\sigma \mapsto \tilde{g}^{-1} \phi' \sigma(\tilde{g})$ have same image in $Z^1(\pi_1^t(X, x), J(\tilde{B}))$. \square

In the same spirit that Lemma 4.8, we have the next fact.

Lemma 6.4. *If $[\phi], [\phi'] \in H^1(\pi_1^t(X, x), \tilde{G}(\tilde{B}))$ have same image in $H^1(K_v, \tilde{G})$, then $[\phi^{ar}] = [\phi'^{ar}] \in H^1(A, \tilde{G})$.*

Proof. Up to extend the scalar to $A_n = \left(A[T_1, \dots, T_r] / (T_1^n - f_1, \dots, T_r^n - f_r) \right) \otimes_A A_D$ for $n \gg 0$, we can assume that ϕ and ϕ' are purely arithmetic. In other words, we deal with $[\phi], [\phi'] \in H^1(\text{Gal}(B/A), \tilde{G}(B))$ for some finite Galois cover B/A having same image in $H^1(K_v, \tilde{G})$. The map $H^1(O_v, \tilde{G}) \rightarrow H^1(K_v, \tilde{G})$ is injective (Thm. 7.2 in the appendix) so that $[\phi], [\phi']$ have already same image in $H^1(O_v, \tilde{G})$. We consider the following diagram

$$\begin{array}{ccc} H^1(\text{Gal}(B/A), \tilde{G}(B)) & \longrightarrow & H^1(O_v, \tilde{G}) \\ \downarrow \wr & & \downarrow \wr \\ H^1(\text{Gal}(l/k), \tilde{G}_k(l)) & \longrightarrow & H^1(k(t_1, \dots, t_{r-1}), \tilde{G}_k) \end{array}$$

where the vertical maps are avatar of the Hensel lemma [50, XXIV.8.1]. Since $H^1(k, \tilde{G}_k) \rightarrow H^1(k((t_1)) \dots, ((t_{r-1})), \tilde{G}_k)$ is injective (Thm. 7.1 in the appendix), a fortiori the bottom horizontal map is injective. Thus the horizontal map is injective. We conclude that $[\phi] = [\phi'] \in H^1(\text{Gal}(B/A), \tilde{G}(B))$ as desired. \square

We say that a tame-loop cocycle $\phi : \text{Gal}(B_n/A_D) \rightarrow \tilde{G}(B)$ is *reducible* if the A -homomorphism $\phi^{geo} : \mu_n^r \rightarrow \phi^{ar} \tilde{G}$ is reducible, that is, normalizes a pair (P, L) where P is a proper parabolic A -subgroup of $\phi^{ar} \tilde{G}$ together with L a Levi subgroup of P .

Proposition 6.5. (1) *The following assertions are equivalent:*

- (i) *The tame-loop cocycle ϕ is reducible;*
- (ii) *The A_D -group ${}_{\phi}G$ is reducible;*
- (iii) *The K_v -group $({}_{\phi}G)_{K_v}$ is reducible.*

If furthermore ϕ is purely geometrical, the above assertions imply the following assertion:

- (iv) *The k -group $(G^{\phi^{geo}})_k$ is isotropic.*

Proof. Once again we can assume that ϕ is purely geometrical. We apply Theorem 6.1 to the A -schemes $Z = \text{Par}^+(G)$ and $Y = Z^{\phi^{geo}}$. We obtain the equivalence between the following statements:

- (i₀) $Y(A) \neq \emptyset$;
- (iii₀) $({}_{\phi}Z)(A_D) \neq \emptyset$;
- (iv₀) $({}_{\phi}Z)(K_v) \neq \emptyset$;
- (i'₀) $Y(k) \neq \emptyset$.

Using Proposition 2.21, we have the equivalences $(i_0) \iff (i)$, $(iii_0) \iff (ii)$ and $(iv_0) \iff (iii)$. It remains only to show the implication $(i'_0) \implies (iv)$. We assume then (i'_0) , i.e. there exists then a proper k -parabolic P of G_k which is normalized by ϕ^{geo} . According to Proposition 2.21, there exists a homomorphism $\lambda : \mathbb{G}_{m,k} \rightarrow (G^{\phi^{geo}})_k$ such that $P = P_{G_k}(\lambda)$. Since P is a proper subgroup of G , λ is non-trivial. Thus $(G^{\phi^{geo}})_k$ is isotropic. \square

We say that the tame-loop cocycle $\phi : \text{Gal}(B_n/A_D) \rightarrow \tilde{G}(B)$ is *isotropic* if the A -homomorphism $\phi^{geo} : \mu_n^r \rightarrow {}_{\phi^{ar}}\tilde{G}$ is isotropic, that is, centralizes a non-trivial split A -subtorus of ${}_{\phi^{ar}}G$. Equivalently the reductive A -group $C_{{}_{\phi^{ar}}G}(\phi^{geo})^0$ is isotropic.

Corollary 6.6. *The following are equivalent:*

- (i) *ϕ is isotropic;*
- (ii) *ϕ is reducible or the torus $({}_{\phi^{ar}}C^{\phi^{geo}})^0$ is isotropic;*

Proof. This is similar with that of Corollary 3.6. \square

Proposition 6.7. *The following assertions are equivalent:*

- (i) *The tame-loop cocycle ϕ is isotropic;*
- (ii) *The A_D -group ${}_{\phi}G$ is isotropic;*
- (iii) *The K_v -group $({}_{\phi}G)_{K_v}$ is isotropic;*

If furthermore ϕ is purely geometrical, the above statements are equivalent to the next ones:

- (i') *The tame-loop cocycle $\underline{\phi}$ (for $G_{\Delta_{\phi,k}}$) is isotropic;*
- (ii') *The \underline{A}_D -group $\underline{\phi}(G_{\Delta_{\phi,k}} \times_k \underline{A}_D)$ is isotropic.*
- (iii') *The $k(t_1, \dots, t_{r-1})$ -group $\underline{\phi}(G_{\Delta_{\phi,k}} \times_k k(t_1, \dots, t_{r-1}))$ is isotropic.*

Proof. Once again we can assume that ϕ is purely geometrical. the equivalences (i') \iff (ii') \iff (iii') follow from Proposition 4.9 , (i) \iff (ii) \iff (iii), applied to the k -group $G_{\Delta_{\phi,k}}$ and the Laurent polynomial ring $k\left[\left(\frac{t_1}{t_r}\right)^{\pm 1}, \dots, \left(\frac{t_{r-1}}{t_r}\right)^{\pm 1}\right] = k[x_1^{\pm 1}, \dots, x_{r-1}^{\pm 1}] = \underline{A}_D$.

(i) \implies (i'). If ϕ is isotropic there exists a non trivial homomorphism $\lambda : \mathbb{G}_{m,A} \rightarrow G^{\phi^{geo}}$ and induces a non trivial homomorphism

$$\mathbb{G}_{m,A} \rightarrow G^{\phi^{geo}} / \phi^{geo}(\mu_n) \subset (G_{\Delta_{\phi}})^{\phi^{geo}}.$$

Its base change to k is non-trivial [50, IX.6.5] whence a non trivial homomorphism

$$\mathbb{G}_{m,k} \rightarrow (G_{\Delta_{\phi,k}})^{\phi^{geo}}$$

so that $\underline{\phi}$ is isotropic.

Summarizing. On the other hand we have the obvious implications (i) \implies (ii) \implies (iii) so this fits in the diagram

$$\begin{array}{ccccccc} (i) & \implies & (ii) & \implies & (iii) & & \\ \Downarrow & & & & & & \\ (i') & \iff & (ii') & \iff & (iii') & . & \end{array}$$

It is then enough to establish the implications (iii) \implies (i') \implies (i).

(i') \implies (i). We assume that tame-loop cocycle $\underline{\phi}$ is isotropic and want to show that ϕ is isotropic as well. There exists a monomorphism $\lambda_k : \mathbb{G}_{m,k} \rightarrow (G_{\Delta_{\phi,k}})^{\phi^{geo}}$. According to Grothendieck's smoothness theorem [50, XI.4.2], it lifts to a monomorphism $\lambda : \mathbb{G}_{m,A} \rightarrow (G_{\Delta_{\phi}})^{\phi^{geo}}$. Since $G^{\phi^{geo}} / \phi^{geo}(\mu_n)$ is open in $(G_{\Delta_{\phi}})^{\phi^{geo}}$ (Lemma 5.5), λ factorizes through $G^{\phi^{geo}} / \phi^{geo}(\mu_n)$. Then λ^n lifts to a non-trivial homomorphism $\mathbb{G}_{m,A} \rightarrow G^{\phi^{geo}}$. Thus ϕ is isotropic.

(iii) \implies (i'). We assume that $(\phi G)_{K_v}$ is isotropic, so that $(\phi G)_{K_v}$ is reducible or its radical is isotropic. If $(\phi G)_{K_v}$ is reducible, Proposition 6.5, (i) \implies (iv), shows that $(G^{\phi^{geo}})_k$ is isotropic. A fortiori $\underline{\phi}$ is isotropic.

The last case is when the radical torus Q of $(\phi G)_{K_v}$ is isotropic. Denoting by $C \subset G$ the radical torus of G , we have $Q = \phi C$. Our assumption is that

$$\mathrm{Hom}_{K_v\text{-gp}}(\mathbb{G}_m, Q) = \mathrm{Hom}_{L_{w_n}\text{-gp}}(\mathbb{G}_m, C)^{\mathrm{Gal}(B_n/A_D)}$$

is not zero where the action is via the twisted action ϕ . Since we have isomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{l(t_1^{1/n}, \dots, t_{r-1}^{1/n})\text{-gp}}(\mathbb{G}_m, C)^{\mathrm{Gal}(B_n/A_D)} & \xleftarrow{\sim} & \mathrm{Hom}_{O_{w_n}\text{-gp}}(\mathbb{G}_m, C)^{\mathrm{Gal}(B_n/A_D)} \\ & & \downarrow \wr \\ & & \mathrm{Hom}_{L_{w_n}\text{-gp}}(\mathbb{G}_m, C)^{\mathrm{Gal}(B_n/A_D)}, \end{array}$$

we obtain then a central non-trivial homomorphism $\theta : \mathbb{G}_{m,k} \rightarrow G_k^{\phi^{geo}}$. As in the proof of (i) \implies (i'), we conclude that $\underline{\phi}$ is isotropic. \square

The next statement proceeds by analogy with Proposition 4.11.

Proposition 6.8. *Let ϕ, ϕ' be purely geometrical loop cocycles given by $\phi^{geo}, \phi'^{geo} : \mu_n^r \rightarrow \tilde{G}$. Assume that ϕ is anisotropic. Then the following are equivalent:*

- (i) ϕ^{geo} and ϕ'^{geo} are $\tilde{G}(A)$ -conjugated;
- (ii) $[\phi] = [\phi'] \in H^1(A_D, \tilde{G})$;
- (iii) $[\phi] = [\phi'] \in H^1(K_v, \tilde{G})$;

Proof. The following implications (i) \implies (ii) \implies (iii) are obvious. It remains to prove the implication (iii) \implies (i). We work at finite level with a basic tame cover B_n of A_D such that G_l is split. Our assumption is that there exists $\tilde{g} \in \tilde{G}(L_w)$ such that

$$(6.2) \quad \phi(\sigma) = \tilde{g}^{-1} \phi'(\sigma) \sigma(\tilde{g}).$$

for all $\sigma \in \Gamma = \mathrm{Gal}(L_w/K_v) = \mu_n(B)^r \rtimes L_w \mathrm{Gal}(B/A)$. The key step is the following.

Claim 6.1.1. $\tilde{g} \in \tilde{G}(O_v)$.

We consider the extended Bruhat-Tits building $\mathcal{B} = \mathcal{B}_e(G_{L_w})$. It comes with an action of $\tilde{G}(L_w) \rtimes \Gamma$ and with the hyperspecial point c which is the unique point fixed by $(DG)^{sc}(O_w)$ [6, 9.1.19.(c)]. According to Lemma 8.2, $\tilde{G}(O_w)$ is the stabilizer of c for the standard action of $\tilde{G}(L_w)$ on \mathcal{B} so that we have to prove that $\tilde{g}c = c$. Denoting by \star the twisted action by ϕ of Γ on \mathcal{B} , we have that

$$(6.3) \quad \mathcal{B}^{\Gamma_\phi} = \{c\}.$$

Indeed ${}_{\phi}G_{K_v}$ is anisotropic according to Proposition 6.7, (iii) \implies (i), that $(\mathcal{B})^{\Gamma\phi}$ consists in one point according to the Bruhat-Tits-Rousseau's theorem [7, 5.1.27]. Since c belongs to $\mathcal{B}^{\Gamma\phi}$, it follows that $\mathcal{B}^{\Gamma\phi} = \{c\}$. For each $\sigma \in \Gamma$, we have

$$\begin{aligned} \sigma \star (\tilde{g} \cdot c) &= \phi(\sigma) \sigma(\tilde{g}) \cdot \sigma(c) \\ &= \phi(\sigma) \sigma(\tilde{g}) \cdot c \quad [c \text{ is invariant under } \Gamma] \\ &= \tilde{g} \cdot \phi'(\sigma) c \quad [\text{relation 6.2}] \\ &= \tilde{g} \cdot c \quad [\phi(\gamma) \in \tilde{G}(B) \subset \tilde{G}(O_w)]. \end{aligned}$$

Thus $\tilde{g} \cdot c = c$ so that $\tilde{g} \in \tilde{G}(O_w)$. We use now the specialization map $\tilde{G}(O_w) \rightarrow \tilde{G}(l(t_1^{1/n}, \dots, t_{r-1}^{1/n}))$. We consider the composite $\psi : \Gamma \xrightarrow{\phi} G(B) \rightarrow G(l)$ and similarly for ψ' . This is a one cocycle and Claim 6.1.1 yields

$$(6.4) \quad \psi(\sigma) = \tilde{g}_0^{-1} \psi'(\sigma) \sigma(\tilde{g}_0).$$

for all $\sigma \in \Gamma = \mu_n(l)^r \rtimes \text{Gal}(l/k)$ with $\tilde{g}_0 \in \tilde{G}(l(t_1^{1/n}, \dots, t_{r-1}^{1/n}))$. Since ψ and ψ' are trivial on $\text{Gal}(l/k)$, the above equation shows that $\tilde{g}_0 \in \tilde{G}(k(t_1^{1/n}, \dots, t_{r-1}^{1/n}))$. Next we consider the transporter

$$X = \{h \in \tilde{G}_k \mid h \psi^{geo} h^{-1} = \psi'^{geo}\}.$$

We have that $X(l(t_1^{1/n}, \dots, t_{r-1}^{1/n})) \neq \emptyset$ so that X is a $(\tilde{G}_k)^{\psi^{geo}}$ -torsor. Theorem 7.1 enables to conclude that $X(k) \neq \emptyset$. Therefore ψ and ψ' are $\tilde{G}(k)$ -conjugated. Since the map $\tilde{G}(A) \rightarrow \tilde{G}(k)$ is onto we can assume that $\psi = \psi'$ without loss of generality.

If \tilde{G} is affine, it follows that ϕ and ϕ' are $G(A)$ -conjugated by applying [50, XI.5.2]. In the general case we reason as in the proof of Corollary 4.12 by working in an affine A -subgroup $\tilde{G}_1 \subset \tilde{G}$. The assertion (i) is established. \square

6.2. The main cohomological result. We define the tame part $H_{tame}^1(K_v, \tilde{G}) \subset H^1(K_v, \tilde{G})$ as the union of $H^1(K'/K_v, \tilde{G})$ for K' running over all finite tamely unramified extensions over K .

Theorem 6.9. *The map $H_{loop, tame}^1(A_D, \tilde{G}) \rightarrow H_{tame}^1(K_v, \tilde{G})$ is injective.*

In the case $r = 1$, A is a henselian DVR of fraction field K and K_v is the completion of K . In this case we know that the map $H^1(K, \tilde{G}) \rightarrow H^1(K_v, \tilde{G})$ is bijective [19, Prop. 3.3.1,(2)] so that the statement holds.

Proof. The proof goes along the same steps than the proof of Theorem 4.6. The injectivity of the map $H_{loop, tame}^1(A_D, \tilde{G}) \rightarrow H^1(K_v, \tilde{G})$ rephases to show that the fiber at the class $[\phi]$ of any loop cocycle ϕ consists in one element. If ϕ is an anisotropic tame-loop cocycle, Proposition 6.8, (iii) \implies (ii), shows that the fiber at $[\phi]$ of $H_{loop, tame}^1(A_D, \tilde{G}) \rightarrow H^1(K_v, \tilde{G})$ is $\{[\phi]\}$.

A first generalization is the irreducible case. We assume then that the tame-loop cocycle ϕ is irreducible. Once again the usual twisting argument enables us to assume that $\phi^{ar} = 1$. Let ϕ' be another tame-loop cocycle such that $[\phi'] = [\phi] \in H^1(K_v, \tilde{G})$. According to Lemma 2.13.(1), the map $H^1(A_D, J) \rightarrow H^1_{tame}(K_v, J)$ is injective; it follows that ϕ and ϕ' have same image in $H^1(A_D, J)$. Next Lemma 6.3 permits to assume without loss of generality that ϕ and ϕ' have same image in $Z^1(\pi_1^t(X, x), J(\tilde{B}))$. We denote by J_1 the image of $\phi^{geo} : \widehat{\mathbb{Z}}'(1)^r \rightarrow J$, this is a finite smooth algebraic A -group of multiplicative type. We put $\tilde{G}_1 = G_1 \times_J J_1$, by construction ϕ and ϕ' have value in $\tilde{G}_1(\tilde{B})$. To avoid any confusion we denote them by ϕ_1 and ϕ'_1 . We consider the commutative diagram of pointed sets

$$\begin{array}{ccccc} (J/J_1)(\tilde{B})^{\pi_1^t(X, x)\phi_1} & \longrightarrow & H^1(\pi_1^t(X, x), \phi_1 G_1(\tilde{B})) & \longrightarrow & H^1(\pi_1^t(X, x), \phi_1 G(\tilde{B})) \\ \downarrow = & & \downarrow & & \downarrow \\ (\phi_1(J/J_1))(K_v) & \longrightarrow & H^1(K_v, \phi_1 \tilde{G}_1) & \longrightarrow & H^1(K_v, \phi_1 \tilde{G}). \end{array}$$

The second one is associated to the exact sequence $1 \rightarrow \phi_1(\tilde{G}_1) \rightarrow \phi_1(\tilde{G}) \rightarrow \phi_1(J/J_1) \rightarrow 1$ of K_v -spaces and the first one is associated to the exact sequences of $\pi_1^t(X, x)$ -sets $1 \rightarrow \phi_1(\tilde{G}_1(\tilde{B})) \rightarrow \phi_1(\tilde{G}(\tilde{B})) \rightarrow \phi_1(J/J_1)(\tilde{B}) \rightarrow 1$. By diagram chase involving the torsion bijection $H^1(\pi_1^t(X, x), \phi_1 \tilde{G}_1(\tilde{B})) \xrightarrow{\sim} H^1(\pi_1^t(X, x), \tilde{G}_1(\tilde{B}))$, we see that we can arrange ϕ'_1 in order that ϕ'_1 has same image than ϕ_1 in $H^1(K_v, \tilde{G}_1)$. We can work then with then \tilde{G}_1 which is generated by G and the image of ϕ_1^{geo} .

Since the A -torus C is central in G , the A -subgroup $C^{\phi^{geo}}$ is central in \tilde{G}_1 . We denote by C_0 the maximal split A -subtorus of $C^{\phi^{geo}}$ and consider the central exact sequence of A -group schemes

$$1 \rightarrow C_0 \rightarrow \tilde{G}_1 \rightarrow \tilde{G}_1/C_0 \rightarrow 1$$

The gain is that the image of ϕ_1 in $Z^1(\pi_1^t(X, x), (\tilde{G}_1/C_0)(\tilde{B}))$ is an anisotropic tame-loop cocycle by applying the criterion of Corollary 6.6. Since $H^1(A_D, C_0) = 1$ (Lemma 5.1), we obtain the following commutative diagram

$$\begin{array}{ccc} H^1(A_D, \tilde{G}_1) & \hookrightarrow & H^1(A_D, \tilde{G}_1/C_0) \\ \downarrow & & \downarrow \\ H^1(K_v, \tilde{G}_1) & \hookrightarrow & H^1(K_v, \tilde{G}_1/C_0) \end{array}$$

where the horizontal maps are injections [28, III.3.4.5.(iv)]. Proposition 6.8 shows that $[\phi_1]$ and $[\phi'_1]$ have same image in $H^1(A_D, \tilde{G}_1/C_0)$. The diagram shows that $[\phi_1] = [\phi'_1] \in H^1(A_D, \tilde{G}_1)$. By pushing in $H^1(A_D, \tilde{G})$ we get that $[\phi] = [\phi'] \in H^1(A_D, \tilde{G})$ as desired.

We deal now with the general case. The above reduction (with \tilde{G}_1) permits to assume that J is finite étale so that \tilde{G} is affine and also that ϕ is purely geometric. Let $(P, L) \subset G$ be an A -parabolic subgroup P of G together with a Levi subgroup L both normalized by ϕ^{geo} and which is minimal for this property. Then the tame-loop cocycle ϕ takes value in $\tilde{L}(B) = N_{\tilde{G}}(P, L)(\tilde{B})$. We have the exact sequence (2.2)

$$1 \rightarrow L \rightarrow \tilde{L} \rightarrow J_{P,L} \rightarrow 1$$

of smooth affine A -group schemes with $L = (\tilde{L})^0$ and such that $J_{P,L}$ is twisted constant. We denote by ψ the image of ϕ in $Z^1(\pi_1^t(X, x), \tilde{L}(\tilde{B}))$. Lemma 6.2.(2) states that ψ is irreducible.

We deal now with the tame-loop cocycle ϕ' having same image in $H^1(A_D, \tilde{G})$ as ϕ . Lemma 6.4 implies that ϕ' is purely geometrical. We consider the A -scheme $X = \tilde{G}/N_{\tilde{G}}(P)$ which is projective according to Lemma 2.19. Since ϕ takes values in $N_{\tilde{G}}(P)(\tilde{B})$, we have $(\phi\tilde{G})_{K_v}/(\phi N_{\tilde{G}}(P))_{K_v}$ so in particular $(\phi X)(K_v) \neq \emptyset$. Since ${}_{\phi'}X \cong {}_{\phi}X$, it follows that $({}_{\phi'}X)(K_v) \neq \emptyset$. Theorem 6.1, (iv) \implies (i), shows that $X^{\phi^{geo}}(A)$ is not empty. We pick $x' \in X^{\phi^{geo}}(A)$ and choose $x \in X^{\phi^{geo}}(A)$ such that $G_x = P$. There exists $\tilde{g} \in \tilde{G}(\tilde{B})$ such that $x' = \tilde{g}.x$. For $\sigma \in \text{Gal}(\tilde{B}/A)$, we have that $x' = \sigma(\tilde{g}).x$ so that $\sigma \rightarrow n_{\sigma} = \tilde{g}^{-1}\sigma(\tilde{g})$ is a 1-cocycle with value in $\tilde{G}_x(\tilde{B}) = N_{\tilde{G}}(P)(\tilde{B})$. For $\sigma \in \text{Gal}(\tilde{B}/A)$, we have $\phi'(\sigma).x' = x'$ so that $\phi'(\sigma).\tilde{g}.x = \tilde{g}.x$. It follows that $\tilde{g}^{-1}\phi'(\sigma)\tilde{g}.x = x$. Since $n_{\sigma}.x = x$ it follows that $\phi''(\sigma) = \tilde{g}^{-1}\phi'(\sigma)$ fixes x . Up to replace ϕ' by the equivalent cocycle ϕ'' we can then assume that ϕ' takes value in $N_{\tilde{G}}(P)(\tilde{B})$. As for ϕ , we can modify further by a coboundary in order that ϕ' has value in $N_{\tilde{G}}(P, L)(\tilde{B})$. We denote by $\psi' : \Gamma \rightarrow N_{\tilde{G}}(P, L)(\tilde{B})$ the result of this reduction. Proposition 2.21 tells us that P admits a Levi subgroup L' normalized by ϕ'^{geo} . Then $L' = {}^g L$ for some $g \in P(A)$. We note that P is minimal for this property (otherwise it will not be minimal for ϕ and equivalently for ϕ_{K_v}). The above argument tells us that the image ψ' of ϕ' in $Z^1(\pi_1^t(X, x), \tilde{L}(\tilde{B}))$ is irreducible. We consider now the commutative diagram

$$\begin{array}{ccc} H^1(A_D, \tilde{G}) & \longrightarrow & H^1(K_v, \tilde{G}) \\ \uparrow & & \parallel \\ H^1(A_D, N_{\tilde{G}}(P, L))_{irr} & \longrightarrow & H^1(K_v, N_{\tilde{G}}(P, L))_{irr} \end{array}$$

The bottom horizontal map is well-defined in view of Proposition 6.5. The right vertical map is injective [21, lemme 4.2.1.(2)]. We have seen that ϕ, ϕ' provide elements $[\psi], [\psi']$ of $H^1(A_D, N_{\tilde{G}}(P))_{irr}$ which give then the same image in $H^1(K_v, N_{\tilde{G}}(P))$. By diagram chase we conclude that $[\phi] = [\phi'] \in H^1(A_D, N_{\tilde{G}}(P))$. Thus $[\phi] = [\phi'] \in H^1(A_D, \tilde{G})$ as desired. \square

7. INJECTIVITY PROPERTY FOR NON-ABELIAN COHOMOLOGY

We start by extending a classical fact on algebraic k -groups which goes back to Bruhat and Tits in the reductive case [31, Prop. 5.5], see [16, Prop. 5.4] and [17, Thm. 5.4] for extensions.

Theorem 7.1. *Let G be a locally algebraic k -group. Then the map $H^1(k, G) \rightarrow H^1(k((t)), G)$ is injective.*

Proof. We have an exact sequence $1 \rightarrow G^0 \rightarrow G \rightarrow J \rightarrow 1$ where G^0 is an algebraic k -group and J is an étale k -group and then a twisted constant k -group scheme [14, §II.5.1]. It follows that the representability facts of Lemma 2.7 apply. In particular the usual twisting argument boils down to establish only the triviality of the kernel of $H^1(k, G) \rightarrow H^1(k((t)), G)$. We consider the commutative diagram of exact sequence of pointed sets

$$\begin{array}{ccccccc} J(k) & \longrightarrow & H^1(k, G^0) & \longrightarrow & H^1(k, G) & \longrightarrow & H^1(k, J) \\ \alpha \downarrow \wr & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow \\ J(k((t))) & \longrightarrow & H^1(k((t)), G^0) & \longrightarrow & H^1(k((t)), G) & \longrightarrow & H^1(k((t)), J). \end{array}$$

The map α is bijective, the map β is injective [17, Thm. 5.4]. On the other hand, the map δ decomposes in

$$H^1(k, J) \rightarrow H^1(k[[t]], J) \rightarrow H^1(k((t)), J).$$

The first map is obviously split injective and the second one is injective (Lemma 2.13.(1)) so that δ is injective. A diagram chase enables us to conclude that $\ker(\gamma) = 1$. \square

For a general DVR, we can extend Nisnevich result as follows [41, Thm. 4.2] [31, Thm. 1.2] from the reductive case to our setting.

Theorem 7.2. *Let R be a semilocal Dedekind ring of fraction field K . Let $1 \rightarrow G \rightarrow \tilde{G} \rightarrow J \rightarrow 1$ be an exact sequence of smooth R -group schemes such that G is reductive and J is twisted constant. Then the map $H^1(R, \tilde{G}) \rightarrow H^1(K, \tilde{G})$ is injective.*

Proof. The usual twisting argument boils down to establish the triviality of the kernel of the map $H^1(R, \tilde{G}) \rightarrow H^1(K, \tilde{G})$. According to Proposition 2.9, we have $J(R) = J(K)$. We consider now the commutative diagram of exact sequences of pointed sets

$$\begin{array}{ccccccc} J(R) & \xrightarrow{\phi} & H^1(R, G) & \longrightarrow & H^1(R, \tilde{G}) & \longrightarrow & H^1(R, J) \\ \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \\ J(K) & \xrightarrow{\phi} & H^1(K, G) & \longrightarrow & H^1(K, \tilde{G}) & \longrightarrow & H^1(K, J) \end{array}$$

where we reported injectivity for G [31, Thm. 1.2] and for J (Lemma 2.13.(1)). Let $\tilde{\gamma}$ be a class in the kernel of $H^1(R, \tilde{G}) \rightarrow H^1(K, \tilde{G})$. A diagram chase shows that $\tilde{\gamma}$ comes from $\gamma \in H^1(R, G)$ such that $\gamma_K = (\phi(x))_K$ for some $x \in J(R)$. We consider the G -torsor $E = p^{-1}(x)$; its class is $\phi(x)$. Then the torsion bijection $\tau : H^1(R, {}^E G) \xrightarrow{\sim} H^1(R, G)$ satisfies that $\tau^{-1}(\gamma) \in \ker(H^1(R, {}^E G) \rightarrow H^1(K, {}^E G))$. Since G is reductive, Nisnevich-Guo's theorem shows that $\tau^{-1}(\gamma) = 1$ so that $\gamma = \phi(x)$. Thus $\tilde{\gamma} = 1$ as desired. \square

8. APPENDIX: (ENLARGED) BRUHAT-TITS THEORY OF A SPLIT REDUCTIVE GROUP

This theory simplifies quite substantially when we deal with the case of a split reductive group G over a field K which is henselian for a discrete valuation. We denote by O the valuation ring of K and by k its residue field (not assumed necessarily perfect). The K -group G is then extended from a Chevalley O -group scheme G itself extended from \mathbb{Z} .

Let C (resp. Q) be the radical torus of G . Both tori are split and the map $p : C \rightarrow Q$ is an isogeny whence an isomorphism $p_* : \widehat{C}^0 \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \widehat{Q}^0 \otimes_{\mathbb{Z}} \mathbb{R}$. Let (B, T) be a Killing couple of G and put $N = N_G(T)$. The enlarged building of G over K is

$$\mathcal{B}_e(G_K) = \mathcal{B}(DG_K) \times (\widehat{C}^0 \otimes_{\mathbb{Z}} \mathbb{R}) = \mathcal{B}(DG_K) \times E$$

where $\mathcal{B}(DG_K)$ is the Bruhat-Tits building of the derived K -group DG which is isomorphic to the building of its simply connected cover $(DG)_K^{sc}$ (and of its adjoint group $G_{ad,K}$).

The valuation induces a map $v_Q : Q(K) = \widehat{Q}^0 \otimes_{\mathbb{Z}} K^\times \rightarrow \widehat{Q}^0$, it gives rise to an action of $Q(K)$ on $\widehat{Q}^0 \otimes_{\mathbb{Z}} \mathbb{R}$ by $q.\lambda = \lambda - v_Q(q)$. It provides as well an action on $Q(K)$ on $E = \widehat{C}^0 \otimes_{\mathbb{Z}} \mathbb{R}$ (and E has to be seen as an affine space) and an action of $G(K)$ through the morphism $G(K) \rightarrow Q(K)$ (see [33, §4.3]).

We have a natural action of $G(K)$ on $\mathcal{B}_e(G_K)$. It acts through the action of $G_{ad}(K)$ on $\mathcal{B}(DG_K)$ and the previous action of $G(K)$ on $(\widehat{C}^0 \otimes_{\mathbb{Z}} \mathbb{R})$. According to [6, 9.1.19.(c)], there exists a unique point $\phi \in \mathcal{B}(DG_K)$ called the center of the building such that $(DG)^{sc}(O) = \text{Stab}_{(DG)^{sc}(K)}(\phi)$; we consider the point $\phi_e = (\phi, 0)$ of the extended building.

Lemma 8.1. $G(O) = \text{Stab}_{G(K)}(\phi_e)$.

Proof. The direct inclusion is clear. Conversely let $g \in G(K)$ fixing ϕ_e and denote by $q \in Q(K)$ its image under the canonical map $G(K) \rightarrow Q(K)$. Then $v_Q(q) = 0$ so that $q \in Q(O)$. We consider the exact sequence $1 \rightarrow DG \rightarrow G \rightarrow Q \rightarrow 1$. According to [13, A.2.7], $T' = T \cap DG$ is a maximal torus of DG so that we have an exact sequence of split tori $1 \rightarrow T' \rightarrow T \rightarrow Q \rightarrow 1$. It follows that the map $G(O) \rightarrow Q(O)$ is split and a fortiori onto. We can pick then $g_0 \in G(O)$ mapping on q ; replacing g by gg_0^{-1} reduces

to the case $g \in DG(K)$. We use now the isogeny $1 \rightarrow \mu \rightarrow (DG)^{sc} \xrightarrow{p} DG \rightarrow 1$. We have a compatible exact sequence $1 \rightarrow \mu \rightarrow T^{sc} \rightarrow T' \rightarrow 1$ where T^{sc} is the inverse image of T' . We have a commutative diagram of exact sequences of pointed sets

$$\begin{array}{ccccccc} (DG)^{sc}(K) & \longrightarrow & DG(K) & \longrightarrow & H_{flat}^1(K, \mu) & \longrightarrow & H^1(K, (DG)^{sc}) \\ \uparrow & & \uparrow & & \wr \uparrow & & \uparrow \\ T^{sc}(K) & \longrightarrow & T'(K) & \longrightarrow & H_{flat}^1(K, \mu) & \longrightarrow & H^1(K, T^{sc}) = 0. \end{array}$$

By diagram chase we get a decomposition

$$(DG)^{sc}(K) = \text{Im}\left((DG)^{sc}(K) \xrightarrow{p} DG(K)\right) T'(K).$$

We deal then with $g = p(g_1) g_2^{-1}$ with $g_1 \in (DG)^{sc}(K)$ and $g_2 \in T'(K)$. It follows that $p(g_1) \cdot \phi = g_2 \cdot \phi$. This point is then of same type than ϕ and belongs to the apartment \mathcal{A} defined by T^{sc} . Since $T^{sc}(K)$ acts transitively on the points of \mathcal{A} of type 0, there exists $g_3 \in T^{sc}(K)$ such that $p(g_1) \cdot \phi = g_2 \cdot \phi = p(g_3) \cdot \phi$. Up to replace g_1 (resp. g_2) by $g_1 g_3^{-1}$ (resp. $g_2 p(g_3)^{-1}$) we are reduced to the case that $p(g_1) \cdot \phi = \phi$ and $g_2 \cdot \phi = \phi$. It follows that $g_1 \in (DG)^{sc}(O)$. On the other hand g_2 acts then trivially on the apartment \mathcal{A} hence $g_2 \in T'(O)$. Thus $g = p(g_1) g_2^{-1}$ belongs to $DG(O)$ as desired. \square

We deal now with an exact sequence of locally algebraic k -groups $1 \rightarrow G \rightarrow \tilde{G} \rightarrow J \rightarrow 1$ such that $G = \tilde{G}^0$. We claim that the action of $G(K)$ on $\mathcal{B}_e(G_K)$ extends naturally to an action on $\tilde{G}(K)$. Pushing the above sequence by $G \rightarrow Q$ gives rise to an exact sequence $1 \rightarrow Q \rightarrow \tilde{Q} \rightarrow J \rightarrow 1$ and to the commutative exact diagram of exact sequences as such that the following diagram commutes

$$\begin{array}{ccccccc} 1 & \longrightarrow & Q(O) & \longrightarrow & \tilde{Q}(O) & \longrightarrow & J(O) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \wr \\ 1 & \longrightarrow & Q(K) & \longrightarrow & \tilde{Q}(K) & \longrightarrow & J(K) \longrightarrow 1 \end{array}$$

by using the triviality of $H^1(O, Q)$ and $H^1(K, Q)$. Pushing one more time by $v_Q : Q(K) \rightarrow (\hat{Q})^0 \otimes_{\mathbb{Z}} \mathbb{R}$ gives rise to the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & Q(K) & \longrightarrow & \tilde{Q}(K) & \longrightarrow & J(K) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \wr \\ 0 & \longrightarrow & (\hat{Q})^0 \otimes_{\mathbb{Z}} \mathbb{R} & \longrightarrow & H & \longrightarrow & J(K) \longrightarrow 1 \end{array}$$

with a canonical decomposition $H = ((\widehat{Q})^0 \otimes_{\mathbb{Z}} \mathbb{R}) \rtimes J(K)$. The group H acts on $(\widehat{Q})^0 \otimes_{\mathbb{Z}} \mathbb{R} \cong E$ as follows:

$$(y, \tau).x = \tau(x) - y.$$

This extends then the opposite translation action of $(\widehat{Q})^0 \otimes_{\mathbb{Z}} \mathbb{R}$ on itself. Composing with the projection $\widetilde{G}(K) \rightarrow H$, we extended then the action of $G(K)$ on E . On the other hand, $\widetilde{G}(K)$ acts by group automorphisms on DG hence acts on $\mathcal{B}(DG_K)$. Altogether we have then an action of $\widetilde{G}(K)$ on $\mathcal{B}_e(G_K)$.

Lemma 8.2. *We have $\widetilde{G}(O) = \text{Stab}_{\widetilde{G}(K)}(\phi_e)$.*

Proof. The direct inclusion is straightforward. Conversely we are given $\tilde{g} \in \widetilde{G}(K)$ fixing ϕ_e . We consider the following diagram of exact sequences of pointed sets

$$\begin{array}{ccccccc} G(K) & \longrightarrow & \widetilde{G}(K) & \longrightarrow & J(K) & \longrightarrow & H^1(K, G) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ G(O) & \longrightarrow & \widetilde{G}(O) & \longrightarrow & J(O) & \longrightarrow & H^1(O, G). \end{array}$$

Using the reduction argument to a Borel subgroup of G , the map $H^1(O, G) \rightarrow H^1(K, G)$ has trivial kernel. By diagram chase it follows that $\tilde{g} = g \tilde{g}_0$ with $\tilde{g}_0 \in \widetilde{G}(O)$ and $g \in G(K)$. According to Lemma 8.1, we have that $g \in G(O)$. Thus \tilde{g} belongs to $\widetilde{G}(O)$. \square

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