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FACTOR-CRITICAL GRAPHS AND DSTAB, ASTAB FOR AN EDGE IDEAL

MARCEL MORALES AND NGUYEN THI DUNG

ABSTRACT. Let G be a simple, connected non bipartite graph and let I_G be the edge ideal of G . In our previous work we showed that L. Lovász's theorem on ear decompositions of factor-critical graphs and the canonical decomposition of a graph given by Edmonds and Gallai are basic tools for the irreducible decomposition of I_G^k . In this paper we use some tools from graph theory, mainly Whitney's theorem on ear decompositions of 2-edge connected graphs in order to introduce a new method to make a graph factor-critical. We can describe the set $\cup_{k=1}^{\infty} \text{Ass}(I_G^k)$ in terms of some subsets of G . We give explicit formulas for the numbers $\text{astab}(I_G)$ and $\text{dstab}(I_G)$, which are, respectively, the smallest number k such that $\text{Ass}(I_G^k) = \text{Ass}(I_G^{k+i})$ for all $i \geq 0$ and the smallest number k such that the maximal ideal belongs to $\text{Ass}(I_G^k)$. We also give very simple upper bounds for $\text{astab}(I_G)$ and $\text{dstab}(I_G)$.

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1. INTRODUCTION

In this paper we will study the sets of associated prime ideals of the powers of the edge ideal $I_G \subset R := K[x_1, \dots, x_d]$ of a graph $G = (V, E)$ with $V = \{P_1, \dots, P_d\}$. There are many articles on this topic, such as [1],[2],[5],[9],[13],[14] and [17]. In [1], Brodmann showed that when R is a Noetherian ring and I is an ideal of R , the sets $\text{Ass}(I_G^k)$ stabilize for large k . In [2] a method is given to construct associated primes of the powers of an edge ideal is given. In [13] it was proved that for edge ideals the sets of associated primes of the powers of I form an ascending chain, known as the persistence property. In [14]

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we proved the strong persistence property, that is an irreducible primary component of a power I_G^k induces an irreducible primary component in I_G^{k+1} with the same associated prime. In [14] we describe the set of irreducible primary components of I_G^k in terms of factor-critical sets and the canonical decomposition of a graph given by Edmonds and Gallai (see for example [3], [6], [7],[10],[11],[12]).

Ear decomposition is an important tool in our work (see [18, Chapter 5]); the key result is Whitney's theorem which states that a graph is 2-edge connected if and only if it has an ear decomposition.

In Section 2 we introduce the work of A.Frank [4] and Z.Szigeti [16] (see also [18, Chapter 5]), who describe the optimal way to make a 2-edge connected graph factor-critical. We extend their work by introducing a generalized ear decomposition for a connected graph and two invariants $\varphi(G)$, the minimum number of even ears in a generalized ear decomposition, and $\psi(G)$ the number of bridges.

In Section 3, inspired by A.Frank [4], Z.Szigeti [16] and Lemma 4.7 in our paper [14], we can describe a new method using duplication of vertices in a non bipartite graph (after A. Schrijver [15]) and a generalized ear decomposition to obtain factor-critical graphs with the minimum number of steps.

In Section 4, by using the description of the irreducible primary components of I_G^k in terms of matching-critical graphs given in [14] and the method developed in Section 3, we can describe the set of associated primes of each power of the edge ideal I_G . Our result extends Proposition 3.3 of [13], where the case of \mathfrak{m}^{1_V} was considered, and theorem 4.1 of [2]. As an application we compute exactly the smallest number k , called $\text{astab}(I_G)$, where $\text{Ass}(I_G^k) = \text{Ass}(I_G^{k+i})$ for all $i \geq 0$, and the smallest number k , called $\text{dstab}(I_G)$, where the maximal ideal belongs to $\text{Ass}(I_G^k)$.

To be more precise let us introduce some notations and definitions. For any vector $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{N}^d$ we set $\mathfrak{m}^{\mathbf{a}}$ the monomial ideal generated by $x_i^{a_i}$. For any subset $U \subset V$ we set $\mathbf{1}_U$ the vector whose i -coordinate is 1 if $P_i \in U$ and 0 otherwise. We will denote by G_U the induced subgraph of G with vertices in U . Our first result is:

Theorem. *Let G be a simple, connected non bipartite graph. Then $\mathfrak{m}^{\mathbf{1}_U} \in \text{Ass}(I_G^{\text{astab}(I_G)})$ if and only if $Z := V \setminus U$ is a co clique set and either $U = N(Z)$ or $U \neq N(Z)$ and every connected component of $G_{U \setminus N(Z)}$ contains an odd cycle.*

Definition. *Let $G = (V, E)$ be a simple, connected non bipartite graph. Let $U \subset V, Z = V \setminus U$. If Z is not an independent set we set $\mathcal{D}^*(U) = \emptyset$. From now on we assume that Z is an independent set. A set $W \subset V \setminus (Z \cup N(Z))$ is called U -dominant if $V = N(W) \cup N(Z) \cup Z$. W is called U -dominant* if either $W = \emptyset$ or it is U -dominant and every connected component of G_W contains an odd cycle. We will denote by $\mathcal{D}^*(U)$ the set of all subsets $W \subset U$ that are U -dominant*.*

For every $W \in \mathcal{D}^(U)$ we define $\nu^*(G_W) = \frac{1}{2}(\text{card}(W) + \varphi(G_W) + \psi(G_W) - t_W)$, where t_W is the number of connected components of G_W .*

First we describe $\text{Ass}(I_G^k)$ for every $k \geq 1$.

Theorem. *Let G be a simple, connected non bipartite graph, then*

$$\text{Ass}(I_G^k) = \{\mathbf{m}^{1U} \mid U \subset V, \mathcal{D}^*(U) \neq \emptyset, \min_{W \in \mathcal{D}^*(U)} \nu^*(G_W) \leq k - 1\}.$$

Then we give explicit formulas for $\text{astab}(I_G)$, $\text{dstab}(I_G)$.

Theorem. *Let G be a simple, connected non bipartite graph. We have*

$$\text{astab}(I_G) = \max_{\{U \subset V \mid \mathcal{D}^*(U) \neq \emptyset\}} \{\min\{1 + \nu^*(G_W) \mid W \in \mathcal{D}^*(U)\}\}.$$

$$\text{dstab}(I_G) = \min\{1 + \nu^*(G_W) \mid W \in \mathcal{D}^*(V)\}.$$

Then we can derive a very simple upper bound for $\text{astab}(I_G)$, $\text{astab}(I_G)$:

Theorem. *Let G be a simple connected non bipartite graph.*

$$\text{dstab}(I_G) \leq \frac{1}{2}(\text{card}(E(G)) + \psi(G)) + 1,$$

$$\text{astab}(I_G) \leq \text{card}(E(G)) - k + 1,$$

where $2k - 1$ is the minimum length of an odd cycle in G .

We then give an example of a graph with $2l + 3$ vertices such that $\text{dstab}(I_G) = l + 1$, $\text{astab}(I_G) = 2l - 2$, proving that the difference between $\text{dstab}(I_G)$, $\text{astab}(I_G)$ can be as large as possible.

Finally, in Section 5 we prove that every 2-edge connected non bipartite graph G has an ear decomposition, with $\varphi(G)$ even ears, starting with an odd cycle.

Recall that the problem of determining dominant sets or independent sets in a graph is a NP-hard, but can be done by hand for a graph with small number of vertices.

2. PRELIMINARIES

Given a non-factor-critical graph, how can we turn it into a factor-critical graph? How can we measure how far a graph is from a factor-critical graph? In [18, Chapter 5] the authors consider three possible operations on 2-edge-connected graphs: contraction, subdivision and ear decomposition. In this section we introduce these operations and extend the results of Szigetı [16], who showed that for 2-edge-connected graphs, all the three approaches are equivalent.

Definition 2.1. ([8]) *An ear decomposition $G_0, G_1, \dots, G_k = G$ of a graph G is a sequence of graphs where the first graph G_0 being a vertex, edge, even cycle, or odd cycle, and each graph G_{i+1} is obtained from G_i by adding an ear.*

Adding an ear is done as follows: take two vertices a and b of G_i and add a path F_i from a to b such that all vertices on the path except a and b are new vertices (present in G_{i+1} but not in G_i). An ear with $a \neq b$ is called *open*, otherwise, *closed*. An ear with F_i having an odd (even) number of edges is called *odd (even)*. The sequence of ears F_0, F_1, \dots, F_s , is also called an ear decomposition.

From Whitney's theorems, (see [18, Chapter 5]), every 2-edge-connected graph has an ear decomposition and by Lovász's theorem [10] a graph is factor-critical if and only if it has an ear decomposition with odd ears.

Theorem 2.2. *(Withney) A graph $G = (V, E)$ is 2-edge-connected if and only if has an ear decomposition. Furthermore G can be constructed from any vertex P by successively adding an ear to the previously constructed graphs.*

Note that 2-edge-connected means bridgeless. Since any connected graph G can be written as a sequence of bridgeless graphs and bridges, we immediately have the following result.

Theorem 2.3. and Definition. *Every connected graph G has a generalized ear decomposition, that is: F_0, F_1, \dots, F_s , such that F_0 is a cycle or a bridge, and F_i is either an ear, or a bridge. If G is not a tree F_0 can be chosen to be a cycle, and if G is a non bipartite graph, F_0 can be chosen to be an odd cycle.*

Let $\varphi(G)$ denote the minimum number of even ears in generalized ear decompositions of G , note that if G_1, G_2, \dots, G_l are the 2-edge connected components of G , then $\varphi(G) = \sum_{i=1}^l \varphi(G_i)$. A generalized ear decomposition of G with $\varphi(G)$ even ears is called optimal. The number of bridges is an invariant, denoted by $\psi(G)$. Clearly, a connected graph G is factor-critical if and only if $\varphi(G) = \psi(G) = 0$.

Lemma 2.4. *The number of ears in a generalized ear decomposition of a connected graph G is independent of the ear decomposition and is equal to $\text{card}(E(G)) - \text{card}(V(G)) + \psi(G) + 1$.*

Proof. The proof follows by induction on the number of bridges in G . Our claim is well known for 2-edge connected graphs. Suppose that G has at least one bridge. Let F_0, F_1, \dots, F_s , be a generalized ear decomposition of G . Note that if F_i is a bridge for some $i < s$ then the graph G' union of $F_{i+1}, F_{i+2}, \dots, F_s$ has $F_{i+1}, F_{i+2}, \dots, F_s$ as a generalized ear decomposition. Let H be the graph with the generalized ear decomposition F_0, F_1, \dots, F_{s-1} . If F_s is a bridge then $\psi(H) = \psi(G) - 1$, $\text{card}(E(H)) = \text{card}(E(G)) - 1$, $\text{card}(V(H)) = \text{card}(V(G)) - 1$, by induction hypothesis we have $s = \text{card}(E(H)) - \text{card}(V(H)) + \psi(H) + 1$ hence by a simple calculation we get $s + 1 = \text{card}(E(G)) - \text{card}(V(G)) + \psi(G) + 1$. Similar arguments apply when F_0 is a bridge. If F_0, F_s are not bridges then there exists $0 < i < s$ such that F_i is a bridge. Let H_1 be the graph with the generalized ear decomposition F_0, F_1, \dots, F_{i-1} and H_2 be the graph with the generalized ear decomposition $F_{i+1}, F_{i+2}, \dots, F_s$. We have $\psi(H_1) + \psi(H_2) = \psi(G) - 1$, $\text{card}(E(H_1)) + \text{card}(E(H_2)) = \text{card}(E(G)) - 1$, $\text{card}(V(H_1)) + \text{card}(V(H_2)) = \text{card}(V(G))$. by induction hypothesis we have $\text{card}(E(H_1)) - \text{card}(V(H_1)) + \psi(H_1) + 1 = i$, $\text{card}(E(H_2)) - \text{card}(V(H_2)) + \psi(H_2) + 1 = s - i$ hence by a simple calculation we get $s + 1 = \text{card}(E(G)) - \text{card}(V(G)) + \psi(G) + 1$. \square

Example 2.5. Let $G = (V, E)$ be a graph with the vertex set $V = \{a, b, c, d, e, f, g, h\}$ and $E(G) = \{ab, bc, ca, cd, de, ef, fg, gd, eh\}$. We have a generalized ear decomposition of G :

$$F_0 = abc, F_1 = cd, F_2 = defgd, F_3 = eh,$$

where F_0 is an odd ear, F_1, F_3 are bridges, F_2 is an even ear. So we have $\varphi(G) = 1, \psi(G) = 2$.

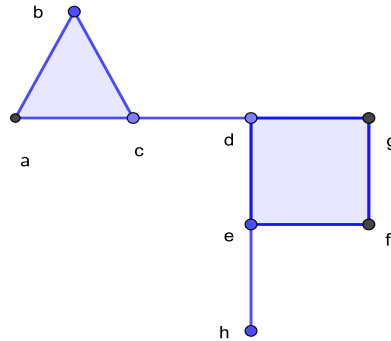


FIGURE 1. G and a generalized ear decomposition

Definition 2.6. The *contraction* of an edge $e = uv$ consists of identifying u and v and deleting the edge e . The *contraction* of an edge set F of G denoted G/F , is obtained from G by contracting each edge of F . For a connected graph G an edge set F is called *critical-making* if the contraction G/F results in a factor-critical graph.

A minimal critical-making set can therefore be used as a measure of how close a graph is to factor criticality.

Example 2.7. Let $G = (V, E)$ be a graph with the vertex set $V = \{a, b, c, d, e, f, g, h\}$ and $E = \{ab, bc, ca, cd, de, ef, fg, gd, eh\}$ (see example 2.5). By contracting the edges $F = \{cd, dg, eh\}$ we have a factor-critical graph in figure 2.

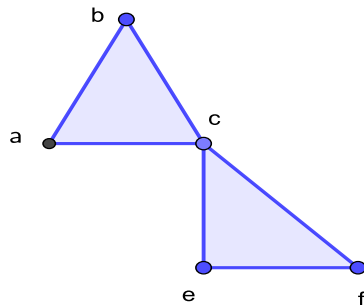


FIGURE 2. G/F contraction of the edges $F = \{cd, dg, eh\}$

Definition 2.8. The *subdivision* (or substitution) of an edge $e = uv$ in a cycle of a graph G means that we add a new vertex w and replace the edge e by two edges

uw, wv . The *subdivision* of a bridge $e = uv$ of a graph G means that we add a new vertex w and two edges uw, wv . The *subdivision* of an edge set F of a graph G means that we subdivide each edge e of F . The resulting graph is denoted by $G \succ F$.

Note that our definition of subdivision is a slight different from the usual one.

Example 2.9. Let $G = (V, E)$ be a graph in the example 2.7. By subdividing the edges $F = \{cd, dg, eh\}$ we have a factor-critical graph in figure 3.

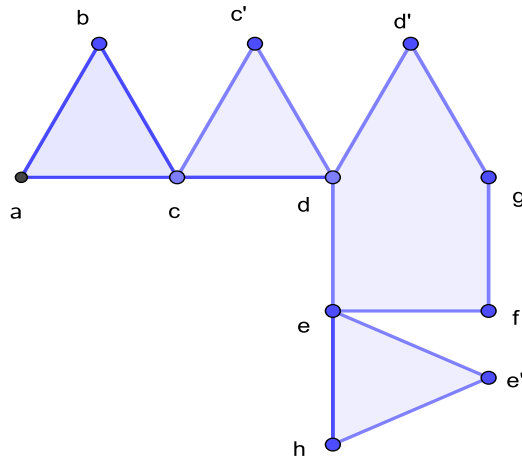


FIGURE 3. $G \succ F$ Subdivision of the edges $F = \{cd, dg, eh\}$

Szigeti [16] showed that for 2-edge-connected graphs, all the three approaches defined above are actually equivalent.

Theorem 2.10. (Szigeti [16]) *Let G be a 2-edge connected graph and k be a positive integer. Then the following statements are equivalent:*

- (1) $\varphi(G) = k$.
- (2) $\min\{\text{card}(F) \mid F \text{ is a critical-making set}\} = k$.
- (3) $\min\{\text{card}(F) \mid G \succ F \text{ is a factor-critical graph}\} = k$.

Since every bridge of G is contained in every critical-making set, we have

Theorem 2.11. *Let G be a connected graph and k be a positive integer. Then the following statements are equivalent:*

- (1) $\varphi(G) + \psi(G) = k$.
- (2) $\min\{\text{card}(F) \mid F \text{ is a critical-making set}\} = k$.
- (3) $\min\{\text{card}(F) \mid G \succ F \text{ is a factor-critical graph}\} = k$.

3. CONSTRUCTING FACTOR-CRITICAL GRAPHS BY REPLICATION

In this section we develop a method to construct factor-critical graphs from a simple connected non bipartite graph in a minimum number of steps by using replication of vertices.

Definition 3.1. (Schrijver [15]) Let $G = (V, E)$ be a graph with $V = \{P_1, \dots, P_d\}$ and $\mathbf{a} \in \mathbb{N}^d$ a non-zero vector. Set $A_i = \{P_i = P_i^{(1)}, \dots, P_i^{(a_i)}\}$ for each $a_i > 0$. The graph $S := p_{\mathbf{a}}(G)$ with the vertex set $V(S) = \cup_{a_i > 0} A_i$ and the edge set $E(S) = \{P_i^{(l)} P_j^{(m)} \mid P_i \in A_i, P_j \in A_j, P_i P_j \in E\}$ is called *the replication (or duplication)* of G by the vector \mathbf{a} . The *support* of S is the set $\text{Supp}(S) := V(S) \cap V = \text{Supp}(\mathbf{a})$. We denote $N_G(S) = N(V(S) \cap V)$. For small values of $a_i \leq 3$ sometimes we will write P_i, P'_i, P''_i instead of $P_i^{(1)}, P_i^{(2)}, P_i^{(3)}$. As usual we set $|\mathbf{a}| = \sum_i a_i$.

Definition 3.2. Let G be simple connected graph. It is well known that G is *non bipartite* if and only if it contains an odd cycle. Let $\varphi_{\text{odd}}(G)$ denote the minimum number of even ears in generalized ear decompositions of G such that the first ear is an odd cycle.

Clearly, a connected graph G is factor-critical if and only if $\varphi_{\text{odd}}(G) = \psi(G) = 0$.

Remark 3.3. Let G be a 2-edge connected non bipartite graph. In Section 5 we will prove that $\varphi_{\text{odd}}(G) = \varphi(G)$. The proof uses the works in graph theory of Frank [4] and Szigeti [16]. From now on we write $\varphi(G)$ instead $\varphi_{\text{odd}}(G)$.

In Lemma 4.7 of [14] we have developed a method for extending factor-critical graphs. We can apply this method to an ear decomposition starting with an odd cycle, since an odd cycle is factor-critical.

Lemma 3.4. *Let G be a connected non bipartite graph. There exists a vector \mathbf{a}_G with support in G such that $p_{\mathbf{a}_G + \mathbf{1}_G}(G)$ is factor-critical and $|\mathbf{a}_G| = \varphi(G) + \psi(G)$. Hence $\nu(p_{\mathbf{a}_G + \mathbf{1}_G}(G)) = \frac{1}{2}(\text{card}(G) + \varphi(G) + \psi(G) - 1)$.*

Proof. Since G is connected non bipartite graph, there exists a generalized ear decomposition of G : F_0, F_1, \dots, F_s , such that F_0 is an odd cycle with $\varphi(G)$ even ears. We set G_{i-1} be the union of F_0, \dots, F_{i-1} .

For each even ear $F_i : P_{i,1}, \dots, P_{i,j_i}$ we duplicate the vertex $P_{i,1}$, let $Q_{i,1} \in G_{i-1}$ a neighbor of $P_{i,1}$ and let F'_i be the path $F'_i : Q_{i,1}, P'_{i,1}, P_{i,2}, \dots, P_{i,j_i}$.

For each bridge $F_i : P_{i,1}, P_{i,2}$, with $P_{i,1} \in F_{i-1}$, we duplicate the vertex $P_{i,1}$. Let $Q_{i,1} \in G_{i-1}$ a neighbor of $P_{i,1}$ and let F'_i be the path $F'_i : Q_{i,1}, P'_{i,1}, P_{i,2}, P_{i,1}$. If F_i is an odd ear we set $F'_i = F_i$. For each vertex P in G , let a_P be the number of times minus one that P is duplicated by the above construction, and $a_P = 0$ if it no duplicated. Let \mathbf{a}_G be the vector with support in G with coordinates $a_P, P \in G$. Then $p_{\mathbf{a}_G + \mathbf{1}_G}(G)$ has an odd ear decomposition F'_0, F'_1, \dots, F'_t . In particular $p_{\mathbf{a}_G + \mathbf{1}_G}(G)$ is factor-critical. Moreover

$$|\mathbf{a}_G| + \text{card}(G) = \text{card}(p_{\mathbf{a}_G + \mathbf{1}_G}(G)) = \text{card}(G) + \varphi(G) + \psi(G),$$

so that $|\mathbf{a}_G| = \varphi(G) + \psi(G)$. Hence

$$\nu(p_{\mathbf{a}_G+1_G}(G)) = \frac{1}{2}(\text{card}(G) + \varphi(G) + \psi(G) - 1).$$

□

Example 3.5. Let $G = (V, E)$ be a graph with the vertex set $V = \{a, b, c, d, e, f, g, h\}$ as in the figure 1. Then we have generalized ear decomposition of G :

$$F_0 = abc, F_1 = cd, F_2 = defgd, F_3 = eh,$$

where F_0 is odd ear, F_1, F_3 are bridges, and F_2 is even ear. Therefore, we have that $\varphi(G) = 1$ and $\psi(G) = 2$. Choose the vector $\mathbf{a}_G = (0, 0, 1, 1, 1, 0, 0, 0) \in \mathbb{N}^8$ and put $S = p_{\mathbf{a}_G+1_G}(G)$. Then we have $V(S) = D(S) = \{a, b, c, c', d', e', f, g, h\}$ and S has an ear decomposition

$$F'_0 = abc, F'_1 = bc'dc, F'_2 = c'd'gd, F'_3 = defg, ehe'f,$$

where they are all odd ears. It is clear that

$$\nu(S) = \frac{1}{2}(\text{card}(G) + \varphi(G) + \psi(G) - 1) = \frac{1}{2}(8 + 1 + 2 - 1) = 5.$$

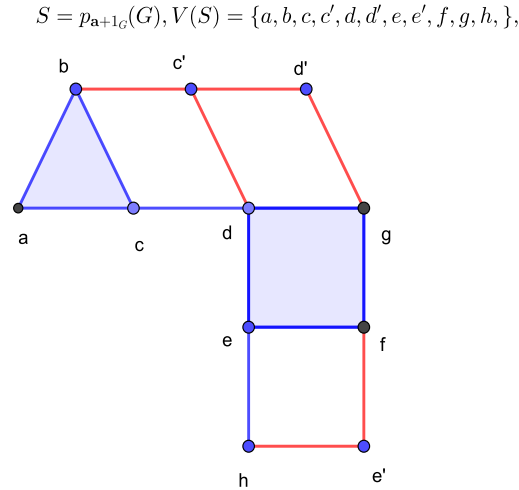


FIGURE 4. G and $p_{\mathbf{a}+1_G}(G)$

Lemma 3.6. Let G be a connected non bipartite graph. If for some vector \mathbf{a} , $p_{\mathbf{a}+1_G}(G)$ is factor-critical then $|\mathbf{a}| \geq \varphi(G) + \psi(G)$. As a consequence

$$\nu(p_{\mathbf{a}+1_G}(G)) \geq \frac{1}{2}(\text{card}(G) + \varphi(G) + \psi(G) - 1).$$

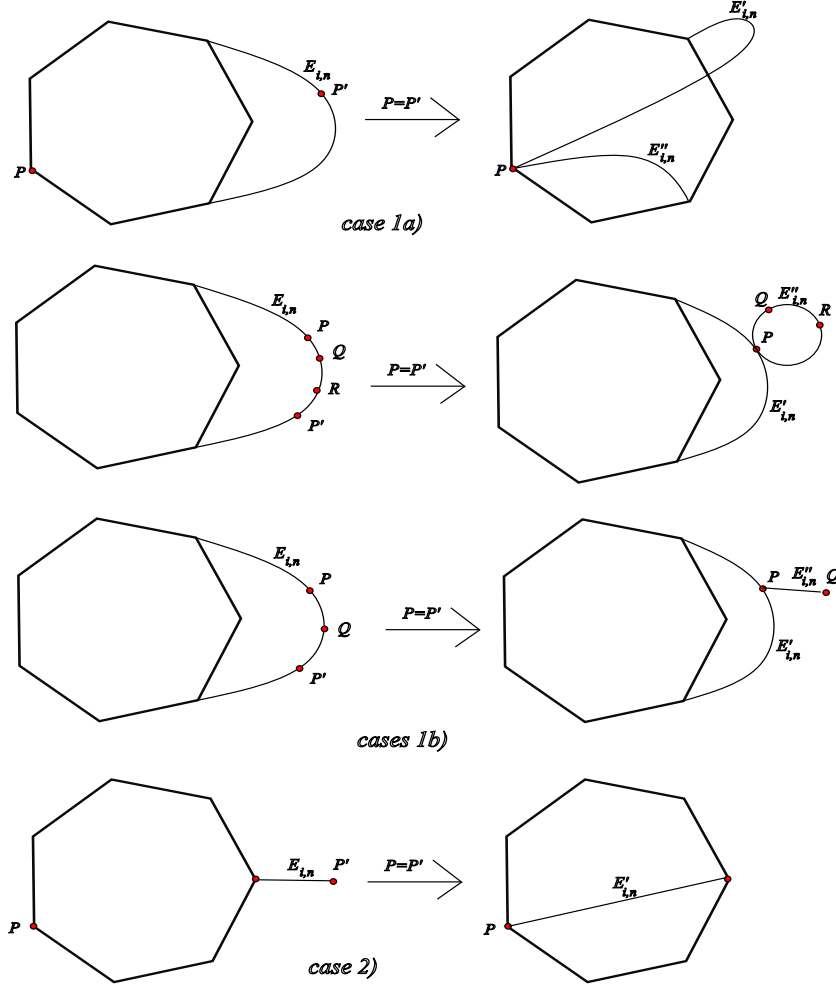


FIGURE 5. Identification of a duplicated vertex

Proof. Let $p_{\mathbf{a}+1_G}(G)$ be factor-critical for some vector \mathbf{a} . For all $n = 0, \dots, |\mathbf{a}|$, we will define by induction a vector \mathbf{a}_n such that $\text{Supp}(\mathbf{a}_n) \subset \text{Supp}(\mathbf{a})$, $|\mathbf{a}_n| = |\mathbf{a}| - n$, $p_{\mathbf{a}_n+1_G}(G)$ has a generalized ear decomposition and $n \geq (\varphi + \psi)(p_{\mathbf{a}_n+1_G}(G))$.

We set $\mathbf{a}_0 = \mathbf{a}$, since $p_{\mathbf{a}+1_G}(G)$ is factor-critical we have $F: F_0, F_1, \dots, F_s$ an odd ear decomposition and $0 = (\varphi + \psi)(p_{\mathbf{a}_0+1_G}(G))$.

Let $n \geq 0$ and suppose that we have a vector \mathbf{a}_n such that $\text{Supp}(\mathbf{a}_n) \subset \text{Supp}(\mathbf{a})$, $|\mathbf{a}_n| = |\mathbf{a}| - n$, $p_{\mathbf{a}_n+1_G}(G)$ has a generalized ear decomposition $E_{0,n}, \dots, E_{t,n}$ and $n \geq (\varphi + \psi)(p_{\mathbf{a}_n+1_G}(G))$. If $n = |\mathbf{a}|$ our claim is done. So suppose $n < |\mathbf{a}|$. Note that in a generalized ear decomposition we can always have $V(E_{0,n}) \subset V$. Let $E_{i,n}: P_{i,1}, \dots, P_{i,j_i}$ be a generalized ear such that all vertices in $E_{0,n} \cup E_{1,n} \cup \dots \cup E_{i-1,n}$ belong to V but there exists $P' \in V(E_{i,n}) \setminus V$ which is the duplicated vertex of $P \in E_{0,n} \cup E_{1,n} \cup \dots \cup E_{i,n}$. By identification of P and P' we get the vector $\mathbf{a}_{n+1} = \mathbf{a}_n - \mathbf{1}_{\{P\}}$. Ears or bridges $E_{j,n}$ for $j > i$ beginning in P' will begin in P . The ear or bridge $E_{i,n}$ will be modified, we have several cases

- (1) $E_{i,n}$ is an ear with several edges.
- (a) If $P' = P_{i,l}$, for some $l \geq 2$ is some internal vertex of $E_{i,n}$ and is the duplication of $P \in E_{0,n} \cup E_{1,n} \cup \dots \cup E_{i-1,n}$ then let $E'_{i,n} : P_{i,1}, \dots, P$, and $E''_{i,n} : P, P_{i,l+1}, \dots, P_{i,j_i}$. We replace the ear $E_{i,n}$ by the two ears $E'_{i,n}, E''_{i,n}$, ears or bridges ending in P' will end in P , so that we get an ear decomposition of $p_{\mathbf{a}_{n+1}+1_G}(G)$. Note that the number of edges in $E'_{i,n}, E''_{i,n}$ equals the number of edges in $E_{i,n}$. If $E_{i,n}$ is an odd ear then one of $E'_{i,n}, E''_{i,n}$ is an even ear and the other is odd so that $n+1 \geq (\varphi + \psi)(p_{\mathbf{a}_{n+1}+1_G}(G)) = (\varphi + \psi)(p_{\mathbf{a}_n+1_G}(G)) + 1$. If $E_{i,n}$ is an even ear then both $E'_{i,n}, E''_{i,n}$ are even or odd ears. If both ears are odd then $(\varphi + \psi)(p_{\mathbf{a}_{n+1}+1_G}(G)) = (\varphi + \psi)(p_{\mathbf{a}_n+1_G}(G)) - 1$. If both ears are even then $(\varphi + \psi)(p_{\mathbf{a}_{n+1}+1_G}(G)) = (\varphi + \psi)(p_{\mathbf{a}_n+1_G}(G)) + 1$. In both cases we have $n+1 \geq (\varphi + \psi)(p_{\mathbf{a}_{n+1}+1_G}(G))$.
- (b) If $P' = P_{i,m}$, for some $m \geq 2$ is some internal vertex of $E_{i,n}$ and $P = P_{i,l} \in E_{i,n}$ then we can suppose $l < m$. We have the following cases:
 If $m = l+2$ then we get the odd path $E'_{i,n} : P_{i,1}, \dots, P_{i,l}, P_{i,l+3}, \dots, P_{i,j_i}$ and the edge $E''_{i,n} : P_{i,l}P_{i,l+1}$. As above we replace the ear $E_{i,n}$ by the two ears $E'_{i,n}, E''_{i,n}$, ears or bridges ending in P' will end in P , so that we get a ear decomposition of $p_{\mathbf{a}_{n+1}+1_G}(G)$ and we still have $n+1 \geq (\varphi + \psi)(p_{\mathbf{a}_{n+1}+1_G}(G))$.
 If $m > l+2$ then we have the path $E'_{i,n} : P_{i,1}, \dots, P_{i,l}, P_{i,m+1}, \dots, P_{i,j_i}$ and $E''_{i,n} : P_{i,l}, \dots, P_{i,m-1}, P_{i,l}$. As above we replace the ear $E_{i,n}$ by the two ears $E'_{i,n}, E''_{i,n}$, ears or bridges ending in P' will end in P , other ears are unchanged, so that we get a generalized ear decomposition of $p_{\mathbf{a}_{n+1}+1_G}(G)$ and we still have $n+1 \geq (\varphi + \psi)(p_{\mathbf{a}_{n+1}+1_G}(G))$.
- (2) $E_{i,n} : P_{i,1}, P_{i,2}$ is a bridge with $P_{i,2} \notin E_{0,n} \cup E_{1,n} \cup \dots \cup E_{i-1,n}$. So that $P' = P_{i,2}$ and $P \in E_{0,n} \cup E_{1,n} \cup \dots \cup E_{i-1,n}$. We have that either $P_{i,1}, P$ is an edge in $E_{0,n} \cup E_{1,n} \cup \dots \cup E_{i-1,n}$ or a chord $E'_{i,n} : P_{i,1}, P$. In this last case we replace the ear $E_{i,n}$ by $E'_{i,n}$. In both cases we get a generalized ear decomposition of $p_{\mathbf{a}_{n+1}+1_G}(G)$ and we have $n+1 \geq (\varphi + \psi)(p_{\mathbf{a}_{n+1}+1_G}(G))$.

□

Definition 3.7. Let $G = G_1 \cup \dots \cup G_t$ be a simple graph, which is a disjoint union of t connected non bipartite graphs $G_i, i = 1, \dots, t$. We set

$$\nu^*(G) = \frac{1}{2}(\text{card}(G) + \varphi(G) + \psi(G) - t).$$

Theorem 3.8. Let $G = G_1 \cup \dots \cup G_t$ be a simple graph, which is a disjoint union of t connected non bipartite graphs $G_i, i = 1, \dots, t$. Then

$$\min\{\nu(p_{\mathbf{a}+1_G}(G)) \mid p_{\mathbf{a}+1_G}(G) \text{ is matching - critical}\} = \nu^*(G).$$

Let $k \in \mathbb{N}$. The following statements are equivalent:

- (1) $\varphi(G) + \psi(G) = k$
- (2) $\min\{|\mathbf{a}| \mid p_{\mathbf{a}+1_G}(G) \text{ is matching - critical}\} = k$

Proof. Let G be a connected non bipartite graph, by lemma 3.6 we have

$$\min\{\nu(p_{\mathbf{a}+1_G}(G)) \mid p_{\mathbf{a}+1_G}(G) \text{ is factor critical}\} = \frac{1}{2}(\text{card}(G) + \varphi(G) + \psi(G) - 1).$$

Let $G = G_1 \cup \dots \cup G_t$ be a simple graph, which is a disjoint union of t connected non bipartite graphs $G_i, i = 1, \dots, t$. For $i = 1, \dots, t$, by lemma 3.6 we have

$$\min\{\nu(p_{\mathbf{a}+1_{G_i}}(G_i)) \mid p_{\mathbf{a}+1_{G_i}}(G_i) \text{ is factor critical}\} = \nu^*(G_i).$$

By lemma 3.4 let \mathbf{a}_{G_i} be a vector such that $p_{\mathbf{a}_{G_i}+1_{G_i}}(G_i)$ is factor-critical and $\nu(p_{\mathbf{a}_{G_i}+1_{G_i}}(G_i)) = \nu^*(G_i)$. Let $\mathbf{a}_G = \sum_{i=1}^t \mathbf{a}_{G_i}$ then $p_{\mathbf{a}_G+1_G}(G)$ is matching-critical, $\nu(p_{\mathbf{a}_G+1_G}(G)) = \sum_{i=1}^t \nu^*(G_i)$. It is clear that

$$\nu(p_{\mathbf{a}_G+1_G}(G)) = \min\{\nu(p_{\mathbf{a}+1_G}(G)) \mid p_{\mathbf{a}+1_G}(G) \text{ is matching - critical}\},$$

and

$$\sum_{i=1}^t \nu^*(G_i) = \frac{1}{2}(\text{card}(G) + \varphi(G) + \psi(G) - t).$$

For a connected non bipartite graph, the proof of the second assertion follows immediately from lemma 3.6. If the graph is disconnected then the proof is similar to the proof of the first assertion. \square

4. STABLE ASSOCIATED PRIMES, DSTAB AND ASTAB

It is known from [1] that $\text{Ass}(I_G^k) = \text{Ass}(I_G^{k+1})$ for k large enough and $\text{Ass}(I_G^k)$ is called *the set of stable associated primes* of I_G . Furthermore if G is a non bipartite graph, then $\mathbf{m}^{1_V} \in \text{Ass}(I_G^k)$ for k large enough. The *persistence property* $\text{Ass}(I_G^k) \subset \text{Ass}(I_G^{k+1})$ for $k \geq 1$ is proved in [13]. Moreover, we have the *strong persistence property* proved by [14, Theorem 5.14] that every irreducible component of I_G^k induces several irreducible components of I_G^{k+1} with the same associated prime. In this section we will describe the sets $\text{Ass}(I_G^k)$ for any $k \geq 1$ in terms of some subsets of V . We denote by $\text{Irr}(I_G^k)$ the set of irreducible components of I_G^k . Note that an irreducible component of I_G^k , whose radical is \mathbf{m}^{1_U} , can be written as $\mathbf{m}^{\mathbf{a}+1_U}$, for some vector \mathbf{a} with $\text{Supp}(\mathbf{a}) \subset U$.

Definition 4.1. Let $G = (V, E)$ be a non bipartite graph.

$$\text{dstab}(I_G) = \min\{k \mid \mathbf{m}^{1_V} \in \text{Ass}(I_G^k)\}.$$

$$\text{astab}(I_G) = \min\{k \mid \text{Ass}(I_G^k) = \text{Ass}(I_G^{k+i}), i \geq 1\}.$$

Let $U \subset V$ such that $\mathbf{m}^{1_U} \in \text{Ass}(I_G^k)$ for some $k \geq 1$. It is well known that $\mathbf{m}^{1_U} \in \text{Ass}(I_G)$ if and only if $V \setminus U$ is a maximal coclique set. From now on we are interested in embedded associated primes, so we can suppose $k \geq 2$. The description of all the embedded irreducible components of I_G^k is given in the following theorems from [14].

Theorem 4.2. *Let $\mathbf{m}^{\mathbf{a}+1_U}$ be an embedded irreducible component of I_G^k and $Z = V \setminus U$. Assume that $\text{Supp}(\mathbf{a}) \cap N(Z) \neq \emptyset$. Let $\mathbf{a} = \mathbf{b} + \mathbf{c}$ with $\mathbf{b}, \mathbf{c} \in \mathbb{N}^d$ the unique decomposition such that $\text{Supp}(\mathbf{b}) = \text{Supp}(\mathbf{a}) \setminus N(Z)$, $\text{Supp}(\mathbf{c}) = \text{Supp}(\mathbf{a}) \cap N(Z)$ and $\delta = |\mathbf{c}|$. Then $\text{Supp}(\mathbf{b}) \cap N(Z) = \emptyset$ and $\mathbf{m}^{\mathbf{b}+1_U}$ is an embedded irreducible component of $I_G^{k-\delta}$.*

Recall the Gallai-Edmonds canonical decomposition of a graph as given by J. Edmonds [3] and T. Gallai [6]. For more details see [18, Theorem 1.5.3] or [14, Section 4]. For any simple graph G , denote by $D(G)$ the set of all vertices in G which are missed by at least one maximum matching of G , and $A(G)$ the set of vertices in $V - D(G)$ that are adjacent to at least one vertex in $D(G)$. Let $C(G) = V - A(G) - D(G)$. More generally let $S \subset V$ and G_S its induced subgraph of G . The induced graphs $A(G_S), D(G_S), C(G_S)$ will be denoted by $A(S), D(S), C(S)$. The graph $D(S)$ is matching-critical if it is a disjoint union of factor-critical graphs.

Theorem 4.3. *Let $k \geq 2$ be an integer, $\mathbf{a} \in \mathbb{N}^d$ be a nonzero vector, $U \subset V$ such that $\text{Supp}(\mathbf{a}) \subset U$. Let denote $Z := V \setminus U$ and $p_{\mathbf{a}}(G)$ the replication of G by \mathbf{a} . Assume that $\text{Supp}(\mathbf{a}) \cap N(Z) = \emptyset$. Then the following statements are equivalent:*

- (i) $\mathbf{m}^{\mathbf{a}+1_U}$ is an embedded irreducible component of I_G^k .
- (ii) $p_{\mathbf{a}}(G)$ and Z satisfy the following properties
 - (1) Z is a coclique set, $\nu(p_{\mathbf{a}}(G)) = k - 1$ and $V = N_G(D(p_{\mathbf{a}}(G))) \cup Z \cup N(Z)$.
 - (2) $C(p_{\mathbf{a}}(G)) = \emptyset$, i.e. $p_{\mathbf{a}}(G) = D(p_{\mathbf{a}}(G)) \cup A(p_{\mathbf{a}}(G))$ in the Gallai-Edmonds's canonical decomposition.

Moreover in [14, Section 5] we also proved that every connected component of $D(p_{\mathbf{a}}(G))$ has at least 3 vertices. Let $\mathbf{m}^{1_U} \in \text{Ass}(I_G^k)$ be an embedded associated prime, we define

$$\text{astab}(I_G, U) := \min\{k \mid \mathbf{m}^{1_U} \in \text{Ass}(I_G^k)\}.$$

Note that $\text{dstab}(I_G) = \text{astab}(I_G, V)$.

Lemma 4.4. *Let $U \subset V$. Then $\mathbf{m}^{1_U} \in \text{Ass}(I_G^k)$ is an embedded associated prime for some $k \geq 2$ if and only if*

- (1) $Z = V \setminus U$ is a coclique set.
- (2) There exists a vector \mathbf{a} such that $\text{Supp}(\mathbf{a}) \cap (N(Z) \cup Z) = \emptyset$, $p_{\mathbf{a}}(G)$ is matching-critical, $V = N_G(p_{\mathbf{a}}(G)) \cup Z \cup N(Z)$ and $\nu(p_{\mathbf{a}}(G)) \leq k - 1$.

As a consequence we have

$$\text{astab}(I_G, U) = \min\{k \mid \mathbf{m}^{\mathbf{a}+1_U} \in \text{Irr}(I_G^k)\} = \min\{\nu(p_{\mathbf{a}}(G)) + 1\},$$

where the vector \mathbf{a} satisfies property (2).

Proof. Let $U \subset V$ and $Z = V \setminus U$. Then $\mathbf{m}^{1_U} \in \text{Ass}(I_G^k)$ if and only if there exists a vector \mathbf{b} such that $\mathbf{m}^{\mathbf{b}+1_U}$ is an embedded irreducible component of I_G^k . By Theorem 4.2 we can assume that $\text{Supp}(\mathbf{b}) \cap (N(Z) \cup Z) = \emptyset$. By Theorem 4.3, we have that Z is a coclique set, $V = N_G(p_{\mathbf{b}}(G)) \cup Z \cup N(Z)$, $\nu(p_{\mathbf{b}}(G)) = k - 1$, and $p_{\mathbf{b}}(G) = D(p_{\mathbf{b}}(G)) \cup A(p_{\mathbf{b}}(G))$. The graph $D(p_{\mathbf{b}}(G))$ is matching-critical and we have by Gallai-Edmonds decomposition theorem that $\nu(D(p_{\mathbf{b}}(G))) = k - 1 - \text{card}(A(p_{\mathbf{b}}(G)))$. Let \mathbf{a} be the vector such that $p_{\mathbf{a}}(G) = D(p_{\mathbf{b}}(G))$. We can see $p_{\mathbf{a}}(G)$ satisfies the conditions of Theorem 4.3, so $\mathbf{m}^{\mathbf{a}+1_U}$ is an embedded irreducible component of $I_G^{k'}$, where $k' = k - \text{card}(A(p_{\mathbf{b}}(G))) \leq k$. \square

Now we can describe the associated primes of I_G^k for $k \geq 2$. Our result extends [13, Proposition 3.3] which considered the case of \mathbf{m}^{1_U} and [2, Theorem 4.1].

Theorem 4.5. \mathbf{m}^{1U} is an embedded associated prime of I_G^k for some k if and only if $Z := V \setminus U$ is a coclique set, eventually the empty set, $U \neq N(Z)$ and every connected component of $G_{U \setminus N(Z)}$ contains an odd cycle.

Proof. If \mathbf{m}^{1U} is an embedded associated prime of I_G^k for some k , then by Lemma 4.4, Z is a coclique set, there exists a vector \mathbf{a} such that $\text{Supp}(\mathbf{a}) \cap (N(Z) \cup Z) = \emptyset$, $p_{\mathbf{a}}(G)$ is matching-critical and $V = N_G(p_{\mathbf{a}}(G)) \cup Z \cup N(Z)$. Let $W = \text{Supp}(\mathbf{a})$ and G_W the induced subgraph of G . Since $p_{\mathbf{a}}(G)$ is matching-critical and every connected component has at least 3 vertices, then every connected component of G_W has an odd cycle. If $T \subset N(W) \setminus W$ then every connected component of $G_{W \cup T}$ has an odd cycle. Note that $U \setminus N(Z) = N(W) \setminus N(Z) = W \cup T$ for some subset $T \subset N(W) \setminus W$. So every connected component of $G_{U \setminus N(Z)}$ has an odd cycle. Reciprocally if $U \neq N(Z)$ and every connected component of $G_{U \setminus N(Z)}$ contains an odd cycle, then by Lemma 3.6, there exists a vector \mathbf{b} , $\text{Supp}(\mathbf{b}) \subset U \setminus N(Z)$ such that $p_{\mathbf{b}+1_{U \setminus N(Z)}}(G)$ is matching-critical and $V = N_G(p_{\mathbf{b}+1_{U \setminus N(Z)}}(G)) \cup Z \cup N(Z)$. Thus by Theorem 4.3 we have that \mathbf{m}^{1U} is an embedded associated prime of I_G^k for some $k = \nu(p_{\mathbf{b}+1_{U \setminus N(Z)}}(G)) + 1$. \square

Once we know a stable associated prime \mathbf{m}^{1U} , finding the smallest k such that \mathbf{m}^{1U} is an associated prime of I_G^k requires a finer analysis of some subsets of U .

Definition 4.6. Let $G = (V, E)$ be a simple, connected non bipartite graph. Let $U \subset V$ and $Z = V \setminus U$. If Z is an independent set then a set $W \subset V \setminus (Z \cup N(Z))$ is called U -dominant if $V = N(W) \cup N(Z) \cup Z$ and U -dominant* if either $W = \emptyset$ or it is U -dominant and every connected component of G_W contains an odd cycle. We will denote by $\mathcal{D}^*(U)$ the set of all subsets $W \subset U$ that are U -dominant*. If Z is not an independent set then we set $\mathcal{D}^*(U) = \emptyset$.

Remark 4.7. With the notation introduced in 4.6, we have that \mathbf{m}^{1U} is a associated prime of I_G if and only if $\mathcal{D}^*(U) = \{\emptyset\}$. Theorem 4.5 can be reformulated as follows: \mathbf{m}^{1U} is an embedded associated prime of I_G^k for some k if and only if $\mathcal{D}^*(U) \neq \emptyset, \{\emptyset\}$ and in this case we have $U \setminus N(Z) \in \mathcal{D}^*(U)$.

In 3.7 we have defined $\nu^*(G_W)$ for each $\emptyset \neq W \in \mathcal{D}^*(U)$. If $W = \emptyset$ then we set $\nu^*(G_W) = 0$.

Example 4.8. Let $G = (V, E)$ be a graph with the vertex set $V = \{a, b, c, d, e, f, g, h\}$ as in the figure 1. Now we look for sets $U \subset V$ such that $\mathcal{D}^*(U) \neq \emptyset$. Note that $Z = V \setminus U$ is an independent set and if $W \neq \emptyset$ is U -dominant*, then G_W is connected and contains the cycle with vertices a, b, c , since there is a unique odd cycle in G . Note also that $N(W) \cap Z = \emptyset$. We can describe the possible cases of the independent set Z such that $\mathcal{D}^*(U) \neq \emptyset$.

- (1) If $\text{card}(Z) = 0$ then $\mathcal{D}^*(U) \neq \emptyset$ if and only if $U_1 = V$ and $W_1 = \{a, b, c, d, e\} \in \mathcal{D}^*(U_1)$, $\nu^*(G_{W_1}) = 3$.
- (2) If $\text{card}(Z) = 1$ then $\mathcal{D}^*(U) \neq \emptyset$ if and only if $Z_2 = \{h\}$. We have $U_2 = \{a, b, c, d, e, f, g\}$ and $W_2 = \{a, b, c, d, g\} \in \mathcal{D}^*(U_2)$, $\nu^*(G_{W_2}) = 3$.

- (3) If $\text{card}(Z) = 2$ then $\mathcal{D}^*(U) \neq \emptyset$ if and only if $Z_3 = \{e, g\}$, $Z_4 = \{f, h\}$, or $Z_5 = \{g, h\}$. We have $U_3 = \{a, b, c, d, f, h\}$, $U_4 = \{a, b, c, d, e, g\}$ or $U_5 = \{a, b, c, d, e, f\}$. and $W_3 = W_4 = W_5 = \{a, b, c\} \in \mathcal{D}^*(U_i)$, $\nu^*(G_{W_i}) = 1$, for $i = 3, 4, 5$.
- (4) If $\text{card}(Z) \geq 3$ then Z is maximal, $\mathcal{D}^*(U) = \{\emptyset\}$.

As a consequence we have:

- (1) $\text{Ass}(I_G^2) \setminus \text{Ass}(I_G) = \{\mathbf{m}^{1U_3}, \mathbf{m}^{1U_4}, \mathbf{m}^{1U_5}\}$, $\text{Ass}(I_G^3) = \text{Ass}(I_G^2)$,
- (2) $\text{Ass}(I_G^4) \setminus \text{Ass}(I_G^3) = \{\mathbf{m}^{1U_1}, \mathbf{m}^{1U_2}\}$, $\text{Ass}(I_G^i) = \text{Ass}(I_G^{i+1})$ for $i \geq 4$,
- (3) $\text{astab}(I_G) = \text{dstab}(I_G) = 4$.

Theorem 4.9. *Let G be a simple, connected non bipartite graph, $U \subset V$ such that \mathbf{m}^{1U} is an embedded associated prime of I_G^k for some k . Then*

$$\text{astab}(I_G, U) = \min\{1 + \nu^*(G_W) \mid W \in \mathcal{D}^*(U)\}.$$

Proof. Let $p_{\mathbf{a}}(G)$ be a matching-critical graph such that $\mathbf{m}^{\mathbf{a}+1U}$ is an embedded irreducible component of I_G^k . Let $W = \text{Supp}(\mathbf{a})$, then $W \in \mathcal{D}^*(U)$. Let W_1, \dots, W_{t_W} be the connected components of G_W and \mathbf{a}_i the restriction of \mathbf{a} to W_i , so that $\mathbf{a} = \sum_{i=1}^{t_W} \mathbf{a}_i$, and $p_{\mathbf{a}_i}(G)$ is factor-critical. There exist by theorem 3.8 the vectors \mathbf{b}_i such that $p_{\mathbf{b}_i+1_{W_i}}(G)$ is factor-critical and

$$\nu(p_{\mathbf{b}_i+1_{W_i}}(G)) = \frac{1}{2}(\text{card}(W_i) + \varphi(G_{W_i}) + \psi(G_{W_i}) - 1) \leq \nu(p_{\mathbf{a}_i}(G)).$$

Let $\mathbf{b}(W) = \sum_{i=1}^{t_W} \mathbf{b}_i$. It follows that $p_{\mathbf{b}(W)+1_W}(G)$ is matching-critical and

$$\nu(p_{\mathbf{b}(W)+1_W}(G)) = \min\{\nu(p_{\mathbf{a}}(G)) \mid W = \text{Supp}(\mathbf{a}), p_{\mathbf{a}}(G) \text{ is matching-critical}\}.$$

Note that $\nu(p_{\mathbf{b}+1_W}(G)) = \nu^*(G_W)$. Hence

$$\text{astab}(I_G, U) = \min\{1 + \nu^*(G_W) \mid W \in \mathcal{D}^*(U)\}.$$

□

Corollary 4.10. *Let G be a simple, connected non bipartite graph. Then*

$$\text{Ass}(I_G^k) = \{\mathbf{m}^{1U} \mid U \subset V, \mathcal{D}^*(U) \neq \emptyset, \min_{W \in \mathcal{D}^*(U)} \nu^*(G_W) \leq k - 1\}.$$

Proof. By the definition of $\text{Ass}(I_G^k) = \{\mathbf{m}^{1U} \mid U \subset V, \text{astab}(I_G, U) \leq k\}$, we can see that the proof follows from the above theorem. □

As an immediate consequence of Theorem 4.5 and Corollary 4.10 we have

Theorem 4.11. *Let G be a simple, connected non bipartite graph. We have*

$$\text{astab}(I_G) = \max_{\{U \subset V \mid \mathcal{D}^*(U) \neq \emptyset\}} \{\min\{1 + \nu^*(G_W) \mid W \in \mathcal{D}^*(U)\}\}.$$

$$\text{dstab}(I_G) = \min\{1 + \nu^*(G_W) \mid W \in \mathcal{D}^*(V)\}.$$

Proof. We have $\text{dstab}(I_G) = \text{astab}(I_G, V)$ and

$$\text{astab}(I_G) = \max_{\{U \subset V \mid \mathcal{D}^*(U) \neq \emptyset\}} \{\text{astab}(I_G, U)\}.$$

□

Now we can give upper bounds for $\text{astab}(I_G)$ and $\text{dstab}(I_G, V)$.

Theorem 4.12. *Let G be a simple connected non bipartite graph. Then we have*

$$\text{dstab}(I_G) \leq \frac{1}{2}(\text{card}(E(G)) + \psi(G) - 1) + 1.$$

$$\text{astab}(I_G) \leq \max_U \left\{ \frac{1}{2}(\text{card}(E(G_{U \setminus N(Z)})) + \psi(G_{U \setminus N(Z)}) - 1) + 1 \right\},$$

where the max is taken over all the sets $U \subset V(G)$ such that $Z := V \setminus U$ is an independent set, $U \neq N(Z)$ and every connected component of $G_{U \setminus N(Z)}$ contains an odd cycle.

Proof. By Lemma 2.4, for every connected non bipartite graph H we have

$$\varphi(H) + \psi(H) \leq \text{card}(E(H)) - \text{card}(V(H)) + \psi(H).$$

Hence

$$\nu^*(H) = \frac{1}{2}(\text{card}(V(H)) + \varphi(H) + \psi(H) - 1) \leq \frac{1}{2}(\text{card}(E(H)) + \psi(H) - 1).$$

Our claim follows immediately from Theorems 4.5 and Theorem 4.11. \square

As a consequence we get a very simple upper bound for $\text{astab}(I_G)$.

Theorem 4.13. *Let G be a simple connected non bipartite graph and $2k - 1$ the minimum length of an odd cycle in G . Then we have*

$$\text{astab}(I_G) \leq \text{card}(E(G)) - k + 1.$$

Proof. With the notation of Theorem 4.12, we have

$$\text{card}(E(G_{U \setminus N(Z)})) \leq \text{card}(E(G)), \psi(G_{U \setminus N(Z)}) \leq \text{card}(E(G)) - (2k - 1).$$

Hence

$$\text{astab}(I_G) \leq \frac{1}{2}(\text{card}(E(G)) + \text{card}(E(G)) - (2k - 1) - 1) + 1 = \text{card}(E(G)) - k + 1,$$

as required. \square

Let G be a cycle of the length $2k - 1$. Then $\text{card}(E(G)) - k + 1 = k$. It is well known that $\text{astab}(I_G) = \text{dstab}(I_G) = k$. Therefore one can see from Theorem 4.13 that our bound is tight.

Example 4.14. Let $G = (V, E)$ be a graph with the vertex set $V = \{a_1, a_2, \dots, a_{2n+1}\}$, which is the union of n triangles with a common vertex a_1 (See Figure 6). Apply Theorem 4.11, let $U \subset V, U \neq V$ such that $\mathcal{D}^*(U) \neq \emptyset$ and $\mathcal{D}^*(U) \neq \{a_1\}$. Note that $Z = V \setminus U$ is non-empty independent set with $a_1 \in N(Z)$. So every subset $W \subset V \setminus (N(Z) \cup Z)$ can not contain an odd cycle. Hence $U = V$ is the unique set such that $\mathcal{D}^*(U) \neq \emptyset, \mathcal{D}^*(U) \neq \{a_1\}$. As a consequence we have that $\text{Ass}(I_G^i) = \text{Ass}(I_G) \cup \{\mathbf{m}\}$ for $i \geq 2$. Hence $\text{astab}(I_G) = \text{dstab}(I_G) = 2$.

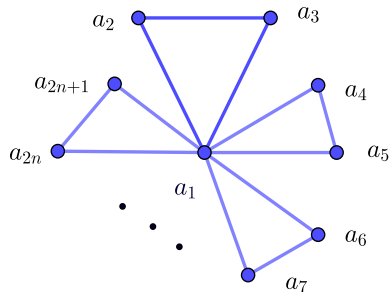


FIGURE 6. Union of triangles

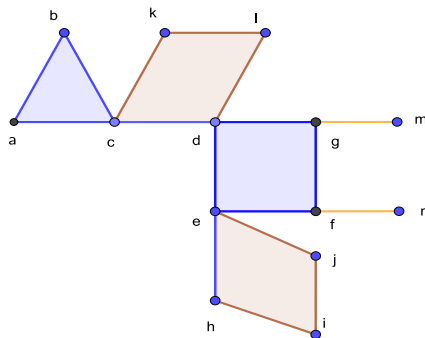


FIGURE 7. G and G_1

Example 4.15. Let $G_1 = (V_1, E_1)$ be a graph as in the Figure 7 with the vertex set $V_1 = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n\}$. Then the graph G of Figure 1 is G_1 -dominant* and gives the minimum in Theorem 4.11. So we have $\text{dstab}(I_{G_1}) = 6$. The irreducible component $(a^2, b^2, c^3, d^3, e^3, f^2, g^2, h^2, i, j, k, l, m, n)$ appears in the irreducible decomposition of $I_{G_1}^6$.

Example 4.16. Let $G = (V, E)$ be the graph with the vertex set $V = \{a_1, a_2, \dots, a_{13}\}$ and $E = \{a_i a_{i+1} \mid i = 1, \dots, 10\} \cup \{a_1 a_{11}, a_1 a_{12}, a_1 a_{13}, a_{12} a_{13}\}$. Note that G is the union of a triangle and a hendecagon represented in Figure 7.

Let $W = \{a_1, a_2, \dots, a_{10}, a_{11}\}$ and G_W be the induced subgraph of G . We can see that G_W is factor-critical and G -dominant* with $\nu(G_W) = 5$. We have by Theorem 4.11 that $\text{dstab}(I_G) = 6$.

Now we look for embedded associated primes strictly contained in the maximal ideal \mathfrak{m}^{1_V} . By Theorem 4.11, we should look for non-maximal independent sets $Z \subset V$ such

that $\mathcal{D}^*(U) \neq \emptyset$ where $U = V \setminus Z$, that is there exists a set $W \subset V \setminus (Z \cup N(Z))$ for which every connected component of G_W contains an odd cycle. In our example, this implies that W is connected and $Z \subset \{a_3, a_4, \dots, a_{10}\}$. Since $\text{astab}(I_G, U)$ is determined by the number of edges in W , we have $\text{astab}(I_G, U)$ will be the largest possible if Z has cardinality one. We can examine all 8 cases where $Z_i = \{a_i\}$, for $i = 3, \dots, 10$.

Indeed, let $U_i = V \setminus Z_i$. We will see by Theorem 4.11 that $\text{astab}(I_G, U_i) = 7$ for $i = 4, \dots, 9$ and $\text{astab}(I_G, U_i) = 8$ for $i = 3, 10$. For instance we consider the cases $Z_6 = \{a_6\}, Z_3 = \{a_3\}$ since the other cases are similar.

- (1) If $Z_6 = \{a_6\}$ then $N(Z_6) = \{a_5, a_7\}$. So $W_6 = \{a_1, a_2, a_3, a_9, a_{10}, a_{11}, a_{12}, a_{13}\}$ is U_6 -dominant* and we have by Theorem 4.11 that $\text{astab}(I_G, U_6) = 7$.
- (2) If $Z_3 = \{a_3\}$ then similarly we have $N(Z_3) = \{a_2, a_4\}$. We can see that the set $W_3 = \{a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}\}$ is U_3 -dominant* and we also have by Theorem 4.11 that $\text{astab}(I_G, U_3) = 8$.

Finally we get $\text{astab}(I_G) = 8$.

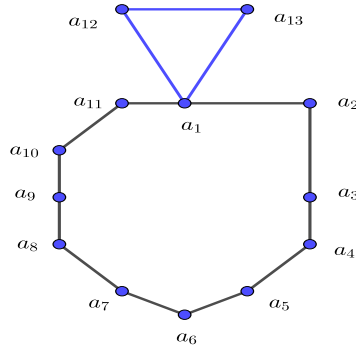


FIGURE 8. graph G

Now we can do a more general case.

Example 4.17. Let $G = (V, E)$ be a graph with the vertex set $V = \{a_1, a_2, \dots, a_{2l+3}\}$ and $E = \{a_i a_{i+1} \mid i = 1, \dots, 2l\} \cup \{a_1 a_{2l+1}, a_1 a_{2l+2}, a_1 a_{2l+3}, a_{2l+1} a_{2l+3}\}$. Then $\text{dstab}(I_G) = l + 1$, $\text{astab}(I_G) = 2l - 2$.

From figure 9, we can see that G is the union of a triangle and a polygon with $2l + 1$ vertices. Let $W = \{a_1, a_2, \dots, a_{2l+1}\}$ and G_W be the induced subgraph of G . Then G_W is factor-critical and G -dominant*, with $\nu(G_W) = l$. By Theorem 4.11 we have $\text{dstab}(I_G) = l + 1$.

Now we look for embedded associated primes strictly contained in the maximal ideal \mathfrak{m}^{1V} . By the theorem 4.11 we should look for non-maximal independent sets $Z \subset V$ such that $\mathcal{D}^*(U) \neq \emptyset$ where $U = V \setminus Z$, that is there exists a set $W \subset V \setminus (Z \cup N(Z))$ for which every connected component of G_W contains an odd cycle, in our example this

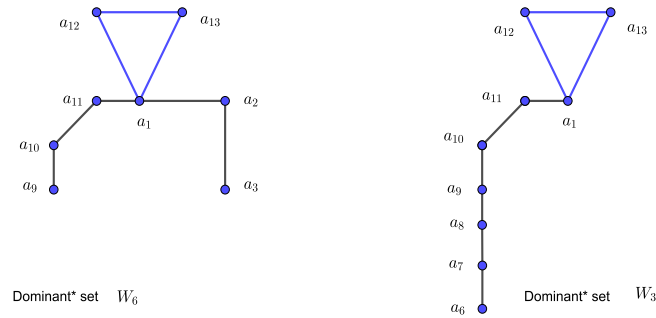


FIGURE 9. Dominants* sets

implies that W is connected and $Z \subset \{a_3, a_4, \dots, a_{2l}\}$. $\text{astab}(I_G, U)$ is determined by the number of edges in W , so that $\text{astab}(I_G, U)$ will be the largest possible when Z has cardinal one. Let examine all cases.

Let $Z_i = \{a_i\}$, $U_i = V \setminus Z_i$, we will see by theorem 4.11 that $\text{astab}(I_G, U_i) = 2l - 3$ for $i = 4, \dots, 2l - 1$ and $\text{astab}(I_G, U_i) = 2l - 2$ for $i = 3, 2l$.

To be more precise let consider $Z_{l+1} = \{a_{l+1}\}$ and $Z_3 = \{a_3\}$, the other cases are similar.

We have $N(Z_{l+1}) = \{a_l, a_{l+2}\}$. So that $W_{l+1} = \{a_1, \dots, a_{l-2}, a_{l+4}, \dots, a_{2l+3}\}$ is U_{l+1} -dominant*, so by theorem 4.11 we have $\text{astab}(I_G, U_{l+1}) = 2l - 3$.

Consider $Z_3 = \{a_3\}$, we have $N(Z_3) = \{a_2, a_4\}$. So that $W_3 = \{a_1, a_6, \dots, a_{2l+3}\}$ is U_3 -dominant*, so by theorem 4.11 we have $\text{astab}(I_G, U_3) = 2l - 2$. Finally we get $\text{astab}(I_G) = 2l - 2$.

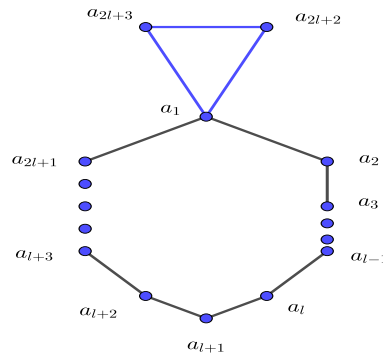


FIGURE 10. Graph G with $2l + 3$ vertices

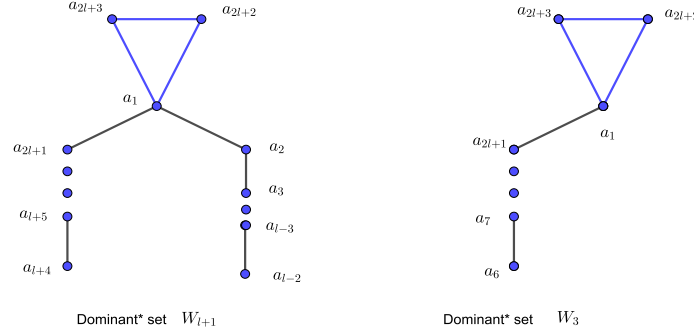


FIGURE 11. Dominants* sets

5. G HAS AN OPTIMAL EAR DECOMPOSITION STARTING WITH AN ODD CYCLE.

The following Lemma is due to Frank [4].

Lemma 5.1. *For every edge f of a 2-edge connected graph G there exists an optimal ear decomposition of G such that the first ear uses f .*

Theorem 5.2. *Let G be a 2-edge connected non bipartite graph. Then there exists an optimal ear decomposition of G starting with an odd cycle. As a consequence $\varphi(G)_{\text{odd}} = \varphi(G)$.*

Proof. The proof will be by induction on the number of vertices of G . Suppose $\text{card}(V(G)) = 3$ then $G = K_3$, our claim is true. Suppose $\text{card}(V(G)) = d$, $d \geq 4$. By induction hypothesis our claim is true for all 2-edge connected graphs G' with $\text{card}(V(G')) < d$. Let F_1, F_2, \dots, F_s be an optimal ear decomposition of G where F_1 is a cycle. For $i = 1, \dots, s$ let G_i be the graph with ear decomposition F_1, \dots, F_i , note that by Whitney's theorem 2.2 G_i is a 2-edge connected graph. If F'_1, \dots, F'_i is an optimal ear decomposition of G_i then $F'_1, \dots, F'_i, F_{i+1}, \dots, F_s$ is an optimal ear decomposition of G . Let χ_i denote the number of even ears among F_{i+1}, \dots, F_s , we have $\varphi(G_i) = \varphi(G) - \chi_i$.

If F_1 is odd our claim is done. So suppose that F_1 is an even cycle, let us recall that an even cycle is bipartite. Since G is non bipartite there exists a small integer $j > 1$ such that G_j is non bipartite. If $\text{card}(V(G_j)) < d$ then by the induction hypothesis G_j has an optimal ear decomposition starting with an odd cycle F'_1, \dots, F'_j . Hence $F'_1, \dots, F'_j, F_{j+1}, \dots, F_s$ is an optimal ear decomposition of G starting with an odd cycle.

If $\text{card}(V(G_j)) = d$ then G_{j-1} is a bipartite graph with a partition A, B . We have two cases either F_j has one or more edges.

- (1) If F_j has one edge f , then $V(G_{j-1}) = V(G)$, $E(G_j) = E(G_{j-1}) \cup \{f\}$ where $f = PQ, P, Q \in A \cup B$. If $P \in A, Q \in B$ then G_j is bipartite, hence we can

assume that $P, Q \in A$. By lemma 5.1 there exists an optimal ear decomposition of G_j : F'_1, \dots, F'_j starting with a cycle F'_1 that uses f . All edges of F'_1 except f belongs to the bipartite graph G_{j-1} , but any path in a bipartite graph between $P, Q \in A$ is even hence F'_1 is an odd cycle. It follows that $F'_1, \dots, F'_j, F_{j+1}, \dots, F_s$ is an optimal ear decomposition of G starting with an odd cycle.

- (2) If $F_j = P_{i_1}P_{i_2} \dots P_{i_l}$ with $l \geq 3$, then $V(G_{j-1}) = V(G), E(G) = E(G_{j-1}) \cup \{P_{i_1}P_{i_2}, P_{i_2}P_{i_3}, \dots, P_{i_{l-1}}P_{i_l}\}$ and $P_{i_1}, P_{i_l} \in A \cup B, \deg_{G_j}(P_t) = 2$ for $1 < t < l$. We have four cases.

- (a) If $P_{i_1} \in A, P_{i_l} \in B$ and l is even then we set $A' = A \cup \{P_{i_1}, P_{i_3}, \dots, P_{i_{l-1}}\}, B' = B \cup \{P_{i_2}, \dots, P_{i_l}\}$ hence G_j is bipartite with partition A', B' , a contradiction.
- (b) If $P_{i_1} \in A, P_{i_l} \in B$ and l is odd then we set $A' = A \cup \{P_{i_j}, j \in 2\mathbb{N}\}, B' = B \cup \{P_{i_j}, j \notin 2\mathbb{N}, j \neq 1\}$. Let G' be the graph with $V(G') = V(G_j), E(G') = E(G_j) \setminus \{P_{i_1}P_{i_2}\}$, hence G' is bipartite with partition A', B' . By lemma 5.1 there exists an optimal ear decomposition of G_j starting with a cycle C that uses $f = P_{i_1}P_{i_2}$. All edges of C except f belongs to the bipartite graph G' , but any path in the bipartite graph G' between $P_{i_1}, P_{i_2} \in A'$ is even hence C is an odd cycle. Now as before we add the ears F_{j+1}, \dots, F_s to the optimal ear decomposition of G_j , so that we get an optimal ear decomposition of G starting with an odd cycle.
- (c) If $P_{i_1}, P_{i_l} \in A$ and l is odd then we set $A' = A \cup \{P_{i_1}, P_{i_3}, \dots, P_{i_l}\}, B' = B \cup \{P_{i_2}, \dots, P_{i_{l-1}}\}$ hence G_j is bipartite with partition A', B' , a contradiction.
- (d) If $P_{i_1}, P_{i_l} \in A$ and l is even then we set $A' = A \cup \{P_{i_j}, j \in 2\mathbb{N}\}, B' = B \cup \{P_{i_j}, j \notin 2\mathbb{N}, j \neq 1\}$. Let G' be the graph with $V(G') = V(G_j), E(G') = E(G_j) \setminus \{P_{i_1}P_{i_2}\}$, hence G' is bipartite with partition A', B' . By lemma 5.1 there exists an optimal ear decomposition of G_j starting with a cycle C that uses $f = P_{i_1}P_{i_2}$. All edges of C except f belongs to the bipartite graph G' , but any path in the bipartite graph G' between $P_{i_1}, P_{i_2} \in A'$ is even hence C is an odd cycle. Now as before we add the ears F_{j+1}, \dots, F_s so that we get an optimal ear decomposition of G starting with an odd cycle.

□

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