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# Advanced Control of Nonlinear Systems, Application to Electric Drives

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# **Advanced Control of Nonlinear Systems Application to Electric Drives**

**A. Glumineau**

Master EPICO M2 2023-2024

ECOLE CENTRALE NANTES

Advanced Control Of Nonlinear Systems , Application to Electric Drives. M2 EPICO

# Outline

Introduction, Examples

Chap. 1 Nonlinear Systems, State representation and Mathematical preliminaries

Chap. 2 Controllability, Accessibility

Chap. 3 Input-Output Linearization and Decoupling by Static State Feedback

Chap. 4 Inversion of NL system (NLS) and Dynamic Decoupling

Chap. 5 Input State Linearization and Flatness

Chap. 6 Observability of NLS

Chap. 7 Observers of NLS

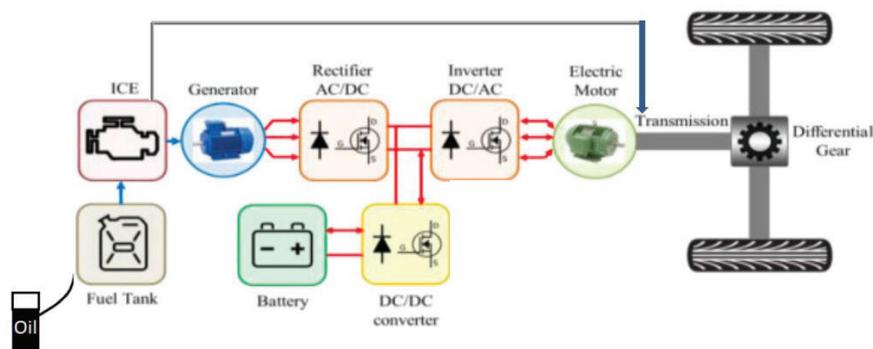
Chap. 8 Introduction to Stability and Robust Control of NLS

Chap. 9 Sliding Mode Control

Chap. 10 Backstepping Control.

Bibliographie

## Example 1 : HEV



A classical motor for HEV and EV, the synchronous motor : nonlinear system.

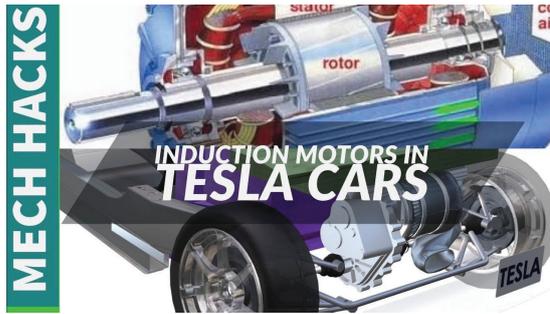
$$\begin{bmatrix} \dot{\theta} \\ \dot{\Omega} \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix} = \begin{bmatrix} \frac{p}{J} ((L_d - L_q) i_d + \Phi_f) i_q - \frac{f_v}{J} \Omega - \frac{C_l}{J} \\ -\frac{R_s}{L_d} i_d + p \frac{L_q}{L_d} \Omega i_q \\ -p \frac{\Phi_f}{L_q} \Omega - p \frac{L_d}{L_q} \Omega i_d - \frac{R_s}{L_q} i_q \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_d} & 0 \\ 0 & \frac{1}{L_q} \end{bmatrix} \begin{bmatrix} V_d \\ V_q \end{bmatrix}$$

Note  $\Phi_f = f(i_r)$  for wound rotor (Zoë, ...)

**Parameters variations** → **sensitivity** → **robust control**

\* M. Ghanes

## Example 2 : Induction motor in Tesla vehicle



Note: front motor of Tesla model S, Audi Q4 e-tron ...;  
rear motor for Mercedes EQA, ...

For an asynchronous machine the two-phase model said in  $\alpha\beta$  is written:

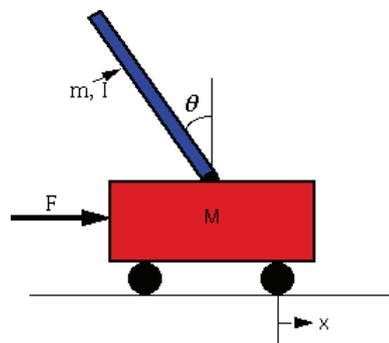
$\dot{x} = f(x) + gu + \xi$  where  $x = [\Omega, \varphi_{r\alpha}, \varphi_{r\beta}, i_{s\alpha}, i_{s\beta}]^T$ ,  $u = [u_{s\alpha}, u_{s\beta}]^T$ , and  $\xi$  is a perturbation (load torque, ...).

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \end{bmatrix} = \begin{bmatrix} (pM_{sr}/JL_r)(\varphi_{r\alpha}i_{s\beta} - \varphi_{r\beta}i_{s\alpha}) - (f_v/J)\Omega \\ -(R_r/L_r)\varphi_{r\alpha} - p\Omega\varphi_{r\beta} + (R_r/L_r)M_{sr}i_{s\alpha} \\ + p\Omega\varphi_{r\alpha} - (R_r/L_r)\varphi_{r\beta} + (R_r/L_r)M_{sr}i_{s\beta} \\ (M_{sr}/\sigma L_s L_r)((R_r/L_r)\varphi_{r\alpha} - p\Omega\varphi_{r\beta}) - \gamma i_{s\alpha} \\ (M_{sr}/\sigma L_s L_r)((R_r/L_r)\varphi_{r\beta} - p\Omega\varphi_{r\alpha}) - \gamma i_{s\beta} \end{bmatrix}, g = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/\sigma L_s & 0 \\ 0 & 1/\sigma L_s \end{bmatrix}, \xi = \begin{bmatrix} -T_l/J \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



**Robust nonlinear control is essential for HEV, EV, ...**

## Inverse Pendulum Stabilisation on a Car: Nonlinear Model



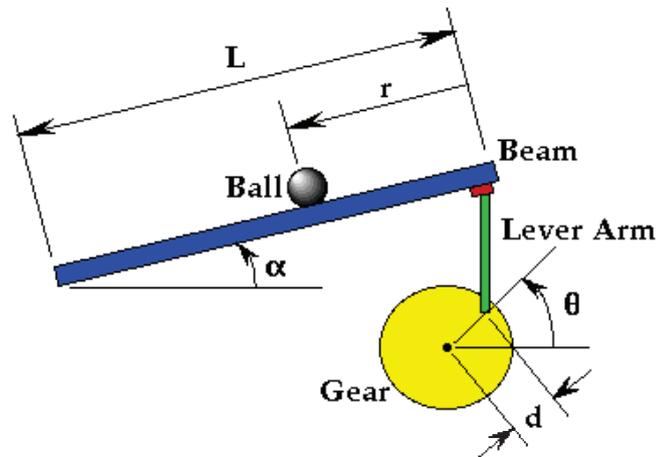
$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F$$

$$(I+ml^2)\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta$$

## Academic Ball & Beam Example

<http://www.engin.umich.edu/group/ctm/examples/ball/ball.html>

Et Robotic Ball Balancing Beam (RBBB)

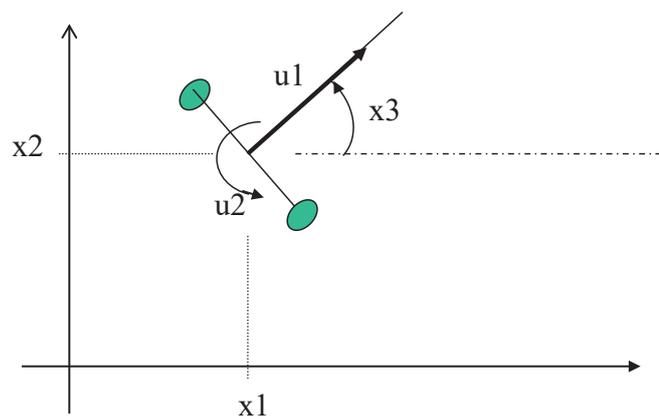


$$0 = \left( \frac{J}{R^2} + m \right) \ddot{r} + mg \sin \alpha - m r (\dot{\alpha})^2$$

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0.7

## Mobil Robot



$$\begin{aligned} \dot{x}_1 &= (\cos x_3) u_1 \\ \dot{x}_2 &= (\sin x_3) u_1 \\ \dot{x}_3 &= u_2 \end{aligned}$$

Nonholonomic system => trajectory limitations (no transversal motion i.e. parallel parking for a car ...)

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0.8

## Serial DC Motor

$$u = (R_s + R_r)i + (L_s + L_r)\frac{di}{dt} + e$$

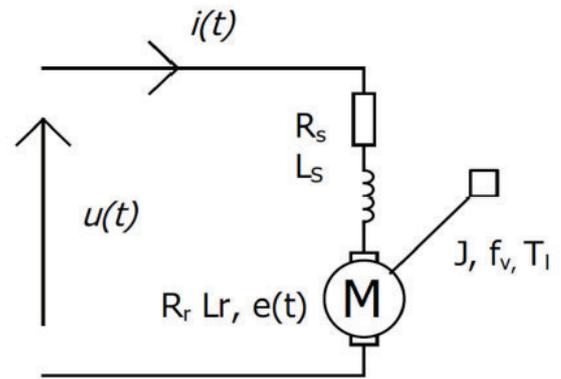
$$e = K\Phi\Omega = KK_f i\Omega \quad *$$

$$J\frac{d\Omega}{dt} = K\Phi i - f_v\Omega - C_l = KK_f i^2 - f_v\Omega - T_l$$

Thus

$$\begin{cases} J\frac{d\Omega}{dt} = KK_f i^2 - f_v\Omega - T_l \\ (L_s + L_r)\frac{di}{dt} = -(R_s + R_r)i - KK_f i\Omega + u \end{cases}$$

$$\begin{cases} \frac{d\Omega}{dt} = \frac{KK_f}{J}i^2 - \frac{f_v}{J}\Omega - \frac{1}{J}T_l \\ \frac{di}{dt} = -\frac{R_s + R_r}{L_s + L_r}i - \frac{KK_f}{L_s + L_r}i\Omega + \frac{1}{L_s + L_r}u \end{cases}$$



\*The stator flux is assumed to be proportionnel to the stator current

## Permanent Magnets Synchronous Motor

$$\begin{bmatrix} \dot{\theta} \\ \dot{\Omega} \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix} = \begin{bmatrix} \Omega \\ \frac{p}{J}((L_d - L_q)i_d + \Phi_f)i_q - \frac{f_v}{J}\Omega - \frac{C_l}{J} \\ -\frac{R_s}{L_d}i_d + p\frac{L_q}{L_d}\Omega i_q \\ -p\frac{\Phi_f}{L_q}\Omega - p\frac{L_d}{L_q}\Omega i_d - \frac{R_s}{L_q}i_q \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_d} & 0 \\ 0 & \frac{1}{L_q} \end{bmatrix} \begin{bmatrix} V_d \\ V_q \end{bmatrix}$$

with  $\theta$  the rotor position;  $\Omega$  the rotor speed;  $i_d, i_q$  the stator currents in the dq frame,  $V_d, V_q$  the voltage control inputs,  $R_s$  the stator resistance,  $L_d, L_q$  the stator inductances,  $f_v, J$  the viscous damping and the inertia of the motor-load system.  $C_l$  is the unmeasured load torque (i.e. a bounded unknown disturbance). The reference position path is noted  $\theta_{ref}$ .

Note  $\Phi_f = f(i_r)$  for wound rotor (Zoë, ...)

## Induction Motor

For an asynchronous machine the two-phase model said in  $\alpha\beta$  is written:

$\dot{x} = f(x) + gu + \xi$  where  $x = [\Omega, \varphi_{r\alpha}, \varphi_{r\beta}, i_{s\alpha}, i_{s\beta}]^T$ ,  $u = [u_{s\alpha}, u_{s\beta}]^T$ , and  $\xi$  is a perturbation (load torque, ...).

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \end{bmatrix} = \begin{bmatrix} (pM_{sr}/JL_r)(\varphi_{r\alpha}i_{s\beta} - \varphi_{r\beta}i_{s\alpha}) - (f_v/J)\Omega \\ -(R_r/L_r)\varphi_{r\alpha} - p\Omega\varphi_{r\beta} + (R_r/L_r)M_{sr}i_{s\alpha} \\ +p\Omega\varphi_{r\alpha} - (R_r/L_r)\varphi_{r\beta} + (R_r/L_r)M_{sr}i_{s\beta} \\ (M_{sr}/\sigma L_s L_r)((R_r/L_r)\varphi_{r\alpha} - p\Omega\varphi_{r\beta}) - \gamma i_{s\alpha} \\ (M_{sr}/\sigma L_s L_r)((R_r/L_r)\varphi_{r\beta} - p\Omega\varphi_{r\alpha}) - \gamma i_{s\beta} \end{bmatrix}, g = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/\sigma L_s & 0 \\ 0 & 1/\sigma L_s \end{bmatrix}, \xi = \begin{bmatrix} -T_l/J \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_s$  et  $R_r$  are the stator and rotor resistances.  $L_s$  and  $L_r$  are the stator and rotor inductances.  $M_{sr}$  is the mutual inductance between stator and rotor.  $J$  is the inertia of the system (motor + load),  $p$  is the number of pole pairs,  $f_v$  is the coefficient of viscous friction and  $T_l$  is the load torque. The parameters  $\sigma$  (coefficient of dispersion) and  $\gamma$  are defined by:

$$\sigma := 1 - \frac{M_{sr}^2}{L_s L_r}, \quad \gamma := \frac{L_r^2 R_s + M_{sr}^2 R_r}{\sigma L_s L_r^2}.$$

### Which kind of models for the Nonlinear Systems:

#### - State Space Equations

Contrary to Linear systems

> No exact frequential representation (as for LS)

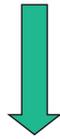
> No transfert matrices

# State Space Representation

**Example : mechanical systems**

→ **1st order differential equations system**

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = F(q)\tau$$



state  $x = (q, \dot{q})$

$$\dot{x} = \begin{pmatrix} x_2 \\ M^{-1}(x_1)[F(x_1)\tau - C(x_1, x_2)x_2 - g(x_1)] \end{pmatrix}$$

## General State Space Représentation of Nonlinear Systems (affine case wrt u)

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \\ x &\in R^n, u \in R^m, y \in R^p \end{aligned}$$

**Problems to solve:**

- Controlability ?
- (Exact) Input Output Linearization ?
- Input Output Decoupling Control?
- Observability ?, Observers ?
- Robust Controls ?

# CHAPTER 1 NONLINEAR SYSTEMS AND MATH PRELIMINARIES <sup>1</sup>

## 1.1 Definition of non-linear systems (NLS)

Consider the non-linear systems described by the models:

$$\begin{cases} \dot{x} = f(x, u) \text{ or } \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \quad (1.1)$$

$$x \in R^n, u \in R^m, y \in R^p$$

1 [Glu92] Chapter 1 and Annex 1

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1.1

The **first approach** is to linearize the system by *approximation* around a point  $(x_0, u_0)$  :

$$\begin{aligned} \dot{z} &= \left[ \frac{\partial f}{\partial x} \right]_{x_0, u_0} z + \left[ \frac{\partial f}{\partial u} \right]_{x_0, u_0} v := Az + Bv \\ y &= \left[ \frac{\partial h}{\partial x} \right]_{x_0} z := Cz \end{aligned}$$

➡ Approximate method around  $(x_0, u_0)$  and possible loss of system properties. For example: a controllable non linear system gives an uncontrollable linearized system. See example 1.1 et 1.4.

### **Example .1.1: Limitations of Approximate Linearization**

Let  $\dot{x}_1 = x_2^3$   
 $\dot{x}_2 = u.$

This nonlinear system *seems to be "controllable"* (we can act on  $x_1$  and  $x_2$  by  $u$ ).

We linearize around  $(x_0, u_0) = ([0, 0]^T, 0)$  :

$$\begin{aligned} \dot{Z}_1 &= 0 \\ \dot{Z}_2 &= v. \end{aligned}$$

We have lost the "controllability" property on the approached system:

**Solution => NL model.**

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1.2

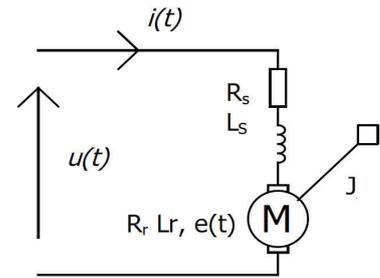
### Exercise 1: The Serial DC motor

A serial DC motor with insignificant viscous damping ( $f_v = 0$ ) and inertia load can be modeled:

$$u = (R_s + R_r)i + (L_s + L_r)\frac{di}{dt} + e$$

$$e = K\Phi\Omega = KK_f i\Omega \quad *$$

$$J\frac{d\Omega}{dt} = K\Phi i = KK_f i^2$$



with  $x_1 = \Omega$ ,  $x_2 = i$ ,  $y = x_1$  a nonlinear SISO state model is:

$$\begin{cases} \dot{x}_1 = \frac{KK_f}{J} x_2^2 \\ \dot{x}_2 = -\frac{R_s + R_r}{L_s + L_r} x_2 - \frac{KK_f}{L_s + L_r} x_1 x_2 + \frac{1}{L_s + L_r} u \end{cases} \quad \text{or} \quad \begin{cases} \dot{x}_1 = ax_2^2 \\ \dot{x}_2 = dx_2 + ex_1x_2 + fu \end{cases}$$

Is  $([0,0]^T, 0)$  an equilibrium point?

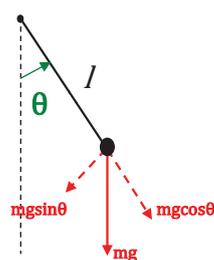
Is the approximate linearized model controllable in this point?

Conclusion?

\* The stator flux is assumed to be proportional to the serial current

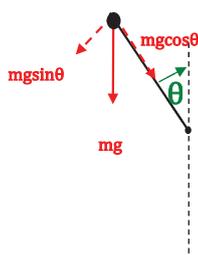
**The right approach** is an exact calculation of the non-linear differential equations and to rewrite the model in a state equation form.

#### Example 1.2: Pendulum



$$\begin{aligned} \ddot{\theta} &= k \sin \theta \\ x &= (\theta, \dot{\theta}) \\ \dot{x} &= \begin{bmatrix} x_2 \\ -k \sin x_1 \end{bmatrix} \\ \text{with } k &= \frac{g}{l} m \end{aligned}$$

#### Example 1.3: Inverse pendulum

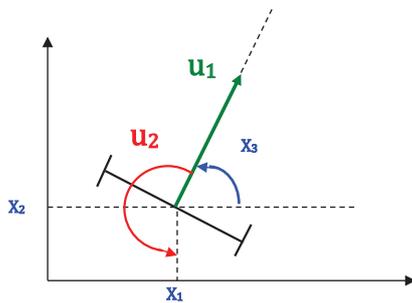


*Problem: 2 equilibrium points*

*Stabilization around an unstable Point (rocket, ...)*

**Example 1.4 : Unicycle<sup>1</sup>**

We consider a mobile cart (airport caddie type: fixed rear wheels, "crazy" front wheels). We note  $u_1$  the longitudinal speed,  $u_2$  the angular speed,  $x_1$  and  $x_2$  the coordinates of the center of the rear wheels.



Model equations

$$\dot{x}_1 = \cos x_3 \cdot u_1$$

$$\dot{x}_2 = \sin x_3 \cdot u_1$$

$$\dot{x}_3 = u_2.$$



Let  $(X_0, u_0) = ([0, 0, 0]^T, [0, 0]^T)$ . By approximate linearization,  $\sin X \approx 0, \cos X \approx 1$ :

$$\begin{aligned} \dot{z}_1 &= u_1 \\ \dot{z}_2 &= 0 * u_1 = 0 \\ \dot{z}_3 &= u_2 \end{aligned} \quad \begin{array}{l} \text{the linearized model is not controllable (z}_2 \text{ non accessible)} \\ \text{To use the NL model.} \end{array}$$

<sup>1</sup> [Glu92] page 51

**Example 1.5 : Car**

The system is described by the following equations :

$$\dot{x} = \dot{x}_1 = u_1 \cos x_3 \quad y_1 = x_1$$

$$\dot{y} = \dot{x}_2 = u_1 \sin x_3$$

$$\dot{\theta} = \dot{x}_3 = u_1 u_2 / l \quad y_2 = x_2$$

with  $u_1$  car speed (=  $F(t)$ ) and  $u_2 = \tan \phi$  "wheel orientation".

Note. The model is non affine wrt inputs.

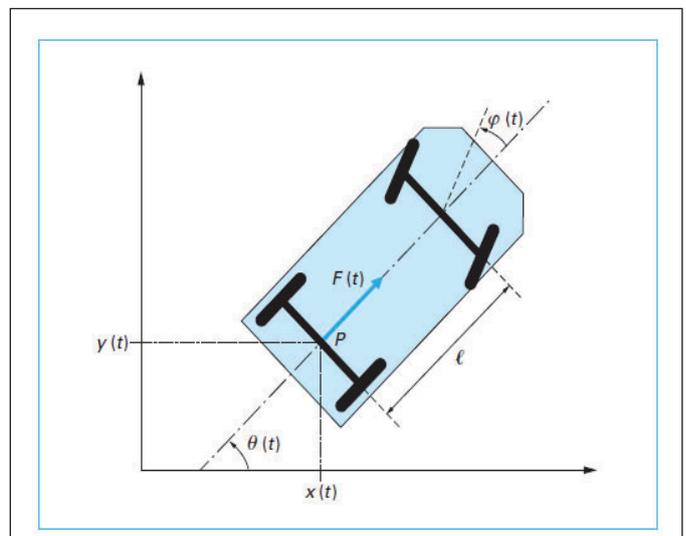


Figure. Car: plan model ([Mar99])

## 1.2 Analytical and meromorphic functions

### 1.2.1 Analytical functions<sup>1</sup>

**Definition:** Analytical functions

$f: \mathbb{R} \rightarrow \mathbb{R}$  is said to be analytical if it is developable in Taylor series in the vicinity of each of its points, i.e.,  $\forall x \in \mathbb{R}$ ,  $f(x)$  is developable in Taylor series.

**Property:** Differentiability ( $C^\infty$ , the derivatives w.r.t the  $x_i$  exist and are continuous) and isolated zeros.

**Example 1.6:** Some examples of analytical functions

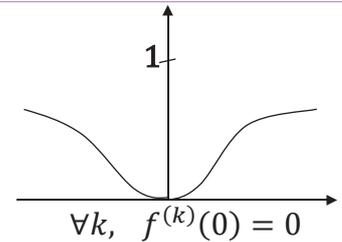
- Trigonometric functions
- Polynomial functions..

**Example 1.7:** A counter-example

Let the function defined by:  $f(x) = e^{-\frac{1}{x^2}}$  if  $x \neq 0$   
if not  $f(x) = 0$ .

This function is really  $C^\infty$  with  $\forall k, f^{(k)}(0) = 0$

However, it is not analytical in  $x = 0$  (non-isolated zeros).



<sup>1</sup> [Glu92] Annexe 1

**Property<sup>1</sup>:** If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is an analytical function, then :

- Either  $f \equiv 0$
- Either the zeroes are isolated.

**Corollary:** The body of analytical function quotients is well defined and :

If  $f_1$  is analytical and  $f_2$  is analytical ( $f_2 \neq 0$ ),

then  $\frac{f_1}{f_2}$  is analytical.

**Corollary:** The set of analytical functions forms an integral ring (there is no divisor of zeros).

If  $f_1 \cdot f_2 = 0$  then  $f_1 = 0$  or  $f_2 = 0$ .

<sup>1</sup> [Glu92] Annexe A1

## 1.2.2 Meromorphic functions<sup>1</sup>

**Definition:** Meromorphic functions

$f: \mathfrak{R} \rightarrow \mathfrak{R}$  is said meromorphic if  $\exists$  analytical  $f_1$  and  $\exists$  analytical  $f_2 \neq 0$  such that  $f = \frac{f_1}{f_2}$ .

**Property:** All the meromorphic functions form a field.

**Example 1.8:** Some examples of meromorphic functions

- Rational functions of  $x$ :  
 $P(x)/Q(x)$  with  $P$  and  $Q$  polynomial functions
- $\tan x = \frac{\sin x}{\cos x}$ .

<sup>1</sup> [Glu92] Chapitre 1 and Annexe A1

## 1.3 Class of the studied NLS

Let the systems of the form : 
$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

with the state  $x \in \mathfrak{R}^n$ , the control  $u \in \mathfrak{R}^m$  and the output  $y \in \mathfrak{R}^p$  :

The components of  $f$ ,  $g$  and  $h$  are meromorphic functions.

Remark: of course this includes linear systems:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

with  $f(x) = Ax$ ,  $g(x) = B$  and  $h(x) = Cx$ .



All the results of this lecture are applicable to linear systems.

### 1.3.1 Functions field <sup>1</sup>

Let the set of variables:

$$\{x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, \dot{u}_1, \dot{u}_2, \dots, \dot{u}_m, \dots, u_1^{(k)}, u_2^{(k)}, \dots, u_m^{(k)}\}.$$

We then define :  $K :=$  body of the meromorphic functions of

$$\{x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, \dot{u}_1, \dot{u}_2, \dots, \dot{u}_m, \dots, u_1^{(k)}, u_2^{(k)}, \dots, u_m^{(k)}\}.$$

- $K$  is a vector space on itself
- $KxK$  is a vector space on  $K$
- $\mathcal{E} = K^{n(m+1)}$  is a vector space on  $K$ .

**Example 1.9:** The following fraction is considered :

$$f(x) = \frac{(x_2 \cdot \sin(x_2)) \cdot u^2 + \ddot{u}}{\dot{u} + u \cdot \ddot{u} \cdot tg(x)} \text{ with } (x_2 \cdot \sin(x_2)) \in K \text{ and } tg(x) \in K$$

then  $f(x) \in K$ .

<sup>1</sup> [Glu92] Chapter 1 and Annexe A1

### 1.3.2 Vector space $\mathcal{E}$

**Definition:** We define the vector space  $\mathcal{E}$  on the field  $K$  by  $\mathcal{E} = K^{n(m+1)}$ .

The unit vectors of  $\mathcal{E}$  are defined as follows:

$$\left\{ \begin{array}{l} (1, 0, \dots, 0) = e_1 = dx_1 \\ (0, 1, 0, \dots, 0) = e_2 = dx_2 \\ \dots \\ (0, \dots, 0, 1, 0, \dots, 0) = e_n = dx_n \\ (0, \dots, 0, 0, 1, 0, \dots, 0) = e_{n+1} = du_1 \\ \dots \\ (0, \dots, 0, \dots, 0, 1, 0, \dots, 0) = e_{n+m} = du_m \\ (0, \dots, 0, \dots, 0, 0, 1, 0, \dots, 0) = e_{n+m+1} = d\dot{u}_1 \\ \dots \\ (0, \dots, 0, 0, \dots, 0, \dots, 1, 0, \dots, 0) = e_{n(m+1)} = du_m^{(n-1)}. \end{array} \right.$$

**Definition:** The unit vectors of the vector space  $\mathcal{E}$  on the  $K$ -field are noted :

$$(dx_1, dx_2, \dots, dx_n, du_1, \dots, du_m, d\dot{u}_1, \dots, d\dot{u}_m^{(n-1)}).$$

Interest of these definitions :

Let's consider  $y = h_1(x)$ , then

$$dy_1 = \frac{\partial h_1}{\partial x_1} dx_1 + \frac{\partial h_1}{\partial x_2} dx_2 + \dots + \frac{\partial h_1}{\partial x_n} dx_n = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \end{bmatrix} \cdot \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} \in K$$

$$\dot{y}_1 = \frac{dy_1}{dt} = \frac{\partial h_1}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial h_1}{\partial x} \dot{x} = \frac{\partial h_1}{\partial x} (f(x) + g(x)u) = h_1^{(1)}(x, u)$$

$$d\dot{y}_1 = \underbrace{\sum_{i=1}^n \frac{\partial h_1^{(1)}}{\partial x_i} dx_i}_{\text{Functions}(x, u)} + \underbrace{\sum_{j=1}^m \frac{\partial h_1^{(1)}}{\partial u_j} du_j}_{\text{Functions}(x)} \in \mathcal{E}.$$

Functions  $(x, u)$   
 $\in K$

Functions  $(x)$   
 $\in K$

Remark.  $\mathcal{E}$  contains the differential of any function  $\phi$  of  $K$ .

**Inverse" problem:** We consider the following function:  $dx_1 + x_3 \cdot dx_2 = (1, x_3, 0) \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \in K$

An important problem in NL is: is there  $\phi(x) \in K$  such that  $d\phi = dx_1 + x_3 \cdot dx_2$ ?

(Searching for differential relationships characteristic of an NLS: linearization by coordinates changing, etc.).

**1.4 Poincaré lemma<sup>1</sup>**

This lemma will help answer the following question:

Given  $\omega \in \mathcal{E}$ , does it exist:

$\phi \in K$  (noted 0 – form) such that  $\omega = d\phi$ .

**Definition : 1-form**

$$\omega \in \mathcal{E} \text{ is a 1-form with } \omega = \underbrace{\sum_{i=1}^n \alpha_i(\cdot) dx_i}_{\in K} + \sum_{j=0}^{n-1} \underbrace{\sum_{k=1}^m \beta_{kj}(\cdot) du_k^{(j)}}_{\in K}.$$

**Definition : exact 1-forme**

Let a 1-form<sup>1</sup>:  $\omega \in \mathcal{E} = Vect_K \{dx, du, \dots, du^{(n-1)}\}$ .

$\omega$  is an exact 1-form if :  $\exists \phi \in K$  such that  $\omega = d\phi$ .

<sup>1</sup>[Ple95] and [Cho89]

**Example 1.10** Example and counter-example of exact 1-formes

- $\omega_1 = dx_1 + x_2 dx_3 + x_3 dx_2$  is exact
- $\omega_2 = x_1 dx_2 - x_2 dx_1$  is not exact.

**Definition : 2-form**

$\Omega$  is 2-form  $\Leftrightarrow \Omega = \sum_{i,j} \alpha_{ij} e_{ij}$  where  $\alpha_{ij} \in K$  et  $e_{ij}$  is a unit vector  $dx_i \wedge dx_j$ , with  $\wedge$  exterior (wedge) product

$$\text{Let } \Omega = \sum_{i,j} \alpha_{ij} dx_i \wedge dx_j.$$

**Example 1.11**  $\phi \in K$  : 0-form

By differentiation of  $\phi$ :  $d\phi = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} dx_i + \sum_{j,k}^m \frac{\partial \phi}{\partial u_j^{(k)}} du_j^{(k)}$  is a 1-form.

By differentiation of  $d\phi \Rightarrow$  a 2-form  $d(d\phi)$  is obtained.

**Reminder: Outdoor product calculation rules ( $\wedge$ ) and differential calculation**

- 1)  $dx_i \wedge dx_j = -dx_j \wedge dx_i$
- 2)  $dx_i \wedge dx_i = 0$
- 3)  $d^2 \equiv 0$ , i. e.  $d(dx) \equiv 0$ .

$$\begin{aligned} \text{Thus } \omega &= \sum_{i=1}^n \alpha_i dx_i, \quad d\omega = \left( \sum_{i=1}^n \frac{\partial \alpha_i}{\partial x_i} dx_i \right) \wedge dx_1 + \dots + \left( \sum_{i=1}^n \frac{\partial \alpha_n}{\partial x_i} dx_i \right) \wedge dx_n \\ &= \sum_{j < i} \sum_{i=1}^n \left( \frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j} \right) dx_i \wedge dx_j. \end{aligned}$$

**Example 1.12**  $\phi = x_1 x_2$ . Calculate the corresponding 2-form.

**Integration problem:**

Let a 1-forme  $\omega = \sum_{i=1}^n \alpha_i dx_i$

does it exist  $\phi \in K$  such that  $\omega = d\phi$  (that is to say that  $\omega$  is an exact 1-form) ?

**Theorem: Poincaré's Lemma <sup>1</sup>**

Given a 1-form  $\omega \in \mathcal{E}$ , then

$\omega$  is an exact 1-Form  $\Leftrightarrow \exists \phi$  such as  $\omega = d\phi \Leftrightarrow d\omega = 0$   
 where  $d\omega = \sum_{i=1}^n d\alpha_i \wedge dx_i$  with the previous calculation rule.

**Example 1.13**  $\omega = dx_1 + x_2 dx_3 + x_3 dx_2$

**Example 1.14** Given  $\omega = x_2 dx_1 - x_1 dx_2$ ,

then

$d\omega = dx_2 \wedge dx_1 - dx_1 \wedge dx_2$  and by antisymmetry:  $d\omega = -2 \cdot dx_1 \wedge dx_2 \neq 0$

So,  $\forall \phi, \omega \neq d\phi$ .

There is no possible integration of this 1-form.

<sup>1</sup> [Ple95] and [Cho89]

**Example 1.15** Given  $\omega = \frac{x_2}{x_1^2} dx_1 - \frac{1}{x_1} dx_2$ , then  $d\omega$

$$d\omega = d\left(\frac{x_2}{x_1^2}\right) \wedge dx_1 - d\left(\frac{1}{x_1}\right) \wedge dx_2 = \left[\frac{1}{x_1^2} dx_2 - 2\frac{x_2}{x_1^3} dx_1\right] \wedge dx_1 - \left[-\frac{dx_1}{x_1^2}\right] \wedge dx_2 + 0$$

$$d\omega = \frac{1}{x_1^2} dx_2 \wedge dx_1 - 2\frac{x_2}{x_1^3} dx_1 \wedge dx_1 + \frac{1}{x_1^2} dx_1 \wedge dx_2 = \left[-\frac{1}{x_1^2} + \frac{1}{x_1^2}\right] dx_1 \wedge dx_2$$

$$\equiv 0$$

A primitive is :  $\phi = -\frac{x_2}{x_1}; \omega = d\left(-\frac{x_2}{x_1}\right)$ .

**Example 1.16** Given  $\omega = \frac{x_2}{x_1^2+x_2^2} dx_1 - \frac{x_1}{x_1^2+x_2^2} dx_2 \Rightarrow d\omega = 0$

$$\text{Then } \begin{cases} \omega = d\left[\text{Arctg}\left(\frac{x_1}{x_2}\right)\right] & \text{if } x_2 \neq 0 \\ \omega = d\left[\text{Arctg}\left(-\frac{x_2}{x_1}\right)\right] & \text{if } x_1 \neq 0 \end{cases}$$

$\phi(x_1, x_2)$ : local solution of  $\omega = d\phi$  (or global on  $\mathbb{R} \times \mathbb{R} - \{0,0\}$ ).

(Conclusion : long live symbolic computation Maple, Mathematica, ...!)

## 1.5 Frobenius Theorem

- **Problem :**

Given a 1-form  $\omega$ , is there  $\phi \in K$  et  $\lambda \in K$  such that  $\omega = \lambda d\phi$ ?  
( $\lambda$  is called the integrating factor).

**Theorem : Frobenius Theorem (first version)**<sup>1</sup>

Let  $\omega \in \varepsilon$ .  $\exists \phi \in K$  and  $\exists \lambda \in K$  such that  $\omega = \lambda d\phi \Leftrightarrow d\omega \wedge \omega = 0$ .

**Example 1.17 :** Let  $\omega = x_2 dx_1 - x_1 dx_2$

$d\omega = -2dx_1 \wedge dx_2 \neq 0$  (no solution by using the Poincaré Theorem)

$$\begin{aligned} d\omega \wedge \omega &= (-2dx_1 \wedge dx_2) \wedge (x_2 dx_1 - x_1 dx_2) \\ &= -2x_2 \cdot \underbrace{dx_1 \wedge dx_2 \wedge dx_1}_0 + 2x_1 \underbrace{dx_1 \wedge dx_2 \wedge dx_2}_0 \end{aligned}$$

So  $d\omega \wedge \omega = 0$

Then,  $\exists \phi \in K$  and  $\exists \lambda \in K$  such that  $\omega = \lambda d\phi$ .

We can notice that:  $\omega = -x_1^2 \left( d\left(\frac{x_2}{x_1}\right) \right)$  i.e.  $\phi = \frac{x_2}{x_1}$  and  $\lambda = x_1^2$  is integrating factor.

<sup>1</sup> [Ple95] and [Cho89]

**Example 1.18 :** Let  $\omega = dx_1 + x_1 dx_2 + x_2 dx_3$

$d\omega = dx_1 \wedge dx_2 + dx_2 \wedge dx_3 \neq 0$

$d\omega \wedge \omega = (dx_1 \wedge dx_2 + dx_2 \wedge dx_3) \wedge (dx_1 + x_1 dx_2 + x_2 dx_3)$

$d\omega \wedge \omega = dx_1 \wedge dx_2 \wedge (x_2 dx_3) + dx_2 \wedge dx_3 \wedge dx_1 =$   
 $(x_2 + 1)dx_1 \wedge dx_2 \wedge dx_3.$

Thus  $d\omega \wedge \omega \neq 0$ . No solution.

**Example 1.19 :** Let  $\omega = x_3 dx_1 + dx_3 \quad d\omega \neq 0$

$$\begin{aligned} d\omega \wedge \omega &= d(x_3 \wedge dx_1 + dx_3) \wedge (x_3 dx_1 + dx_3) \\ &= (dx_3 \wedge dx_1 + 0) \wedge (x_3 dx_1 + dx_3) = 0 \end{aligned}$$

It can be shown that there are several solutions for the terms  $\lambda$  et  $\phi$  :

In fact, there are an infinite number of solutions.

**Theorem: Frobenius Theorem (2<sup>nd</sup> version)**<sup>1</sup>

Be  $\omega_1, \omega_2, \dots, \omega_s \in \mathcal{E}$ ,

There exist  $\phi_1, \phi_2, \dots, \phi_s \in K$  such that

$$\begin{aligned} \text{span}\{\omega_1, \dots, \omega_s\} &= \text{span}\{d\phi_1, \dots, d\phi_s\} \\ \Leftrightarrow \forall i = 1 \dots s, \quad &\underbrace{d\omega_i \wedge \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_s}_{(s+2)\text{-Forme}} = 0. \end{aligned}$$

(span = vector subspace generated by ...).

**Note:** the constraints have been relaxed compared to the first version.

<sup>1</sup>[Ple95] and [Cho89]

**Exercise 2, example, application:**

$$\begin{aligned} \text{Be } \omega_1 &= x_3 dx_1 + dx_2 \\ \omega_2 &= dx_3 \\ \text{and } \Omega &= \text{span}\{\omega_1, \omega_2\} \end{aligned}$$

Find  $d\phi_1, d\phi_2$  such that  $\Omega = \text{span}\{d\phi_1, d\phi_2\}$

**Frobenius theorem conditions:**

$$\begin{aligned} d\omega_2 \wedge \omega_1 \wedge \omega_2 &= 0 \\ d\omega_1 \wedge \omega_1 \wedge \omega_2 &= (dx_3 \wedge dx_1) \wedge (x_3 dx_1 + dx_2) \wedge dx_3 = (dx_3 \wedge dx_1 \wedge dx_2) \wedge dx_3 \\ &= 0 \end{aligned}$$

Let's search  $\mu$  and  $\lambda$  such that  $\mu\omega_1 + \lambda\omega_2$  has an "obvious" integration.

For example,  $\mu = 1$  et  $\lambda = x_1$  then  $\mu\omega_1 + \lambda\omega_2 = \omega_1 + x_1\omega_2 = d(x_1x_3) + d(x_2)$ .

Thus  $\text{span}\{\omega_1, \omega_2\} = \text{span}\{d(x_1x_3 + x_2), dx_3\}$ ;  $\phi_1 = x_1x_3 + x_2$ ;  $\phi_2 = x_3$ . If we set an invertible and differentiable (NL) coordinate change (: diffeomorphism)

$$\begin{aligned} \tilde{x}_1 &= x_1x_3 + x_2; \tilde{x}_2 = x_3; \tilde{x}_3 = x_1; \\ \text{then } \text{span}\{\omega_1, \omega_2\} &= \text{span}\{d\phi_1, d\phi_2\} = \text{span}\{d\tilde{x}_1, d\tilde{x}_2\}. \end{aligned}$$

Application: research of nonlinear "linearizing" transformation (e.g. classical: rotation matrix in robotics, electrical machines, ...).

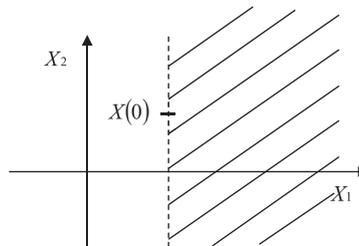
# CHAPTER 2 CONTROLLABILITY AND ACCESSIBILITY<sup>1</sup>

## 2.1 Some characteristic examples of the "controllability" for NLS

*Example 2.1*<sup>2</sup>

$$\begin{cases} \dot{x}_1 = x_2^2 \\ \dot{x}_2 = u \end{cases}$$

 is the set of points reachable from the initial condition  $X(0)$



➡ ACCESSIBLE, BUT NON-CONTROLLABLE

Rq1 What about the approximate linear model in  $(0,0)$  ?

Rq2 Accessibility ~~➡~~ Stabilisability (to  $(0,0)$  the equilibrium point)

<sup>1</sup>[Glu97] and [Ara94], <sup>2</sup>[Glu95] and [Glu97]

*Example 2.2:* Let system  $\Sigma$   $\begin{cases} \dot{x}_1 = \cos(x_2) \\ \dot{x}_2 = u \end{cases}$

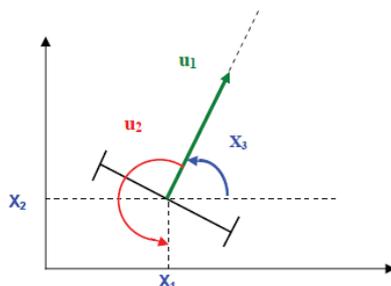
Whatever the control  $u$ ,  $\dot{x}_1$  will vary only between -1 and 1:  $x_1$  is constrained.

➡  $\Sigma$  ACCESSIBLE, NOT CONTROLLABLE.

*Example 2.3: Unicycle [Sam91]*

Consider a mobile cart: fixed rear wheels, "crazy" front wheels.

*Note*  $u_1$  the longitudinal speed,  $u_2$  the angular speed,  $x_1$ , and  $x_2$  the coordinates of the center of the rear wheels and  $x_3$  the angular velocity.



System equation:

$$\begin{cases} \dot{x}_1 = \cos(x_3) u_1 \\ \dot{x}_2 = \sin(x_3) u_1 \\ \dot{x}_3 = u_2 \end{cases}$$

?  $x_3=0$  or  $x_3=\pi/2$ : singularities ➡ **Generic property:** true except in (isolated) singularities.

## 2.2 Reminder on linear systems

Let a linear system (SL) of the form:  $\dot{x} = Ax + Bu$ .

### Definition: Output relative degree

The relative degree of the output is equal to the degree of the denominator of the transfer function minus the degree of the numerator of the transfer function:

$$r = d^\circ \text{Denominator} - d^\circ \text{Numerator}.$$

Another definition: the relative degree is the minimum order  $k$  of derivation of the output such as:

$$\frac{\partial y^{(k)}}{\partial u} \neq 0.$$

### Structure of the commandability matrix (and duality):

The controllability matrix is of the form:  $[B \ AB \ \dots \ A^{n-1}B]$ .

**Remark:** let a fictitious output  $y = Cx$  then  $\dot{y} = C\dot{x} = CAx + CBu$ .

If the relative degree is 1 then  $\partial \dot{y} / \partial u \neq 0 \Leftrightarrow CB \neq 0$

$(B^\perp = \{\omega \mid \omega \cdot B = 0\})$  characterizes all "outputs" having a relative degree  $\geq 2$

$$\frac{\partial \dot{y}}{\partial u} = 0 \Leftrightarrow C \perp B$$

then  $\dot{y} = C(Ax + Bu) = CAx$  et  $\ddot{y} = CA^2x + CABu$

thus

$$[B \ AB]^\perp = \text{span} \{ \omega \mid \omega \cdot B = 0 \text{ et } \omega \cdot AB = 0 \}$$

characterizes the "outputs" having a relative degree greater than or equal to 3:

$$\left. \begin{array}{l} \frac{\partial \dot{y}}{\partial u} = 0 \\ \frac{\partial \ddot{y}}{\partial u} = 0 \end{array} \right\} \Leftrightarrow C \perp [B \ AB]$$

- In a more general way:

$$[B \ AB \ \dots \ A^{n-1}B]^\perp = \{ \omega \mid \omega B = 0, \omega AB = 0, \dots, \omega A^{n-1}B = 0 \}$$

describes the set of "outputs" having a relative degree greater than or equal to  $n+1$ :

$$dr(y) \geq k \Leftrightarrow "C" \perp [B \ AB \ \dots \ A^{k-2}B]$$

**i.e. information on the system structure with respect to the input.**

*Theorem: Controllability Criterion:*

- $Rang([B \ AB \ \dots \ A^{n-1}B]) = n \Rightarrow dim[B \ AB \ \dots \ A^{n-1}B]^\perp = 0$
- $\Leftrightarrow$  no "output" having a relative degree  $\geq n+1$  (i.e.  $\frac{\partial y^{(n)}}{\partial u} \neq 0$ )
- $\Leftrightarrow$  notation: no "output" with a relative degree  $\infty$
- $\Leftrightarrow$  In the system, there is no "output" that is solution of a differential equation independent of  $u$  ("autonomy").

To conclude, if the linear system is not controllable,

then  $\exists w \neq 0$  such as a fictitious output  $y = wX$   
is of relative degree greater than or equal to  $n+1$

$\Leftrightarrow$  the relative degree of "output" is said to be  $\infty$

$\Leftrightarrow$  the "output"  $y$  is the solution of an autonomous linear differential equation  
(i.e independent of  $u$ ).

*Example 2.4: let the system*

$$\dot{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, [B \ AB] = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \text{ of rank one: uncontrollable system.}$$

It can be remark that  $[-2 \ 1] \in B^\perp$  and that the "fictitious output"  
 $y = -2x_1 + x_2$  has a relative degree  $\infty$  ( $\dot{y} = 0, \dots, y^{(k)} = 0$ )

The "output"  $y$  is the solution of the autonomous linear differential equation (independent of  $u$ )  
 $\dot{y} = 0$

**2.3 Definitions for non-linear systems**

*Definition: Relative degree of a function  $\varphi(x)$*

*The relative degree of  $\varphi(x)$  is defined by the minimum order  $k$  of derivation such that:*

$$\frac{\partial \varphi^{(k)}}{\partial u} \neq 0 \quad \text{i.e.} \quad r = \min \left\{ k \in \mathbb{N} \mid \frac{\partial \varphi^{(k)}}{\partial u} \neq 0 \right\}.$$

Example 2.5: Consider a mechanical system such as:

$$\underline{x} = (\text{position, speed}) \text{ and } u = \text{Force or torque}$$

then the relative degree (position) = 2 (otherwise under-actuated mechanical system)

Definition<sup>1</sup>: Relative Degree of a 1-Form  $\omega \in \mathcal{E}$

Let  $\omega := \sum_{i=1}^n \alpha_i dx_i$  with  $\forall i, \alpha_i \in K$

The relative degree  $r$  of  $\omega \in \mathcal{E}$  is defined by:

$$r = \min \left\{ k \in \mathbb{N} \mid \omega^{(k)} \notin \text{span} \{ dx \} \right\}$$

where  $\dot{\omega} = \sum_{i=1}^n \dot{\alpha}_i dx_i + \alpha_i d\dot{x}_i$ .

Proposition:  $\varphi(x) \in K$  is autonomous

$$\Leftrightarrow \text{the relative degree of } \varphi \text{ is infinite} \Leftrightarrow \forall k \in \mathbb{N}, \frac{\partial \varphi^{(k)}}{\partial u} \equiv 0.$$

<sup>1</sup>[Glu97] page 11

Sketch of proof

- (only if)  $\Leftarrow$  Suppose that the relative degree of  $\varphi$  is infinite ( $\frac{\partial \varphi^{(k)}}{\partial u} = 0, \forall k$ )

$x \in \mathbb{R}^n$ then	$d\varphi = \begin{matrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \dot{\varphi}}{\partial x} \\ \dots \\ \frac{\partial \varphi^{(n)}}{\partial x} \end{matrix} dx$	
	$d\dot{\varphi} = \begin{matrix} \frac{\partial \dot{\varphi}}{\partial x} \\ \dots \\ \frac{\partial \dot{\varphi}^{(n)}}{\partial x} \end{matrix} dx$	
	$d\varphi^{(n)} = \begin{matrix} \dots \\ \frac{\partial \varphi^{(n)}}{\partial x} \end{matrix} dx$	$\leftarrow$ linear combination (over $K$ ) of the previous $n$ rows
	$\swarrow$ Jacobian Matrix (n+1) rows	$\Downarrow$ $\exists F/F(\varphi, \dot{\varphi}, \dots, \varphi^{(n)}) = 0$
n columns rank $\leq n$		by the implicit function theorem then $\varphi$ is autonomous.

- (if)  $\Rightarrow$  By contradiction, let us show that if  $d^\circ$  relative of  $\varphi < \infty$ , then  $\varphi$  is not autonomous.

$\exists r < \infty$  such that  $\frac{\partial \varphi^{(r)}}{\partial u} \neq 0$

then,  $\varphi^{(r+1)} = * + \frac{\partial \varphi^{(r)}}{\partial u} \dot{u}$  independent of  $\varphi, \dot{\varphi}, \dots, \varphi^{(r)}$  etc,

$$\varphi^{(r+k)} = * + \frac{\partial \varphi^{(r)}}{\partial u} u^{(k)}$$

$\Rightarrow \varphi$  is not autonomous.

## 2.4 Controllability and accessibility of nonlinear systems

*Definition: Commandability of a non-linear system*

Let the NLS  $\dot{x} = f(x) + g(x)u$ .

This system is said to be "controllable", if

$\forall x_0$  (initial state) and  $\forall x_1$  (any state),  $\exists u(t)$  and  $T < \infty$  such that  
 $x(x_0, u(t), T) = x_1$

- *Problem: There is no characterization of commandability for SNL (see introductory example)*

*Definition: Accessibility of a NLS*

The system  $\dot{x} = f(x) + g(x)u$  is said to be "accessible" if there is no autonomous element.

*Property: For Linear Systems only.*

$LS$  is controllable  $\Leftrightarrow$   $LS$  is accessible.

*Example 2.6:* Let the following non-linear system 
$$\begin{cases} \dot{x}_1 = x_2^2 \\ \dot{x}_2 = u \end{cases}$$

This system is not controllable.

However, it is accessible because:

$$\forall \varphi(x_1, x_2), dr(\varphi) < \infty \text{ (max 2)}$$

*Property:* For non-linear systems :

A NLS is accessible  $\Leftrightarrow \forall \varphi(x) \neq cte, dr(\varphi) < \infty$   
(no autonomous element)

### Algorithm to test the accessibility property <sup>1</sup>

- - Characterization of 1-Form  $\omega \in \varepsilon$  such that the relative degree of  $\omega$  be  $\geq 2$  :
  - $H_0 := \text{span}\{dx, du\} \Rightarrow \omega \in H_0 \Leftrightarrow$  relative degree of  $\omega \geq 0$
  - $H_1 := \text{span}\{dx\} \subset H_0 \Rightarrow \omega \in H_1 \Leftrightarrow$  relative degree of  $\omega \geq 1$
  - $H_2$  is the vector space of all 1-Forms  $\omega (\in H_1)$  such that the relative degree of  $\omega \geq 2$   
 $H_2 \subset H_1$
- Computation of  $H_2$  :
  - Let  $\omega \in H_1, \omega = \sum_{i=1}^n \alpha_i dx_i \Rightarrow \dot{\omega} = \sum_{i=1}^n (\dot{\alpha}_i dx_i + \alpha_i d\dot{x}_i)$   
 $\Rightarrow$  Compute the  $\alpha_i$  such that  $\dot{\omega} \in H_1$  (not for  $H_0$  otherwise  $\omega$  is degree 1!)
  - $\dot{\omega} = \sum_{i=1}^n \alpha_i d\dot{x}_i + \sum_{i=1}^n \dot{\alpha}_i dx_i = [\alpha_1, \dots, \alpha_n] d(f(x) + g(x)u) + (" \in H_1 ")$
  - Thus  $\dot{\omega} \in H_1 \Leftrightarrow [\alpha_1, \dots, \alpha_n] [g(x)] du = 0 \Leftrightarrow [\alpha_1, \dots, \alpha_n] \cdot [g(x)] = 0$
  - Result :  $H_2 = g(x)^\perp = \text{span}\{\omega \in H_1 \mid w \cdot g \equiv 0\}$ .

<sup>1</sup>[Ara94] page 24 et [Glu97] page 11

*Definition: Chain of vector subspaces (VS) of  $\mathcal{E}$*

$$H_0 \supset H_1 \supset H_2 \supset \dots \supset H_k \supset H_{k+1} \supset \dots H_\infty$$

$$H_0 = \text{span}\{dx, du\}$$

$$H_1 = \text{span}\{dx\}$$

$$H_2 = g^\perp$$

$$\vdots$$

$H_k = \text{VS of all 1-Linear forms having a relative degree greater than or equal to } k$

$$H_{k+1} = \text{span}\{\omega \in H_k \mid \dot{\omega} \in H_k\}$$

$\exists$  a limit  $k^*$  such that  $H_{k^*-2} = H_{k^*-1} = H_\infty$

*Theorem: Accessibility of a NLS*

A NLS satisfies the accessibility condition if and only if  
 $H_\infty = 0$

i.e. there are no autonomous elements (generalization of the Kalman criterion for linear systems)

*Example 2.7:* Let the system  $\dot{x} = Ax + Bu$

Then:

$$H_2 = B^\perp$$

$$H_3 = [B \ AB]^\perp$$

$$\vdots$$

$$H_k = [B \ \dots \ A^{k-2}B]^\perp$$

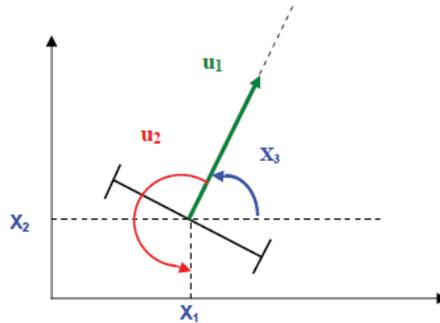
*Example 2.8:* Let the nonlinear system:

$$\dot{x} = \begin{bmatrix} -x_2^3 \\ u \end{bmatrix}.$$

Is it accessible?

Example 2.9: Mobile cart (Exercise)<sup>1</sup>

We consider a mobile cart: fixed rear wheels, "crazy" front wheels. We note  $u_1$  the longitudinal speed,  $u_2$  the angular speed,  $x_1$  and  $x_2$  the coordinates of the center of the rear wheels. The model equation is:



$$\dot{x} = \begin{bmatrix} \cos(x_3) & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad f(x) \equiv 0,$$

$$H_0 = \text{span}\{dx, du\}, \quad H_1 = \text{span}\{dx\}, \quad H_2 = g(x)^\perp$$

<sup>1</sup>[Glu92] p. 51 and [Sam91]

$$\dot{x} = \begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad f(x) \equiv 0,$$

Computation of  $H_2 = g(x)^\perp$

$$\omega = [\alpha_1 \quad \alpha_2 \quad \alpha_3] \text{ and } \omega \cdot g \equiv 0$$

$$\Rightarrow 1) \quad \omega \cdot g_2 \equiv 0 \quad \Leftrightarrow \quad \alpha_3 = 0$$

$$2) \quad \omega \cdot g_1 \equiv 0 \quad \Rightarrow \quad \text{for example } \omega = [\sin x_3 \quad -\cos x_3 \quad 0] \\ \omega = \sin x_3 dx_1 - \cos x_3 dx_2$$

$$\text{Thus } H_2 = \text{span}\{\sin x_3 dx_1 - \cos x_3 dx_2\}.$$

Computation of  $H_3$ :  $H_3 = \{\omega \in H_2 \mid \dot{\omega} \in H_2\}$

$$\begin{aligned} \dot{\omega} &= \dot{\alpha}[\sin x_3 dx_1 - \cos x_3 dx_2] + \alpha \frac{d}{dt}[\sin x_3 dx_1 - \cos x_3 dx_2] \\ &= \dot{\alpha}[\sin x_3 dx_1 - \cos x_3 dx_2] + \alpha u_2 \cos x_3 dx_1 + \alpha u_2 \sin x_3 dx_2 \\ &\quad + \alpha \sin x_3 d(u_1 \cos x_3) - \alpha \cos x_3 d(u_1 \sin x_3) \\ &= (\in H_2) - \alpha u_1 \sin x_3^2 dx_3 - \alpha u_1 \cos x_3^2 dx_3 = * -\alpha u_1 dx_3 \notin H_2 \Rightarrow H_3 = \{0\} \end{aligned}$$

$\Rightarrow$  the system is accessible.

*Canonical representation of accessibility in "linear form".*

=> Search for functions  $\varphi(x)$  such as  $\omega = d\varphi$

Definition: Integrable vector subspace

A vector subspace is said to be integrable when it admits a base of the form:  $\{d\varphi_1, \dots, d\varphi_n\}$

• Example 2.10: Test the integrability of  $H_2$  for the mobile cart

- Computation of  $d\omega$  (Poincaré Lemma):

$$d\omega = \cos x_3 dx_3 \wedge dx_1 + \sin x_3 dx_3 \wedge dx_2 \neq 0 \Rightarrow H_2 \text{ is not integrable}$$

- then computation of  $d\omega \wedge \omega$  (Frobenius Theorem) :

$$\begin{aligned} d\omega \wedge \omega &= [\cos x_3 dx_3 \wedge dx_1 + \sin x_3 dx_3 \wedge dx_2] \wedge [\sin x_3 dx_1 - \cos x_3 dx_2] \\ &= dx_3 \wedge dx_2 \wedge dx_1 \neq 0 \end{aligned}$$

Thus by Frobenius Theorem application:

$$d\omega \wedge \omega = dx_3 \wedge dx_2 \wedge dx_1 \neq 0 \Rightarrow H_2 \text{ is not integrable.}$$

Example 2.11: ACCESSIBILITY; INTEGRABILITY of  $H_k$  (exercise)

Let the following system:

$$\dot{x} = \begin{bmatrix} x_2(1-u) \\ x_3 \\ x_2 u \end{bmatrix}$$

Is it accessible?  
if yes, can the  $H_k$  be integrated?

# CHAPTER 3

## INPUT/OUTPUT LINEARIZATION <sup>1</sup>

### BY STATE FEEDBACK AND STATIC DECOUPLING

### 3.1 Introduction

#### 3.1.1 System Structure, relative degree

*Definition: relative degree*

The relative degree  $r$  of a system output is the smallest order of derivation (temporal) showing explicitly a control input.

Remark:

in the case of linear systems, the relative degree  $r$  is the order of the system

*Exercise: relative degree computation*

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = u \quad y = x_2 - x_1 \qquad \dot{y} = \dot{x}_2 - \dot{x}_1 = u - x_2$$

$u$  explicitly appears **the relative degree of the output  $y$  is  $r=1$ .**

<sup>1</sup> [Glu92] Chapter 3

#### 3.1.2 Introductive examples

Example 3.1 Monovariate Case Let system  $\begin{cases} \dot{x} = -x^2 + u \\ y = x \end{cases}$

With a nonlinear control  $u = x^2 + v$ , then the closed loop system is written:

$$\dot{x} = v$$

$\Rightarrow$  linear system with a new control  $v$ .

$\frac{Y(s)}{V(s)}$  is a linear system  $\Rightarrow$  exact input/output linearization

**Interest:** no approximation, allows then to use the whole linear theory for example with  $v = -\frac{1}{T}[y - y_r]$ : pole placement + output reference !

*Check the Input/Output relationship?*

<sup>1</sup> [Glu92] Chapter 3.

Example 3.2 Multivariable case, *Input/Output Coupling*

$$\left\{ \begin{array}{l} \dot{x}_1 = -x_2 + u_1 \\ \dot{x}_2 = -2x_2 + u_2 \\ y_1 = x_1 \\ y_2 = x_2 \end{array} \right. \quad \text{There is a Input/ Output Coupling (non-symmetrical) :}$$

$u_1$  have an effect on  $x_1$  and thus  $y_1$

$u_2$  have an effect on  $x_1$  and  $x_2$ , and thus  $y_1$  and  $y_2$ .

Which control can provide input/output decoupling?

**PROBLEM to solve: In a general non-linear (multivariable) framework, how to calculate a static compensator that will give an exact input-output linearization of the system?**

Definition: Static Feedback

$$u = F(x) + G(x)v$$

- Linear case:  $u = Fx + Gv$
- In closed loop :  $\dim(\text{System} + \text{Controller}) = \dim(\text{System})$

How to linearize? 2 complementary tools

1) Reversible static feedback:

$$u = F(x) + G(x)v \quad \text{with } G \text{ an invertible square matrix}$$

2) Transformation of state coordinates:

$$z = \phi(x), \text{ locally invertible i.e. out of singularities}$$

Example 3.3 . See the following monovariable NLS

$$\begin{array}{l} \dot{x}_1 = x_3 \\ \dot{x}_2 = \sin(x_3) \\ \dot{x}_3 = u \\ y = x_2 \end{array}$$

An idea for a solution, i.e. to obtain an Input-Output linearization, a solution approach is to examine the Input-Output structure (relative degree, ...) by **Inversion Analysis**.

=> Problem formulation: Input/Output Linearization of a NLS

Given a non-linear system, to find if possible :

- a state feedback  $u = F(x) + G(x)v$
- a bijective coordinates transformation of state variables  $z = \phi(x)$

such that after feedback and coordinates transformation, the NL System can be written:

$$\begin{aligned} \dot{z}_1 &= A z_1 + B v && \text{linear dynamics} \\ \dot{z}_2 &= f_2(z_1, z_2) + g_2(z_1, z_2). v && \text{zero dynamics (linear or nonlinear)} \\ y &= C z_1 \end{aligned}$$

with  $z = (z_1, z_2), v = (v_1, v_2)$ .

and the pair (A,B) is controllable, the pair (C,A) is observable, with :

- **Monovisible case:**

$$A = \begin{bmatrix} 0 & 1 & 0 \\ & & 1 \\ 0 & & 0 \end{bmatrix}_{R \times r}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{r \times 1}, C = [1 \quad 0 \quad \dots \quad 0]_{1 \times r}$$

- **Multivariable case:**

$$y_1 = z_{1,1}, \quad y_2 = z_{1,r_1+1}, \dots$$

$$A = \begin{bmatrix} A_{11} & & 0 \\ & A_{12} & \\ 0 & & \end{bmatrix}, \quad B = [B_1 \quad B_2 \quad \dots]$$

$$\text{with } A_{11} = \begin{bmatrix} 0 & 1 & 0 \\ & & 1 \\ 0 & & 0 \end{bmatrix}_{r_1 \times r_1}, \dots, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{r_1 \times 1}$$

where  $r_i$  is the relative degree of the output  $y_i$ , etc.

- After coordinate transformation and static feedback the I/O relations are:

$$y_1^{(r_1)} = v_1, \dots, y_p^{(r_p)} = v_p: p \text{ decoupled integrator chains of length } r_i^1$$

-  $z_2$  corresponds to the zero dynamics:

Generalization of the notion of zeros in linear

(sometimes pay attention to the stability of the zero dynamics)

- The transfer is then written  $\frac{Y(s)}{V(s)} = C(sI - A)^{-1}B$

- In addition, a complementary linear loop allows to place the poles of the transfer:

This then allows the application of robust linear controllers for example, (exemple 1.1)

• *Limitations of the problem:*

- *Loss of "commandability" (accessibility, singularities)*

- *the zero dynamics  $z_2$  can be unstable (not always critical)*

- *Equivalent in linear to "hide poles" ( $\Rightarrow$  dynamic looping, stability)*

<sup>1</sup>[Isi89] Normal form

## 3.2 Input Output Linearization

### 3.2.1 Single output case (dim $y = p = 1$ )

#### Theorem<sup>1</sup>

The input/output linearization of a NL single-input single-output system admits a solution  
 $\Leftrightarrow$  the relative degree of  $y$  is finite, ie  $dr(y) := r \leq n$ .

Proof:

- Sufficient condition:

$$\text{If } dr(y) := r = n \text{ then } y^{(n)} = F'(x) + G'(x)u := v$$

And  $u = -G'^{-1}(x)F'(x) + G'^{-1}(x)v$  is a linearizing input/output control

$$\text{with } z_1 = [y, \dots, y^{(r-1)}]$$

and  $z_2$ : arbitrary function of  $x$  such as  $z = \varphi(x)$  is locally invertible.

- Necessary condition: otherwise no input ! (the output cannot be controlled).

<sup>1</sup>[Isi89] Chapter 4 and [Mar95] section 4.2

Example 3.4 Let system: 
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_2 - x_1 \end{cases}$$

- $\dot{y} = \dot{x}_2 - \dot{x}_1 = u - x_2$   
 $\Rightarrow$  the relative degree  $r$  of the output is:  $r = 1$

- Goal:  $\dot{y} = v$  thus

1)  $u = x_2 + v$

- 2) Coordinates transformation:

$$z_1 = y = x_2 - x_1 \Rightarrow \dot{z}_1 = v$$

$$z_2 = x_2 \Rightarrow \dot{z}_2 = u = x_2 + v = z_2 + v$$

Finally: 
$$\begin{cases} \dot{z}_1 = v \\ \dot{z}_2 = z_2 + v : \text{zero dynamics (? stability)} \end{cases}$$

- The canonical form is well found:

$$A = [0], B = [1], C = [1 \ 0], f_2(z) = z_2, g_2(z) = 1.$$

Note: Therefore pay attention to the zero dynamics.

a) A solution: change the output

Example 3.5:

Example 3.4 with

$$\bar{y} = x_2 + x_1, \dot{\bar{y}} = \dot{x}_2 + \dot{x}_1 = x_2 + u, \ddot{\bar{y}} = \dot{x}_2 + \dot{u} = u + \dot{u}$$

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s^2} \text{ (stable zero)}$$

To obtain a pure integrator  $\Rightarrow u = -x_2 + v$ ,

Choose  $z_1 = x_1 + x_2, z_2 = x_2 \Rightarrow x_1 = z_1 - z_2$

then 
$$\begin{cases} \dot{z}_1 = v \\ \dot{z}_2 = -z_2 + v : \text{stable now!} \end{cases}$$

To keep the tracking of  $y_{ref}$  as  $\bar{y} = x_2 + x_1 = y + 2x_1 = y + 2(z_1 - z_2)$

$$\Rightarrow \bar{y}_{ref} = x_2 + x_1 = y_{ref} + 2x_1$$

b) It is not always significant

Example : A control to constant speed  $\Rightarrow$  the position increases!

## Application to trajectory tracking

Given that : 
$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$
 and a differentiable reference trajectory  $y_{ref}(t)$ .

Find  $u$  such that:  $y_{ref}(t) - y(t) \rightarrow 0$  when  $t \rightarrow \infty$ .

- **Mono output case:** Let the error  $e = y_{ref}(t) - y(x)$

Solve in  $u$  :  $e^{(r)} + \sum_{k=0}^{r-1} \lambda_k e^{(k)} = 0$  with the  $\lambda_k \in R$  the closed loop tuning parameters.

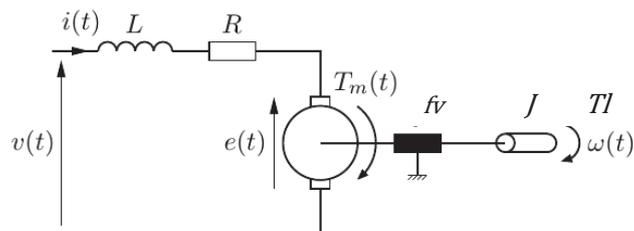
$$y_{ref}^{(r)}(t) + \sum_{k=0}^{r-1} \lambda_k e^{(k)}(t) = y^{(r)}(x, u) := a(x) + b(x)u$$

$$\Rightarrow u = \frac{1}{b(x)} \left[ -a(x) + y_{ref}^{(r)}(t) + \sum_k \lambda_k e^{(k)}(t) \right]$$

$e(t)$  is asymptotically stable :  $e(t) \rightarrow 0$  when  $t \rightarrow \infty$  according to the choice of the  $\lambda_k$ .

### Example 3.6 a perturbed linear system : the constant flux DC motor

Constant flux (as the permanent magnetic flux), the rotor scheme is:



Electric equation  $v = R i + L \frac{di}{dt} + e$  with  $e = K_{em} \omega$  i.e.  $L \frac{di}{dt} = -R i - K_{em} \omega + v$

Mechanical equation  $J \frac{d\omega}{dt} = T_m - f_v \omega - T_l$  with  $T_m = K_{em} i$  and  $T_l$  a load torque.

With  $x_1 = \omega, x_2 = i, u = v$  and the output  $y = x_1$ , the state model of the DC motor can be read as:

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 + c T_l \\ \dot{x}_2 = dx_2 + ex_1 + fu, y = x_1 \end{cases}$$

It is a disturbed SISO linear state model. The disturbance is the load torque.

? **Analysis of the system** : Relative degree, perturbation ?

Pole placement, Trajectory tracking ....

Exercise a perturbed nonlinear system : the serial DC motor

$$v = (R_s + R_r)i + (L_s + L_r)\frac{di}{dt} + e$$

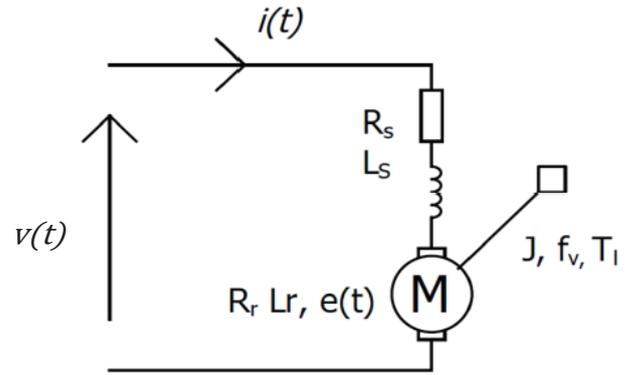
$$e = K\Phi\Omega = KK_f i\Omega^*$$

$$J\frac{d\Omega}{dt} = K\Phi i - f_v\Omega - C_l = KK_f i^2 - f_v\Omega - T_l$$

Thus

$$\begin{cases} J\frac{d\Omega}{dt} = KK_f i^2 - f_v\Omega - T_l \\ (L_s + L_r)\frac{di}{dt} = -(R_s + R_r)i - KK_f i\Omega + u \end{cases}$$

$$\begin{cases} \frac{d\Omega}{dt} = \frac{KK_f}{J}i^2 - \frac{f_v}{J}\Omega - \frac{1}{J}T_l \\ \frac{di}{dt} = -\frac{R_s + R_r}{L_s + L_r}i - \frac{KK_f}{L_s + L_r}i\Omega + \frac{1}{L_s + L_r}u \end{cases}$$



? **Analysis of the system** : Relative degree, perturbation ?  
Pole placement, Trajectory tracking ....

**3.2.2 Multi-outputs case (MIMO) (dim y = p > 1)**

If  $(r_1, \dots, r_p)$  are the relative degrees, then  $\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_p^{(r_p)} \end{bmatrix} = A_0(x) + B_0(x)u := v$

Theorem (Sufficient Condition):<sup>1</sup>

Input/Output linearization via static state feedback admits a solution if

$$\text{rang} \frac{\partial(y_1^{(r_1)}, \dots, y_p^{(r_p)})}{\partial(u_1, \dots, u_m)} := \text{rang} B_0 = p$$

Proof : SC A linearizing Input/Output control  $\Rightarrow u = B_0^{-1}(x)[-A_0(x) + v]$

Remark: a decoupled system can be obtained after feedback.

<sup>1</sup>[Isi89] Chapter 5

NC Counter-example to necessity

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ x_3 + u_1 \\ u_2 \end{bmatrix} \text{ et } y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{y}_1 = u_1 \quad \Rightarrow r_1 = 1$$

$$\dot{y}_2 = x_3 + u_1 \quad \Rightarrow r_2 = 1$$

$$\text{rang} \frac{\partial(y_1^{(r_1)}, \dots, y_p^{(r_p)})}{\partial(u_1, \dots, u_m)} = \text{rang} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1 < p$$

and yet the system is linear in input-output !

### 3.3 New problem: the static decoupling and Input/Output linearization, multivariable case

Goal : Find - a state feedback  $u = F(x) + G(x)v$

- a bijective change of state variables  $z = \phi(x)$

such that after feedback and transformation the NL system can be written :

$$\dot{z}_1 = A. z_1 + B. v \quad \text{Linear dynamics}$$

$$\dot{z}_2 = f_2(z_1, z_2) + g_2(z_1, z_2). v$$

$$y = C. z_1 \quad \text{with}$$

$z = (z_1, z_2), v = (v_1, v_2), \text{ pair}(A, B) \text{ est controllable, pair}(C, A) \text{ is observable,}$

with  $y_1 = z_{1,1}, \dots, y_2 = z_{1, \sum r_i + 1}, \dots$

$$A = \begin{bmatrix} A_{11} & & 0 \\ & A_{12} & \\ 0 & & \end{bmatrix}, \quad B = [B_1 \quad B_2 \quad \dots]$$

$$\text{with } A_{11} = \begin{bmatrix} 0 & 1 & 0 \\ & & 1 \\ 0 & & 0 \end{bmatrix}_{r_1 * r}, \dots, B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{r_1 * 1} \quad \text{where } r_i \text{ is the relative degree of } y_i, \text{ etc}$$

Theorem:

The NL System can be decoupled by static Feedback  $\Leftrightarrow \text{rang} \frac{\partial(y_1^{(r_1)}, \dots, y_p^{(r_p)})}{\partial(u_1, \dots, u_m)} = p$

Definition: Decoupling matrix

The Jacobian Matrix  $\frac{\partial(y_1^{(r_1)}, \dots, y_p^{(r_p)})}{\partial(u_1, \dots, u_m)}$  is called "decoupling matrix" ( $:=B_0(x)$ )

After feedback, we get  $p$  independent Input/Output subsystems that are easy to linearize

Sketch of proof:

- Sufficient condition: the  $p$  decoupled systems can be obtained by feedback.
- Necessary condition: otherwise, the accessibility of the "Input/Output" dynamics after feedback is not satisfied (fall of the rank of  $B_0$ ).

• Remark 1:

If the theorem is satisfied ( $B_0(x)$  invertible), the system can be written:

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_p^{(r_p)} \end{bmatrix} = A_0(x) + B_0(x)u = v$$

and  $u = B_0^{-1}(x)[-A_0(x) + v] \Rightarrow$   **$p$  decoupled integrator chains of length  $r_i$**

• Remark 2:

2.1 Coordinates transformation:  $z_{1,1} = y_1, \dots, z_{1,r_1+1} = y_2, \dots$

2.2 The  $p+1$  dynamic is unobservable (noted the zero dynamics)

2.3 We obtain controllable linear subsystems of relative degree identical to those before feedback (because we have used a static state feedback).

Example 3.7 Mobile Cart (Exercise)

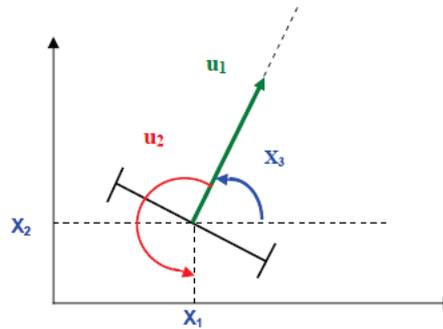
We consider a mobile cart (rear drive wheels, front "crazy" wheels).

Let  $u_1$  the longitudinal speed,

$u_2$  the angular speed,

$x_1$  and  $x_2$  the coordinates of the center of the rear wheels.

$x_3$  the rotation angle wrt the center of the rear wheels.



The model equation is:

$$\begin{cases} \dot{x}_1 = u_1 \cos x_3 \\ \dot{x}_2 = u_1 \sin x_3 \\ \dot{x}_3 = u_2 \end{cases}$$

- 2 study cases:     a) outputs      $y_1 = x_1, y_2 = x_3$   
                               b) outputs      $y_1 = x_1, y_2 = x_2$

Example 3.7 Mobile Cart with  $y_1 = x_1, y_2 = x_2$  (case b)

- no solution by static compensator

- ? dynamic compensator: for example,  $u_1 = x_4, \dot{x}_4 = v_1, u_2 = v_2$

with  $v_1$  : longitudinal acceleration

=> extended system dynamics = system dynamics + compensator dynamics:

$$\begin{aligned} \dot{x}_1 &= \cos(x_3)x_4 & y_1 &= x_1 \\ \dot{x}_2 &= \sin(x_3)x_4 & y_2 &= x_2 \\ \dot{x}_3 &= v_2 \\ \dot{x}_4 &= v_1 \end{aligned}$$

Compute  $v = \alpha(x) + \beta(x)w$  such that  $(y_1, w_1)$  et  $(y_2, w_2)$  are decoupled.

- Calculate the decoupling matrix

- Linearization Input/Output:  $v, \xi$

➡ First: Input-Output Inversion

$$\begin{aligned} \dot{y}_1 &= \cos(x_3)x_4, & \ddot{y}_1 &= -v_2 \sin(x_3)x_4 + v_1 \cos(x_3) := w_1 \\ \dot{y}_2 &= \sin(x_3)x_4, & \ddot{y}_2 &= v_1 \sin(x_3) + v_2 x_4 \cos(x_3) := w_2 \end{aligned}$$

Example 3.7 continuation Mobile cart (case b)

$$\text{Decoupling matrix : } \frac{\partial(y_1^{(2)}, y_2^{(2)})}{\partial(v_1, v_2)} = \begin{bmatrix} \frac{\partial y_1^{(2)}}{\partial v_1} & \frac{\partial y_1^{(2)}}{\partial v_2} \\ \frac{\partial y_2^{(2)}}{\partial v_1} & \frac{\partial y_2^{(2)}}{\partial v_2} \end{bmatrix} = \begin{bmatrix} \cos x_3 & -x_4 \sin x_3 \\ \sin x_3 & x_4 \cos x_3 \end{bmatrix}$$

Its determinant is  $x_4 (\neq 0) \Rightarrow$  row matrix 2

So we can decouple.

Then,  $v_1 = w_1 \cos x_3 + w_2 \sin x_3$

$$v_2 = \frac{-w_1 \sin x_3 + w_2 \cos x_3}{x_4}$$

We choose  $z_{1,1} = x_1, z_{1,2} = x_4 \cos x_3, z_{1,3} = x_2, z_{1,4} = x_4 \sin(x_3)$ .

Note : no zero dynamics.

### 3.4 Application of static decoupling to trajectory tracking<sup>1</sup>

Given that :  $\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$  and a differentiable reference trajectory  $y_{ref}(t)$ .

Find  $u$  such that:  $y_{ref}(t) - y(t) \rightarrow 0$  when  $t \rightarrow \infty$

- Single output case (reminder): Let the error  $e = y_{ref}(t) - y(x)$

Solve in  $u$  :  $e^{(r)} + \sum_{k=0}^{r-1} \lambda_k e^{(k)} = 0$  with the  $\lambda_k \in R$  the closed loop tuning parameters.

$$y_{ref}^{(r)}(t) + \sum_{k=0}^{r-1} \lambda_k e^{(k)}(t) = y^{(r)}(x, u) = a(x) + b(x)u$$

$$\longrightarrow u = \frac{1}{b(x)} \left[ -a(x) + y_{ref}^{(r)}(t) + \sum_k \lambda_k e^{(k)}(t) \right]$$

$e(t)$  is asymptotically stable :  $e(t) \rightarrow 0$  when  $t \rightarrow \infty$  according to the choice of the  $\lambda_k$ .

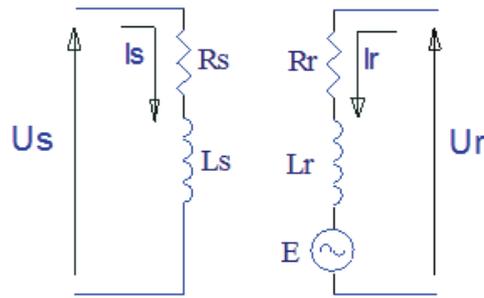
- Multi outputs case: after decoupling and linearization, we have  $p$  single-output systems:

$$y_i^{(r)}(t) = v_i(t)$$

- Remark: if the NSC of the theorem "Static Decoupling" is satisfied, one can:

a) Decouple, b) linearize and c) make trajectory tracking in a single feedback!

• Example 3.8 DC motor, with a separately excited flux stator coil



Electrical scheme of a separately excited flux DC Motor

Electrical equations:

$$u_s = R_s i_s + L_s \frac{di_s}{dt}, \quad u_r = R_r i_r + L_r \frac{di_r}{dt} + K\Phi\Omega = R_r i_r + L_r \frac{di_r}{dt} + KK_f i_s \Omega *$$

Mechanical equation:  $J \frac{d\Omega}{dt} = K\Phi i_r - f_v \Omega - T_l = KK_f i_s i_r - f_v \Omega - T_l$

\*The stator flux is assumed to be proportional to the stator current :  $\Phi = K_f i_s$

$$\begin{cases} J \frac{d\Omega}{dt} = KK_f i_s i_r - f_v \Omega - T_l \\ L_s \frac{di_s}{dt} = -R_s i_s + u_s \\ L_r \frac{di_r}{dt} = -R_r i_r - KK_f i_s \Omega + u_r \end{cases}$$

Or

$$\begin{cases} \frac{d\Omega}{dt} = \frac{KK_f}{J} i_s i_r - \frac{f_v}{J} \Omega - \frac{1}{J} T_l \\ \frac{di_s}{dt} = -\frac{R_s}{L_s} i_s + \frac{1}{L_s} u_s \\ \frac{di_r}{dt} = -\frac{R_r}{L_r} i_r - \frac{KK_f}{L_r} i_s \Omega + \frac{1}{L_r} u_r \end{cases}$$

with  $x_1 = \Omega$ ,  $x_2 = i_s$ ,  $x_3 = i_r$ ,  $u_1 = u_s$ ,  $u_2 = u_r$  the DC motor with controllable stator flux is a nonlinear MIMO model:

$$\begin{cases} \dot{x}_1 = \frac{KK_f}{J} x_2 x_3 - \frac{f_v}{J} x_1 - \frac{1}{J} T_l \\ \dot{x}_2 = -\frac{R_s}{L_s} x_2 + \frac{1}{L_s} u_s \\ \dot{x}_3 = -\frac{R_r}{L_r} x_3 - \frac{KK_f}{L_r} x_1 x_2 + \frac{1}{L_r} u_r \end{cases}$$

or

$$\begin{cases} \dot{x}_1 = a x_2 x_3 + b x_1 + c T_l \\ \dot{x}_2 = d x_2 + e u_1 \\ \dot{x}_3 = f x_3 + g x_1 x_2 + h u_2 \end{cases}$$

The outputs to control could be  $y_1 = x_1, y_2 = K_f x_2$ .

( $y_2$  is the rotor flux)

Is the system decouplable?

- Calculate the decoupling matrix
- If decouplable, compute  $v = \alpha(x) + \beta(x)w$  such that  $(y_1, w_1)$  et  $(y_2, w_2)$  are decoupled.
- Linearization Input/Output:  $v, \xi$ , Trajectory tracking ...

### 3.5 Application to AC machines

Example 3.9 Currents controlled Induction motor

Here the **inputs are the two currents**:  $u_1 = i_{sd}, u_2 = i_{sq}$

The induction motor reduced model is then

$$\begin{aligned} \dot{x}_1 &= m x_2 u_2 - c x_1 \\ \dot{x}_2 &= -a x_2 + a M_{sr} u_1 \\ \dot{x}_3 &= p x_1 + a \frac{M_{sr}}{x_2} u_2 \quad \text{with } x_1 = \Omega, x_2 = \Phi_{rd}, x_3 = \rho \text{ (rotor flux angle).} \end{aligned}$$

$$a = \frac{Rr}{Lr}, m = \frac{pM_{sr}}{JLr}, c = \frac{f_v}{J}, p = \text{pole pair number}$$

Case 1  $y_1 = x_1 = \Omega, y_2 = x_2 = \Phi_{rd}$

Case 2  $y_1 = x_1 = \Omega, y_2 = x_3 = \rho$

Is the induction motor controlled by stator currents is decouplable by static feedback ?

Exercice 3.10 Synchronous motor

Exercice 3.11 Induction motor

# CHAPTER 4 SYSTEMS INVERSION AND DYNAMIC DECOUPLING <sup>1</sup>

## 4.1 Introduction to system inversion

Let the system:

$$\Sigma \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

find, if possible, the inverse system:

$$\Sigma^{-1} \begin{cases} \dot{z} = F(z, y, \dot{y}, \dots) \\ u = H(z, y, \dot{y}, \dots) \end{cases}$$

such that  $(u(t), y(t))$  a solution of  $\Sigma$  is also a solution of  $\Sigma^{-1}$  and vice versa.

**Method:** Differentiation of outputs + algebraic elimination of input variables  $u$ .

<sup>1</sup>[Glu92]

**Example 4.1<sup>1</sup>:** Current in dipoles

- Let  $u$  be the voltage across a resistor  $R$  through which the current flows  $y$ . Then

$$\Sigma \left\{ y = \frac{1}{R} u \quad \text{and} \quad \Sigma^{-1}: u = Ry \text{ hence the control } \dots \right.$$

- Now let's take a resistor  $R$  and an inductance  $L$  in series ( $u$  voltage,  $x$  current)

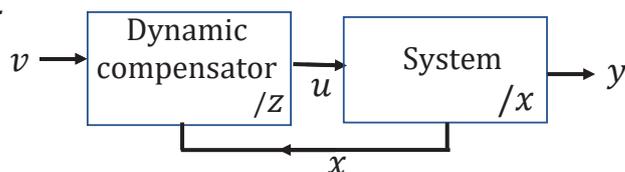
$$\Sigma \begin{cases} \dot{x} = -\frac{R}{L}x + \frac{1}{L}u \\ y = x \end{cases} \text{ a proper system.}$$

$\Sigma^{-1}: u = L\dot{y} + Ry$  the inverse system is not proper (depends on the derivative of  $y$ ).

By this inversion analysis, this is the information on the system structure that is researched

**Definition:** Dynamic compensator

$$\begin{cases} \dot{z} = F(x, z) + G(x, z)v \\ u = H(x, z, v) \end{cases}$$



In closed loop

*Dim of (system + compensator) = system dimension + compensator dimension*

<sup>1</sup>[Glu95]

## 4.2 Inversion algorithm<sup>1</sup>: to find p independent I/O equations.

### Step 1

- Compute  $\dot{y} = \frac{\partial h}{\partial x} [f(x) + g(x)u] = \alpha_1(x) + \beta_1(x)u$

- Let  $\rho_1 = \text{rank} [\beta_1(x)]$

- Choose  $\rho_1$  independent rows of  $\beta_1$  such that the  $\rho_1$  first rows of  $\beta_1$  are independent (swap the order of the outputs if necessary).

*Important notation:* it is defined:  $\dot{y} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix}$  where  $\dot{y}_1$  contains the  $\rho_1$  independent equations. Step 1

$$\begin{aligned} \dot{y}_1 &= \tilde{\alpha}_1(x) + \tilde{\beta}_1(x)u \\ \dot{y}_2 &= \hat{\alpha}_1(x) + \hat{\beta}_1(x)u \quad \text{with the rank of } \beta_1 = \rho_1^* \end{aligned}$$

- Eliminate u between the  $\dot{y}_1$  and  $\dot{y}_2$ :

$$\begin{aligned} \dot{y}_1 &= \tilde{\alpha}_1(x) + \tilde{\beta}_1(x)u \\ \dot{y}_2 &= \hat{y}_2(x, \dot{y}_1) \end{aligned}$$

Remark:  $\{dx, d\dot{y}_1\} = \text{basis of } \{dx, d\dot{y}\}$

*\*Vocabulary: The system has  $\rho_1$  zeros at infinity of order 1 (step 1).*

<sup>1</sup>[Glu92] page 27 and annex 4.

### Step 2

- Compute  $\frac{d}{dt} \dot{y}_1$

$$\begin{aligned} \ddot{y}_1 &= \frac{\partial \dot{y}_1}{\partial \dot{y}_1} \ddot{y}_1 + \frac{\partial \dot{y}_1}{\partial x} [f(x) + g(x)u] \\ \ddot{y}_2 &= \alpha_2(x, \dot{y}_1, \ddot{y}_1) + \beta_2(x, \dot{y}_1)u \end{aligned}$$

- Let  $\rho_2 = \text{rang} \begin{bmatrix} \tilde{\beta}_1(\cdot) \\ \beta_2(\cdot) \end{bmatrix} \geq \rho_1$ , total number of independent equations in step 2.

- If  $\rho_2 < p$  swap if necessary, the order of the outputs to obtain:

$$\begin{aligned} \ddot{y}_1 &= \begin{bmatrix} \ddot{y}_2 \\ \ddot{y}_2 \end{bmatrix}, \quad \dim \ddot{y}_2 = \rho_2 - \rho_1, \quad \dim \ddot{y}_2 = p - \rho_2, \\ \text{Let } \begin{cases} \ddot{y}_2 = \tilde{\alpha}_2(\cdot) + \tilde{\beta}_2(\cdot)u \\ \ddot{y}_2 = \hat{\alpha}_2(\cdot) + \hat{\beta}_2(\cdot)u \end{cases} & \text{with } \rho_2 = \text{rang} \begin{bmatrix} \tilde{\beta}_1(\cdot) \\ \beta_2(\cdot) \end{bmatrix} \\ & \text{and } \hat{\beta}_2 = L_1(\cdot)\tilde{\beta}_1(\cdot) + L_2(\cdot)\beta_2(\cdot). \end{aligned}$$

- After elimination of u we obtain  $\ddot{y}_2 = \ddot{y}_2(x, \dot{y}_1, \ddot{y}_1, \ddot{y}_2)$   
after this step, there remain  $p - \rho_2$  dependent equations.

*Vocabulary: The system has  $\rho_2 - \rho_1$  zeros at infinity of order 2 (i.e.  $n_i'$ )*

Summary in step 2 of the independent equations found :

$$\begin{aligned}\dot{\hat{y}}_1 &= \tilde{\alpha}_1(x) + \tilde{\beta}_1(x)u \\ \ddot{\hat{y}}_2 &= \tilde{\alpha}_2(x, \dot{\hat{y}}_1, \ddot{\hat{y}}_1) + \tilde{\beta}_2(x, \dot{\hat{y}}_1)u\end{aligned}$$

Step k+1

- Compute  $\frac{d}{dt} \hat{y}_k^{(k)}$ :

$$\hat{y}_k^{(k)} = \hat{y}_k^{(k)}(x, \dot{\hat{y}}_1, \dots, \tilde{y}_k^{(k)})$$

$$\hat{y}_k^{(k+1)} = \alpha_{k+1}(\cdot) + \beta_{k+1}(\cdot)u$$

- So:

$$\rho_{k+1} = \text{rank} \begin{bmatrix} \tilde{\beta}_1(\cdot) \\ \vdots \\ \tilde{\beta}_k(\cdot) \\ \tilde{\beta}_{k+1}(\cdot) \end{bmatrix} \geq \rho_k$$

- Decompose  $\hat{y}_k^{(k+1)}$  into  $\begin{bmatrix} \tilde{y}_{k+1}^{(k+1)} \\ \hat{y}_{k+1}^{(k+1)} \end{bmatrix}$  with  $\tilde{y}_{k+1}^{(k+1)}$  having  $\rho_{k+1} - \rho_k$  independent equations.

- Eliminate u :

$$\begin{aligned}\tilde{y}_{k+1}^{(k+1)} &= \tilde{\alpha}_{k+1}(\cdot) + \tilde{\beta}_{k+1}(\cdot)u \\ \hat{y}_{k+1}^{(k+1)} &= \hat{y}_{k+1}^{(k+1)}(x, \dot{\hat{y}}_1, \dots, \tilde{y}_{k+1}^{(k+1)})\end{aligned}$$

$$\dim \hat{y}_{k+1} = p - \rho_{k+1}$$

Vocabulary: the system has  $\rho_{k+1} - \rho_k$  zeros at infinity (of order k+1)

Result at step k+1: the independent input-output equations retained are:

$$\begin{aligned}\dot{\hat{y}}_1 &= \tilde{\alpha}_1(x) + \tilde{\beta}_1(x)u \\ &\vdots \\ \tilde{y}_{k+1}^{(k+1)} &= \tilde{\alpha}_{k+1}(x, \dot{\hat{y}}_1, \ddot{\hat{y}}_1, \dots) + \tilde{\beta}_{k+1}(x, \dot{\hat{y}}_1, \dots)u\end{aligned}$$

**The inversion algorithm yields to compute:**

$$\rho_1 \leq \rho_2 \leq \dots \leq \rho_n = \rho_{n+1} = \rho \text{ limit with } \rho \leq m \quad \rho \leq p$$

- Vocabulary:
  - $\rho$  is the rank of the system
  - the orders of derivative of the highest outputs in the equations considered independent are called "essential orders" ( $n_{ie}$ ) i.e. they are the minimal orders that it is necessary to reach in the inversion to have independent differential relations Input-Output (necessary and sufficient condition for the decoupling problem).
  
- Remark 2:
  - The inversion of the system will allow the Input-Output Decoupling control by dynamic compensator
  - In the case of a linear system,  $\rho$  is the rank of the transfer matrix.

**4.3 Conclusion: decoupling by dynamic state compensator via Inversion algorithm**

Theorem:

The dynamic decoupling admits a solution  $\iff \rho$  the rank of the system is equal to  $p$ .

Note: the essential order  $n_{ie}$  is the largest derivation order of  $y_i$  in the inversion equations.

Construction of the dynamic compensator: the inversion algorithm gives the equations

$$\tilde{y}_k^{(k)} = \tilde{\alpha}_k(x, y_i^{(j)}) + \tilde{\beta}_k(x, y_i^{(j)})u$$

We solve in  $u$  the following equations:

$$y_i^{(n_{ie})} = v_i, \quad y_i^{(n_i+j-1)} = z_{ij} \quad 1 \leq j \leq n_{ie}-n_i$$

The dynamic compensator is then:

$$\begin{aligned} \dot{z}_{i1} &= z_{i2} \\ &\vdots \\ \dot{z}_{i, n_{ie}-n_i} &= v_i \quad i = 1, \dots, p \\ u &= H(x, z_{ij}, v_i) \end{aligned}$$

Remark 3: generalizes the rank of the transfer matrix of a linear system

Notation: if rank= $p$ , then the SNL is right invertible, if rank= $m$ , then the SNL is left invertible.

**Example 4.2: Unicycle <sup>1</sup>**

The system is described by the following equations:

$$\begin{aligned} \dot{x}_1 &= u_1 \cos(x_3) & y_1 &= x_1 \\ \dot{x}_2 &= u_1 \sin(x_3) \\ \dot{x}_3 &= u_2 & y_2 &= x_2 \end{aligned}$$

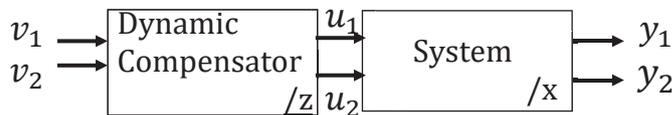
- Equations of the inversion algorithm:

$$\begin{aligned} \dot{y}_1 &= \dot{y}_1 = u_1 \cos x_3 \\ \dot{y}_2 &= \tan(x_3) \dot{y}_1 + (1 + \tan^2 x_3) \dot{y}_1 u_2 \quad \text{STOP} \\ &\Rightarrow n_{1e} = 2, n_{2e} = 2 \quad \text{ordre minimal de découplage} \end{aligned}$$

- Construction of the dynamic compensator:

We solve in u :

$$\begin{aligned} \ddot{y}_1 &= v_1 & \dot{z}_{11} &= v_1 & u_1 &= \frac{z_{11}}{\cos x_3} \\ \ddot{y}_2 &= v_2 & u_2 &= \frac{v_2 - \tan(x_3)v_1}{z_{11}(1 + \tan^2 x_3)} \end{aligned}$$



<sup>1</sup>[Glu92] page 51

**Exercise 4.1 Induction motor controlled by currents (reduced model)**

The inputs are the two currents  $u_1 = i_{sd}, u_2 = i_{sq}$

with  $x_1 = \Omega, x_2 = \Phi_{rd}, x_3 = \rho$ , the induction motor reduced model is:

$$\begin{aligned} \dot{x}_1 &= m x_2 u_2 - c x_1 & y_1 &= x_1 = \Omega, y_2 = x_3 = \rho \quad (\text{case 2}) \\ \dot{x}_2 &= -a x_2 + a M_{sr} u_1 \\ \dot{x}_3 &= p x_1 + a \frac{M_{sr}}{x_2} u_2 & a &= \frac{Rr}{Lr}, m = \frac{pM_{sr}}{JLr}, c = \frac{f_v}{J}, p = \text{pole pair number} \end{aligned}$$

Is the induction motor controlled by stator currents is decouplable by static feedback?

If not, is a dynamic feedback is possible ? If yes, what is the minimum size of the Decoupling Compensator ?

**Inversion Step 1**

$$\begin{aligned} \dot{y}_1 &= m x_2 u_2 - c x_1 \Rightarrow r_1 = 1 \quad (\text{relative degree of } y_1) \\ \dot{y}_2 &= p x_1 + a \frac{M_{sr}}{x_2} u_2 \Rightarrow r_2 = 1 \quad (\text{relative degree of } y_2) \end{aligned}$$

$$\beta_1(x) = \begin{bmatrix} 0 & m x_2 \\ 0 & a \frac{M_{sr}}{x_2} \end{bmatrix} \Rightarrow \text{rang } \beta_1 = 1$$

➡ Static feedback cannot be used for decoupling control ...

**Example 4.3: Car (Exercise)**

The system is described by the following equations:

$$\begin{aligned} \dot{x} &= \dot{x}_1 = u_1 \cos x_3 & y_1 &= x_1 \\ \dot{y} &= \dot{x}_2 = u_1 \sin x_3 \\ \dot{\theta} &= \dot{x}_3 = u_1 u_2 / l & y_2 &= x_2 \end{aligned}$$

with  $u_1$  car speed (=  $F(t)$ )  
and  $u_2 = \tan \varphi$ .

Note: it is a non-affine system!

Coordinates inputs change ?

Is it possible to decouple and Input-Output linearize this system?

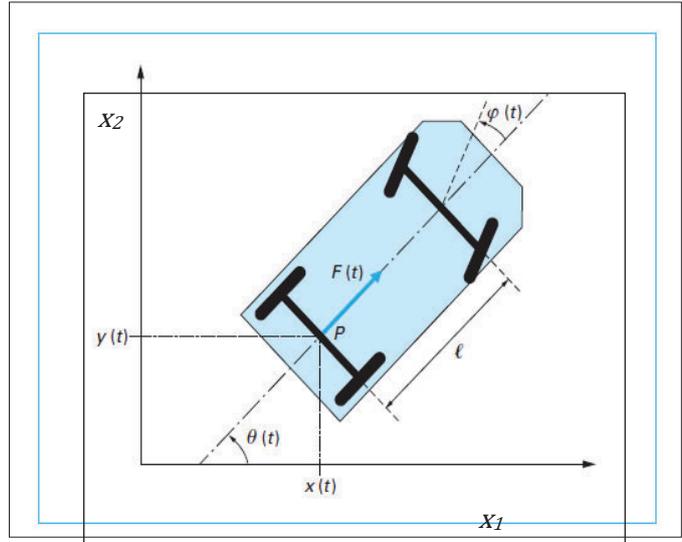


Figure : Car ([Mar99].)

**Inversion Step 1**

Compute the relative degree of each output:

$$\begin{aligned} \dot{y}_1 &= u_1 \cos x_3 & r_1 &= 1 \\ \dot{y}_2 &= u_1 \sin x_3 & r_2 &= 1, \quad \text{---} \rightarrow \quad n_2 = 1 \end{aligned}$$

Choose  $\tilde{y}_1 = y_1$  et  $\hat{y}_1 = y_2$   
rang  $\beta_1 = 1$

Step 2: we differentiate  $\hat{y}_1 = y_2 \rightarrow \ddot{y}_2$  explicitly depends of  $u_2 \rightarrow$  STOP

• So the equations of the inversion algorithm are:

$$\begin{aligned} \dot{y}_1 &= \dot{\tilde{y}}_1 = u_1 \cos x_3 \\ \ddot{y}_2 &= \ddot{y}_1 \tan x_3 + \frac{(1 + \tan^2 x_3) \dot{y}_1^2}{l \cos x_3} u_2 \end{aligned}$$

**→**  $n_1 = 2$   
 $n_{1e} = 2, n_{2e} = 2$  the minimal orders for input-output decoupling.

$$\sum n_i = 3, \sum n_{ie} = 4 \quad \rightarrow \text{dim of the DC} = 1$$

- Construction of the dynamic compensator: we solve in u ....

## CHAPITRE 5

### INPUT STATE LINEARIZATION « FLATNESS »<sup>1</sup>

#### 5.1 Remainder: Limits of input-output linearization

- The unobservable part of the closed loop system may be unstable
- Corresponds to hiding zeros by poles (for linear systems)

Example 5.1: Note  $\Sigma$ : 
$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_2 - x_1 \end{aligned} \right\} \frac{Y(s)}{Us} = \frac{s-1}{s^2} \text{ (of degree 1)}$$

Input Output linearization:

$$\Sigma \text{ Inversion} \Rightarrow \dot{y} = u - x_2 \Rightarrow u = x_2 + v \Rightarrow \dot{y} = v$$

$\Sigma$  en BF

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2 + v \\ y &= x_2 - x_1 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} z_1 &= y = x_2 - x_1 \\ z_2 &= x_2 \end{aligned} \right. \text{ with } \left\{ \begin{aligned} \dot{z}_1 &= x_2 + v - x_2 = v \rightarrow \text{observable} \\ \dot{z}_2 &= z_2 + v \rightarrow \text{unobservable unstable} \\ y &= z_1 \end{aligned} \right.$$

[Ara94] Chapitre 4

M2 EPICO ACEV Chap. 5: Input state Linearization or flatness

5.1

- Alternative:  
Search for a fictitious output noted  $y$  with minimal phase (stable zeros)

#### Remarks:

- If there is only one input ( $m=1$ ) and the relative degree of  $y = n$ , then the Input-Output linearization obviously involves the complete Input-State linearization.
- If  $m > 1$ ,  $\sum_{i=1}^p r_i = n$ , and the decoupling matrix is invertible, then the whole state can be linearized by feedback

Then It exists  $\varphi$  et  $\psi$  such that:

$$\begin{cases} x = \varphi(y, \dot{y}, \dots, y^{(n-1)}) \\ u = \psi(y, \dot{y}, \dots, y^{(n)}) \end{cases}$$

(Flatness)

## 5.2 Problem 1 Input State Linearization by static state feedback

### Problem 1 Input State Linearization by static state feedback

Given  $\dot{x} = f(x) + g(x)u, x \in R^n, u \in R^m,$

Find if possible :

$$u = \alpha(x) + \beta(x)v, \quad \beta(x) \text{ square invertible and coordinates } z = \Phi(x)$$

such that the closed loop system is written

$$\dot{z} = Az + Bv \quad \text{with } (A,B) \text{ controllable.}$$

#### Theorem:<sup>1</sup>

The Input-State linearization by static feedback admits a solution

$$\Leftrightarrow \begin{aligned} H_\infty &= 0 \\ H_k &\text{integrable } \forall k \geq 1 \end{aligned}$$

*The  $H_k$ 's have been defined in the chapter on the accessibility of nonlinear systems. Their integrability is verified with the Frobenius theorem.*

<sup>1</sup> [Ara94] Chapitre 4 page 39

#### - Proof: Case $m=1$

- Sufficient condition, if  $H_\infty = 0$  no autonomous elements then

$$\dim H_1 = n, \dim H_2 = n - 1, \dim H_3 = n - 2, \dim H_k = n - k + 1,$$

$$\dim H_n = 1, \dim H_{n+1} = \dim H_\infty = 0$$

$$\text{Integrability of } H_n: H_n = \text{span}\{d\varphi(x)\}$$

The « output »  $y = \varphi(x)$  has a relative degree =  $n$

$$\text{New state variables: } z = \Phi(x) = \begin{bmatrix} \varphi(x) \\ \frac{d}{dt} \varphi(x) \\ \vdots \\ \frac{d^{n-1}}{dt^{n-1}} \varphi(x) \end{bmatrix}$$

Solve equation

$$y^{(n)} = \varphi^{(n)}(x, u) = a(x) + b(x)u := v \Rightarrow u = -b^{-1}(x)a(x) + b^{-1}(x)v$$

- Necessary condition:

If  $H_\infty \neq 0$  then an autonomous nonlinear element can exist.

Example 5.2: With  $\Sigma$ :  $\dot{x} = \begin{bmatrix} x_3 \\ x_3^2 \\ u \end{bmatrix}$

- $H_2 = \text{span}\{dx_1, dx_2\}$   
 $H_3 \quad \omega = \alpha_1 dx_1 + \alpha_2 dx_2 \quad \dot{\omega} = * + (\alpha_1 + 2\alpha_2 x_3) dx_3$   
 $\Rightarrow \alpha_1 + 2\alpha_2 x_3 = 0 \quad \text{a solution: } \alpha_2 = -1, \quad \alpha_1 = 2x_3$   
 $H_3 = \text{span}\{2x_3 dx_1 - dx_2\}$  which is not integrable (Frobenius theorem):  
 $\Rightarrow$  No Input-State linearization for this system.
- $H_4 = 0 \quad (= H_\infty) \Rightarrow \Sigma$  is accessible but not Input-State linearizable by static feedback.

Example 5.3 (m=1):

$$\dot{x} = \begin{bmatrix} x_2^2 + \frac{x_1 x_3}{x_2} \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad \text{Search for an output with a relative degree} = 3$$

Example 5.4 (m=2): Mobil cart

$$\dot{x} = \begin{bmatrix} \cos(x_3) & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad f(x) \equiv 0,$$

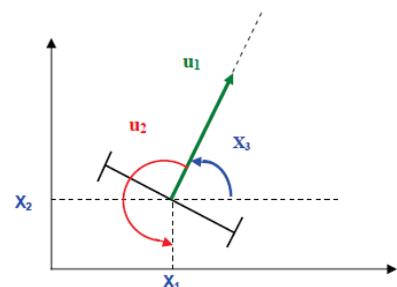
Computation of  $H_2 = g(x)^\perp$

$$H_2 = \text{span}\{(\sin x_3) dx_1 - (\cos x_3) dx_2\}$$

$$H_3 = \{0\}$$

$\Rightarrow$  The system is accessible.

Is it Input State linearizable by static feedback?



## Problem 2 Input State Linearization by dynamic feedback

### Problem 2:

With  $\dot{x} = f(x) + g(x)u$ ,  $x \in R^n, u \in R^m$ ,

Find a dynamic feedback: 
$$\begin{cases} \dot{\eta} = F(x, \eta) + G(x, \eta)v \\ u = \alpha(x, \eta) + \beta(x, \eta)v \end{cases} \eta \in R^q$$

and  $z = \varphi(x, \eta)$ ,  $z \in R^{n+q}$  such that  $\dot{z} = Az + Bv$  with  $(A, B)$  controlable.

- Goal: search for  $m$  outputs  $y$  (called flat outputs) such that the system is invertible (thus decouplable by dynamic feedback) and  $\dim(sp\{dx\} \cap sp\{dy, \dots, dy^{(k)}\}) = n$

Canonic basis of  $H_k$ :

$$H_1 \supset H_2 \supset \dots \supset H_{k^*} \supseteq H_{k^*+1} = 0 \text{ with } k^* = \{\max k : H_k \neq 0\}$$

Let  $H_{k^*} = sp\{\omega_{k^*}\}$ ,

and  $H_k = (H_{k+1} + \dot{H}_{k+1}) \oplus sp\{\omega_k\}$  with  $\oplus$  direct sum, i.e.

$$(H_{k+1} + \dot{H}_{k+1}) \cap sp\{\omega_k\} = sp\{0\}$$

The Input State linearization by dynamic feedback requires the search for integrable forms generating the  $H_k$ .

### Theorem:

If  $\{\omega_1, \dots, \omega_k\}$  is integrable then there is a solution to the Input-State linearization by dynamic feedback.

**Rq 1.** There is no constructive Necessary and Sufficient Condition to the total linearization problem by dynamic feedback.

**Rq 2** The constraint can be relaxed a little if a non-controllable dynamic is linear !

Example 5.5: Mobile Cart

$$\dot{x} = \begin{bmatrix} \cos(x_3) & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad f(x) \equiv 0,$$

This system is not Input-State linearizable by static feedback.

Is it linearizable by dynamic feedback?

Reminder:

It is known that there is a dynamic feedback which realizes a "total" linearization of the outputs  $y_1 = x_1 ; y_2 = x_2$ .

We can therefore associate to the unicycle the model "increased" by the addition of an integrator in front of the input  $u_1$ !

Examples of systems that can be Input-State linearized (from SIAM News, 1995!)



In addition, there is also a catalog of systems that cannot be linearized, for example

- the "ball and beam" system
- the pendulum of variable length
- Car with 2 trailers and axis of rotation
- the double inverted pendulum
- ...

System	Picture	Flat Output
Mobile robot, car		Rear wheel position
Car with 1 trailer		Function of hitch angle/position
Car with N trailers, special hitching		Last trailer wheel position
Hopping robot		Position of end of leg
Ducted fan/ PVTOL		Center of oscillation
Ducted fan with ideal stand		Quasi-center of oscillation
Planar rigid body chain		Center of oscillation, last rigid body
Planar satellite with actuated robotic arm		Body fixed point plus linear functions of joint angles
Simplified planar crane		Position of the load
Rigid body with S' symmetry and body fixed forces in R'		Axisymmetric center of oscillation
Towed cable system		Position of end of cable

Table 1. Partial catalog of differentially flat mechanical systems.

**CHAPTER 6**  
**NON LINEAR SYSTEMS**  
**OBSERVABILITY <sup>1</sup>**

**6.1 Introduction**

*Definition: Observability*

It is the property to reconstruct the state from the measured output, the input and their temporal derivatives.

- *Remark:* For linear, the observability can be verified by the criterion:  $rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$

The observability of linear systems is therefore :

- independent of the input
- characterized by the possibility to express x as a function of y and its time derivatives at an order lower than n-1.

<sup>1</sup> [Ple95] Chapter 3

Typical examples of the observability of nonlinear systems:

Example 6.1<sup>1</sup>:

Let the system:  $\dot{x} = 0$

$$y = x^2$$

This system is not "observable" because one cannot distinguish the sign of with only the knowledge of y.

We needs complementary information as  $x \in \mathfrak{R}^+ \dots$

Example 6.2: Let the system  $\dot{x}_1 = x_2^4; \dot{x}_2 = x_2^2/x_1$   
 $y = x_1 \quad (n=2)$

We can write  $x_2 = \begin{cases} 0 & \text{if } \dot{y} = 0 \\ \frac{\dot{y}y}{4\ddot{y}} & \text{otherwise} \end{cases}$

To distinguish the sign of  $x_2$ , one also needs the knowledge of  $\dot{y} = y^{(n)}$  !  
 (to be compared to the linear case).

<sup>1</sup> [Ple95] Chapter 3 p.27

Example 6.3: Let the system :

$$\begin{cases} \dot{x}_1 = x_2 u \\ \dot{x}_2 = 0 \\ y = x_1 \end{cases}$$

The system is observable for  $u \neq 0$

=> observability is dependent on the input unlike the linear case

- *Conclusion.* The observability of nonlinear systems is a generic notion: there may be singularities in the state or in the input that may cause observability to be locally lost.

*Remark:* if the input is not permanently singular, then the notion of input persistence is introduced (observability is obtained in "average") [Bes96].

## 6.2 Formal characterization of generic observability for nonlinear systems

Notation:

$$\begin{aligned} X &= \text{span}_K\{dx\} \\ U &= \text{span}_K\{du^{(j)}, j \geq 0\} \\ Y^i &= \text{span}_K\{dy^{(j)}, 0 \leq j \leq i\} \\ Y &= \bigcup_{i \geq 0} Y^i \end{aligned}$$

Definition. Observability filtration<sup>1</sup>

$$0 \subset O_0 \subset \dots \subset O_k \subset \dots$$

with  $O_k := X \cap (Y^k + U)$  et  $O_{-1} = 0$

The limit of this filtration:  $O_\infty = X \cap (Y + U)$  is called "observability space".

- Remark<sup>1</sup>: For the linear case:  $O_k = \text{span}_K\{Cdx, CA^k dx, \dots, CA^{k-1} dx\}$

<sup>1</sup> [Ple95] Chapter 3 p.28 and [Glu97]

Definition<sup>1</sup>

$\sigma_i = \dim(O_{i-1}/O_{i-2}) :=$  number of observability indices  $k_i$  that are greater than or equal to  $i$ .

The list of observability indices  $k_i$  is then defined by:

$$k_i = \text{card}\{\sigma_j \geq i\} \text{ i.e. the number of } \sigma_j \geq i. \\ \text{with } k_1 > k_2 > \dots > k_p$$

Theorem<sup>2</sup> A system is generically observable  $\Leftrightarrow \dim O_\infty = n$

Example 6.4. Check the observability of the system:

$$\begin{cases} \dot{x}_1 = x_2 u \\ \dot{x}_2 = 0 \\ y = x_1 \end{cases}$$

<sup>1</sup>[Glu97] Tr.13, <sup>2</sup>[Ple95] Chapter 3 p.28

Example 6.4 Solution

$$\dot{x}_1 = x_2 \cdot u \quad \dot{x}_2 = 0 \quad y = x_1$$

Calculation of output derivatives:

$$\begin{array}{ll} y = x_1 & \text{thus } dy = dx_1 \\ \dot{y} = \dot{x}_1 = x_2 \cdot u & d\dot{y} = x_2 \cdot du + u \cdot dx_2 \end{array}$$

$$X = \text{span}_K\{dx_1, dx_2\}, U = \text{span}_K\{du, d\dot{u}, \dots\}$$

$$Y^0 = \text{span}_K\{dx_1\}, Y^1 = \text{span}_K\{dx_1, x_2 \cdot du + u \cdot dx_2\}$$

$$\begin{aligned} O_0 &= X \cap \{Y^0 + U\} = \text{span}_K\{dx_1, dx_2\} \cap \text{span}_K\{dx_1, du, d\dot{u}, \dots\} \\ &= \text{span}_K\{dx_1\} \end{aligned}$$

$$O_1 = \text{span}_K\{dx_1, dx_2\} \cap \text{span}_K\{dx_1, dx_2, du, d\dot{u}, \dots\} = \text{span}_K\{dx_1, dx_2\}$$

$$O_\infty = O_1 = X \quad \dim = 2, \text{ the system is generically observable}$$

$$\sigma_1 = \dim\{O_0/\{0\}\} = 1 \Rightarrow \text{an observability index } \geq 1$$

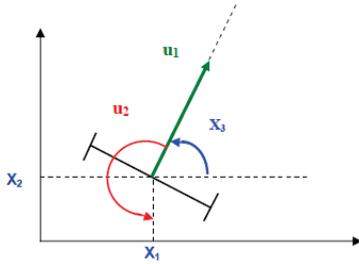
$$\sigma_2 = \dim\{O_1/O_0\} = 1 \Rightarrow \text{an observability index } \geq 2$$

Conclusion:  $k_1 = 2$  (only one output here!)

**Example 6.5. Mobile cart (MIMO).**

We consider a mobile cart.

Note  $u_1$  the longitudinal speed,  $u_2$  the angular speed,  $x_1$  and  $x_2$  the coordinates of the center of the rear wheels.



The system equation is:

$$\begin{cases} \dot{x}_1 = u_1 \cos x_3 \\ \dot{x}_2 = u_1 \sin x_3 \\ \dot{x}_3 = u_2 \end{cases} \quad y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**Example 6.5 solution: Mobile cart, Characterization of the observability**

$$y_1 = x_1, \quad \dot{y}_1 = \dot{x}_1 = \cos(x_3)u_1, \quad \ddot{y}_1 = \ddot{x}_1 = -\sin(x_3)u_1u_2 + \cos(x_3)\dot{u}_1,$$

$$y_2 = x_2, \quad \dot{y}_2 = \dot{x}_2 = \sin(x_3)u_1,$$

$$\ddot{y}_2 = \ddot{x}_2 = \dot{x}_3 \cos(x_3)u_1 + \dot{u}_1 \sin(x_3) = \cos(x_3)u_1u_2 + \sin(x_3)\dot{u}_1$$

$$X = \text{span}_K\{dx_1, dx_2, dx_3\}$$

$$U = \text{span}_K\{du_1, du_2, \dots\} \quad Y^0 = \text{span}_K\{dx_1, dx_2\},$$

$$Y^1 = \text{span}_K\{dx_1, dx_2, d[\cos(x_3)u_1], d[\sin(x_3)u_1]\}$$

$$O_0 = X \cap \{Y^0 + U\} = \text{span}_K\{dx_1, dx_2\}$$

$$O_1 = X \cap \{Y^1 + U\} = \text{span}_K\{dx_1, dx_2, dx_3\}$$

$$O_\infty = O_1 \Rightarrow \dim O_\infty = \dim O_1 = 3 \Rightarrow \text{The system is observable.}$$

$$\sigma_1 = \dim(O_0/O_{-1}) = 2 \Rightarrow 2 \text{ observability indices } \geq 1$$

$$\sigma_2 = \dim(O_1/O_0) = 1 \Rightarrow 1 \text{ observability indice } = 2$$

$$\sigma_3 = 0$$

Conclusion: the system is generically observable, with an observability index  $k_1=2$  and an observability index  $k_2=1$  (decreasing order by convention).

Remark: the association of the observability indices to the outputs is not always unique.

### 6.3 Practical characterization of generic observability for nonlinear systems

*Reminder* For linear system, the observability can be verified by the criterion:

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

For nonlinear systems, the characterization of generic observability can also be seen as the generalization of the linear criterion:

$$\text{rank} \left[ \frac{\partial(y, \dot{y}, \dots, y^{(n-1)})}{\partial(x)} \right]$$

- A "canonical" form based on observability can be defined. It allows the decomposition of the system into subsystems whose dimensions are equal to the observability indices (obviously linked to the choice of the observability indices).

- Exemple 6.6: Mobile cart practical characterization of generic observability

Reminder

$$y_1 = x_1, \quad \dot{y}_1 = \dot{x}_1 = \cos(x_3)u_1, \quad \ddot{y}_1 = \ddot{x}_1 = -\sin(x_3)u_1u_2 + \cos(x_3)\dot{u}_1,$$

$$y_2 = x_2, \quad \dot{y}_2 = \dot{x}_2 = \sin(x_3)u_1,$$

$$\ddot{y}_2 = \ddot{x}_2 = \dot{x}_3 \cos(x_3)u_1 + \dot{u}_1 \sin(x_3) = \cos(x_3)u_1u_2 + \sin(x_3)\dot{u}_1$$

Thus  $\left[ \frac{\partial(y, \dot{y}, \dots, y^{(n-1)})}{\partial(x)} \right] \Rightarrow$

$$\begin{array}{c} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{array}{ccc} x_1 & x_2 & x_3 \\ y_1 & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\sin(x_3)u_1 \\ 0 & 0 & \cos(x_3)u_1 \end{bmatrix} \end{array} \end{array}$$

*Note: There is an observability singularity for  $u_1 = 0$  ! High order derivatives have to be analysed!*

*Generically (outside the singularity), the observability indices can be chosen as:*

$$\{k_1, k_2\} = \{2, 1\}$$

## 6.4 « Canonical » form of observability

We can then define a state coordinate transformation:

$$z = (z_1^T(x, \bar{u}), \dots, z_p^T(x, \bar{u}))^T \quad \text{where } \bar{u} = \{u, \dot{u}, \dots, u^{(n-2)}\},$$

with for  $1 \leq i \leq p$ : 
$$z_i = \begin{bmatrix} y_i \\ \dot{y}_i \\ \dots \\ y_i^{(k_i-1)} \end{bmatrix} \quad \text{We obtain:}$$

$$\dot{z} = A.z + \chi(z, u, \dot{u}, \dots, u^{(n-2)}) = \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & A_p \end{bmatrix} .z + \begin{bmatrix} \chi_1(z, u, \dot{u}, \dots, u^{(n-2)}) \\ \chi_2(z, u, \dot{u}, \dots, u^{(n-2)}) \\ \chi_3(z, u, \dot{u}, \dots, u^{(n-2)}) \\ \dots \\ \chi_p(z, u, \dot{u}, \dots, u^{(n-2)}) \end{bmatrix}$$

$$y = C.z = \begin{bmatrix} C_1 & 0 & 0 & \dots & 0 \\ 0 & C_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & C_p \end{bmatrix} .z$$

with

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the matrices  $A_i$  and the vectors  $C_i$  ( $1 \leq i \leq p$ ) are defined by:

$$A_i = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \dots & \dots & 0 \end{bmatrix}_{k_i \times k_i} \quad \text{and } C_i = [1 \quad 0 \quad \dots \quad 0]_{1 \times k_i}$$

For  $1 \leq i \leq p$ , the functions  $\chi_i$ , are defined by:

$$\chi_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h_i^{(k_i)}(x(z, \bar{u})) \end{bmatrix}.$$

## The system description

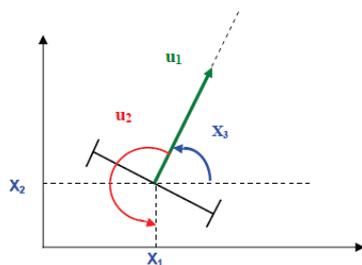
$$\dot{z} = \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & A_p \end{bmatrix} \cdot z + \begin{bmatrix} \chi_1(z, u, \dot{u}, \dots, u^{(n-2)}) \\ \chi_2(z, u, \dot{u}, \dots, u^{(n-2)}) \\ \chi_3(z, u, \dot{u}, \dots, u^{(n-2)}) \\ \dots \\ \chi_p(z, u, \dot{u}, \dots, u^{(n-2)}) \end{bmatrix}$$

is called the canonical observability form.

Each nonlinear function  $h_i^{(k_i)}(x(z, \bar{u}))$  is a function of  $z$  and of the first  $n-1$  derivatives of  $u$ .

### Example 6.7: Mobile cart

Note  $u_1$  the longitudinal speed,  $u_2$  the angular speed,  $x_1$  and  $x_2$  the coordinates of the center of the rear wheels.



The model equation is:

$$\begin{cases} \dot{x}_1 = u_1 \cos x_3 \\ \dot{x}_2 = u_1 \sin x_3 \\ \dot{x}_3 = u_2 \end{cases}, \quad y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observability indices:  $k_1 = 2$ ,  $k_2 = 1$

Reminder:  $k_1$  is not necessarily associated to the output 1 (idem condition of rank).

=> writing in canonical form:

Example 6.7 following: Observability canonical form for the mobil cart ( $k_1=2, k_2=1$ )

$$z_{11} = y_1 = x_1$$

$$z_{12} = \dot{y}_1 = u_1 \cos x_3 \Rightarrow \cos x_3 = \frac{z_{12}}{u_1}, x_3 = \text{Ar} \cos\left(\frac{z_{12}}{u_1}\right)$$

$$z_{21} = y_2 = x_2$$

$$\dot{z}_{11} = \dot{y}_1 = z_{12}$$

$$\dot{z}_{12} = -\sin x_3 \cdot u_2 \cdot u_1 + \cos x_3 \cdot \dot{u}_1 = -\sin \left[ \text{Arcos} \left( \frac{z_{12}}{u_1} \right) \right] u_2 \cdot u_1 + z_{12} \frac{\dot{u}_1}{u_1}$$

$$\dot{z}_{21} = \sin x_3 \cdot u_1 = \sin \left[ \text{Arcos} \left( \frac{z_{12}}{u_1} \right) \right] u_1$$

$$\dot{z}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z_1 + \begin{bmatrix} 0 \\ -\sin \left[ \text{arcos} \left( \frac{z_{12}}{u_1} \right) \right] u_1 \cdot u_2 + \frac{z_{12}}{u_1} \dot{u}_1 \end{bmatrix}$$

$$\dot{z}_2 = 0 * z_2 + \sin \left( \text{arcos} \left( \frac{z_{12}}{u_1} \right) \right) u_1$$

# CHAPTER 7

## OBSERVERS FOR NONLINEAR SYSTEMS

### OUTLINE

- CHAPTER 7 PART 1 INTRODUCTION TO OBSERVERS FOR NONLINEAR SYSTEM
- CHAPTER 7 PART 2 OBSERVER DESIGN BY LINEARIZATION VIA GENERAL INPUT-OUTPUT INJECTION
- CHAPTER 7 PART 3 OBSERVER DESIGN BY TRANSFORMATION INTO AN AFFINE SYSTEM

## CHAPTER 7 PART 1

### INTRODUCTION TO OBSERVERS FOR NONLINEAR SYSTEMS

#### 7.1 Introduction

Linear case: **Observability**  $\Rightarrow$  observer Type Luenberger.

Time-variant case :  $\dot{x} = A(t)x + Bu$

Observability  $\Rightarrow$  Deterministic extended Kalman type observer  
 $\Rightarrow$  Variable gain observer given by solving a Riccati equation

**Nonlinear case:** the observability property does not imply the "design" of an observer.

## Position of the problem:

$$\text{Given } \Sigma \quad \dot{x} = f(x, u), \quad y = h(x) \quad (1)$$

- Determine  $\hat{\Sigma}$  such that  $\hat{x} \rightarrow x$  is not a solved problem for nonlinear systems in general.
- Moreover, the separation principle does not exist for NLS ( $\Rightarrow$  no "Observer-Controller superposition" without stability check!)

## 7.2 Different solutions for establishing Observers

- Luenberger type observer modulo injection of known quantities
- Observers for state affine systems (Deterministic extended Kalman type with a Riccati differential equation)
- Observers with large gains
- Observers for interconnected systems
- Sliding mode observers (with discontinuous feedback) for triangular systems.
- Observers with finite time convergence, ....

### **7.2.1 Luenberger type observers modulo a generalized input injection [Lue64]**

**Reminder for the linear case:** Luenberger type observer (pole placement of the dynamics of the deviation ( $\dot{e} = (A - KC)e$ ) :

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$

#### **Generalization:**

If a nonlinear system (1) can be written in the form (2) by a (nonlinear) change of state and output coordinates and modulo an input-output injection  $\varphi$ ,

$$\dot{\zeta} = A\zeta + \varphi(y, u), \quad \tilde{y} = C\zeta \quad (2)$$

then a "Luenberger like" observer can be written:

$$\dot{\hat{\zeta}} = A\hat{\zeta} + \varphi(y, u) + KC(\zeta - \hat{\zeta}), \quad \hat{\tilde{y}} = C\hat{\zeta}$$

#### **Problem 1 to solve:**

Is there a transformation such that the NLS  $\dot{x} = f(x) + g(x)u$ , can by change of state and/or output coordinates, and modulo an input-output injection  $\varphi$ , be "linearized" i.e. be written:  $\dot{\zeta} = A\zeta + \varphi(y, u), \quad \tilde{y} = C\zeta$ ?

## 7.2.2 Observers for state affine systems [Ham90]

If a nonlinear system (1) is written in the form:

$$\dot{\zeta} = A(u, y)\zeta + \varphi(y, u), \quad \tilde{y} = C\zeta$$

then under certain conditions (depending on  $u$ ) a "Kalman like" observer can be written :

$$\begin{aligned} \dot{\hat{\zeta}} &= A(u, y)\hat{\zeta} + \varphi(y, u) - S^{-1}C^T(C\hat{\zeta} - \tilde{y}), \quad \hat{\tilde{y}} = C \cdot \hat{\zeta} \\ \dot{S} &= -\theta S - A(u, y)^T S - SA(u, y) + C^T C \\ S_0 &> 0 \text{ et } \exists \lambda > 0 \text{ tel que} \\ \|\hat{\zeta} - \zeta\| &\leq \lambda \exp(-\tau t), \tau > 0. \end{aligned}$$

### ➡ Problem 2 to solve:

Is there a transformation such that  $\dot{x} = f(x) + g(x)u$  can be written by change of state coordinates and output modulo an input-output injection:

$$\dot{\zeta} = A(y, u) \cdot \zeta + \varphi(y, u), \quad \tilde{y} = C \zeta \quad (\text{A State Affine System}) ?$$

## 7.2.1 High Gain Observers [Esf92]

If the transformations defined above (problem 1 and 2) do not exist, a solution is possible for the uniformly observable system (no singularity of the observation because of the inputs). The writing of the system in the canonical observability form with  $z = \varphi(x)$ :

$$\begin{aligned} z_i &= \begin{bmatrix} y_i \\ \dot{y}_i \\ \dots \\ y_i^{(k_i-1)} \end{bmatrix} \\ \dot{z} &= A \cdot z + \chi(z, u, \dot{u}, \dots, u^{(n-2)}) = \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & A_p \end{bmatrix} \cdot z + \begin{bmatrix} \chi_1(z, u, \dot{u}, \dots, u^{(n-2)}) \\ \chi_2(z, u, \dot{u}, \dots, u^{(n-2)}) \\ \chi_3(z, u, \dot{u}, \dots, u^{(n-2)}) \\ \dots \\ \chi_p(z, u, \dot{u}, \dots, u^{(n-2)}) \end{bmatrix} \\ y &= C \cdot z = \begin{bmatrix} C_1 & 0 & 0 & \dots & 0 \\ 0 & C_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & C_p \end{bmatrix} \cdot z \end{aligned}$$

allows the writing of a High Gain Observer if the system can be written (monovariable case):

$$\dot{z} = Az + \varphi(z) + \Psi(z)u, \quad y = Cz \quad \text{with } A \text{ canonical observability matrix,}$$

$$C = [1, 0, \dots, 0] \text{ and } \Psi(z) = \begin{bmatrix} \Psi_{1, \cdot}(z_1) \\ \Psi_{2, \cdot}(z_1, z_2) \\ \vdots \\ \Psi_{n, \cdot}(z_1, \dots, z_n) \end{bmatrix} \text{ (triangular structure with respect to } z)$$

- with the functions  $\varphi(z)$  et  $\Psi(z)$  globally Lipschitz with respect to  $z$ ,
- and the inputs  $u$  are measurable and bounded.

A High Gain Observer is then written:

$$\dot{\hat{z}} = A\hat{z} + \varphi(\hat{z}) + \Psi(\hat{z})u - \theta \Delta_\theta^{-1} S^{-1} C^T (C\hat{z} - y), \quad \hat{y} = C\hat{z}$$

with  $\theta > 0$  a setting parameter,  $S$  is the unique solution of the algebraic Lyapunov equation

$$S + A^T S + SA = C^T C \text{ and}$$

$$\Delta_\theta = \text{diag}\left(1, \frac{1}{\theta}, \frac{1}{\theta^2}, \dots, \frac{1}{\theta^{n-1}}\right). \text{ The observed state is then } \hat{x} = \Phi^{-1}(\hat{z})$$

In the original coordinates, the observer can also be written :

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u - \theta \left( \frac{\partial \Phi(\hat{x})}{\partial \hat{x}} \right)^{-1} \Delta_\theta^{-1} S^{-1} C^T (h(\hat{x}) - y).$$

**Nonlinear Observers. Part 2: OBSERVER DESIGN By  
LINEARISATION VIA GENERAL INPUT-OUTPUT INJECTION**

### OUTLINE

- Reminders and problem introduction
  - Nonlinear systems assumptions
  - Observability - Rank condition
  - Observers
  - Problems to solve
  - State of the art - Objective - Ways
- A definition of the Input-output Injection
- Linearization via generalized input-output injection
- Linearization via generalized input-output injection with output derivatives
- Extensions, Conclusions and Prospects

### Assumptions

---

Nonlinear Systems Model :

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{1}$$

The state is  $x \in M$ , the input  $u \in R^m$  and the output  $y \in R^p$ .  
 $M$  is a dense open subset of  $R^n$

$f(.,.)$  and  $h(x)$  are analytics over  $M$ ;  $u(t)$  is admissible.

The body of the meromorphic functions in  $\{x, u, \dot{u}, \dots, u^{(q)}, q \geq 0\}$  is noted  $K$ . A space generated on this body is noted  $\text{Span}_K$ .

**Example 1** Let the system:

$$\begin{aligned} \dot{x}_1 &= x_2 u, \quad \dot{x}_2 = 0, \quad y = x_1 \\ \longrightarrow x_1 &= y, \quad \dot{x}_2 = y/u \end{aligned}$$

Introduce the Generic Observability Space:  $O$  with

$$O = X \cap (Y + U)$$

where  $U = \text{Span}_K \{du^{(q)}, q \geq 0\}$ ,  $Y = \text{Span}_K \{dy^{(s)}, s \geq 0\}$ ,  $X = \text{Span}_K \{dx\}$ .

**Definition 1** System (1) is generically observable if and only if  $\dim O = n$ .

→ Rank condition of generic Observability

**Definition 2** An asymptotic Observer of system (1) is a system in the form:

$$\begin{aligned} \dot{z} &= \hat{f}(z, y, u) \\ \hat{x} &= \hat{h}(z, y, u) \end{aligned} \quad (2)$$

Such that:

- $\|e(t)\| = \|\hat{x}(t) - x(t)\| \rightarrow 0$  when  $t \rightarrow \infty$
- If, for  $t = t_0$ ,  $\hat{x}(t_0) = x(t_0)$ , then for every  $t \geq t_0$ , we have  $\hat{x}(t) = x(t)$ .

Goal: to determine an output coordinates transformation  $\tilde{y} = T(y)$  and a generalized state coordinates transformation  $\zeta = \phi(x, u, \dot{u}, \dots, u^{(q-1)})$  such that the system (1) is equivalent to:

$$\begin{aligned} \dot{\zeta} &= A \zeta + \varphi(y, u, \dot{u}, \dots, u^{(q)}) \\ \tilde{y} &= C \zeta \end{aligned} \quad (3)$$

Hypothesis :  $A$  and  $C$  are in observability canonical form.

An observer for (3) is:

$$\dot{\hat{\zeta}} = A \hat{\zeta} + \varphi(y, u, \dot{u}, \dots, u^{(q)}) + K C (\zeta - \hat{\zeta})$$

→ Exponential convergence function of  $K$ .

→ Observed State of system (1) is computed by

$$\hat{x} = \phi^{-1}(\hat{\zeta}, u, \dot{u}, \dots, u^{(q-1)}).$$

- Exact linearization
  - Geometric approach: [Kre83], [Kre85], [Xia89], ...
  - Algebraic approach: [Kel87], [Pro93], [Glu96], ...
- Approximate Linearisation : [Bes83], [Zei87], ...

### Objective

To obtain a streamlined and unified solution for the exact linearisation problem via a generalized input-output injection.

### Means

- To study the structure of the input-output differential equations of nonlinear system → Realisation problem.
- By means of the differential systems theory.

### Linearization of MISO systems by generalized state transformation and generalized input-output injection: Example

---

**Hypothesis:** initial problem with  $p = 1$ , and no output coordinates transformation

**Example 2** Let the nonlinear system:

$$\dot{x}_1 = x_2, \dot{x}_2 = 0, y = x_1 \quad (4)$$

The input-output differential equation is:  $y^{(2)} = 2\dot{y}\frac{\dot{u}}{u}$  If the system is locally equivalent to:

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 + \varphi_1(y, u, \dot{u}, \ddot{u}) \\ \dot{\zeta}_2 &= \varphi_2(y, u, \dot{u}, \ddot{u}) \\ y &= \zeta_1, \text{ then } y^{(2)} = \varphi_1^{(1)} + \varphi_2 \text{ with a solution:} \end{aligned}$$

$$\varphi_1 = \frac{2y\dot{u}}{u}, \varphi_2 = 2y \left( \frac{\dot{u}^2}{u^2} - \frac{\ddot{u}}{u} \right)$$

- NSC for the existence of a generalized state coordinates transformation
- Design of an observer
- Errors dynamics
- Coordinates transformation

## MISO systems linearisation by generalized state transformation and generalized input-output injection: Algorithm

---

System input-output differential equation (1) :

$$y^{(n)} = P(y, \dot{y}, \dots, y^{(n-1)}, u, \dot{u}, \dots, u^{(n-1)})$$

### G.I.O.I.A.m. algorithm

Let  $P_0 := P$  and  $\varphi_0(y, u, \dot{u}, \dots, u^{(q)}) := 0$ . For  $k = 1$  to  $n$ , let define:

$$P_k := P_{k-1} - [\varphi_{k-1}(y, u, \dot{u}, \dots, u^{(q)})]^{(n-k+1)}$$

The differential form  $\omega_k$  is defined by:

$$\omega_k = \frac{\partial P_k}{\partial y^{(n-k)}} dy + \sum_{j=1}^m \frac{\partial P_k}{\partial u^{(n-k+q)}} du_j^{(q)}$$

- If  $d\omega_k \wedge du \wedge d\dot{u} \wedge \dots \wedge du^{(q-1)} \neq 0$ , then the problem has no solution.
- IF  $d\omega_k \wedge du \wedge d\dot{u} \wedge \dots \wedge du^{(q-1)} = 0$ , then the function  $\varphi_k$  is solution of:

$$\frac{\partial \varphi_k}{\partial y} dy + \sum_{j=1}^m \frac{\partial \varphi_k}{\partial u_j^{(q)}} du_j^{(q)} = \omega_k \quad (1 \leq k \leq n-1)$$

$$\varphi_n(y, u, \dot{u}, \dots, u^{(q)}) = P_n$$

Note :  $q = 0 \Rightarrow d\omega_k \wedge du^{(-1)} := d\omega_k$ .

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## MISO systems linearisation by generalized state transformation and generalized input-output injection: NSC

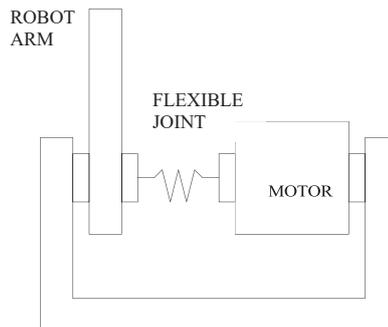
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**Theorem 1 (IEEE TAC)** [Glu96] Nonlinear system (1) (with  $p = 1$ ) is locally equivalent to system (3) by  $\zeta = \phi(x, u, \dot{u}, \dots, u^{(q-1)})$  and  $T(y) = y$  if and only if:

$$d\omega_k \wedge du \wedge d\dot{u} \wedge \dots \wedge du^{(q-1)} = 0$$

for  $1 \leq k \leq n$ .

**Application** : **Flexible joint robot** with the measurement of the arm position.



- $x_1$  arm position,  $x_2$  arm speed,
- $x_3$  motor position,  $x_4$  motor speed.

## MISO systems linearisation by generalized state transformation and generalized input-output injection: Application

---

Flexible joint robot model.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{mgl}{J_l} \sin x_1 - \frac{k}{J_l} (x_1 - x_3) - \frac{f_1}{J_l} x_2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{k}{J_m} (x_1 - x_3) - \frac{f_2}{J_m} x_4 + \frac{u}{J_m}\end{aligned}$$

Equivalent system ( $q=0$ ) :

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2 - \left(\frac{f_1}{J_l} + \frac{f_2}{J_m}\right)y \\ \dot{\zeta}_2 &= \zeta_3 - \frac{mgl}{J_l} \sin y - \left(\frac{k}{J_l} + \frac{k}{J_m} + \frac{f_1 f_2}{J_l J_m}\right)y \\ \dot{\zeta}_3 &= \zeta_4 - \frac{f_2}{J_m} \frac{mgl}{J_l} \sin y - \frac{k}{J_l J_m} (f_1 + f_2)y \\ \dot{\zeta}_4 &= -\frac{k}{J_m} \frac{mgl}{J_l} \sin y - \frac{k}{J_l J_m} u \quad \zeta_1 \\ y &= \zeta_1\end{aligned}$$

via the state space coordinates transformation :

## MISO systems linearisation by generalized state transformation and generalized input-output injection: Application

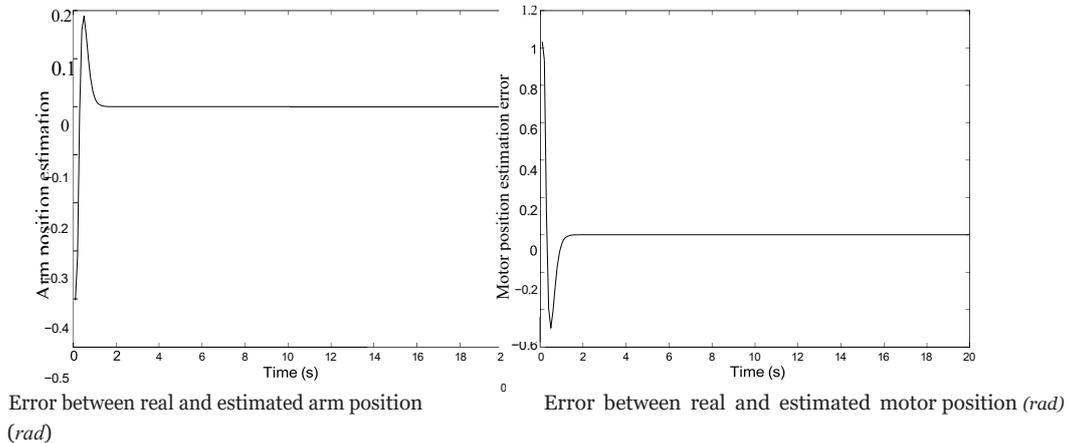
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via the state space coordinates transformation :

$$\zeta = \left[ x_1, x_2 + \left(\frac{f_1}{J_l} + \frac{f_2}{J_m}\right)x_1, \frac{k}{J_l}x_3 + \frac{f_2}{J_m}x_2 + \left(\frac{k}{J_m} + \frac{f_1 f_2}{J_l J_m}\right)x_1, \frac{k}{J_l}x_4 + \frac{k}{J_l J_m}f_2 x_3 + \frac{k}{J_m}x_2 + \frac{k}{J_l J_m}f_1 x_1 \right]^T$$

→ An observer with a linear error dynamics can be designed.

Simulation results:



**Hypothesis** : Initial problem with  $p = 1$ .

**Objective** : to enlarge the class of the linearisable systems.

**Example 3** Let the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2^2 + x_1^2 x_2 \\ y &= x_1 \end{aligned}$$

The NSC of Theorem 3 is not satisfied  $\rightarrow$  This system is not linearisable.

The NSC of Theorem 3 is satisfied with  $\tilde{y} = e^{-y} \rightarrow$  This system is linearisable.

**MISO systems linearisation by generalized state transformation,  
output transformation and generalized input-output injection: N.C and N.S.C**

---



**Theorem 2 (Respondek)** [Kre85] If nonlinear system (1) (with  $p = 1$ ) is locally equivalent to system (3) with  $\zeta = \phi(x, u, \dot{u}, \dots, u^{(q-1)})$  and with  $\tilde{y} = T(y)$ , then:

$$d\left[\frac{\partial^2 P}{\partial \dot{y} \partial y^{(n-1)}}\right] \wedge dy = 0$$

If this condition is satisfied then  $T(y)$  verifies:

$$\frac{dT}{dy} \frac{\partial^2 P}{\partial \dot{y} \partial y^{(n-1)}} + n \frac{d^2 T}{dy^2} = 0$$

N.S.C  $\rightarrow$  By applying Theorem 3 to system (1) with the new output  $\tilde{y}$ .

**Theorem 3 (IEEE TAC)** [Glu96] Nonlinear system (1) (with  $p = 1$ ) is locally equivalent to system (3) by  $\zeta = \phi(x, u, \dot{u}, \dots, u^{(q-1)})$  and  $\tilde{y} = T(y)$  if and only if :

$$d\tilde{\omega}_k \wedge du \wedge d\dot{u} \wedge \dots \wedge du^{(q-1)} = 0$$

for  $1 \leq k \leq n$ .

with  $\tilde{\omega}_k$  a differential form deduced from the G.I.O.A.m. algorithm applied to system (1) with the new output.

**MISO systems linearisation by generalized state transformation,  
output transformation and generalized input-output injection: Application**

---

**Serial DC Motor.** Measured Motor Current [Chi94].

$$\dot{x}_1 = -k_1 x_1 x_2 - \frac{R_a + R_f}{K} x_1 + u \quad y = x_1$$

$$\dot{x}_2 = -k_2 x_2 - x_3 + \frac{k_1}{J} K x_1^2$$

$$\dot{x}_3 = 0$$

- $x_1$  inductor flux,  $x_2$  motor speed,
- $x_3$  load torque/inertia assumed to be constant.

System equivalent to ( $q=0$ )

$$\dot{\zeta}_1 = \zeta_2 + \frac{1}{y} (\chi_1(y) + u) k_2 \ln y \quad \tilde{y} = \zeta_1$$

$$\dot{\zeta}_2 = \zeta_3 + \frac{k_2}{y} (\chi_1(y) + u) - k_1 \chi_2(y)$$

$$\dot{\zeta}_3 = 0$$

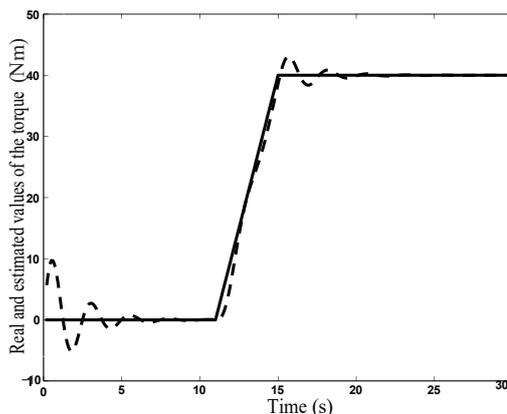
with output coordinates transformation  $\tilde{y} = \ln y$  and state coordinates transformation  $\zeta = [\ln x_1, -k_1 x_2 + k_2 \ln x_1, k_1 x_3]^T$ .

→ Design of an observer with linear dynamics errors.

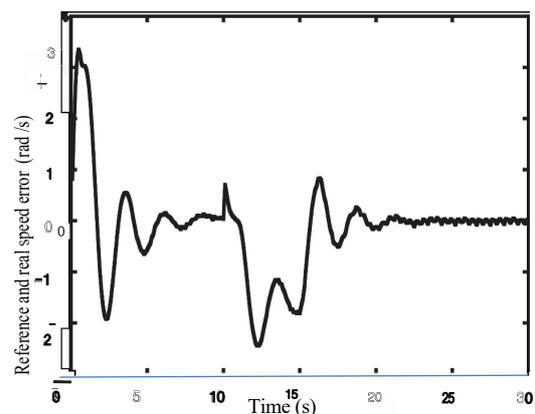
**MISO systems linearisation by generalized state transformation,  
output transformation and generalized input-output injection: Application**

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Simulation results:



Error between real and estimated arm position (rad)



Error between real and estimated motor position(rad)

**MIMO systems linearisation by generalized state transformation  
and generalized input-output injection**

---

**Hypothesis** : Initial problem for any  $p$ , and no output transformation.

**Problem** : the input-output (I/O) differential equations are coupled.

**Example 4** let the nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 & y_1 &= x_1 \\ \dot{x}_2 &= x_2 x_3 \\ \dot{x}_3 &= x_1 & y_2 &= x_3 \end{aligned}$$

The I/O differential equations (with observability indices  $k_1 = 2, k_2 = 1$ ) are:

$$\boxed{y_1^{(2)} = \dot{y}_1 y_2 = y_2^{(2)} y_2!, \quad y_2^{(1)} = y_1}$$

If this system is linearisable, then :

$$\boxed{y_1^{(2)} = \varphi_{11}^{(1)} + \varphi_{12} = \varphi(y_2^{(2)}, \dots), \quad y_2^{(1)} = \varphi_{21}}$$

→ **The I/O differential equation associated to  $y_1$  is coupled to the I/O differential equation associated to  $y_2$ .**

→ **Use independant variables  $y_i^{(j)}, j < k_i$**

A solution:  $\varphi_{11} = y_1 y_2, \varphi_{12} = -y_1^2, \varphi_{21} = y_1$ .

**MIMO systems linearisation by generalized state transformation  
and generalized input-output injection: C.N.**

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I/O differential equation associated to each output:

$$y_i^{(k_i)} = P_i(y_1, \dot{y}_1, \dots, y_1^{(k_1-1)}, \dots, y_p, \dot{y}_p, \dots, y_p^{(k_p-1)}, u, \dot{u}, \dots, u^{(k_1-1)})$$

**Theorem 4** If nonlinear system (1) is locally equivalent to system (3) with  $\zeta = \phi(x, u, \dot{u}, \dots, u^{(q-1)})$  and  $T(y) = y$ , then:

$$\frac{\partial P_i}{\partial y_j^{(\alpha)}} = 0 \quad \frac{\partial P_i}{\partial u_k^{(\alpha+q)}} = 0 \quad (8)$$

for  $\alpha \geq k_i$  ( $1 \leq i \leq p$ ), for any output  $y_j$  ( $1 \leq j \leq p$ ) with an observability index greater than  $k_i$  and for any input  $u_k$  ( $1 \leq k \leq m$ ).

**Proof.** If the system (1) is linearisable, then for any output  $y_i$  ( $1 \leq i \leq p$ ), the I/O differential equation is of the form:

$$y_i^{(k_i)} = \varphi_i^{(k_i-1)} + \varphi_{i2}^{(k_i-2)} + \dots + \varphi_{ik_i}$$

→ **Necessary condition.**

**MIMO systems linearisation by generalized state transformation  
and generalized input-output injection: Algorithm**

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**Algorithm G.I.O.I.A.**

For  $i = 1$  to  $p$ ,  $P_i^0 := P_i$  and  $\varphi_{io}(y, u, \dots, u^{(q_i)}) := 0$ .

For  $k = 1$  to  $k_i$ :  $P_i^k = P_i^{k-1} - [\varphi_{ik-1}(y, u, \dot{u}, \dots, u^{(q_i)})]^{(k_i-k+1)}$

The number of outputs whose observability index is strictly greater than  $k_i - k$  is noted  $d_i^k$ . The differential form  $\omega_i^k$  is defined by:

$$\omega_i^k = \sum_{j=1}^{d_i^k} \frac{\partial P_i^k}{\partial y_j^{(k_i-k)}} dy_j + \sum_{j=1}^m \frac{\partial P_i^k}{\partial u_j^{(k_i-k+q_i)}} du_j^{(q_i)}$$

■ If  $d_i^k < p$  then:

– If  $\omega_i^k \wedge dy_{d_i^{k+1}} \wedge \dots \wedge dy_p \wedge du \wedge d\dot{u} \wedge \dots \wedge du^{(q_i-1)} = 0$ , then  $\varphi_{ik}$  is

solution of  $\sum_{j=1}^{d_i^k} \frac{\partial \varphi_{ik}}{\partial y_j} dy_j + \sum_{j=1}^m \frac{\partial \varphi_{ik}}{\partial u_j^{(q_i)}} du_j^{(q_i)} = \omega_i^k$  for  $1 \leq k \leq k_i - 1$

$$\varphi_{ik_i}(y, u, \dot{u}, \dots, u^{(q_i)}) = P_i^{k_i}$$

– Otherwise, the system is not linearisable.

■ – If  $d_i^k = p$ , then if  $d\omega_i^k \wedge du \wedge d\dot{u} \wedge \dots \wedge du^{(q_i-1)} = 0$ , then  $\varphi_{ik}$  is solution of:

$$\sum_{j=1}^p \frac{\partial \varphi_{ik}}{\partial y_j} dy_j + \sum_{j=1}^m \frac{\partial \varphi_{ik}}{\partial u_j^{(q_i)}} du_j^{(q_i)} = \omega_i^k \text{ for } 1 \leq k \leq k_i - 1$$

$$\varphi_{ik_i}(y, u, \dot{u}, \dots, u^{(q_i)}) = P_i^{k_i}$$

– Otherwise, the system is not linearisable.

**MIMO systems linearisation by generalized state transformation  
and generalized input-output injection: N.S.C.**

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**Theorem 5** [Ple96] The nonlinear system (1) is locally equivalent to system (3) with  $\zeta = \phi(x, u, \dot{u}, \dots, u^{(q_i-1)})$  and  $T(y) = y$  if and only if the condition (8) of Theorem 4 is satisfied and:

-  $d\omega_i^k \wedge dy_{d_i^{k+1}} \wedge \dots \wedge dy_p \wedge du \wedge d\dot{u} \wedge \dots \wedge du^{(q_i-1)} = 0$ , if  $d_i^k < p$ ,

-  $d\omega_i^k \wedge du \wedge d\dot{u} \wedge \dots \wedge du^{(q_i-1)} = 0$ , if  $d_i^k = p$ ,

with  $1 \leq i \leq p$ ,  $1 \leq k \leq k_i$  and  $q = \text{Max}\{q_1, \dots, q_p\}$ .

**Example 5** Let the nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 & y_1 &= x_1 \\ \dot{x}_2 &= x_2 x_3 \\ \dot{x}_3 &= x_1 & y_2 &= x_3 \end{aligned}$$

The Input-Output differential equations are:  $y_1^{(2)} = y_1$   $y_2$   $y_2^{(1)} = y_1$

- Output  $y_1$  :  $\omega_1^1 = y_2 dy_1 \rightarrow d\omega_1^1 \wedge dy_2 = 0 \rightarrow \varphi_{11} = y_1 y_2$   
 $\omega_1^2 = -2y_1 dy_1 \rightarrow d\omega_1^2 = 0 \rightarrow \varphi_{12} = -y_1^2$

- output  $y_2$  :  $\omega_2^1 = dy_1 \rightarrow d\omega_2^1 = 0 \rightarrow \varphi_{21} = y_1$

The necessary and sufficient condition is satisfied.

**Hypothesis:** Initial problem.

**Output Transformation results:** [ Kre85].

**N.S.C.** Theorem 4 and 5 are applied to system (1) with the new outputs  $\tilde{y}$ .

- $\tilde{P}_i$ , the input-output differential equation associated to each output  $\tilde{y}_i$  ( $1 \leq i \leq p$ ).
- $\tilde{\omega}_i^k$  a differential form derived from the algorithm G.I.O.I.A. applied to system (1) with the new output coordinates.

**Theorem 6** The nonlinear system (1) is locally equivalent to system (3) by

$$\zeta = \phi(x, u, \dot{u}, \dots, u^{(q_i-1)}) \text{ and } \tilde{y} = T(y) \text{ if and only if } \frac{\partial \tilde{P}_i}{\partial y_j^{(\alpha)}} = 0 \quad \frac{\partial \tilde{P}_i}{\partial u_k^{(\alpha+q)}} = 0$$

for  $\alpha \geq k_i$  ( $1 \leq i \leq p$ ), for any output  $y_j$  ( $1 \leq j \leq p$ ) with an observability index greater than  $k_i$  and for all input  $u_k$  ( $1 \leq k \leq m$ ), and:

- $d\tilde{\omega}_i^k \wedge dy_{d_i^{k+1}} \wedge \dots \wedge dy_p \wedge du \wedge d\dot{u} \wedge \dots \wedge du^{(q_i-1)} = 0$ , if  $d_i^k < p$ ,
- $d\tilde{\omega}_i^k \wedge du \wedge d\dot{u} \wedge \dots \wedge du^{(q_i-1)} = 0$ , if  $d_i^k < p$ ,  
with  $1 \leq i \leq p$ ,  $1 \leq k \leq k_i$  and  $q = \text{Max}\{q_1, \dots, q_p\}$ .

**MIMO systems linearisation by generalized state transformation,  
output transformation and generalized input-output injection: Application**

**Application** Stepper motor - Speed observer ([Chi93])

**Motor model**

$$\begin{aligned} \dot{x}_1 &= x_2 & y_1 &= x_1 \\ \dot{x}_2 &= -K_1 \cdot x_3 \cdot \sin(N_r x_1) + K_3 \cdot x_4 \cdot \cos(N_r x_1) - K_4 \cdot x_2 & y_2 &= x_3 \\ \dot{x}_3 &= -K_1 \cdot x_3 + K_2 \cdot x_2 \cdot \sin(N_r x_1) + u_1 & y_3 &= x_4 \\ \dot{x}_4 &= -K_1 \cdot x_4 - K_2 \cdot x_2 \cdot \cos(N_r x_1) + u_2 \end{aligned}$$

with  $x_1$  rotor position,  $x_2$  rotor speed,  $x_3$  and  $x_4$  phasis currents.

**System equivalent to ( $q = 0$ ) :**

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 - K_4 \cdot y_1 \\ \dot{\zeta}_2 &= -K_3 \cdot y_1 \cdot \sin(N_r \cdot y_1) + K_3 \cdot y_2 \cdot \cos(N_r \cdot y_1) \\ \dot{\zeta}_3 &= -K_1 \cdot y_2 + u_1 \\ \dot{\zeta}_4 &= -K_1 \cdot y_3 + u_2 \end{aligned}$$

with output coordinates transformation

$$\tilde{y} = [y_1, y_2 + \frac{K_2}{N_r} \cos(N_r y_1), y_3 + \frac{K_2}{N_r} \sin(N_r y_1)]^T$$

and state coordinates transformation

$$\zeta = [x_4, x_3 + \frac{K_2}{N_r} \cos(N_r x_4), x_2 + \frac{K_2}{N_r} \sin(N_r x_4)]^T$$

→ **Design of an observer with linear errors dynamics.**

## GENERALISED PROBLEM

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Goal: To determine a generalized state coordinates transformation

$\zeta = \phi(x, u, \dot{u}, \dots, u^{(q)})$  such that the system (1) (with  $p = 1$ ) is equivalent to:

$$\begin{aligned}
 \dot{\zeta}_1 &= \zeta_2 \\
 \dot{\zeta}_2 &= \zeta_3 \\
 &\vdots \\
 \dot{\zeta}_s &= \zeta_{s+1} \\
 \dot{\zeta}_{s+1} &= \zeta_{s+2} + \varphi_{s+1}(y, \dot{y}, \dots, y^{(s)}, u, \dot{u}, \dots, u^{(q)}) \\
 \dot{\zeta}_{s+2} &= \zeta_{s+3} + \varphi_{s+2}(y, \dot{y}, \dots, y^{(s)}, u, \dot{u}, \dots, u^{(q)}) \\
 &\vdots \\
 \dot{\zeta}_{n-1} &= \zeta_n + \varphi_{n-1}(y, \dot{y}, \dots, y^{(s)}, u, \dot{u}, \dots, u^{(q)}) \\
 \dot{\zeta}_n &= \varphi_n(y, \dot{y}, \dots, y^{(s)}, u, \dot{u}, \dots, u^{(q)}) \\
 y &= \zeta_1
 \end{aligned} \tag{9}$$

An observer for (9) is:

$$\dot{\hat{\zeta}} = A' \cdot \hat{\zeta} + \varphi(y, \dots, y^{(s)}, u, \dots, u^{(q)}) + K \cdot C \cdot (\zeta - \hat{\zeta})$$

→ Use of outputs derivatives: Diop-Grizzle-Moraal-Stefanopoulou (1994).

### Linearisation by generalized Input-Output injection with outputs derivatives: Example

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**Hypothesis:** Generalized problem.

**Objective:** Propose a solution using a limited number of output derivatives, taking maximum advantage of the system's structure.

**Example 6** Let the nonlinear system:

$$\begin{aligned}
 \dot{x}_1 &= x_2 & y &= x_1 \\
 \dot{x}_2 &= x_3 \\
 \dot{x}_3 &= x_2 \sin x_2 + x_3 + x_1^2 + x_1 x_2
 \end{aligned}$$

Input-Output differential equation  $y^{(3)} = \dot{y} \sin \dot{y} + \ddot{y} + y^2 + y \dot{y}$

→ **The NSC of theorem 1 is not satisfied.**

→ **Use of an output derivative:** if the system is locally equivalent to:

$$\begin{aligned}
 \dot{\zeta}_1 &= \zeta_2 & y &= \zeta_1 \\
 \dot{\zeta}_2 &= \zeta_3 + \varphi_2(y, \dot{y}) \\
 \dot{\zeta}_3 &= \varphi_3(y, \dot{y})
 \end{aligned}$$

The Input-Output differential equation is:  $y^{(3)} = \varphi_2^{(1)} + \varphi_3$

A solution of the problem is:  $\varphi_2 = \dot{y}$   $\varphi_3 = \dot{y} \cdot \sin \dot{y} + y^2 + y \dot{y}$

**Linearisation by generalized Input-Output injection with outputs  
derivatives: Algorithm**

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Input-Output differential equation of system (1)

$$y^{(n)} = P(y, \dot{y}, \dots, y^{(n-1)}, u, \dot{u}, \dots, u^{(n-1)})$$

**G.I.O.I.A.d. Algorithm**

Note  $P_s := P$  and  $\varphi_s(y, \dot{y}, \dots, y^{(s)}, u, \dot{u}, \dots, u^{(q)}) := 0$ . For  $k = s + 1$  to  $n$ , define:

$$P_k := P_{k-1} - [\varphi_{k-1}(y, \dot{y}, \dots, y^{(s)}, u, \dot{u}, \dots, u^{(q)})]^{(n-k+1)}$$

The differential form  $\omega_k$  is defined by:

$$\omega_k = \frac{\partial P_k}{\partial y^{(n-k+s)}} dy^{(s)} + \sum_{j=1}^m \frac{\partial P_k}{\partial u_j^{(n-k+q)}} du_j^{(q)}$$

- If  $d\omega_k \wedge dy \wedge \dot{y} \wedge \dots \wedge dy^{(s-1)} \wedge du \wedge \dot{u} \wedge \dots \wedge du^{(q-1)} \neq 0$ , there is no solution to the problem.

- If  $d\omega_k \wedge dy \wedge \dot{y} \wedge \dots \wedge dy^{(s-1)} \wedge du \wedge \dot{u} \wedge \dots \wedge du^{(q-1)} = 0$ , then  $\varphi_k(y, \dot{y}, \dots, y^{(s)}, u, \dot{u}, \dots, u^{(q)})$  is solution of:

$$\frac{\partial \varphi_k}{\partial y^{(s)}} dy^{(s)} + \sum_{j=1}^m \frac{\partial \varphi_k}{\partial u_j^{(q)}} du_j^{(q)} = \omega_k \quad \text{for } s + 1 \leq k \leq n - 1,$$

$$\varphi_n(y, \dot{y}, \dots, y^{(s)}, u, \dot{u}, \dots, u^{(q)}) = P_n$$

**Linearisation by generalized Input-Output injection with outputs  
derivatives: C.N.S.**

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**Theorem 7** [Ple96] Nonlinear system (1) is locally equivalent to system (9) by  $\zeta = \phi(x, u, \dot{u}, \dots, u^{(v-1)})$  if and only if:

$$d\omega_k \wedge dy \wedge \dot{y} \wedge \dots \wedge dy^{(s-1)} \wedge du \wedge \dot{u} \wedge \dots \wedge du^{(q-1)} = 0$$

for  $s + 1 \leq k \leq n$  and  $v = \text{Max}(s - 2, q - 1)$ .

**Application: Flexible joint robot** - measured motor position [Ple94].

Flexible joint robot model.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{mgl}{J_l} \sin x_1 - \frac{k}{J_l} (x_1 - x_3) - \frac{f_1}{J_l} x_2$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{k}{J_m} (x_1 - x_3) - \frac{f_2}{J_m} x_4 + \frac{u}{J_m}$$

$$y = x_3$$

## Linearisation by generalized Input-Output injection with outputs derivatives: Application

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System equivalent to ( $s = 2, q = 0$ )

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2 - \left(\frac{f_1}{J_l} + \frac{f_2}{J_m}\right)y \\ \dot{\zeta}_2 &= \zeta_3 - \left(\frac{k}{J_m} + \frac{k}{J_l} + \frac{f_1 f_2}{J_l J_m}\right)y + \frac{1}{J_m} u \\ \dot{\zeta}_3 &= \zeta_4 - \frac{k}{J_l J_m} (f_1 + f_2)y + \frac{f_1}{J_l J_m} u \\ \dot{\zeta}_4 &= \frac{k}{J_l J_m} u - \frac{k}{J_l} \frac{mgl}{J_m} \sin\left(\frac{J_m}{k} \dot{y} + \frac{f_2}{k} \dot{y} + y - \frac{u}{k}\right) \\ y &= \zeta_1\end{aligned}$$

with the state coordinates transformation :

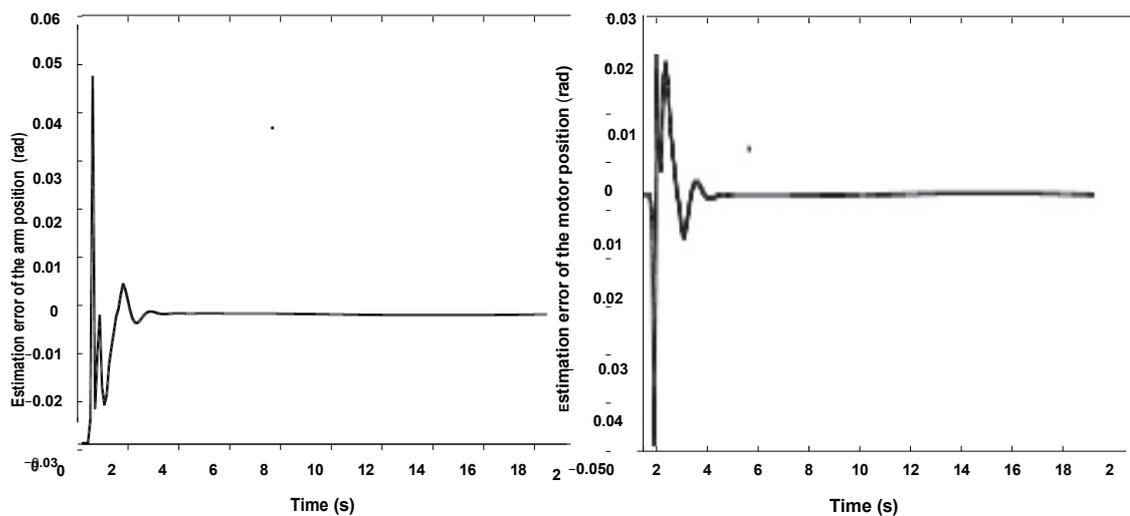
$$\begin{aligned}\zeta &= \left[ x_3, x_4 + \left(\frac{f_1}{J_l} + \frac{f_2}{J_m}\right) x_3, \frac{k}{J_m} x_1 + \frac{f_1}{J_l} x_4 + \left(\frac{k}{J_l} + \frac{f_1 f_2}{J_l J_m}\right) x_3, \right. \\ &\quad \left. k \left(\frac{x_2}{J_m} + \frac{x_4}{J_l}\right) + \frac{k}{J_l J_m} (f_1 x_1 + f_2 x_3) \right]^T\end{aligned}$$

→ Design of an observer with linear errors dynamics.

## Linearisation by generalized Input-Output injection with outputs derivatives: Application

---

Sliding Modes Control with a sinusoidal reference + encoder noise measurement.



Error between the estimated position and the real position of the arm (rad).

Error between the estimated position and the real position of the motor (rad).

**Summary - Extensions "Linearisation" by generalized  
input-output injection**

---

System	Approach	Applications
$\zeta' = A\zeta + \varphi(y, u)$	Geometric [Kre83, Kre85, Mar91, Xia89] Algebraic [Glu96, Ham88, Lop97, Ple95]	DC Shunt motor Serial DC motor: [Chi94, Ple95]. Stepper motor: [Chi93]. Flexible joint robot: [Ple94].
$\zeta' = A\zeta + \varphi(y, u, \dot{u})$	Geometric [Wil77, Kel87] Algebraic [Pro93]	Biological system: [Wil77]. Bilinear system: [Pro93].
$\zeta' = A\zeta + \varphi(y, u, \dots, u^{(w)})$	Algebraic: [Ple93]	Numerical Differentiation: [Dio94]
$\zeta' = A\zeta + \varphi(y, \dots, y^{(s)}, u, \dots, u^{(w)})$	Algebraic [Ple97]	Numerical Differentiation:[Dio94]

**Conclusions**

- Approach based on the structural study of nonlinear systems.
- Using the theory of exterior differential systems.

**Supplements**

- Generalization of the theory to MIMO systems without calculating I/O equations([Lop97])
- State affine Case ([Lop98, Sou03, Sou07])

$\zeta' = A(y, u)\zeta + \varphi(y, u)$	Algebraic [Bes96, Ham92, Ham90, Lop98]	Inverse pendulum [Bes96] Distillation column [Vie94]
---	--	---

**Prospects**

- Necessary and sufficient condition for the existence of a transformation of output coordinates using an algebraic approach.
- Non-additive input-output injection linearization study
- ...

## Chapter 7 part 3

### Observer design by Transformation into an Affine System

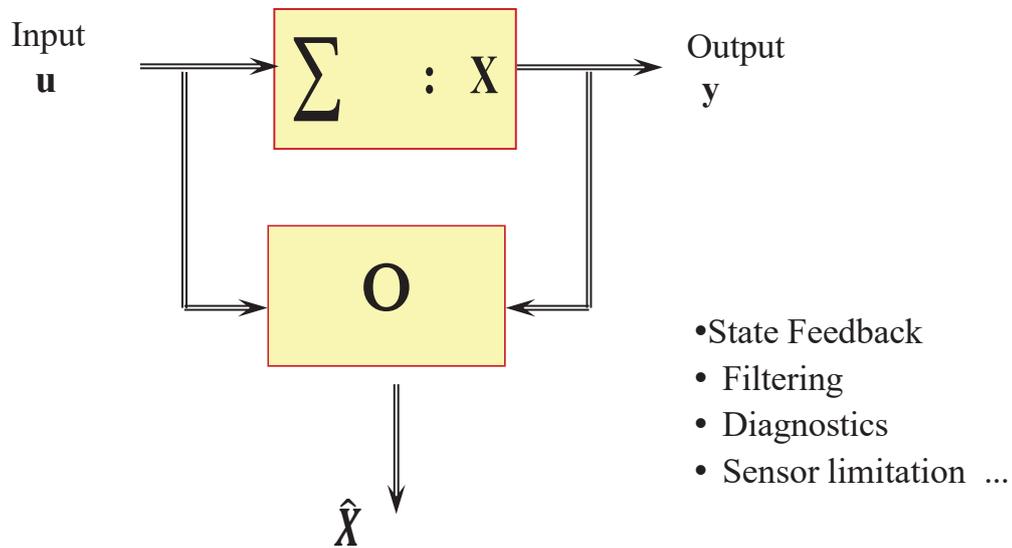
#### Outline

- I Introduction
- II State of the art
- III Problem statement
- IV Transformation  
NLS  $\longleftrightarrow$  SAS (State Affine System)
- V Application to the Inverse Pendulum
- VI Conclusions, other results

#### Introduction, Objectives

- To expand the class of observers for non-linear systems
- Focus on the applicability of solutions: algorithms and symbolic calculation
- Implementation on experimental site (pendulum)

## Observer



## II State of the art: Observer Class for NLS

References	Systems	Hypothesis	Solutions
Bornard, <i>et al</i> [Bor88] Hammouri De Léon [Ham90]	$z = A(y, u) + \phi(y, u)$ $y = Cz$	<ul style="list-style-type: none"> <li>• Observability</li> <li>• <math>u</math>-persistent</li> </ul>	<ul style="list-style-type: none"> <li>• Exponential local observers</li> <li>• High gains</li> </ul>
Hammouri, Busawon [Ham93]	Observable for all $u$ $x' = Ax + \varphi(x) + \phi(x)u$ $y = Cx$ MIMO	<ul style="list-style-type: none"> <li>• Globally Stabilisable</li> <li>• <math>\varphi(x), \phi(x)</math> Lipschitz</li> </ul>	<ul style="list-style-type: none"> <li>• SC</li> <li>• Exp local Obs.</li> <li>• High gains</li> </ul>
Gauthier, Kupka [Gau92]	$x' = Ax + u(Bx + b)$ $y = Cx$ Observable Dissip. MIMO	<ul style="list-style-type: none"> <li>• Accessible</li> </ul>	<ul style="list-style-type: none"> <li>• Asymptotic observers.</li> <li>• High gains</li> </ul>
Marques [Mar93]	$x' = f(x) + g(x)u$ $y = h(x)$ MIMO Dissipative	<ul style="list-style-type: none"> <li>• Observability</li> <li>• Global stability</li> <li>• <math>\exists</math> Lyapunov function</li> </ul>	<ul style="list-style-type: none"> <li>• Asymptotic Observers</li> <li>• High gains</li> </ul>

## Possible techniques

### Linearisation Modulo one Injection

$$\sum_L : \begin{aligned} \dot{\xi} &= A\xi + \varphi(y, u), \\ y &= C\xi \end{aligned}$$

- Krener, Respondek [Kre85]
- Xia, Gao [Xia89]
- Plestan, Glumineau [Ple97]

### Affine's Modulo Injection system (SAS)

$$\sum_A : \begin{aligned} \dot{z} &= A(y, u)z + \varphi(y, u), \\ y &= Cz \end{aligned}$$

- Bornard, *et al* [Bor88]
- Hammouri, De Leon [Ham91]
- Hammouri, Gauthier [Ham92]
- Tornambé [Tor93]
- Besancon, Bornard [Bes96]

## Outline

I Introduction

II State of the art

III **Problem statement**

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NLS  $\leftrightarrow$  SAS (State Affine System)

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IV Conclusions, other results.

**Problem Statement**  
**SNL ↔ SAS ?**

Is there a transformation  $T(\cdot) : \Sigma \rightarrow \Sigma_A$  such that

$$\Sigma : \begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \quad (\text{Observable } \forall u)$$

is equivalent to: (SAS)

$$\Sigma_A : \begin{cases} \dot{z} = A(y, u)z + \phi(y, u), \\ y = Cz \end{cases}$$

If  $\exists T(\cdot) : \Sigma \rightarrow \Sigma_A$  then with  $\mathfrak{G} = (y, u)$

$$\Sigma_A : \begin{cases} \dot{z} = A(\mathfrak{G})z + \phi(\mathfrak{G}), \\ y = Cz, \end{cases}$$

A Kalman like observer can be designed (Hammouri, DeLeon, [Ham91]):

$$\hat{\Sigma}_A : \begin{cases} \dot{\hat{Z}} = A(\mathfrak{G}) \hat{Z} + \phi(\mathfrak{G}) - R^{-1} C^T C (\hat{Z} - Z) \\ \dot{R} = -\theta R - A^T(\mathfrak{G})R - RA(\mathfrak{G}) + C^T C \end{cases}$$

such that:

$$\|e(t)\|^2 \leq \lambda \exp^{-\theta t}$$

## Applications examples

$$\begin{aligned} \dot{z} &= A(y,u)z + \varphi(y,u), \\ y &= Cz \end{aligned}$$

- Synchronous generator: de Léon M., Acha Daza [Leon98]
- Inverse pendulum : Besançon, Bornard [Bes96]
- Chemical reactor: Guillaume, Rouchon [Gui97 ], Viel [Vie94]
- Hydraulic process diagnostics: Kaboré, Hammouri, Othman [Kab97]

## Outline

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NLS  $\longleftrightarrow$  SAS (State Affine System )

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**Affine system, MISO case (m>1,p=1)**

$$\phi(\cdot): \Sigma \rightarrow \Sigma_A \quad \Sigma: \begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned}$$

where  $x \in \mathbb{R}^v, \psi \in \mathbb{R}, u \in \mathbb{R}^\mu, \text{observable } \forall u$

$$\Sigma_A: \dot{z} = \begin{bmatrix} 0 & A_1(u) & 0 & \dots & 0 \\ 0 & 0 & A_2(y, u) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{n-1}(y, u) \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} z + \begin{bmatrix} \varphi_1(y, u) \\ \varphi_2(y, u) \\ \vdots \\ \varphi_{n-1}(y, u) \\ \varphi_n(y, u) \end{bmatrix}$$

$$y = C z = z_1$$

**Preliminary example  
I/O differential equation**

$$\begin{aligned} \dot{z}_1 &= A_1(y, u)z_2 + \varphi_1(y, u), & y^{(2)} &= \underbrace{\dot{y}\hat{A}_1}_{F_2} + \underbrace{\dot{\varphi}_1 - \varphi_1\hat{A}_1}_{F_1} + \underbrace{A_1\varphi_2}_{F_0} \\ \dot{z}_2 &= \varphi_2(y, u), \\ y &= Cz = z_1 \end{aligned} \quad \text{where } \hat{A}_1 = \frac{\dot{A}_1}{A_1}$$

$$\Rightarrow y^{(2)} = \partial_y \hat{A}_1 \dot{y} + \partial_u \hat{A}_1 \dot{u} + \Delta(A_1, \varphi_1, \varphi_2, \hat{A}_1, \dot{\varphi}_1)$$

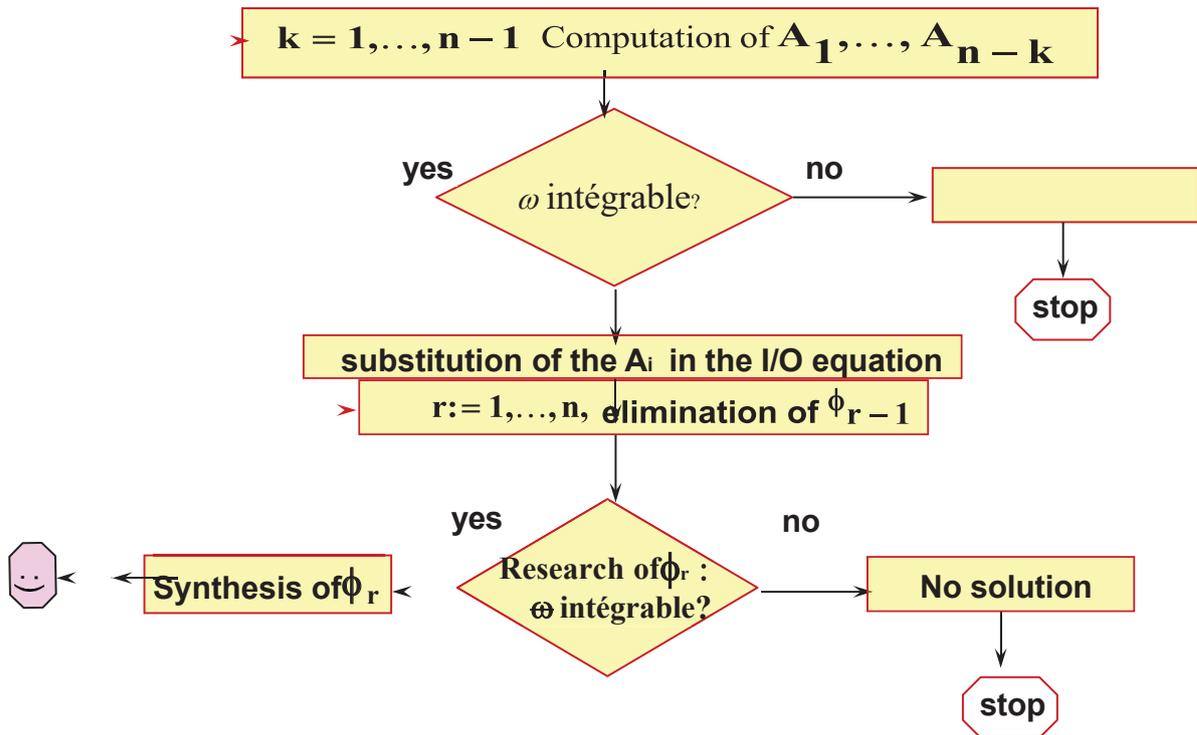
$$\omega = \frac{\partial^2 y^{(2)}}{\partial \dot{y} \partial \dot{y}} dy + \frac{\partial^2 y^{(2)}}{\partial \dot{y} \partial \dot{u}} du = \frac{\partial \hat{A}_1}{\partial y} dy + \frac{\partial \hat{A}_1}{\partial u} du$$

$\omega$  is computed from the I/O differential equation.

$$\omega \text{ int\`egrable} \Rightarrow \exists A_1(y, u) \quad (\text{First step})$$

Second step: to check the existence of functions  $\varphi_i(y, u)$ ,

### Algorithm: MISO case



Nonlinear Observers Chap 7 Part. 3

7.3.13

### Conclusions

- Equivalence SNL to SAS
  - MISO
    - I/O Differential equation approach
- Necessary and Sufficient Conditions
- Constructive method
- Synthesis of a "Kalman like" observer

## Outline

I Introduction

II State of the art

III Problem statement

IV Transformation

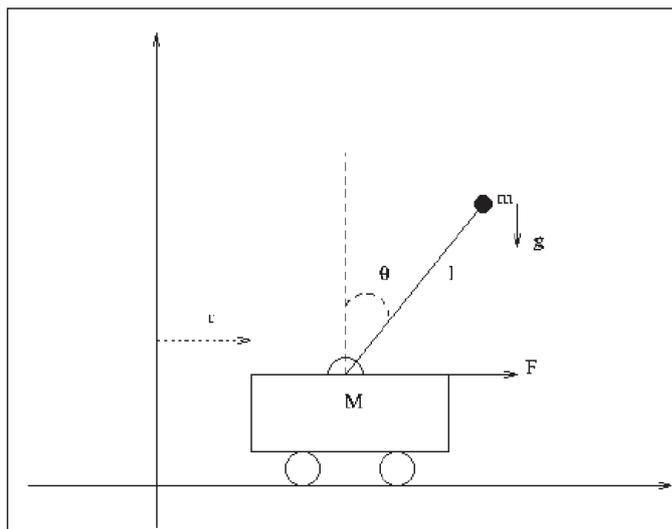
NLS  $\leftrightarrow$  SAS (State Affine System)

V Application to Inverse Pendulum

- SAS transformation
- Pendulum control
- Results : Simulation and experiments

VI Conclusions, other results.

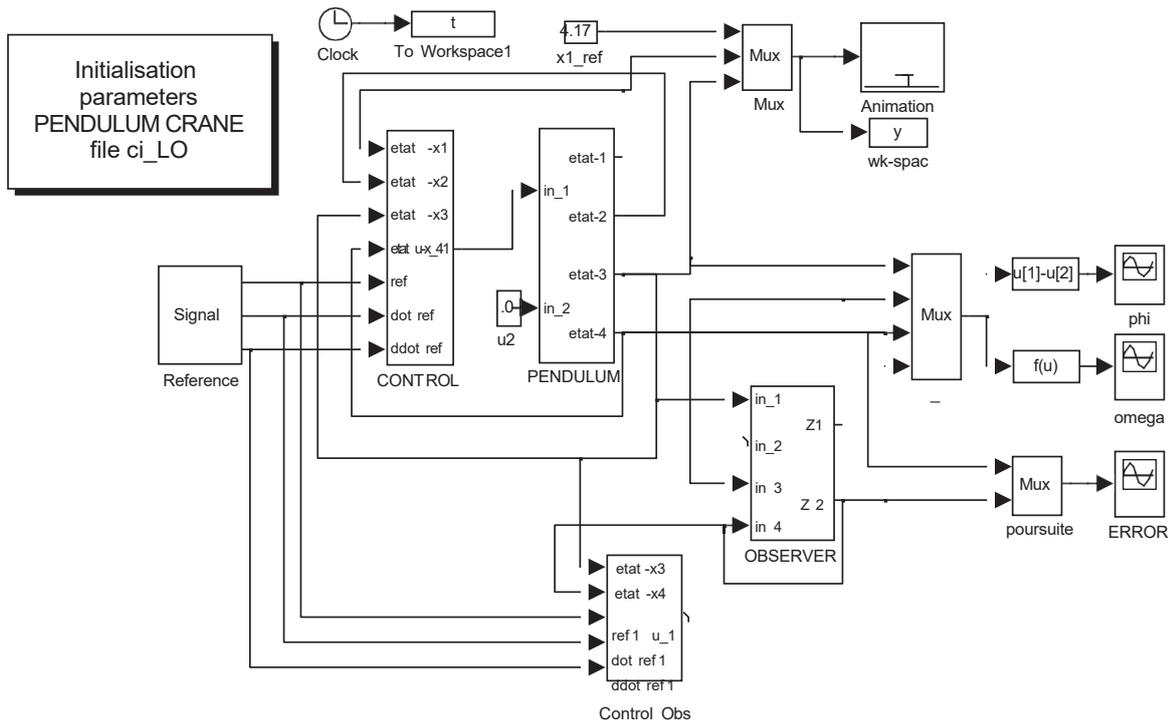
## Inverse pendulum or Crane



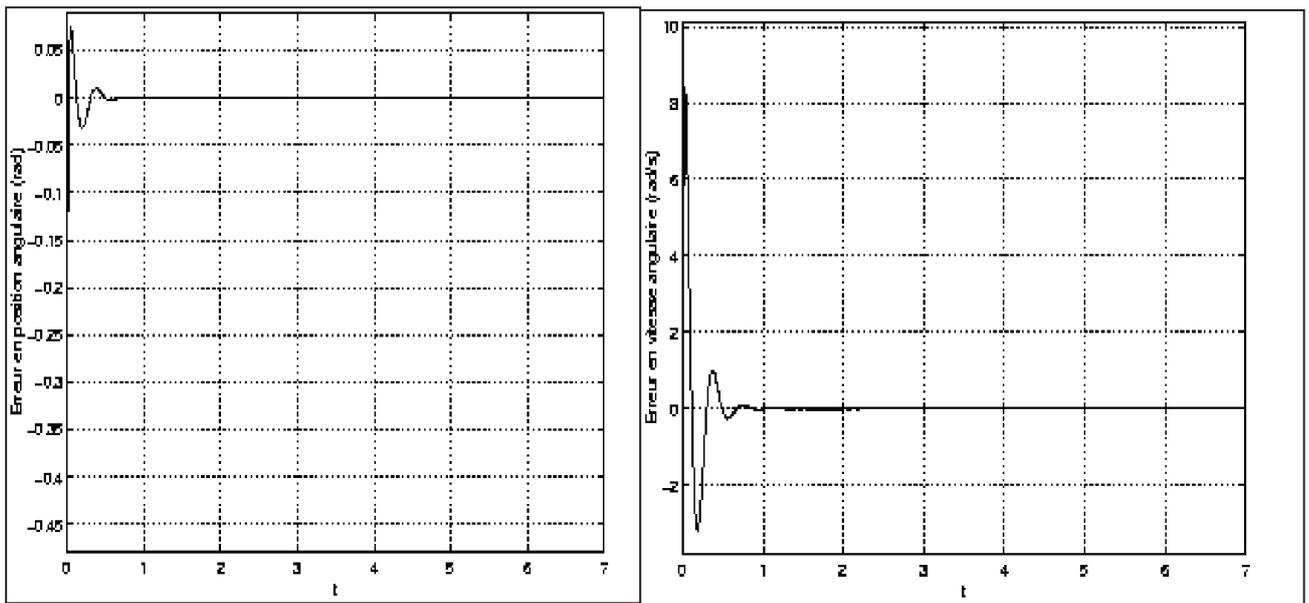
$$\begin{bmatrix} M + m & ml \cos \theta \\ ml \cos \theta & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & ml \sin \theta \dot{\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ -mgl \sin \theta \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

## Simulation Scheme: Inverse Pendulum observer

### INVERSE PENDULUM (OR CRANE) CONTROL and OBSERVER



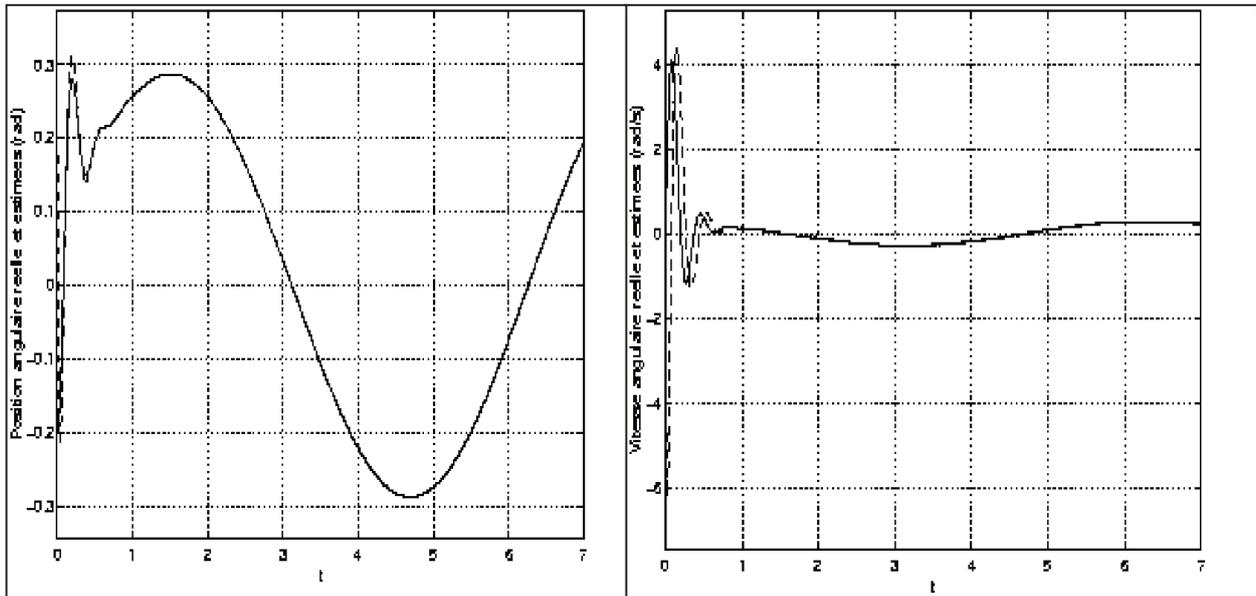
## Simulation Inverse Pendulum observer



Angle position error

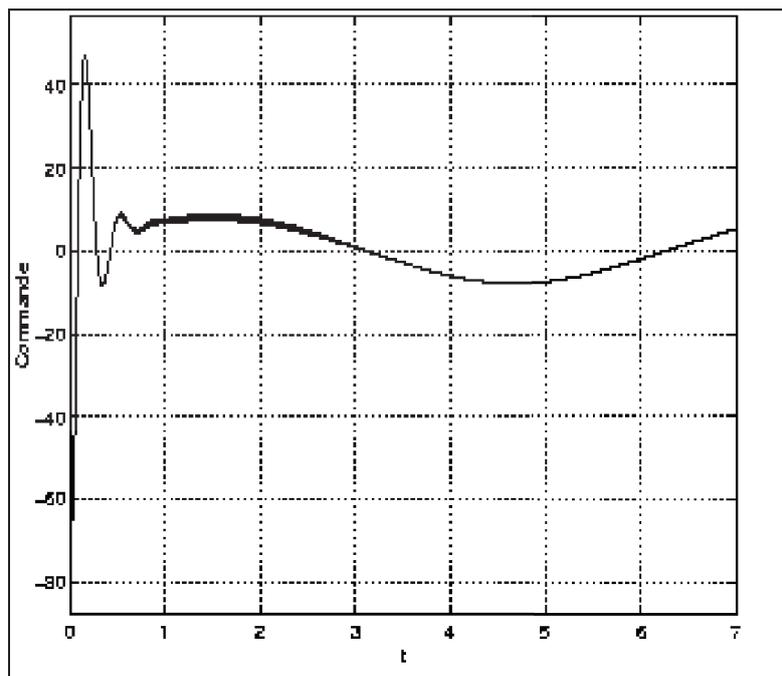
Speed angle error

## Simulation Inverse Pendulum observer



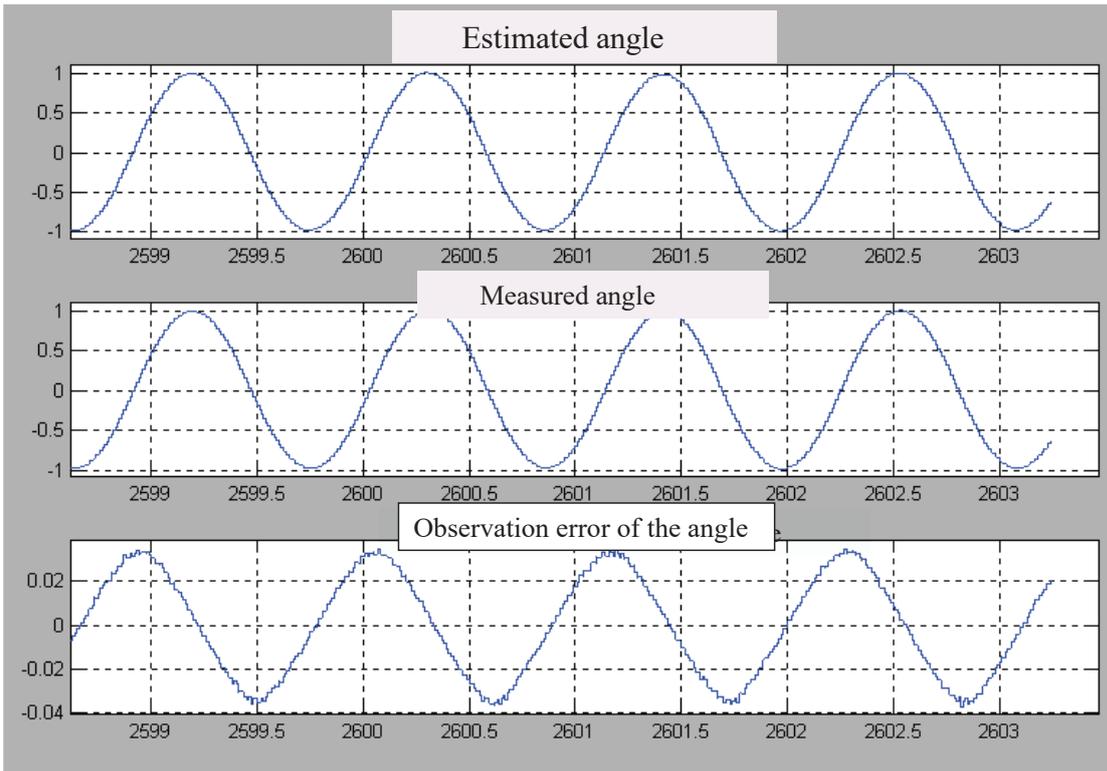
Position trajectory

Speed trajectory



Control with state estimation

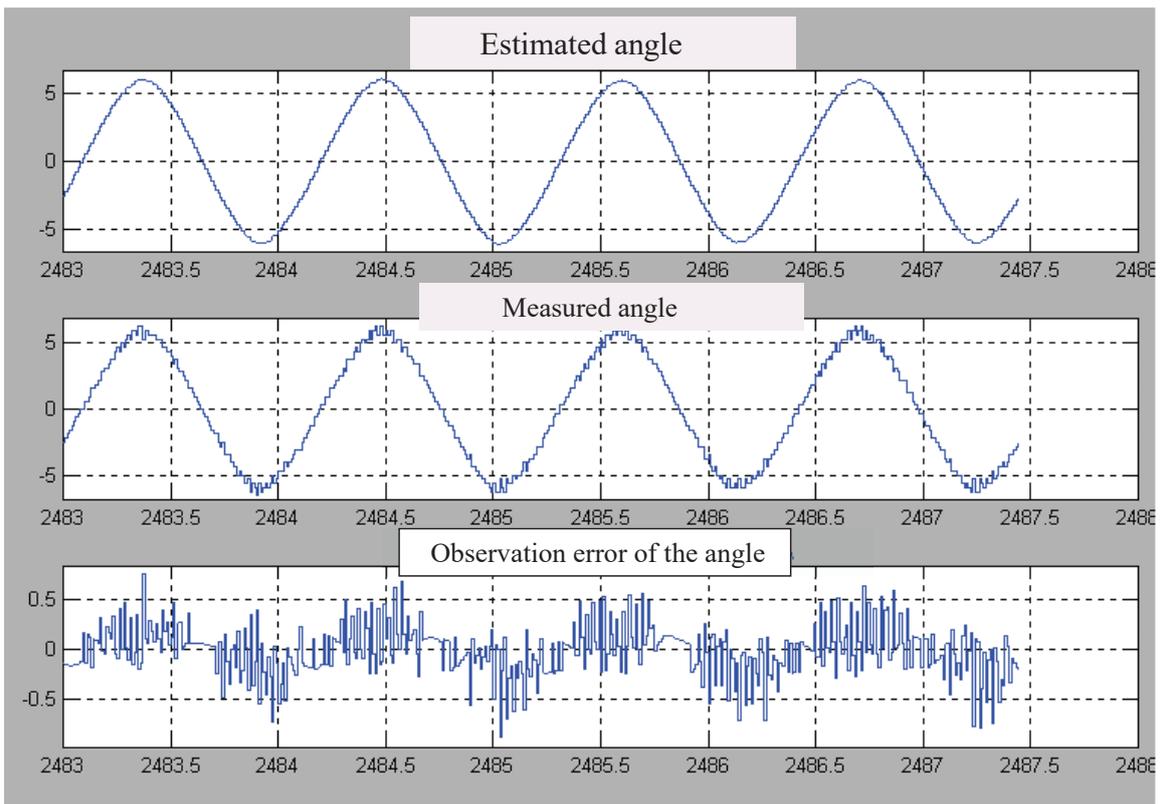
## Experimental Validation in Crane Mode Angular position



Nonlinear Observers Chap 7 Part. 3

7.3.21

## Experimental Validation in Crane Mode Angular Speed



Nonlinear Observers Chap 7 Part. 3

7.3.22

## **General conclusions**

- Observer for a class of NLS
- Equivalence NLS  $\leftrightarrow$  SAS
- Necessary and sufficient Conditions
- Constructive method
- Application to a real system

## **Other available results**

- NLS  $\leftrightarrow$  SAS Transformation in the MIMO Case
- Direct approach (without calculation of the I/O equations)
- General matrix  $A(y,u)$  and generalised injection (derivatives of  $y$  and  $u$  if necessary)
- Stability: Controller - Observer (Lyapunov)

## CHAPTER 8 INTRODUCTION TO STABILITY OF NONLINEAR SYSTEMS

Purpose: Introduction to two robust controls of nonlinear systems:

- Sliding mode control
- Backstepping control

Common points of these two techniques :

- Deterministic approach
- Stabilization by controlled Lyapunov functions

Outline :

- Chap. 8 Definitions and examples of Lyapunov stability
- Chap. 9 Sliding Modes control
- Chap. 10 Backstepping control

### 8. Notion of stability and Stabilization of Nonlinear Systems (NLS) [Zin89]

#### 8.1 Preliminary definitions

**NLS equilibrium point :**

For  $\dot{x} = f(x) + g(x)u \rightarrow$  any point  $(x_0, u_0)$  solution of  $\dot{x} = 0$ .

**Remark 1** In general we talk about the equilibrium points of the autonomic system

$$\Sigma : \dot{x} = f(x)$$

**Example 8.1 :** the pendulum has two points of equilibrium,  $\theta=0^\circ$  or  $180^\circ$  (stable or unstable ?)

**Definition 1. Stability (in the sense of Lyapunov)** The equilibrium point  $x_0$  is said to be stable if

$$\forall R > 0, \exists r > 0: \|x(0)\| < r \Rightarrow \|x(t)\| < R, \forall t > 0.$$

**Definition 2. Asymptotic stability (in the Lyapunov sense)**

The equilibrium point  $x_0$  is said to be asymptotically stable if

$$\exists r > 0: \|x(0)\| < r \Rightarrow \|x(t)\| \rightarrow 0 \text{ when } t \rightarrow \infty.$$

## Sufficient conditions for stability around an equilibrium point

Using linearization by first order approximation of the SNL around the equilibrium point:

$$\{\dot{x} = f(x) \rightarrow \dot{\eta} = \left. \frac{\partial f}{\partial x} \right|_{x=x_0} \eta \text{ or } \dot{\eta} := A\eta.$$

### Theorem 1. Stability by means of the first order approximation (around a point)

- 1) If the linearized system is asymptotically stable (the eigenvalues of A have real part  $< 0$ ) then the SNL is asymptotically stable.
- 2) If the linearized system is unstable (at least one eigenvalue of A with real part  $> 0$ ) then the SNL is unstable
- 3) If the linearized system is marginally stable (one real part eigenvalue = 0), then we cannot conclude on the stability of the SNL.

**Exercices** Calculation of the stability of  $\begin{cases} 1) \dot{x} = \pm x + x^2 \text{ for } x = 0 \text{ (2 cases)} \\ 2) \dot{x} = kx^2, k > 0 \text{ for } x = 0 \end{cases}$

## 8.2 Study of the equilibrium for NLS by using Lyapunov functions

**Definition 3.** A continuously differentiable function  $V(x)$  is called a **Lyapunov function** if

$$V(0) = 0$$

$$V(x) > 0 \quad \forall x \neq 0 \text{ (positive definite)}$$

$$\dot{V}(x) \leq 0 \quad \forall x \neq 0 \text{ (negative semidefinite).}$$

### Theorem 2. Lyapunov Stability

For an NLS with an equilibrium point at 0, if there is a function  $V(x)$  such that:

$$V(0) = 0$$

$$\text{and for } x \neq 0, V(x) > 0 \text{ and } \dot{V}(x) < 0$$

then the equilibrium point is **asymptotically stable**.

**Remark 2.** Stability in the sense of Lyapunov is a mathematical translation of the fact that if the total energy of a system (linear or not, stationary or not) dissipates continuously then this system tends to return to an equilibrium state.

**Remark 3.** There is not only one function that can be Lyapunov!

**Example of conservatism of Lyapunov functions:**

For the system : 
$$\begin{cases} \dot{x}_1 = 2x_1(x_2^2 - 1) \\ \dot{x}_2 = -x_2(x_1^2 + 1) \end{cases}$$

Let the candidate Lyapunov function  $V_1(x_1, x_2) = \frac{x_1^2 + x_2^2}{2}$ , then its time derivative

$$\dot{V}_1(x_1, x_2) = x_1^2 x_2^2 - 2x_1^2 - x_2^2 \text{ could be } < 0 \text{ in a domain of } \mathbb{R}^2.$$

Example in (1,1) or (2,2) ?

For the same system, the function  $V_2(x_1, x_2) = \frac{x_1^2 + 2x_2^2}{2}$  has the time derivative:

$$\dot{V}_2(x_1, x_2) = -2x_1^2 - 2x_2^2 \text{ that is } < 0 \text{ in all } \mathbb{R}^2!$$

The first Lyapunov function  $V_1$  is conservative because it gives a **pessimistic** result on the stability domain of the system

### Exercises: Stability of NL systems

Ex 1

$$\dot{x} = -x^3$$



Ex 2



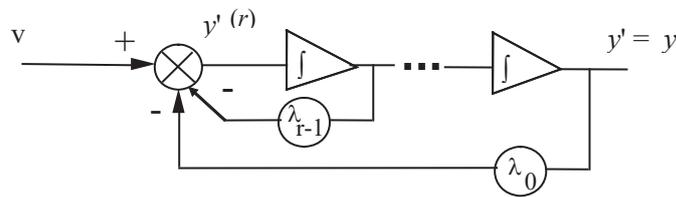
$$\dot{x} = -x^2$$

### 8.3 Reminders on deterministic controls.

#### Exact Input/Output linearization problem + pole placement

$y' := y$  is the output of the system after feedback

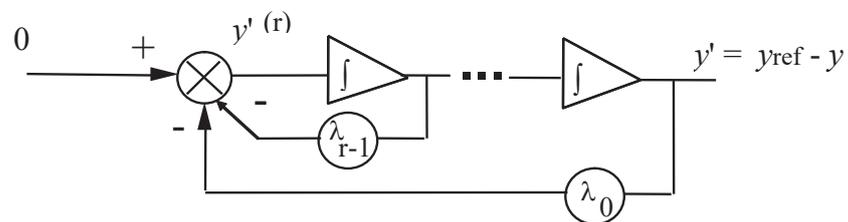
So in closed loop:



#### Trajectory tracking problem i.e. exact linearization of the error dynamics

$y' := y_{ref} - y$  is now the output.

So in closed loop:



## ROBUST CONTROL OF NON-LINEAR SYSTEMS APPLICATION TO ELECTRIC DRIVES

#### Contents :

- Chap. 9 Sliding Modes Control
- Chap. 10 Backstepping Control

# CHAPTER 9 SLIDING MODES CONTROL

## 9.1 Sliding Modes Control of order 1: Seminal works and principle

- Historic framework: control of systems with discontinuous inputs:
  - A. G. Phillipov Differential equations with discontinuous right-hand sides, 1960 (Russian), 1988 (English), SPRINGER, Mathematics and its applications.
  - Vadim Utkin, Variable structure systems with sliding modes, IEEE Transactions on Automatic Control, May 1977.
- Principle. Suppose that the nonlinear system to be controlled is written in the form:

$$\begin{cases} \dot{x} = f(x) + g(x).u \\ y = h(x) \end{cases} \quad \text{with } x \in R^n, u \in R^m, y \in R.$$

The control by sliding modes finds its justification by using the notion of stability according to Lyapunov. The stability is determined using a differentiable function  $V(x)$  from  $R^n$  to  $R^+$  so-called Lyapunov function ("picture" of the system energy) which satisfies the following conditions:

$$\begin{cases} V(x) > 0 \text{ pour } x \neq 0, V(0) = 0 & (1) \\ \dot{V}(x, u) \leq 0 \text{ ou } < 0 & (2) \end{cases}$$

$V(x)$  is positive definite (1), and  $\dot{V}(x, u)$  is negative semidefinite (or definite) (2).

This function will be determined from a "pseudo-output" of the system  $S(x)$  called sliding variable because in closed loop, the system will be forced to slide on the equation surface

$$S(x) = 0.$$

### A particular choice of pseudo-output $S(t, x)$

With the trajectory to be followed  $y_{ref}(t)$ , define:

$$S(t, x) = \sum_{i=0}^{r-1} \lambda_i \cdot (y_{ref}(t) - y(x))^{(i)}$$

with  $\lambda_{r-1} = 1$  and  $r$  the relative degree of the output  $y$ .

**First-order sliding mode control purpose:**

To bring the dynamics of the system onto the surface  $S(t, x) = 0$  and make them slide on it. The convergence dynamics of the error is ensured by the choice of the coefficients  $\lambda_i$ . By taking the candidate function of Lyapunov  $V(t, x) = \frac{1}{2}S^2$  then  $\dot{V} = S\dot{S}$

with

$$\begin{aligned} \dot{S} &= \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} [f(x) + g(x).u] \\ &:= S_1(x, t) + S_2(x).u \\ u_{eq} &= -\frac{S_1(x)}{S_2(x)} \text{ et } v_n = \frac{u_n}{S_2(x)} \end{aligned}$$

By choosing  $u = u_{eq} + v_n$

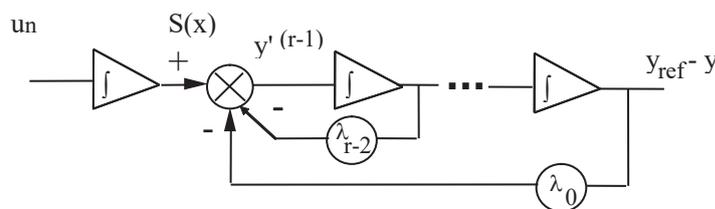
where  $u_{eq}$  is the so-called equivalent control (a linearizing control of the pseudo output  $S!$ ), then after this first feedback :

$$\dot{V} = Su_n.$$

To force  $\dot{V} < 0$  for  $S \neq 0$ , it is sufficient to design  $u_n = -k \cdot \text{Signe}(S)$ , with  $k > 0$ .

This last part of the control  $u_n = -k \cdot \text{Signe}(S)$ ,  $k > 0$  (of discontinuous type) forces the system dynamics to converge towards the surface, even if the system parameters are not perfectly identified, and/or if the system varies slowly over time, and/or if a bounded perturbation acts according to certain conditions (such as an action at the same level as the control i.e. "matching condition"). The dynamics of the looped system is shown in Figure 1:

We obtain a dynamic equation of the error which is "forced" to  $S=0$  by  $u_n$  and which is autonomous for  $u_n = 0$ .



$$u_n = -k \cdot \text{Signe}(S), \quad k > 0$$

Figure. 1. Dynamics error after SM feedback

## 9.2 Chattering phenomenon.

Due to disturbances or poorly known or varying system parameters, the "closed loop system" does not slide perfectly over the surface, it leaves it. The discontinuous control returns it to the surface.

=> Switching phenomenon around the surface (chattering).

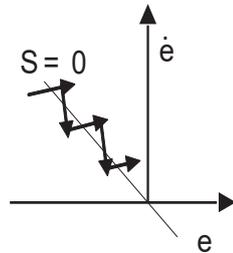


Figure 2. Switching phenomenon around the surface (chattering).

This can be detrimental to some actuators. To limit this problem, one solution is to use softened sign functions.

### Softened sign functions: examples.

The basic softened sign functions are:

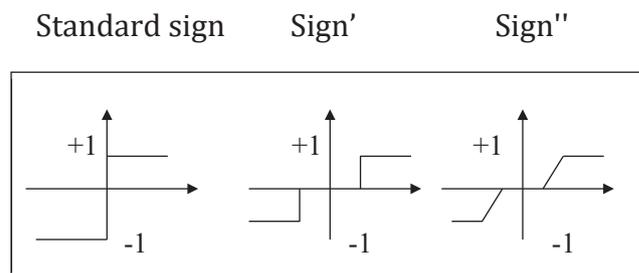


Figure 3. Softened sign functions

Remark: for digital practical implementation (real sliding mode), the frequency of switching is limited by the sampling period of the controller.

More complex "sign" functions are also available :

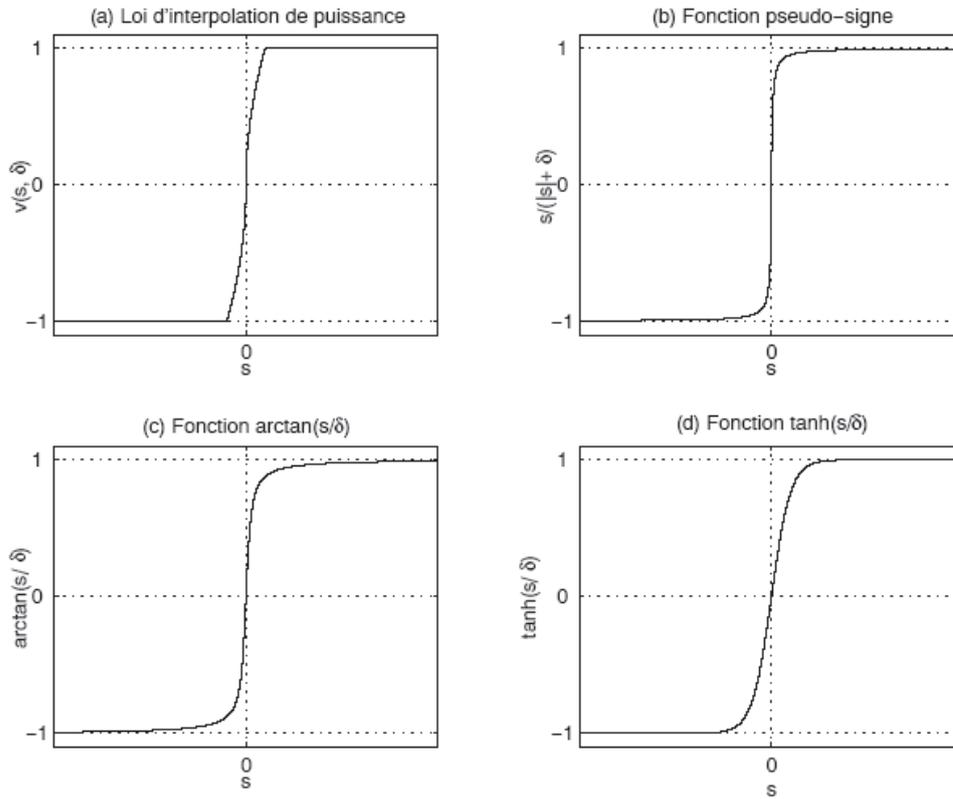


Figure 4. Softened sign functions " (following)

### 9.3 Example 1 : Sliding Mode Control design (monovariable case)

$$\dot{x}_1 = x_1 x_2$$

$$\dot{x}_2 = u$$

$$y = x_1$$

#### 9.4 Uncertain case, robustness analysis.

$$\text{Reminder } \dot{S} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} [f(x) + g(x) \cdot u] := S_1(x, t) + S_2(x) \cdot u$$

If  $S_1(x, t), S_2(x)$  poorly known, let's pose:

$$S_1(x, t) := S_{1n}(x, t) + \Delta S_1(x, t) \text{ et } S_2(x) := S_{2n}(x) + \Delta S_2(x).$$

with  $\Delta S_1(x, t)$  the  $S_1$  uncertainties and with  $|\Delta S_1(x, t)| \ll |S_{1n}(x, t)|$

and  $\Delta S_2(x)$  the  $S_2$  uncertainties and with  $|\Delta S_2(x)| \ll |S_{2n}(x)|$ .

$$\text{then } \dot{S} = S_{1n}(x, t) + \Delta S_1(x, t) + (S_{2n}(x) + \Delta S_2(x))u$$

$$\text{and with } u = -\frac{S_{1n}}{S_{2n}} - \frac{K \text{Signe}(S)}{S_{2n}} \Rightarrow \dot{S} = \Delta_1 - (\Delta_2 + 1)K \text{sign}(S)$$

$$\text{where } |\Delta_1| = |\Delta S_1(x) - \Delta S_2(x)S_{1n}(x)/S_{2n}(x)|$$

$$|\Delta_2| = |\Delta S_2(x)/S_{2n}(x)| \ll 1$$

$$\text{By choosing } \dot{S} = a - b \text{sign}(S),$$

a Sufficient Condition to have  $\dot{S} < 0$  is  $b > |a|$

$$\Rightarrow \text{choice of } K \Rightarrow \text{SM robustness.}$$

#### 9.5 Sliding Mode Control: Multivariable Case

If the system  $\Sigma$  to control has  $p$  outputs  $\rightarrow p$  switching surfaces of relative order 1.

If  $\Sigma$  has  $m$  ( $\geq p$ ) inputs, then

$$[\dot{S}(t, x)] = \left[ \frac{\partial S}{\partial t} \right] + \left[ \frac{\partial S}{\partial x} \right] [f(x)] + \left[ \frac{\partial S}{\partial x} \right] [g(x)]u$$

$$:= A(t, x) + B(x)u$$

$\rightarrow$  Non-linear static decoupling problem

$\rightarrow$  Is Matrix  $B(x)$  invertible?

$\rightarrow$  If yes, then the "equivalent control" is a decoupling and linearizing control.

$$\begin{bmatrix} \dot{S}_1(t, x) \\ \vdots \\ \dot{S}_p(t, x) \end{bmatrix} = \begin{bmatrix} u_{1n} \\ \vdots \\ u_{pn} \end{bmatrix}$$

$\rightarrow$  If not, is dynamic decoupling possible?

**Example 2.** a Multivariable case: unicycle

$$\begin{aligned}\dot{x}_1 &= (\cos x_3) u_1 & y_1 &= x_1 \\ \dot{x}_2 &= (\sin x_3) u_1 & y_2 &= x_2 \\ \dot{x}_3 &= u_2\end{aligned}$$

**Exercise 1**

$$\dot{x}_1 = x_2^2 + u_1 + \Delta_1, \quad y_1 = x_1 \quad \text{with } \Delta_1, \Delta_2 \text{ unknown bounded uncertainties.}$$

$$\dot{x}_2 = x_1 + u_2 + \Delta_2, \quad y_2 = x_2$$

**Particular Sliding Surface**  **internal variable limitation.**

Example: Direct Current motor.

## 9.6 Application to the Permanent Magnet Synchronous Motor

The Sliding Modes Control of Order 1 is applied to the permanent magnet synchronous motor: From the equations of the machine, after successive application of the Concordia and Park transformations, we obtain the model of the permanent magnet synchronous machine of the form:

$$\dot{x} = f(x) + g(x)u,$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{\Omega} \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix} = \begin{bmatrix} \Omega \\ \frac{p}{J} \left( (L_d - L_q) i_d + \Phi_f \right) i_q - \frac{f_v}{J} \Omega \\ -\frac{R_s}{L_d} i_d + p \frac{L_q}{L_d} \Omega i_q \\ -p \frac{\Phi_f}{L_q} \Omega - p \frac{L_d}{L_q} \Omega i_d - \frac{R_s}{L_q} i_q \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \frac{1}{L_d} & 1 \\ 0 & L_q \end{bmatrix} \begin{bmatrix} V_d \\ V_q \end{bmatrix} + \begin{bmatrix} -C_l \\ J \end{bmatrix}$$

$C_l$  is the load torque and is considered as an unmeasurable disturbance.

**Outputs to control:**  $\theta \rightarrow \theta_{ref}$ ,  $i_d \rightarrow 0$  (linearization of the motor torque by elimination of the magnetic salient effect ( $L_d \neq L_q$ )).

### Design of the sliding modes control of order 1.

From the state equations and considering the motor without load ( $C_l = 0$ ).

#### First output: rotor position of relative degree 3

Note  $e_p = \theta_{ref} - \theta$ . The surface is defined by  $S_p = \lambda_0 \cdot e_p + \lambda_1 \cdot \dot{e}_p + \ddot{e}_p$  to which is associated the candidate function of Lyapunov  $V_p = \frac{1}{2} S_p^2$ . By time differentiation of this function:

$$\dot{V}_p = S_p \cdot \dot{S}_p = S_p \cdot (S_1 + S_2 \cdot V_q)$$

$$\text{with } \begin{cases} S_1 = -\lambda_0 \cdot (\Omega - \dot{\theta}_{ref}) + \lambda_1 \ddot{\theta}_{ref} + \left( \frac{f_v}{J} - \lambda_1 \right) \left[ \frac{p}{J} [(L_d - L_q) i_d + \varphi_f] \cdot i_q - \frac{f_v}{J} \Omega \right] \\ \quad + \frac{p}{J} [(L_d - L_q) i_d + \varphi_f] \cdot \left( \frac{p \varphi_f}{L_q} \Omega + \frac{p L_d}{L_q} \Omega i_d + \frac{R_s}{L_q} i_q \right) \\ S_2 = -\frac{p}{J \cdot L_q} [(L_d - L_q) i_d + \varphi_f] \end{cases}$$

The term  $\frac{di_d}{dt}$  appearing in the calculation of  $\dot{S}_p$  has been neglected as weak, but it could be formally taken into account by using the analytic equation.

The control  $V_q$  to be taken is then  $V_q = u_{q,eq} + v_{q,n}$  where:

$$\begin{cases} u_{q,eq} = -\frac{S_1}{S_2} \\ v_{q,n} = -\frac{u_n}{S_2} \text{ with } u_n = -K_q \text{Signe}(S_p), \quad K_q > 0. \end{cases}$$

This control implies  $\dot{V}_p = -K_q S_p \text{Signe}(S_p) \leq 0$ .

Second output objective: set  $i_d$  to 0 i.e. linearization of the torque by suppressing the saliency effects ( $i_{dref}=0$ ). Its relative degree is 1:

We define the surface  $S_i = -i_d$  associated with the candidate Lyapunov function

$$V_i = \frac{1}{2} S_i^2.$$

Let's compute  $\dot{V}_i$ :

$$\dot{V}_i = S_i \cdot \dot{S}_i \quad \text{with } \dot{S}_i = \frac{R_s}{L_d} \cdot i_d - \frac{p \cdot L_q}{L_d} \cdot \Omega \cdot i_q - \frac{1}{L_d} \cdot V_d.$$

Then define  $V_d = u_{d,eq} + v_{d,n}$ , with

$$\begin{cases} u_{d,eq} = R_s \cdot i_d - p L_q \cdot \Omega \cdot i_q \\ v_{d,n} = -L_d u_n \text{ avec } u_n = -K_d \text{Signe}(S_i), \quad K_d > 0. \end{cases}$$

And after feedback, we finally get for the synchronous machine a Lyapunov function  $V = V_p + V_i$

such that:

$$\dot{V} = -K_q S_p \text{Signe}(S_p) - K_d S_i \text{Signe}(S_i) \leq 0.$$

### Practical implementation of SM of order 1

- Chattering: softening the Sign function
- Robustness tests with respect to parametric variations.

## 9.7 Speed and flux control of an asynchronous machine (Induction machine) \*

For an asynchronous machine the  $\alpha\beta$  two-phase model is:

$$\dot{x} = f(x) + gu + \xi \quad \text{where } x = [\Omega, \varphi_{r\alpha}, \varphi_{r\beta}, i_{s\alpha}, i_{s\beta}]^T, \quad u = [u_{s\alpha}, u_{s\beta}]^T,$$

and  $\xi$  is a perturbation (load torque, ...).

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \end{bmatrix} = \begin{bmatrix} (pM_{sr}/JL_r)(\varphi_{r\alpha}i_{s\beta} - \varphi_{r\beta}i_{s\alpha}) - (f_v/J)\Omega \\ -(R_r/L_r)\varphi_{r\alpha} - p\Omega\varphi_{r\beta} + (R_r/L_r)M_{sr}i_{s\alpha} \\ +p\Omega\varphi_{r\alpha} - (R_r/L_r)\varphi_{r\beta} + (R_r/L_r)M_{sr}i_{s\beta} \\ (M_{sr}/\sigma L_s L_r)((R_r/L_r)\varphi_{r\alpha} - p\Omega\varphi_{r\beta}) - \gamma i_{s\alpha} \\ (M_{sr}/\sigma L_s L_r)((R_r/L_r)\varphi_{r\beta} - p\Omega\varphi_{r\alpha}) - \gamma i_{s\beta} \end{bmatrix}, \quad g = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/\sigma L_s & 0 \\ 0 & 1/\sigma L_s \end{bmatrix}, \quad \xi = \begin{bmatrix} -T_l/J \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_s$  et  $R_r$  are the stator and rotor resistances.  $L_s$  and  $L_r$  are the stator and rotor inductances.  $M_{sr}$  is the mutual inductance between stator and rotor.  $J$  is the inertia of the system (motor + load),  $p$  is the number of pole pairs,  $f_v$  is the coefficient of viscous friction and  $T_l$  is the load torque. The parameters  $\sigma$  (coefficient of dispersion) and  $\gamma$  are defined by:

$$\sigma := 1 - \frac{M_{sr}^2}{L_s L_r}, \quad \gamma := \frac{L_r^2 R_s + M_{sr}^2 R_r}{\sigma L_s L_r^2}.$$

\* Sliding Modes Control of the Induction Motor: a Benchmark Experimental Test, A. Glumineau, L. C. De Souza Marques, and R. Boisliveau, Book Sliding Mode Control in Engineering, Ed. Marcel Dekker, 2002, ISBN ISBN 0-8247-0671-4 .

### Sliding Mode Control Design

The outputs to control are  $y_1 = \Omega$  the rotor speed and  $y_2 = \varphi^2 = \varphi_{r\alpha}^2 + \varphi_{r\beta}^2$  the flux squared. Note that for  $y_1$  and  $y_2$  the relative degrees are 2. With respect to the previous SM control introduction, the selected sliding surfaces are defined by:

$$S_O = \dot{\Omega}_{ref} - \dot{y}_1 - l_1(y_{1,ref} - y_1) = \dot{\Omega}_{ref} - \dot{\Omega} - l_1(\Omega_{ref} - \Omega) \quad \text{for the speed tracking}$$

$$S_F = (\dot{y}_{2,ref} - \dot{y}_2) - l_2(y_{2,ref} - y_2) = (\dot{\varphi}_{ref}^2 - \dot{\varphi}^2) - l_2(\varphi_{ref}^2 - \varphi^2) \quad \text{for the flux tracking}$$

So we can write the dynamics of the pseudo-output  $S_O(x, t)$  as :

$$\dot{S}_O = \ddot{\Omega}_{ref} - \ddot{y}_1 - l_1(\dot{\Omega}_{ref} - \dot{\Omega}).$$

If the load torque is not taken into account by considering it as an unknown disturbance, the

$$\text{equation takes the form } \begin{cases} \dot{S}_O(x, u, t) & = \ddot{\Omega}_{ref} - \dot{f}_1(x, u) - l_1(\dot{\Omega}_{ref} - \dot{f}_1(x)) \\ & := S_{O1}(x, t) + S_{O21}(x)u_{s\alpha} + S_{O22}(x)u_{s\beta} \end{cases}$$

With the same technique, we obtain for the dynamics of the pseudo-output  $S_F(x, t)$ :

$$\begin{aligned} \dot{S}_F(x, u, t) = & \ddot{\varphi}_{ref}^2 - 2(\varphi_{r\alpha}\dot{f}_2(x, u) + (f_2(x))^2 + \varphi_{r\beta}\dot{f}_3(x, u) + (f_3(x))^2) \\ & - l_2(\dot{\varphi}_{ref}^2 - 2(\varphi_{r\alpha}f_2(x) + \varphi_{r\beta}f_3(x))) \\ \dot{S}_F(x, u, t) = & S_{F1}(x, t) + S_{F21}(x)u_{s\alpha} + S_{F22}(x)u_{s\beta}. \end{aligned}$$

Thus the control is written as :

$$u = \begin{bmatrix} u_{s\alpha} \\ u_{s\beta} \end{bmatrix} = \begin{bmatrix} S_{O21}(x) & S_{O22}(x) \\ S_{F21}(x) & S_{F22}(x) \end{bmatrix}^{-1} \begin{bmatrix} -S_{O22}(x, t) \\ -S_{F22}(x, t) \end{bmatrix} - \begin{bmatrix} k_1 * \text{signe}(S_O) \\ k_2 * \text{signe}(S_F) \end{bmatrix}$$

where the  $k_j$  are the switching gains.

## 9.8 Sliding Modes Control of order 2 (Twisting Algorithm principle)

### 9.8.1 Goal

- Reduce the switching of the control in case of Sliding Modes Control of order 1 (Chattering)
- Improve sliding accuracy (i.e. to limit the switching zone around the sliding surface).

### 9.8.2 Way

To obtain by feedback  $S = 0$  and  $\dot{S} = 0$  with  $S(x, t) = \sum_{i=0}^{r-1} \lambda_i \cdot (y_{ref} - y)^{(i)}$  where  $y_{ref}$  is the reference trajectory (i.e. S is a pseudo output of relative degree one).

Let the candidate Lyapunov function  $V(x, t) = \frac{1}{2}S^2 + \frac{1}{2}\dot{S}^2$ .

To have an asymptotic stability the Lyapunov conditions are: 
$$\begin{cases} V(0) = 0, \\ V(x, t) > 0, x \neq 0 \\ \dot{V}(x, t, u) < 0, x \neq 0 \end{cases}$$

Its time derivative is

$$\dot{V}(x, t, u) = S \cdot \dot{S} + \dot{S} \cdot \dot{S} = \dot{S} \cdot (S + \dot{S}).$$

In a **first step**, the control  $u$  can be calculated as for Sliding Modes of order 1:

$$\begin{aligned}\dot{S} &= S_1(x, t) + S_2(x, t).u \\ u &= u_{eq} + v_n.\end{aligned}$$

Thus,  $u_{eq} = -\frac{S_1(x,t)}{S_2(x,t)}$  and  $v_n = \frac{u_n}{S_2(x,t)}$  with  $u_n$  is a new control to finally reach:

$$S \rightarrow 0, \frac{dS}{dt} \rightarrow 0 \text{ i.e. } \frac{dV}{dt} < 0.$$

After applying the control  $u$ , it is obtained:

$$\begin{aligned}\dot{S} &= u_n \text{ and } \ddot{S} = \dot{u}_n \\ \text{i.e. } \dot{V} &= u_n(S + \dot{u}_n).\end{aligned}$$

Or in equivalent ways the stabilization of the closed loop system is obtained by:

$$\begin{aligned}S + \dot{u}_n &\leq 0 \text{ if } u_n > 0 \\ S + \dot{u}_n &> 0 \text{ if } u_n \leq 0.\end{aligned}$$

In the first order Sliding Modes method,  $u_n$  is a discontinuous function of type

$$u_n = -k \cdot \text{Signe}(S).$$

Here, it is considered a discontinuity on  $\dot{u}_n$  with a limitation of  $u_n$  to 1:

$$\begin{aligned}\dot{u}_n &= -\alpha_M \text{Signe}(S) \quad \text{if } |u_n| \leq 1 \text{ and } S \cdot u_n > 0, \text{ with } \alpha_M > |S| \\ \dot{u}_n &= -\alpha_m \text{Signe}(S) \quad \text{if } |u_n| \leq 1 \text{ and } S \cdot u_n < 0, \text{ with } \alpha_m < |S| \\ \dot{u}_n &= -u_n \quad \text{if } |u_n| > 1\end{aligned}$$

This control allows to satisfy the condition  $\dot{V}(x, u) \leq 0$ .

The choice of these two positive constants  $\alpha_m$  and  $\alpha_M$  define the trajectory of convergence towards the surfaces

$$S = 0 \text{ et } \dot{S} = 0.$$

### 9.8.3 Phase plane

Whatever the quadrant,

$$\dot{V}(x, u) \leq 0$$

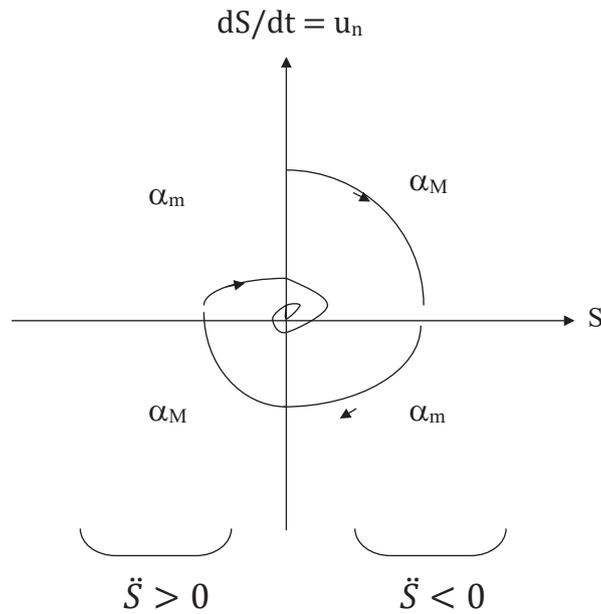


Figure 5. Phase Plane for SM of order 2

An alternative to the calculation of the equivalent control is possible with the original Twisting Algorithm.

### 9.8.4 Original Twisting Algorithm (Levant Int. Journal of Control, 1993)

The control is:

$$\dot{u} = \begin{cases} -u & |u| > 1 \\ -\alpha_M \text{Signe}(S) & |u| \leq 1, \quad S \cdot v_n > 0, \quad \alpha_M \geq |S| \\ -\alpha_m \text{Signe}(S) & |u| \leq 1, \quad S \cdot v_n \leq 0, \quad 0 < \alpha_m \leq |S| \end{cases}$$

with the two positive constants  $\alpha_m$  and  $\alpha_M$  that must check the following constraints :

$$\alpha_M > \alpha_m > 0$$

$$\alpha_m > \frac{4\Gamma_M}{s_0}$$

$$\alpha_M > \frac{\Phi}{\Gamma_m}$$

$$\Gamma_m \alpha_M - \Phi > \Gamma_M \alpha_m + \Phi \quad \text{where} \quad \begin{cases} |S| \leq s_0 \\ 0 < \Gamma_m \leq \left| \frac{\partial \dot{S}}{\partial u} \right| \leq \Gamma_M \\ \left| \frac{\partial \dot{S}}{\partial t} + \frac{\partial \dot{S}}{\partial x} \dot{x} \right| \leq \Phi \end{cases}$$

Remark: the constants  $\alpha_m$  and  $\alpha_M$  are thus defined according to the maximum variations of  $S$  and its derivatives, which depends on the trajectories, so the first method is simpler.

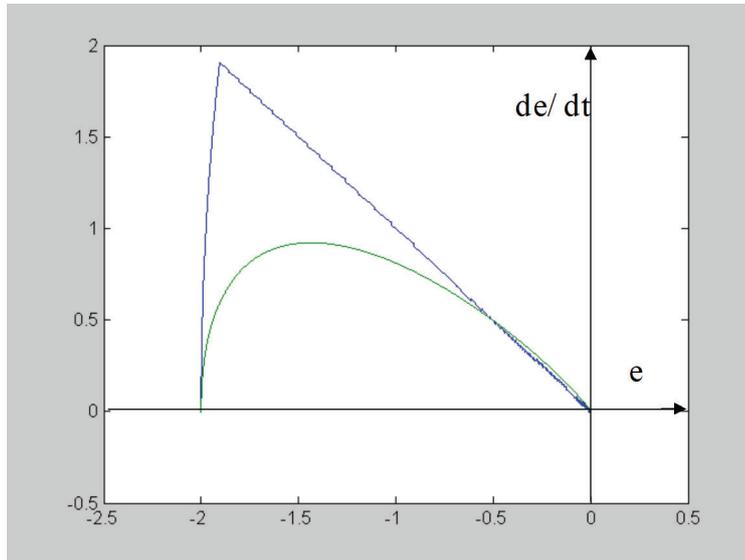


Figure 6. Trajectory in the phase plane: SM of order 1 (blue) and order 2 (green) comparison.

**Exercise 2.** SM of order 2  $\dot{x}_1 = x_1 x_2$ ,  $\dot{x}_2 = u$ ,  $y = x_1$

### 9.8.5 Application to the permanent magnet synchronous machine.

From the exposed method for the sliding modes of order 1, it can be defined:

$$\begin{aligned}
 e_p &= \theta_{ref} - \theta \\
 S_p &= \lambda_0 \cdot e_p + \lambda_1 \cdot \dot{e}_p + \ddot{e}_p \\
 \dot{S}_p &= S_1 + S_2 \cdot V_q \\
 \left\{ \begin{array}{l}
 S_1 = -\lambda_0 \cdot (\Omega - \dot{\theta}_{ref}) + \lambda_1 \cdot \ddot{\theta}_{ref} + \left( \frac{f_v}{J} - \lambda_1 \right) \cdot \left[ \frac{p}{J} \cdot [(L_d - L_q) \cdot i_d + \varphi_f] \cdot i_q - \frac{f_v}{J} \cdot \Omega \right] \\
 \quad + \frac{p}{J} \cdot [(L_d - L_q) \cdot i_d + \varphi_f] \cdot \left( \frac{p \cdot \varphi_f}{L_q} \cdot \Omega + \frac{p \cdot L_d}{L_q} \cdot \Omega \cdot i_d + \frac{R_s}{L_q} \cdot i_q \right) \\
 S_2 = -\frac{p}{J \cdot L_q} \cdot [(L_d - L_q) \cdot i_d + \varphi_f]
 \end{array} \right.
 \end{aligned}$$

Let the candidate function of Lyapunov  $V_1 = \frac{1}{2}S_p^2 + \frac{1}{2}\dot{S}_p^2$ .

Let  $V_q$  control defined by  $V_q = u_{q,eq} + v_{q,n}$  where:

$$\begin{cases} u_{q,eq} = -\frac{S_1}{S_2} \\ v_{q,n} = \frac{1}{S_2} \cdot u_{q,n} \end{cases}$$

$$\text{with } \dot{u}_{q,n} = \begin{cases} -u_{q,n} & |u_{q,n}| > 1 \\ -\alpha_{Mp} \cdot \text{Signe}(S_p) & |u_{q,n}| \leq 1, \quad S_p \cdot v_{q,n} > 0, \quad \alpha_{Mp} \geq |S_p| \\ -\alpha_{mp} \cdot \text{Signe}(S_p) & |u_{q,n}| \leq 1, \quad S_p \cdot v_{q,n} \leq 0, \quad 0 < \alpha_{mp} \leq |S_p| \end{cases}$$

Identically, in order to control the current  $i_d$  :

$$\begin{aligned} S_i &= -i_d \\ \dot{S}_i &= \frac{R_s}{L_d} \cdot i_d - \frac{p \cdot L_q}{L_d} \cdot \Omega \cdot i_q - \frac{1}{L_d} \cdot V_d \end{aligned}$$

Let's consider Lyapunov's candidate function  $V_2 = V_1 + \frac{1}{2}S_i^2 + \frac{1}{2}\dot{S}_i^2$ .

The control  $V_d$  is then  $V_d = u_{d,eq} + v_{d,n}$  where:

$$\begin{cases} u_{d,eq} = R_s \cdot i_d - pL_q \cdot \Omega \cdot i_q \\ v_{d,n} = -L_d \cdot u_{d,n} \end{cases}$$

with

$$\dot{u}_{d,n} = \begin{cases} -u_{d,n} & |u_{d,n}| > 1 \\ -\alpha_{Mi} \cdot \text{Signe}(S_i) & |u_{d,n}| \leq 1, \quad S_i v_{d,n} > 0, \quad \alpha_{Mi} \geq |S_i| \\ -\alpha_{mi} \cdot \text{Signe}(S_i) & |u_{d,n}| \leq 1, \quad S_i v_{d,n} \leq 0, \quad 0 < \alpha_{mi} \leq |S_i| \end{cases}$$

## II.2. Application

Gains :  $\alpha_{Mi} = 3$ ;  $\alpha_{mi} = 1$   $\alpha_{Mp} = 3$ ;  $\alpha_{mp} = 1$

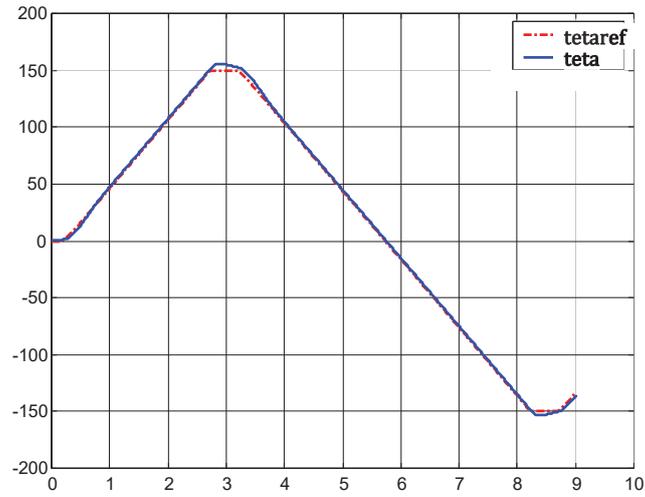


Figure 7. tetaref (rad) and teta (rad) wrt time (s)

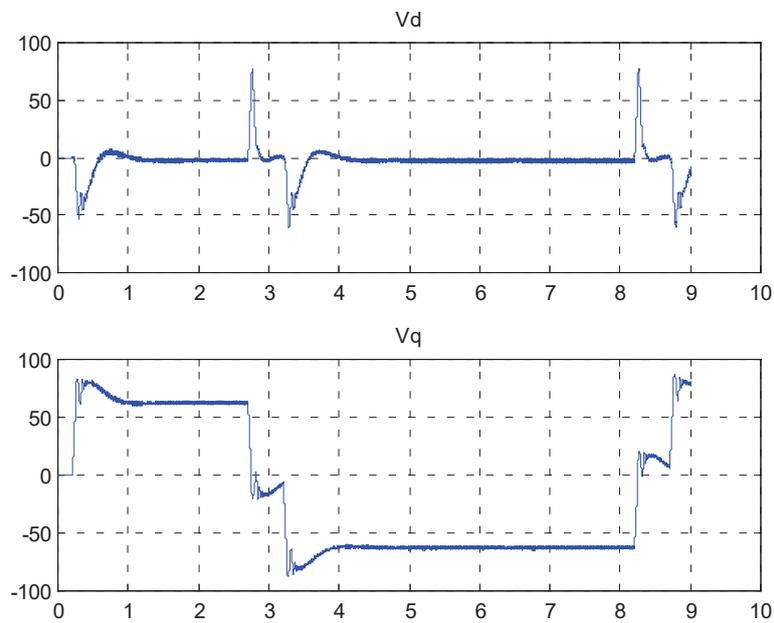


Figure 8.  $V_d$  (V),  $V_q$  (V) wrt time (s)

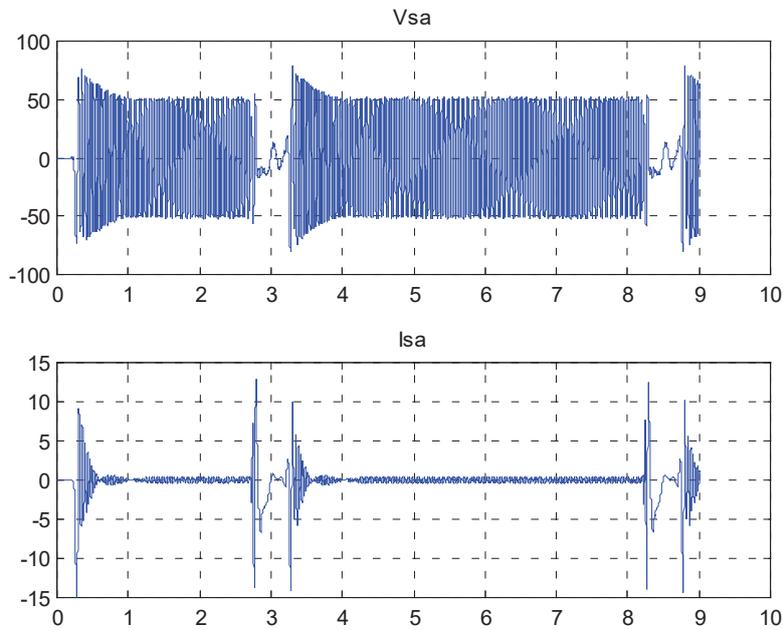


Figure 9. Vsa (V), Isa (A) wrt time (s)

**Comments:** Very satisfactory performance in position control. Moreover, chattering is almost totally eliminated. Controls remain acceptable despite a few peaks in Isa above 10.8 A.

### Robustness and disturbance rejection tests

#### Rejection of load torque disturbances:

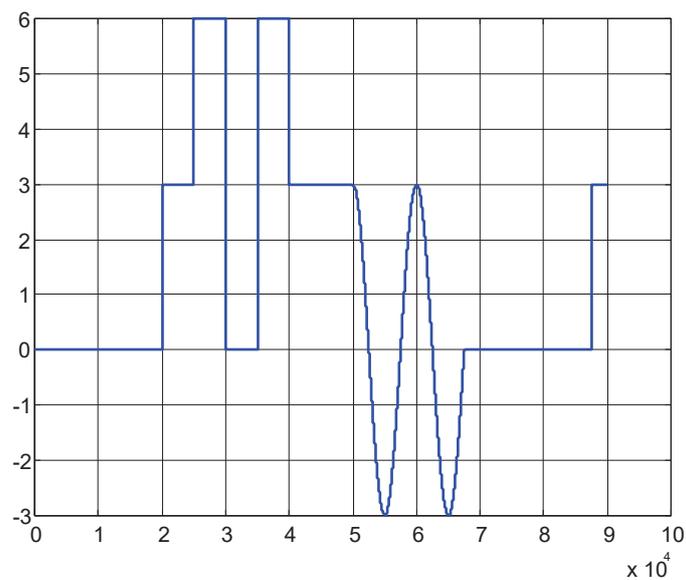


Figure 10. Load torque disturbance CL(Nm) wrt time (s)

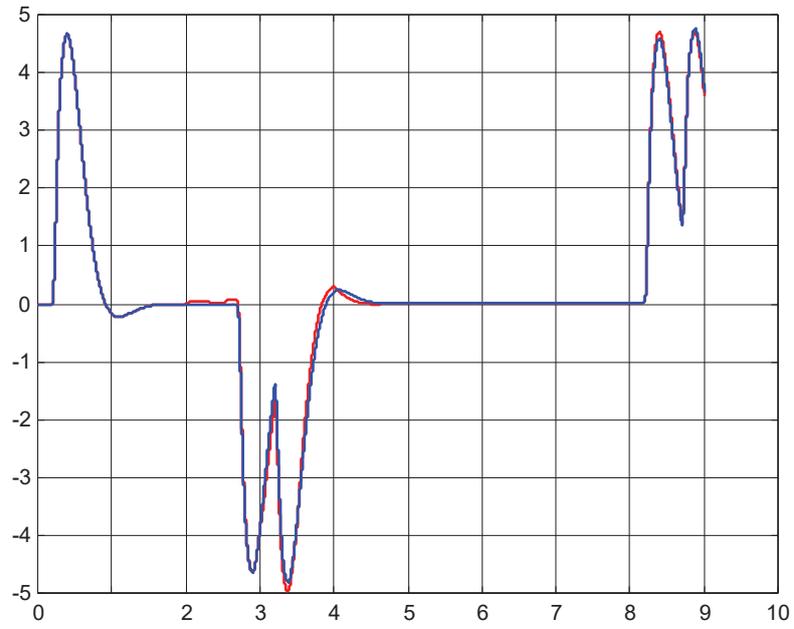


Figure 11. Angle error (rad, blue) and disturbed angle error (rad, red) wrt time (s).

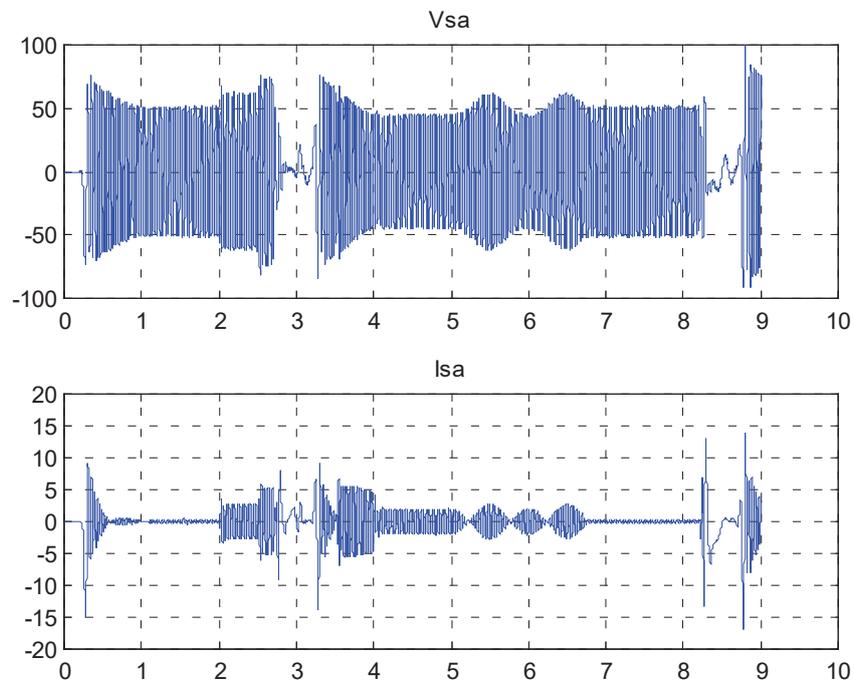


Figure 12.  $V_{sa}$  (V) and  $I_{sa}$  (A) wrt time (s)

**Comments:** Disturbance rejection is almost perfect; however, there are larger peaks in  $I_{sa}$ .

### Robustness tests :

First the resistors are varied by +50% and the inductors by +20%. The result is:

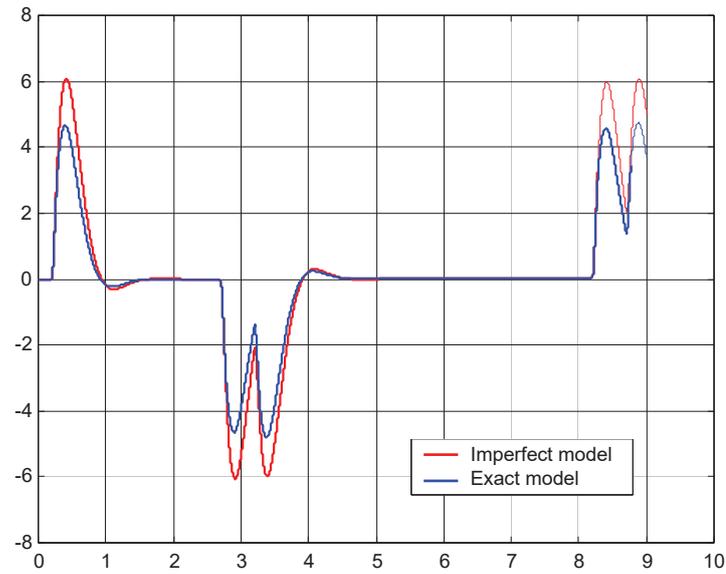


Figure 13. Angle error (rad) versus time (s).

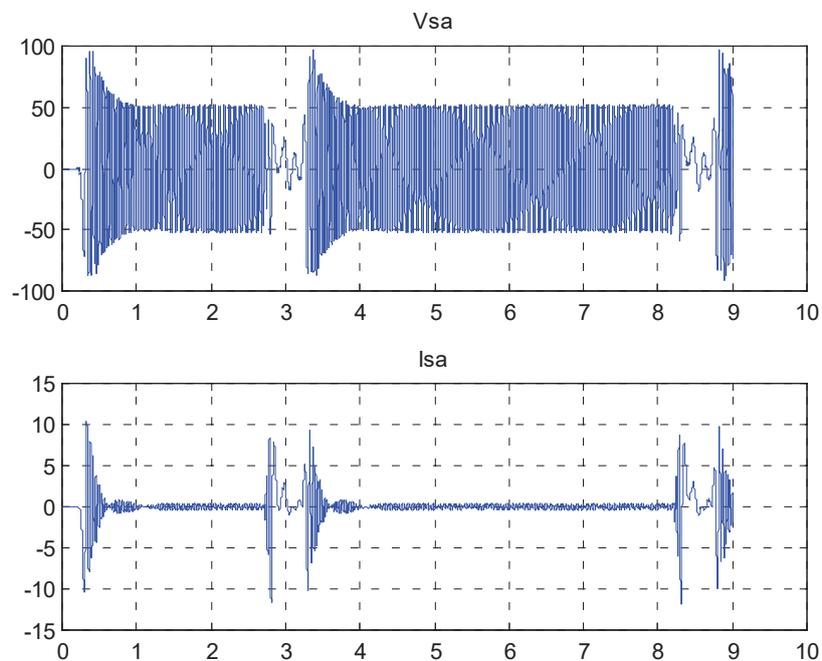


Figure 14.  $V_{sa}$  (V) and  $I_{sa}$  (A) versus time (s).

**Comments:** This control is very robust even if the errors in the model cause position deviation peaks to be increased.

## Comparison of order 1 and order 2 sliding modes

For the two methods compared, draw the following 3D parametric curve:

$$X(t) = Ep(t) := Ep, \text{ (position error)}$$

$$Y(t) = \frac{d}{dt}(\text{position error}) := dEp$$

$$Z(t) = \frac{d^2}{dt^2}(\text{position error}) := d2Ep$$

as well as the projections in the  $Z=0$  and  $X=0$  plane.

The curve corresponding to method 1 ( order1 + Sign'' ) is drawn in **green** and that of method 2 (order2) in **red**.

The plotted part corresponds to the response to the second ramp.

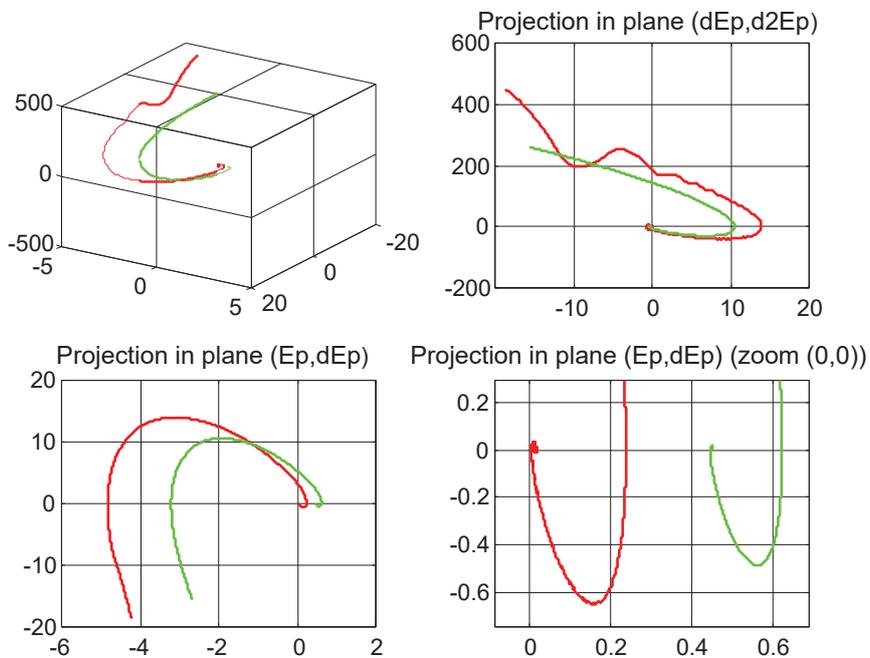


Figure 15. Error planes responses:  $Ep$ ,  $\frac{d}{dt}(\text{error}) := dEp$  and  $\frac{d^2}{dt^2}(\text{error}) := d2Ep$ .

## Conclusion Sliding Mode of order 2 applied to the synchronous machine

It can be noticed that only a little chattering is present on the controls.  
The performance goal is obtained.

### 9.8.6 Conclusion SM order 2 :

- Finite time pour  $S$ ,  $\dot{S}$ , robustness and precision without "chattering"!
- What about sliding modes higher than two? ( $S = 0$ ,  $\dot{S} = 0$ ,  $\ddot{S} = 0, \dots$ ) ?

**Yes it exists** (precision proportional to the order).

Experimental results: Sensorless control of a MAS by higher-order sliding modes (IFAC08).

- Complementary technique: "Integral Sliding mode": the initial surface "passes" through the initial conditions of the system and is modified to go to the "objective surface".
- others development: to automatically compute the gain of the discontinuous control (real time) i.e. Adaptive Sliding Mode Control.

# CHAPTER 10

## ROBUST CONTROL VIA BACKSTEPPING

### 10.1. Introduction to Backstepping Control

**Principle:** the Backstepping Control is a recursive methodology for the construction of robust nonlinear control laws from Lyapunov functions. The reference documents on Backstepping are [KANE92], [KRST95], [SEPU97]; see also [FOSS98].

**Goal:** To establish Robust Control Laws with respect to the "quality" of the model and / or to the bad knowledge of the parameters. This method is complementary to the deterministic nonlinear controls (Linearization, Decoupling Control, Trajectory tracking, ...).

**How:** The main property of the Backstepping Control is to make the looped systems equivalent to stable cascading subsystems of order 1 within the meaning of Lyapunov functions.

The control is therefore based on criteria of Lyapunov stability with respect to an equilibrium point or a reference to follow.

**Note 1:** This objective can be obtained by controlling only the "bad nonlinearities" i.e. by limiting the constraints on the control input.

**Note 2:** The structure of the system must allow this recursive methodology (usually the case): output controllability condition.

## 10.2. Integrator Backstepping Method (basic method)

Consider system  $\Sigma$ :

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + x_2 \\ \dot{x}_2 &= u \\ y &= x_1\end{aligned}$$

The purpose of the control will be to force  $y$  to 0 when  $t \rightarrow \infty$  and to make the equilibrium point "Globally Asymptotically Stable" or "Globally Exponentially Stable".

The only equilibrium point with  $y = 0$  is:  $(x_1, x_2) = (0, -f(0))$   
corresponding to  $\dot{x}_1 = f(x_1) + x_2 = 0$ .

**Preliminary notation:** Change of coordinates:

$$z = \Phi(x) := [z_1, z_2]^T$$

where  $\Phi$  is a diffeomorphism (invertible with  $\Phi(x)$  and  $\Phi^{-1}(x)$  differentiable).

### Integrator Backstepping Algorithm

#### Step 1.

Note the error  $z_1 = x_1 - 0$ , and consider a subsystem defined by the first equation:

$$\Sigma_1 \quad \dot{z}_1 = f(z_1) + x_2$$

Consider :

- $x_2$  as the "control" of this subsystem (a pseudo or virtual control)
- the candidate Lyapunov function  $V_1(z_1) = \frac{1}{2}z_1^2$ , then  $\dot{V}_1 = \dot{z}_1 z_1 = (f(z_1) + x_2)z_1$ .

To stabilize the subsystem  $\Sigma_1$ , a sufficient condition is that  $V_1$  is a Lyapunov function. For this, just take as a control

$$x_2 = -f(z_1) - k_1 z_1 \text{ with } k_1 > 0. \text{ Then } \dot{V}_1(z_1) = -k_1 z_1^2 !$$

But  $x_2$  is not a real control ! It is a "pseudo control". So we note:

$$x_{2d} := \alpha_1 = -f(z_1) - k_1 z_1 \text{ the desired control to stabilize the dynamics of } \Sigma_1$$

and we note the error  $z_2 = x_2 - x_{2d}$  between  $x_2$  and the desired control.

$$\longrightarrow x_2 = z_2 + x_{2d} = z_2 + \alpha_1$$

Then  $\Sigma_1 \quad \dot{z}_1 = f(z_1) + \alpha_1 + z_2$  and  $\dot{V}_1 = (f(z_1) + \alpha_1 + z_2)z_1 = -k_1 z_1^2 + z_1 z_2$  (\*)  
 with  $k_1$  the closed loop gain of dynamics  $\Sigma_1$ . The control term is now  $z_2$  whose dynamics are

$$\dot{z}_2 = \dot{x}_2 - \dot{x}_{2d} = u - \dot{x}_{2d} \quad (\text{and } \Sigma_1 \quad \dot{z}_1 = f(z_1) + z_2)$$

**Step 2.**

Consider the function  $V_2(z) = V_1(z_1) + \frac{1}{2}z_2^2$  (stacked Lyapunov functions).

then with (\*)

$$\dot{V}_2 = \dot{V}_1 + \dot{z}_2 z_2 = -k_1 z_1^2 + z_2(u - \dot{x}_{2d} + z_1) \text{ and } \dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = u - \dot{x}_{2d}$$

To stabilize the complete system (i.e.  $V_2$  is a Lyapunov function), it is sufficient to take as a control

$$u = \dot{x}_{2d} - z_1 - k_2 z_2 \text{ with } k_2 > 0. \text{ Then, } \dot{V}_2 = -k_1 z_1^2 - k_2 z_2^2 < 0 \text{ for } z_1 \neq 0, z_2 \neq 0.$$

**The z dynamics of the closed loop system are stable:  $y \longrightarrow 0$**

**Note 3:** Computation of  $\dot{x}_{2d}$

To avoid temporal derivatives terms in the control, one can express the temporal derivatives according to the dynamics of the state:

$$\text{As } x_{2d} = -f(z_1) - k_1 z_1, \dot{x}_{2d} = -\frac{\partial f(x_1)}{\partial x_1} \dot{x}_1 - k_1 \dot{x}_1 = -\left(\frac{\partial f(x_1)}{\partial x_1} + k_1\right) \cdot [f(x_1) + x_2].$$

**Note 4:** The closed loop system is made up of stable "cascading" subsystems.

**Exercise 1:** for  $f(x_1) = -x_1$  (linear case), in original coordinates, the integral backstepping algorithm leads to the following control law:

$$u = -(2 + k_1 k_2 + k_1 + k_2)x_1 - (k_1 + k_2 + 1)x_2 := -K_1 x_1 - K_2 x_2$$

(Poles placement, ...).

### 10.3. Backstepping coordinates transformation

The new coordinates are the differences between the references (output or virtual controls) and their values.

$$z = \Phi(x), \quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - f(x_1) - k_1 x_1 \end{bmatrix}$$

$$x = \Phi^{-1}(z), \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 + f(z_1) + k_1 z_1 \end{bmatrix}$$

The dynamics written in the new coordinates  $(z_1, z_2)$  are

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = - \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\text{i.e. } \dot{z} = -Kz + Sz$$

with  $K$  positive definite diagonal matrix and  $S$  antisymmetric matrix ( $S = -S^T$  and  $z^T S z = 0$ ).

### 10.4. Generalization: Backstepping closed loop stability study

With these notations and for  $n$  dynamics, the general Lyapunov function  $V_n$  is written

$$V_n = \frac{1}{2} z^T z$$

thus

$$\dot{V}_n = \frac{1}{2} \frac{d}{dt} (z^T z) + \frac{1}{2} z^T (-Kz + Sz)$$

$$\frac{d}{dt} (z^T) = \frac{d}{dt} (z)^T = (-Kz + Sz)^T = -z^T K^T + z^T S^T$$

$$\dot{V}_n = \frac{1}{2} [-z^T K^T z + z^T S^T z] - \frac{1}{2} z^T K z + \frac{1}{2} z^T S z$$

$$\text{with } K \text{ is diagonal} \Rightarrow K^T = K, \text{ and } S = -S^T$$

$$\Rightarrow \dot{V}_n = -z^T K z < 0 \text{ with } K \text{ diagonal} > 0.$$

so the equilibrium point of the closed loop system is **globally asymptotically stable** (GAS).

## 10.5. Integral Backstepping and linearizing control.

**Initial note:** The integral backstepping method is a linearizing control since the objective is to exactly "compensated" for the nonlinear function  $f(x_1)$  but this implies:

- a perfect knowledge of  $f(x_1)$
- a control sensitivity to modeling errors
- a maximum energy penalization of the control

**Solution:** the Backstepping gives the possibility of exploiting the "good" non-linearities ".

Example: consider

$$f(x_1) = -a_0 \cdot x_1 - a_1 \cdot x_1^2 - a_2 \cdot |x_1| \cdot x_1$$

where  $a_0$ ,  $a_1$  et  $a_2$  are positive constants,

→ the only destabilizing term in the expression of  $f(x_1)$  is  $a_1 x_1^2$ .

→ to stabilize the system, it is therefore sufficient to eliminate this term with,

$$\text{for example: } x_{2d} = a'_1 z_1^2 - k_1 z_1 \quad \text{with } a'_1 := a_1.$$

→ the control law is simplified and "minimized": the control is only "used" for stabilization.

→ In addition, it is no longer necessary to perfectly know  $a_0$  and  $a_2$ .

→ Moreover, to "desensitize" the command of the knowledge of  $a_1$ , one can add a stabilizing

nonlinear damping overstating the term  $a_1 x_1^2$  →  $a'_1 > a_{1max} > a_1$ .

→ More robustness.

### 10.6. Another solution: Backstepping Control with a nonlinear damping.

For the example above, the destabilizing term  $a_1 x_1^2$  (with  $a_1 < a_{1max}$ ) can be "dominated" by adding a nonlinear damping term proportional to  $x_1^3$ :

$$x_{2d} = -[k_1 + \eta_1(z_1)]z_1$$

with  $k_1 > 0$  and

$$\eta_1(z_1) = \beta_1 z_1^2 \text{ avec } \beta_1 > 0$$

So,

$$\begin{aligned} \dot{z}_1 &= f(z_1) + x_{2d} + z_2 \\ &= -a_0 z_1 - a_1 z_1^2 - a_2 |z_1| z_1 - (k_1 + \beta_1 z_1^2) z_1 + z_2 \\ &= -(a_0 + a_2 |z_1| + k_1) z_1 - a_1 z_1^2 - \beta_1 z_1^3 + z_2 \end{aligned}$$

The calculation of the Lyapunov function for subsystem 1 gives:

$$V_1(z_1) = \frac{1}{2} z_1^2$$

$$\dot{V}_1 = -(a_0 + a_2 |z_1| + k_1) z_1^2 - a_1 z_1^3 - \beta_1 z_1^4 + z_1 z_2$$

The time derivative of the candidate Lyapunov function  $V_2(z) = V_1(z_1) + \frac{1}{2} z_2^2$  is:

$$\dot{V}_2 = -(a_0 + a_2 |z_1| + k_1) z_1^2 - a_1 z_1^3 - \beta_1 z_1^4 + z_2 (z_1 + u - \dot{\alpha}_1)$$

By taking  $u = \dot{x}_{2d} - k_2 z_2 - z_1$ , with  $k_2 > 0$  then

$$\dot{V}_2 = -(a_0 + a_2 |z_1| + k_1) z_1^2 - a_1 z_1^3 - \beta_1 z_1^4 - k_2 z_2^2,$$

$$\dot{V}_2 = -a_2 |z_1| z_1^2 - k_2 z_2^2 - z_1^2 (a_0 + k_1 + a_1 z_1 + \beta_1 z_1^2).$$

As  $a_2 > 0, \beta_1 > 0, k_2 > 0$ , by choosing the gain  $k_1 > \frac{a_{1max}^2}{4\beta_1} - a_{0min}$  implies  $\dot{V}_2 \leq 0$ .

Thus the controller does not require the exact knowledge of the parameters of the model  $a_0, a_1$  and  $a_2$ .

## 10.7. Integral Backstepping: mismatched perturbation rejection

In the case where unknown terms bounded and described by piecewise constant functions (or with weak temporal variations compared to the dynamics of the system) come to disturb the behavior of the system, it is possible to reject these classes of disturbances.

**Principle:** Let a system 
$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 + b_1 & y &= x_1 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u + b_2\end{aligned}$$

with  $b_1$  and  $b_2$  unknown bounded disturbances that are piecewise constant.

$b_1$  is a mismatched perturbation. Only  $b_2$  is directly "accessible" by the control (matching condition). The standard calculation of the Backstepping Control gives:

**Step 1** Let  $z_1 = x_1 - x_{1ref}$ ,  $\Sigma_1 \quad \dot{z}_1 = -\dot{x}_{1ref} + f_1(x_1) + g_1(x_1)x_2 + b_1$ .

with  $x_{2d} = g_1(x_1)^{-1} [\dot{x}_{1ref} - f_1(x_1) - k_1 z_1]$  and  $z_2 = x_2 - x_{2d}$

then  $\Sigma_1 \quad \dot{z}_1 = -k_1 z_1 + g_1(x_1)z_2 + b_1$ .

Because of the disturbance  $b_1$ , the convergence of  $z_1$  is no longer assured when  $z_2$  is forced to 0 by step 2 of the Backstepping !

**Solution: add an integral term in  $z_2$  (i.e an observer of the perturbation)**

$$z_2 = x_2 - x_{2d} + \frac{1}{T_{i1}} \int_0^t z_1 d\tau, \text{ so } x_2 = z_2 + x_{2d} - \frac{1}{T_{i1}} \int_0^t z_1 d\tau .$$

Then  $\Sigma_1 \quad \dot{z}_1 = -\frac{1}{T_{i1}} z_1 + g_1(x_1)[z_2 - \frac{1}{T_{i1}} \int_0^t z_1 d\tau] + b_1$

If Backstepping step 2 forces  $z_2$  to zero, then the integral term makes it possible to reject  $b_1$  that is not "directly accessible" by the input (mismatched disturbance).

**Step 2**

$$\Sigma_2 \quad \dot{z}_2 = \dot{x}_2 - \dot{x}_{2d} + \frac{1}{T_{i1}} z_1 = -\dot{x}_{2d} + \frac{1}{T_{i1}} z_1 + f_2(x_1, x_2) + g_2(x_1, x_2)u + b_2 .$$

From where a control

$$u = g_2(x_1, x_2)^{-1} \left[ \dot{x}_{2d} - \frac{1}{T_{i1}} z_1 - k_2 z_2 - f_2(x_1, x_2) - \frac{1}{T_{i2}} \int_0^t z_2 d\tau \right]$$

gives the dynamics:  $\Sigma_2 \quad \dot{z}_2 = -k_2 z_2 - \frac{1}{T_{i2}} \int_0^t z_2 d\tau + b_2$

with the integral term which makes it possible to reject  $b_2$  a disturbance directly "accessible" by the input (matched disturbance).

## 10.8. Conclusion

A robust controller is obtained with the Backstepping approach

- complementary to exact linearization requiring exact knowledge of the model.
- The basic method presented above can be generalized to cascading systems (Nonlinear block Backstepping) and to systems with bounded unknown nonlinearities (Nonlinear Damping or Adaptive Backstepping).

**Example:** Speed control of the induction motor (reduce model)

## 10.9. Example: Application to a synchronous machine with permanent magnets (European CRAFT contract)

The application of the Integrator Backstepping method to the permanent magnet synchronous motor was carried out in [KRST95], [GLUM15] in order to control the motor position.

The motor model state equations are established in the dq reference frame linked to the rotor position:

$$\dot{x} = f(x) + g(x)u$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{\Omega} \\ \dot{i}_d \\ \dot{i}_q \end{bmatrix} = \begin{bmatrix} \frac{p}{J}((L_d - L_q)i_d + \Phi_f)i_q - \frac{f_v}{J}\Omega - \frac{C_l}{J} \\ -\frac{R_s}{L_d}i_d + p\frac{L_q}{L_d}\Omega i_q \\ -p\frac{\Phi_f}{L_q}\Omega - p\frac{L_d}{L_q}\Omega i_d - \frac{R_s}{L_q}i_q \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_d} & 0 \\ 0 & \frac{1}{L_q} \end{bmatrix} \begin{bmatrix} V_d \\ V_q \end{bmatrix}$$

with  $\theta$  the rotor position;  $\Omega$  the rotor speed;  $i_d, i_q$  the stator currents in the dq frame,  $V_d, V_q$  the voltage control inputs,  $R_s$  the stator resistance,  $L_d, L_q$  the stator inductances,  $f_v, J$  the viscous damping and the inertia of the motor-load system.  $C_l$  is the unmeasured load torque (i.e. a bounded unknown disturbance). The reference position path is noted  $\theta_{\text{ref}}$ .

**Step 1:** Note the difference between the position of the motor and its reference:

$$z_1 = \theta - \theta_{\text{ref}} = \xi_1 - \alpha_0, \text{ with } \xi_1 = \theta \text{ and } \alpha_0 = \theta_{\text{ref}}.$$

By differentiation and using the backstepping symbols, we get:

$$\begin{aligned} \dot{z}_1 &= \dot{\xi}_1 - \dot{\alpha}_0 \\ &= \xi_2 + \beta_1 \end{aligned}$$

with  $\xi_2 = \dot{\xi}_1 = \Omega$  and  $\beta_1 = -\dot{\alpha}_0$ .

Consider the candidate Lyapunov function  $V_1 = \frac{1}{2}z_1^2$ .

Then  $\dot{V}_1 = z_1 \cdot (\xi_2 + \beta_1)$ . If  $\xi_2$  was a real control input, we would take

$$\xi_2 = -\omega_1 \cdot z_1 - \beta_1 \omega_1 > 0$$

to obtain  $\dot{V}_1 \leq 0$ .

Since this is not the case, note  $\alpha_1 = -\omega_1 \cdot z_1 - \beta_1$  and the error  $z_2 = \xi_2 - \alpha_1$ . Then

$$\begin{aligned} \dot{z}_1 &= z_2 - \omega_1 \cdot z_1 \\ \dot{V}_1 &= -\omega_1 \cdot z_1^2 + z_1 \cdot z_2 \end{aligned}$$

with  $z_2$  is the virtual control of the first subsystem.

**Step 2:**

By differentiation of this second error  $z_2$ :

$$\begin{aligned} \dot{z}_2 &= \dot{\xi}_2 - \dot{\alpha}_1 \\ &= \frac{p \cdot \varphi_f}{J} i_q + \frac{p}{J} (L_d - L_q) i_d i_q - \frac{f_v}{J} \Omega - \dot{\alpha}_1 \\ &= \xi_3 + \beta_2 \end{aligned}$$

with  $\xi_3 = \frac{p \cdot \varphi_f}{J} i_q$  and  $\beta_2 = \frac{p}{J} (L_d - L_q) i_d i_q - \frac{f_v}{J} \Omega - \dot{\alpha}_1$  (without load torque)  $\Omega$ .

Consider the candidate Lyapunov function  $V_2 = V_1 + \frac{1}{2}z_2^2$ . Then

$$\dot{V}_2 = -\omega_1 \cdot z_1^2 + z_2 \cdot (z_1 + \xi_3 + \beta_2).$$

If  $\xi_3$  was a real control input, we would take  $\xi_3 = -z_1 - \omega_2 \cdot z_2 - \beta_2 \omega_2 > 0$  to force  $\dot{V}_2 \leq 0$ .

Then we note  $\alpha_2 = -z_1 - \omega_2 \cdot z_2 - \beta_2$  and the error  $z_3 = \xi_3 - \alpha_2$ . This gives:

$$\begin{aligned} \dot{z}_2 &= z_3 - \omega_2 z_2 - z_1 \\ \dot{V}_2 &= -\omega_1 z_1^2 - \omega_2 z_2^2 + z_3 z_2 \end{aligned}$$

**Step 3:** By differentiation of this third error:

$$\begin{aligned}\dot{z}_3 &= \dot{\xi}_3 - \dot{\alpha}_2 \\ &= \frac{p \cdot \varphi_f}{J} \left( -\frac{p \cdot \varphi_f}{L_q} \Omega - \frac{p \cdot L_d}{L_q} \Omega i_d - \frac{R_s}{L_q} \cdot i_q \right) - \dot{\alpha}_2 + \frac{p \cdot \varphi_f}{J L_q} \cdot V_q \\ &= \beta_3 + K V_q \\ \text{with } \beta_3 &= \frac{p \cdot \varphi_f}{J} \left( -\frac{p \cdot \varphi_f}{L_q} \Omega - \frac{p \cdot L_d}{L_q} \Omega i_d - \frac{R_s}{L_q} i_q \right) - \dot{\alpha}_2 \text{ and } K = \frac{p \cdot \varphi_f}{J \cdot L_q}.\end{aligned}$$

We notice that a real command has appeared ( $V_q$ ). Consider the candidate Lyapunov function:

$$V_3 = V_2 + \frac{1}{2} \cdot z_3^2.$$

Then  $\dot{V}_3 = -\omega_1 \cdot z_1^2 - \omega_2 \cdot z_2^2 + z_3 \cdot (z_2 + \beta_3 + K \cdot V_q)$ . In order to force  $\dot{V}_3 \leq 0$ , it is sufficient to apply the control:

$$V_q = \frac{1}{K} (-z_2 - \omega_3 z_3 - \beta_3) \omega_3 > 0.$$

Thus, the first three components of the state are controlled by the Backstepping feedback.

**Step 4:** The fourth component  $i_d$  must be stabilized around 0 (minimization of the effects of reluctance on the torque). Its dynamics are described by the differential equation:

$$\dot{z}_4 = \dot{i}_d = -\frac{R_s}{L_d} i_d + \frac{p \cdot L_q}{L_d} \Omega i_q + \frac{1}{L_d} V_d.$$

Consider the candidate Lyapunov function:  $V_4 = V_3 + \frac{1}{2} i_d^2$ . This gives:

$$\dot{V}_4 = \dot{V}_3 + i_d \left( -\frac{R_s}{L_d} i_d + \frac{p \cdot L_q}{L_d} \Omega i_q + \frac{1}{L_d} V_d \right) \dot{V}_3 < 0.$$

And  $V_d = -p L_q \Omega i_q + R_s i_d - L_d \omega_4 i_d \omega_4 > 0$ , yields  $\dot{V}_4 \leq 0$ .

## Conclusion

We can write our loop system in the form:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = - \begin{bmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 0 & \omega_4 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

which is the form provided by the "Integrator Backstepping" method.

### 10.10 Motion Control Benchmark: Experimental results (European Contract CRAFT Auto Drive)

The controller is tested on three cases of inertia in the framework of the Motion Control Benchmark (CRAFT Contrat): **Load inertia = [ 0 (unloaded drive) ; 5\* drive inertia ; 10\* drive inertia]**.

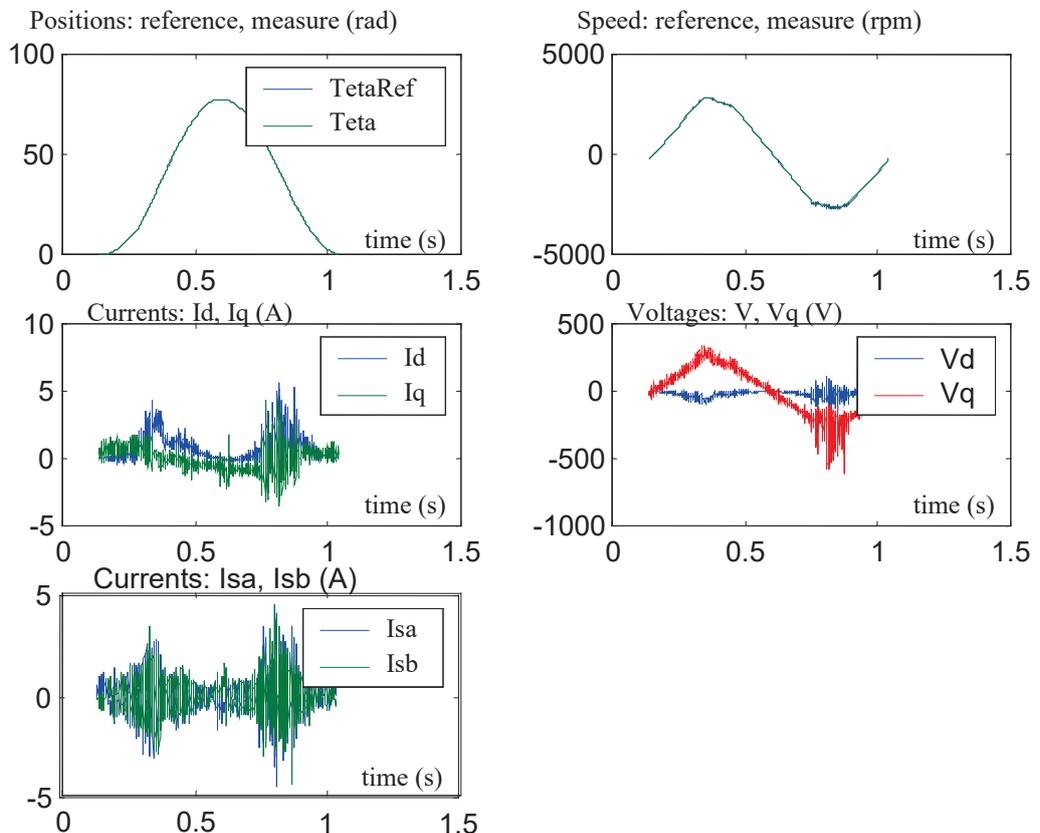
**Step 1** with some basic measurements the open loop time constant of the currents ( $T_{ol\_id}$ ,  $T_{ol\_iq}$ ) and the mechanical time constant ( $T_{ol\_o}$ ) are **approximated**.

**Step 2** The desired closed loop time constants are then chosen  $T_{cl\_id}$ ,  $T_{cl\_iq}$ ,  $T_{cl\_o}$  (i.e the choices of  $\omega_1, \omega_2, \omega_3, \omega_4$ ):

i <sub>d</sub> control		i <sub>q</sub> control		Speed & Position Control	
T <sub>ol_id</sub>	0.0036 s	T <sub>ol_iq</sub>	0.004 s	T <sub>ol_o</sub>	70 s
r <sub>id</sub>	1.11	r <sub>iq</sub>	1.	r <sub>o</sub> = r <sub>p</sub>	350
T <sub>cl_id</sub>	<b>0.004</b>	T <sub>cl_iq</sub>	<b>0.004</b>	T <sub>cl_o</sub>	<b>0.2</b>
$\omega_3$	250	$\omega_4$	250	$\omega_2$	15

The tuning is computed with the estimated parameters and applied to case 1 and 2 of the Motion Benchmark.

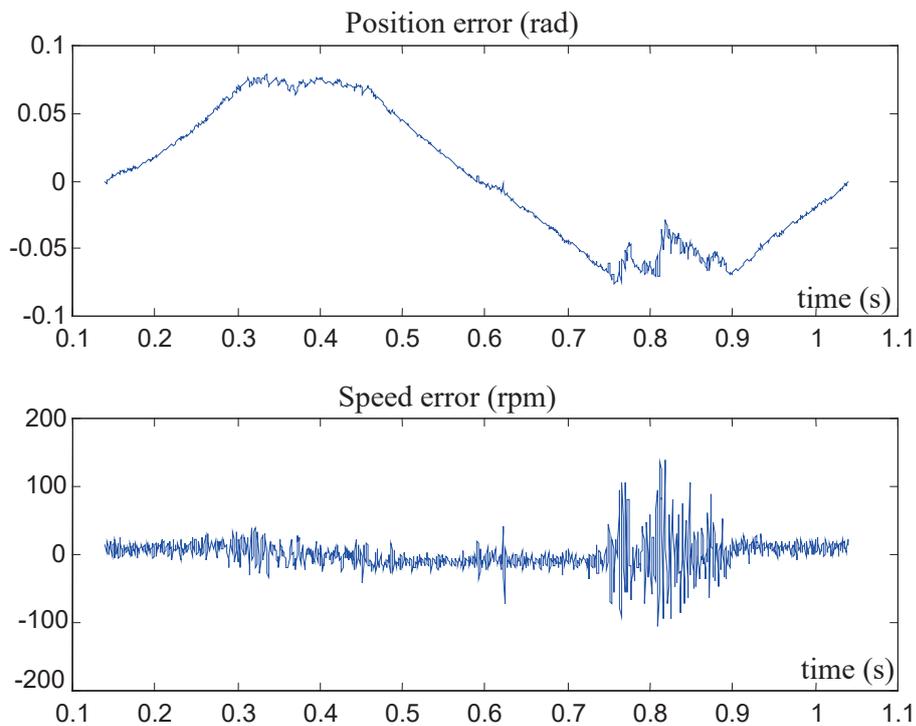
**Case 1.**  
**Inertia load = 0**



These measurements show that the max values for currents respect the motor limitations.

**Case 1.**

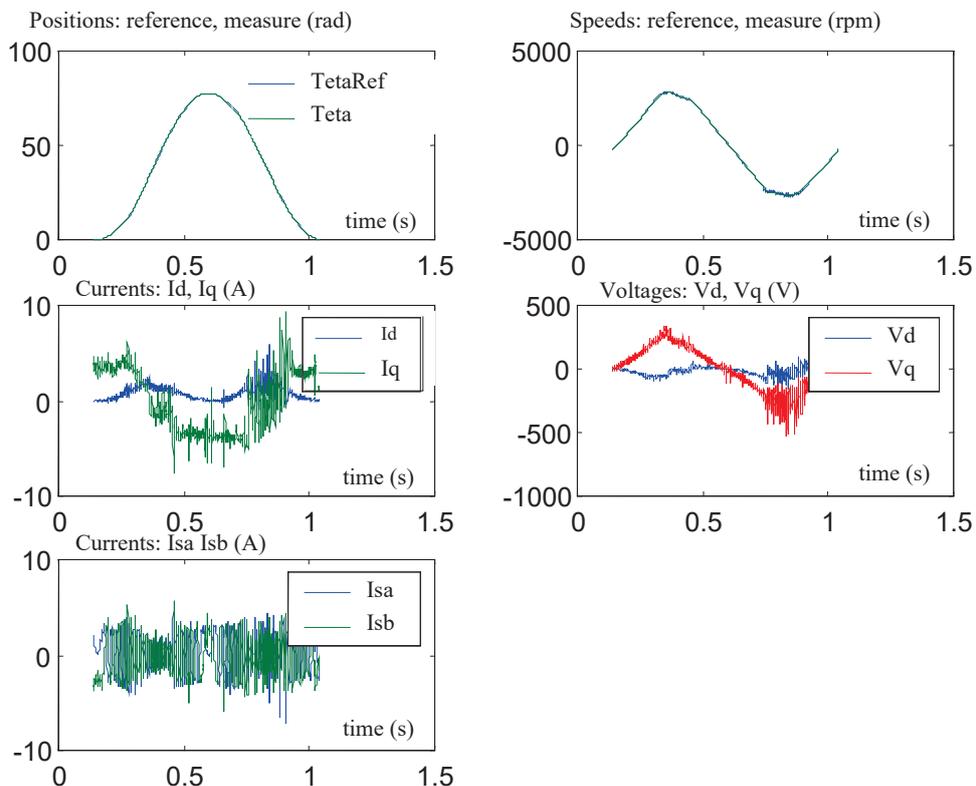
**Inertia load = 0**



Experimental results case 2

**Case 2.**

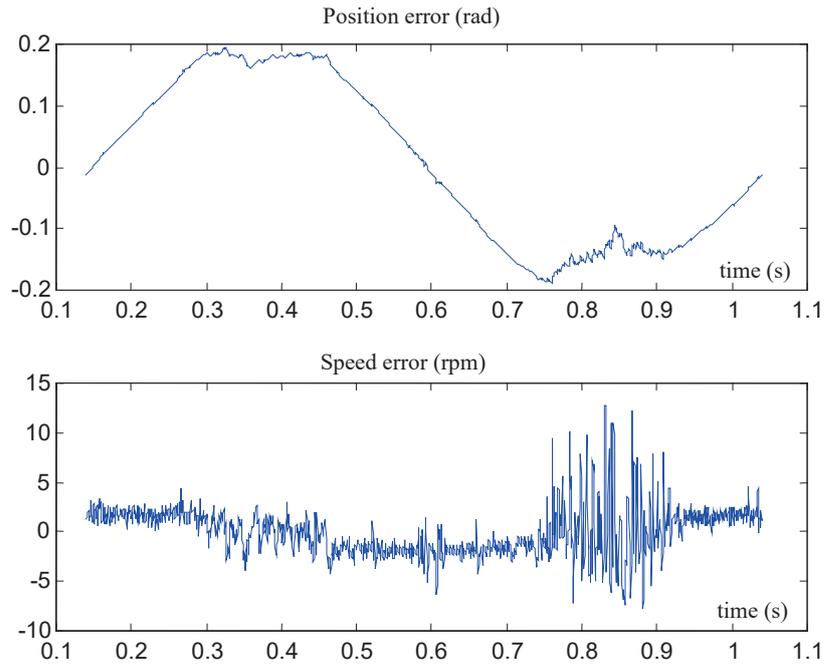
**Inertia load =  
5 \* drive inertia**



Experimental results case 2

**Case 2.**

**Inertia load =  
5 \* drive inertia**



Experimental results case 2

**Case 3. Inertia load = 10 \* drive inertia.**

**Step1 Drive and process parameters estimation.**

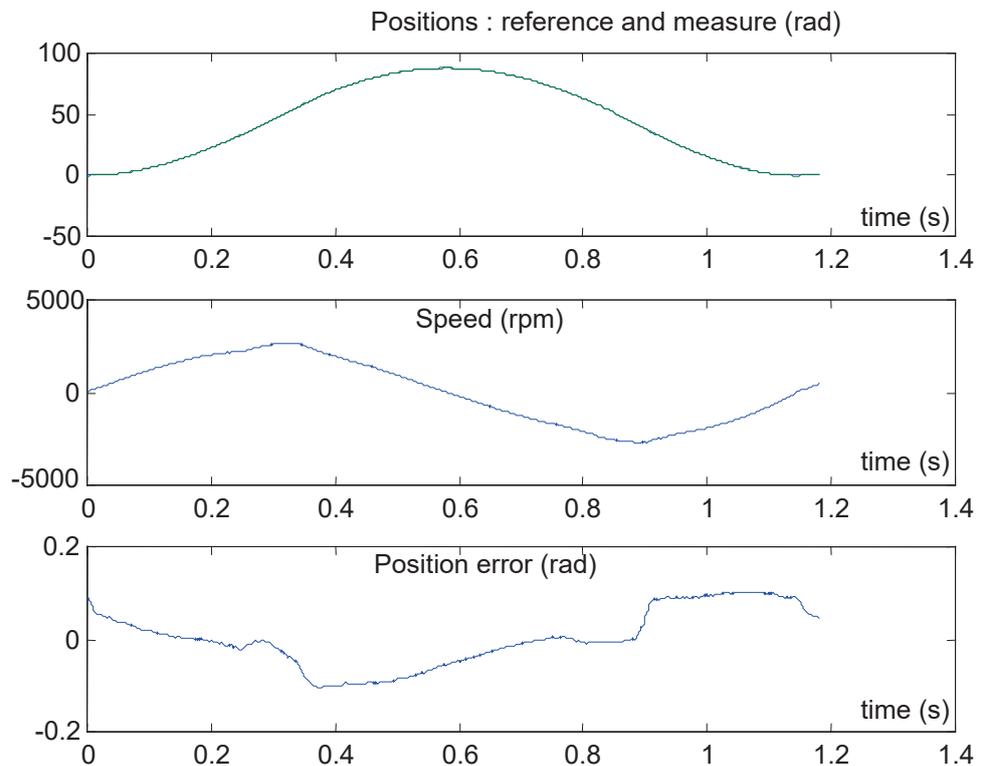
Drive manufacturer parameters		Identified parameters	
<b>R (Ph - Ph)</b>	6.5 Ω	<b>Rs</b>	5.61 Ω
<b>L (Ph - Ph)</b>	30 mH	<b>Ld</b>	18 mH
<b>Is_eff</b>	3.8 A	<b>Lq</b>	20 mH
<b>Jm</b>	0.37g m <sup>2</sup>	<b>Phif</b>	0.4 Wb
<b>J<sub>total</sub></b>	4 gm <sup>2</sup>	<b>fv</b>	0.013 10 <sup>-3</sup> kg m <sup>2</sup> s <sup>-1</sup>
		<b>J<sub>total</sub></b>	5 10 <sup>-3</sup> kg m <sup>2</sup>

Step 2 Design of Backstepping controller (Case 3. Inertia load = 10 \* drive inertia).

i <sub>d</sub> control		i <sub>q</sub> control		Speed Control		Position Control	
T <sub>ol_i<sub>d</sub></sub>	3.2 ms	T <sub>ol_i<sub>q</sub></sub>	3.56 ms	T <sub>ol_ω</sub>	384.6 s	T <sub>ol_p</sub>	384.6 s
r <sub>i<sub>d</sub></sub>	5.42	r <sub>i<sub>q</sub></sub>	6.03	r <sub>ω</sub>	38460	r <sub>p</sub>	96150
T <sub>cl_i<sub>d</sub></sub>	0.59 ms	T <sub>cl_i<sub>q</sub></sub>	0.59 ms	T <sub>cl_ω</sub>	0.01s	T <sub>cl_p</sub>	0.004 s
ω <sub>3</sub>	1700 rad/s	ω <sub>4</sub>	1700 rad/.s	ω <sub>2</sub>	100 rad/s	ω <sub>1</sub>	250 rad/s

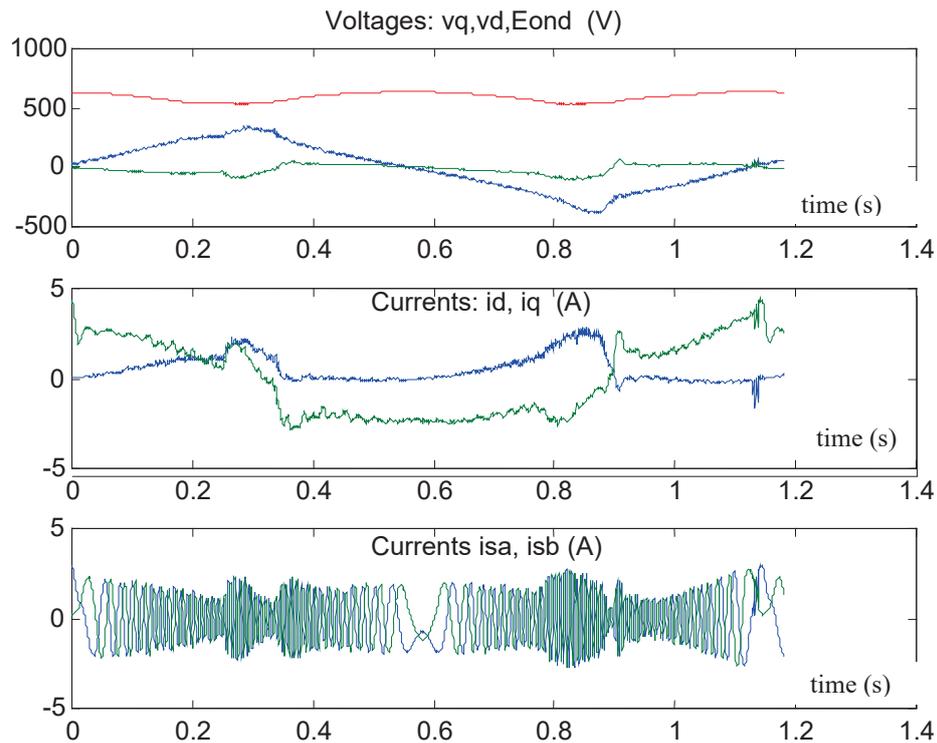
Case 3.

Inertia load =  
10 \* drive inertia.



Experimental results case 3

Case 3. Inertia load =  
10 \* drive inertia.



Experimental results case 3: Electrical voltages and currents

### Experimental tests conclusion (CRAFT CONTRACT)

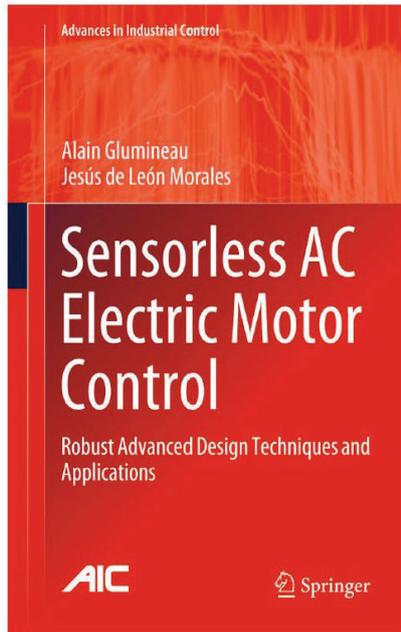
The previous sections results show that the auto-tuning (identification and controller design) is realised for the *Motion Control* problem when using a Backstepping Control. Nevertheless, for some industrial applications, it is necessary for the users (control technician) to have the possibility to adjust the controllers with a minimum of control theory knowledge. That must be carried out by using tuning *meta-parameters* as Faster tuning, Slower tuning, ... . These meta-parameters are used to modified the closed-loop time response over open-loop time response ratio given by  $r_{id}$ ,  $r_{iq}$ ,  $r_o$ ,  $r_p$ .

The initial values of tuning have to be adapted to the performance of hardware: sampling time computation of the board processor, inverter frequency, noise sensitivity ... . This implies the initial ratio has to be chosen with respect to these constraints. The best result could be obtained for best initial manufacturing tuning with the knowledge of the controlled motor drive.

Roughly speaking, accurate data is the key of good performance. Thus, the user has to privilege the knowledge of the motor and load data. If these data are not available, the user has to choose the most significant plant conditions in order to start the auto-tuning procedure to obtain the best results.

Many others results for:

- Permanent Magnet Synchronous Motor (PMSM,  $L_d = L_q$ )
- Induction motor are in the book:



Springer

Engineering : Control

Glumineau, Alain, de Leon Morales, Jesús

# Sensorless AC Electric Motor Control

Robust Advanced Design Techniques and Applications

- Explains how partial system knowledge can be avoided by making electric motor control robust
- Provides specific detailed treatment of the observability problem for trajectories of specific hard-to-measure system parameters
- Details genuine experimental results rather than being confined to simulations

This monograph shows the reader how to avoid the burdens of sensor cost, reduced internal physical space, and system complexity in the control of AC motors. Many applications fields—electric vehicles, wind- and wave-energy converters and robotics, among them—will benefit.

## Comparison of Robust Non linear Controls of AC machines with observers

- *Surface Permanents Magnets Synchronous Motor (Ezzat 2011)*

	<i>Control laws</i>		
	<i>HOSM Pre Compute trajectories</i>	<i>Backstepping</i>	<i>Homogeneous HOSM</i>
<i>Simplicity of development</i>	***	*****	****
<i>Simplicity of tuning</i>	**	***	****
<i>Iteration Time Dspace DS 1103 en</i>	<i>36 μs</i>	<i>16 μs</i>	<i>36 μs</i>

- Interior Permanent Magnets Synchronous Motor [HAMI13].

	<b>Control laws</b>		
	<i>Backstepping</i>	<i>HOSM Precalculated traj.</i>	<i>Homogenous HOSM</i>
<b>Experimental results</b>	<i>Yes</i>	<i>No</i>	<i>Yes</i>
<b>+50 % <math>R_s</math></b>	****	****	****
<b>-50 % <math>R_s</math></b>	****	****	****
<b>+20% <math>L_d, L_q</math></b>	*****	*	***
<b>-20% <math>L_d, L_q</math></b>	*****	*	***
<b>Simplicity of tuning</b>	****	***	****
<b>Convergence</b>	<i>Exponential</i>	<i>Finite time</i>	<i>Finite time</i>
<b>Computation time by iteration</b>	<i>11 <math>\mu</math>s</i>	<i>26 <math>\mu</math>s</i>	<i>16 <math>\mu</math>s</i>

- Induction Motor [TRA008].

	<b>Controls</b>			
	PI controller (FOC)	SM of order 1	Backstepping	HOSM order 3 (PT)
Observable area	Stable	Stable	Stable	Stable
Unobservable area	Stable	Stable	Stable	Stable
Nominal case	Convergence	Convergence	Convergence	Convergence
Robustness $\pm 50\%$ $R_r$	*****	*****	*****	*****
Robustness +10% $L_r$	**	**	*****	*****
Robustness +10% $L_s$	**	**	*****	*****
Computation time Control+Observer $\mu$ s	10 + 20*	11 + 20*	12 + 34 **	35+34**

\* Interconnected Observer, \*\* Adaptive interconnected observer

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