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# Comparison of several approximation schemes on the Cubed Sphere



## Goals

- Compare interpolation, weighted least squares and hyperinterpolation on the Cubed Sphere grid.
- Get an efficient scheme for spherical approximation using spherical harmonics.

## Cubed Sphere grid

Equiangular Cubed Sphere with step  $\frac{\pi}{2N}$ ,  $CS_N \subset \mathbb{S}^2$

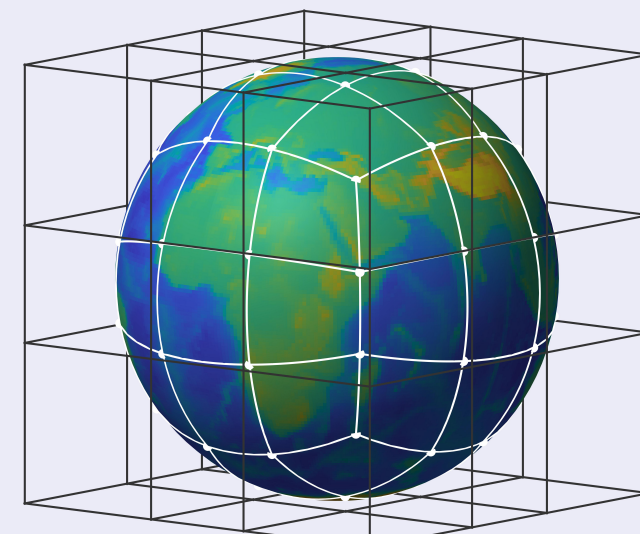


Fig.  $CS_N$  (white dots) is obtained by projection of a meshed circumscribed cube ( $N = 3$ ).

## Spherical harmonics

Space of spherical harmonics with degree  $\leq D$ ,

$$\mathcal{Y}_D = \text{Span}\{Y_n^m, |m| \leq n \leq D\}$$

- $f \in \mathcal{Y}_D$  admits a unique expansion

$$f = [Y_n^m(\cdot)]_{|m| \leq n \leq D} \hat{f} = \sum_{|m| \leq n \leq D} \hat{f}_n^m Y_n^m$$

- Fourier-like coefficients:

$$\hat{f} = [\hat{f}_n^m]_{|m| \leq n \leq D} \in \mathbb{R}^{(D+1)^2}$$

- Vandermonde matrix on the Cubed Sphere

$$A_N^D = [Y_n^m(x)]_{\substack{x \in CS_N \\ |m| \leq n \leq D}} \in \mathbb{R}^{(6N^2+2) \times (D+1)^2}$$

## Data approximation

Find  $f \in \mathcal{Y}_D$  such that  $f(x) \approx y(x)$ ,  $x \in CS_N$ .

- Data:  $y = [y(x)]_{x \in CS_N} \in \mathbb{R}^{6N^2+2}$
- Unknown:  $f \in \mathcal{Y}_D$ , with Fourier coefficients  $\hat{f} \in \mathbb{R}^{(D+1)^2}$
- Matrix writing: find  $\hat{f} \in \mathbb{R}^{(D+1)^2}$  such that  $A_N^D \hat{f} \approx y$
- Application:  $F : \mathbb{S}^2 \rightarrow \mathbb{R}$  is approximated by  $f$  from  $y = F|_{CS_N}$

## Lexicographical interpolation

$$\inf_{\substack{f \in \mathcal{Y}_{4N-1} \\ f|_{CS_N=y}} (\|\hat{f}_{4N-1}\|, \|\hat{f}_{4N-2}\|, \dots, \|\hat{f}_0\|), \quad \text{with } \hat{f}_n := [\hat{f}_n^m]_{|m| \leq n} \in \mathbb{R}^{2n+1} \quad (\text{LI})$$

For  $D = 4N - 1$ , there are spherical harmonics  $f \in \mathcal{Y}_D$  which interpolate  $y$ ; among those spherical harmonics, one selects the one which minimizes  $(\|\hat{f}_D\|, \|\hat{f}_{D-1}\|, \dots, \|\hat{f}_0\|)$  for the lexicographical ordering in  $\mathbb{R}^{D+1}$  [2, Chapter 6]. In practice, the numerical degree is  $D = 3N$  and the computation is based on an orthogonal factorization of  $A_N^{3N}$  under a suitable echelon form, as in [1].

## Weighted least squares

$$\inf_{f \in \mathcal{Y}_{2N-1}} \sum_{x \in CS_N} \omega(x) |f(x) - y(x)|^2, \quad \text{with } \omega : CS_N \rightarrow (0, +\infty) \quad (\text{WLS})$$

We assume that  $\omega$  is the interpolatory quadrature weight from [3]. The choice  $D = 2N - 1$  is a numerical guarantee that the problem has a unique solution and a low condition number [4]. The normal equation associated to (WLS) is

$$(A_N^{2N-1})^T \Omega A_N^{2N-1} \hat{f} = (A_N^{2N-1})^T \Omega y, \quad \text{with } \Omega = \text{diag}(\omega(x), x \in CS_N). \quad (*)$$

## Hyperinterpolation

$$f \in \mathcal{Y}_{2N-1}, \quad \text{with } \hat{f} = (A_N^{2N-1})^T \Omega y \quad (\text{HI})$$

In this way,  $\hat{f}_n^m \approx \int_{\mathbb{S}^2} F(x) Y_n^m(x) d\sigma$  if  $y = F|_{CS_N}$  and the quadrature rule  $\omega$  is accurate enough, as in [5]. The approximation (HI) can be understood as the zero-th order term of a Neumann series corresponding to (\*).

## Approximation of test functions

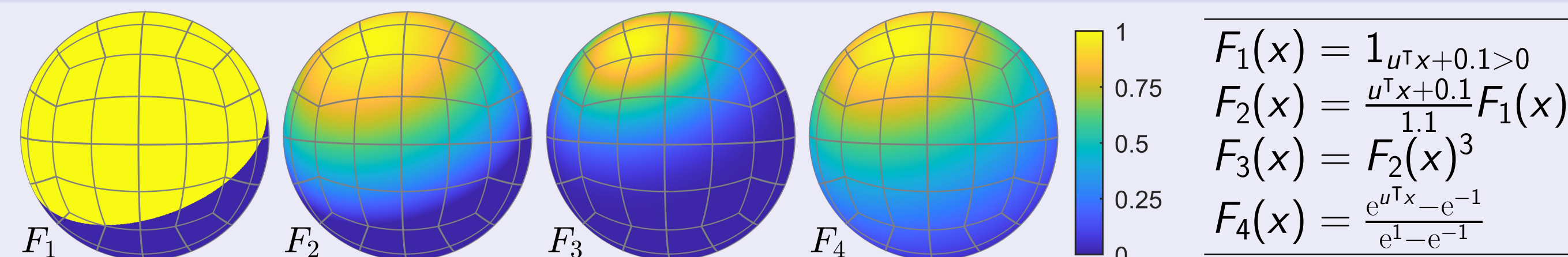


Fig. Four zonal functions, with axis  $u = \frac{1}{\sqrt{14}}(1, 2, 3)$ , and with various smoothness properties. (The grid  $CS_4$  is superposed on the plot.)

## Numerical results

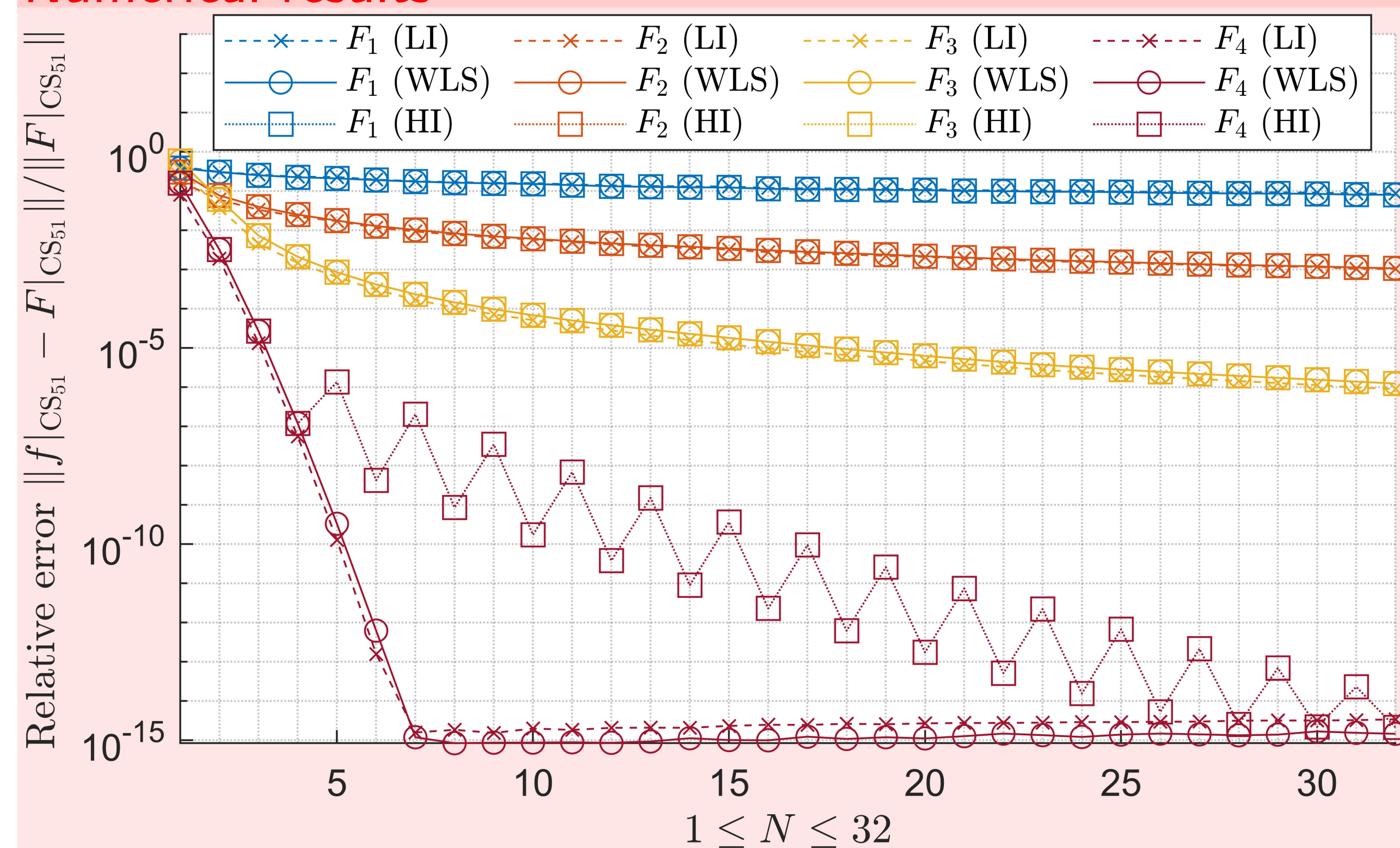


Fig. Comparison of (LI), (WLS) and (HI) on the four test functions.

## Properties of the normal matrix

$$(A_N^{2N-1})^T \Omega A_N^{2N-1} = I - E_N \quad \begin{cases} \bullet E_N \text{ is block diagonal (12 blocks)} \\ \bullet E_N \text{ is symmetric, sparse and "small"} \end{cases}$$

N	1	2	3	4	5	6	7	8	9	10	11	12	16	24	32
Sparsity score (%)	0	0	0	0	6.5	5.1	6.9	5.9	7.1	6.3	7.2	6.6	6.8	6.8	6.6
Spectral radius	4e-16	7e-16	2e-15	2e-15	2e-2	9e-3	8e-3	9e-3	7e-3	7e-3	4e-3	4e-3	3e-3	1e-3	9e-4

Tab. Sparsity score (with threshold  $10^{-14}$ ), and spectral radius of  $E_N$ .

## Convergence of the Neumann series for (WLS)

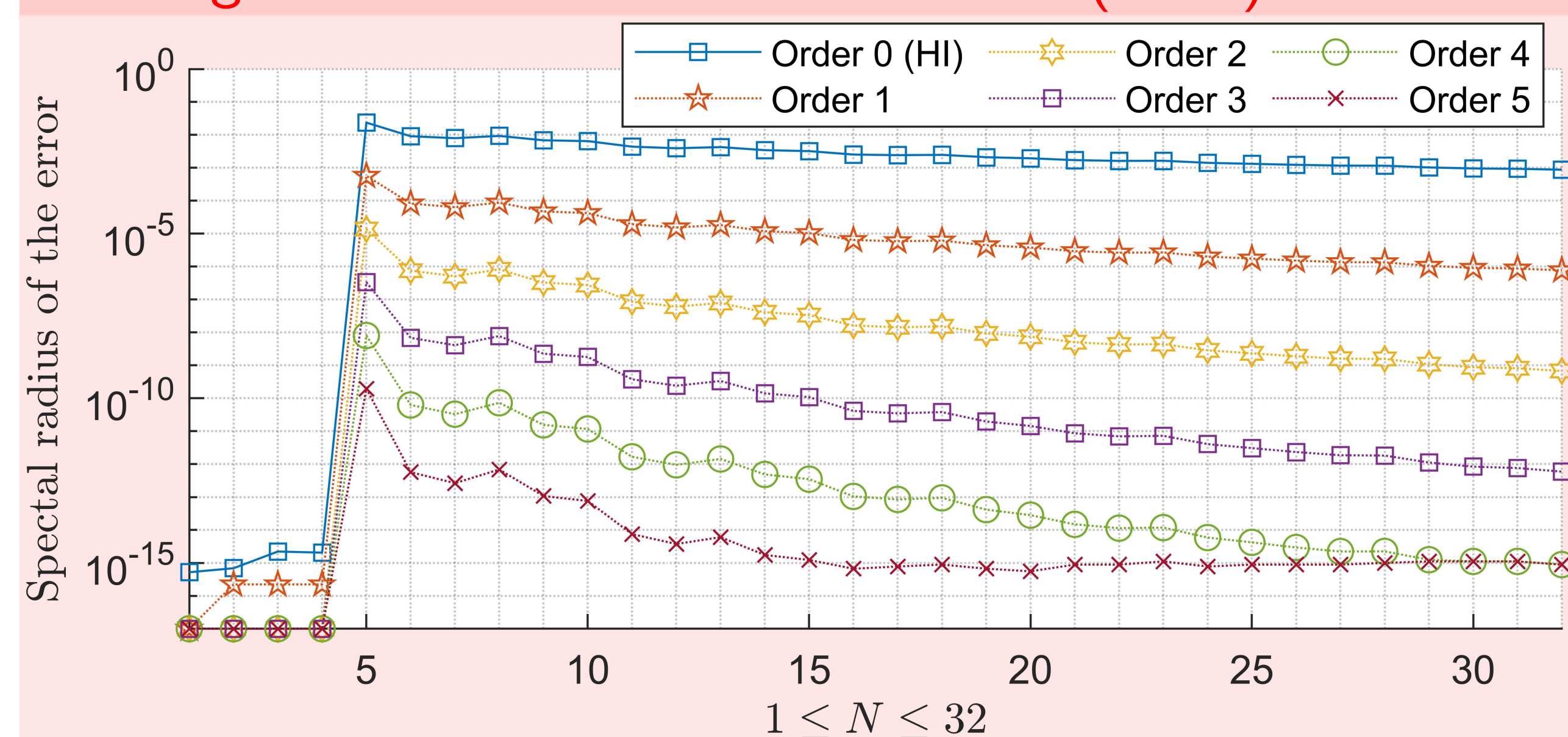


Fig. Approximation by a  $k$ -th order Neumann series: spectral radius of  $[(A_N^{2N-1})^T \Omega A_N^{2N-1}]^{-1} - (I + E_N + \dots + (E_N)^k)$  (thresholded at  $10^{-17}$ ).

## Theorem (Optimal quadrature rule on low resolution grids)

Let  $1 \leq N \leq 4$ . There is exactly one grid function  $\omega : CS_N \rightarrow \mathbb{R}$  that defines an optimal quadrature rule with respect to the degree of accuracy. It has the octahedral symmetry, degree  $4N - 1$ , and is given as follows.

## Algebraic expression of optimal quadrature weights

N	$x_1$	$x_2$	$x_3$	$\omega(x_1, x_2, x_3)$	N	$x_1$	$x_2$	$x_3$	$\omega(x_1, x_2, x_3)$
1	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{\pi}{2}$	1	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{729\pi}{20020}$
2	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{9\pi}{70}$	2	$\frac{1}{\sqrt{2+s^2}}$	$\frac{1}{\sqrt{2+s^2}}$	$\frac{s}{\sqrt{2+s^2}}$	$\frac{2053\pi}{51480} - \frac{183\pi\sqrt{2}}{80080}$
3	$\frac{1}{\sqrt{2+t^2}}$	$\frac{1}{\sqrt{2+t^2}}$	$\frac{t}{\sqrt{2+t^2}}$	$\frac{16\pi}{105}$	4	$\frac{1}{\sqrt{1+2s^2}}$	$\frac{1}{\sqrt{1+2s^2}}$	$\frac{s}{\sqrt{1+2s^2}}$	$\frac{2053\pi}{51480} + \frac{183\pi\sqrt{2}}{80080}$
	$\frac{1}{\sqrt{1+2t^2}}$	$\frac{t}{\sqrt{1+2t^2}}$	$\frac{t}{\sqrt{1+2t^2}}$	$\frac{4\pi}{21}$		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{512\pi}{15015}$
	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{9\pi}{140}$		$\frac{1}{\sqrt{1+s^2}}$	$\frac{1}{\sqrt{1+s^2}}$	0	$\frac{2048\pi}{45045}$
	$\frac{1}{\sqrt{2+t^2}}$	$\frac{1}{\sqrt{2+t^2}}$	$\frac{t}{\sqrt{2+t^2}}$	$\frac{61\pi}{840} - \frac{3\pi\sqrt{3}}{560}$		1	0	0	$\frac{736\pi}{15015}$
	$\frac{1}{\sqrt{1+2t^2}}$	$\frac{t}{\sqrt{1+2t^2}}$	$\frac{t}{\sqrt{1+2t^2}}$	$\frac{61\pi}{840} + \frac{3\pi\sqrt{3}}{560}$					with $s = \sqrt{2} - 1$

Tab. Weights of the theorem, computed by integration of a Lagrange-kind basis from [2, Lemma 6.1].

## Conclusion and discussion

- (LI), (WLS) and (HI) look similar for the non-smooth functions  $F_i$ ,  $i \leq 3$ .
- (LI) and (WLS) look similar for the very smooth function  $F_4$ , and they outperform (HI) for  $5 \leq N \leq 25$ .
- For  $1 \leq N \leq 4$ , (WLS) reduces to (HI), because the chosen weight has the degree of accuracy  $4N - 1$  (it is the optimal one).
- More generally, (WLS) can be solved efficiently by a few iterations of suitable iterative solvers, due to the structure of the normal matrix.
- For instance, compromises between (HI) and (WLS) have been achieved using truncated Neumann series.
- Perspectives include the use of fast spherical transforms [6] to reduce the computational cost.

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## Acknowledgements

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