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Christophe Berthon, Victor Michel-Dansac, Andrea Thomann. An entropy-stable and fully well-balanced scheme for the Euler equations with gravity. 2024. hal-04620125

**HAL Id: hal-04620125**

**<https://hal.science/hal-04620125v1>**

Preprint submitted on 21 Jun 2024

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# An entropy-stable and fully well-balanced scheme for the Euler equations with gravity

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## Abstract

The present work concerns the derivation of a numerical scheme to approximate weak solutions of the Euler equations with a gravitational source term. The designed scheme is proved to be fully well-balanced since it is able to exactly preserve all moving equilibrium solutions, as well as the corresponding steady solutions at rest obtained when the velocity vanishes. Moreover, the proposed scheme is entropy-preserving since it satisfies all fully discrete entropy inequalities. In addition, in order to satisfy the required admissibility of the approximate solutions, the positivity of both approximate density and pressure is established. Several numerical experiments attest the relevance of the developed numerical method.

## 1 Introduction

The present paper is devoted to designing a numerical scheme to approximate the weak solutions of the well-known Euler equations with a given gravitational source term [54, 38, 52, 50, 21, 37, 46]. From a numerical point of view, such a system poses three main challenges:

- (a) the physical admissibility of the approximate solutions,
- (b) satisfying the discrete entropy stability,
- (c) the suitable capture of the steady solutions.

The admissibility of the discrete solutions, namely preserving the positivity of both density and pressure, is the basic property to be satisfied by any first-order numerical scheme. For instance, the Godunov scheme, the HLL and HLLC schemes, as well as relaxation schemes are usual numerical methods which satisfy this essential robustness property, see [33, 48, 35, 19, 15, 47], but the list is clearly not exhaustive. However, it is worth noting that the set of physically admissible states is not convex. Hence, specific techniques, like entropy inequalities, are required to get a positive approximate pressure, see for instance [3].

Moreover, entropy inequalities are essential to rule out some non-physical solutions. Indeed, since the system under consideration is hyperbolic, discontinuous solutions may form in finite time [27, 20]. The physical admissibility of the discontinuities are thus governed by the entropy inequalities. The homogeneous Euler system contains a large range of entropy inequalities, since they are defined up to a smooth function, according to prescribed restrictions [44].

In this work, we are concerned with a non-homogeneous Euler system, supplemented by a gravitational source term. It is thus very interesting to remark that the gravitational source term is compatible with the entropies, as shown in [21], since it contains the same set of entropy functions as the usual homogeneous Euler system. From a numerical point of view, this essential entropy stability must be preserved. The establishment of discrete entropy inequalities is known to be very challenging for usual homogeneous hyperbolic systems. For instance, the Godunov scheme, the HLL and HLLC schemes, or the Suliciu and Xin-Jin relaxation schemes are known to be entropy-satisfying for the Euler equations, see [33, 27, 9, 3, 15, 21]. Here, however, we have to take into account additional source terms. In order to recover this crucial stability property, we adopt some recent Godunov-type techniques [39, 40] which are well-designed to approximate source terms. In particular, in one of our main results,

we show that the specific entropy given by the celebrated approximate Riemann solver by Harten, Lax and van Leer [33] satisfies a decreasing principle when a convex function composition is applied. This valuable and useful principle is the main argument to prove that the approximate solutions generated by our new approximate Riemann solver satisfy the discrete entropy inequalities.

The last challenge that we need to address is posed by accurately capturing steady solutions. Indeed, due to the source term, the system admits non-obvious steady solutions [9, 42, 21]. These particular equilibrium states are of strong interest from a numerical point of view. Indeed, huge errors can be introduced by the source term discretization, and resulting spurious waves can be propagated over the whole computational domain. This numerical failure has been underlined a long time ago in [2, 12, 31, 29] for shallow water simulations. For over two decades, much work was devoted to the derivation of schemes preserving steady solution, called well-balanced schemes. The reader is referred, for instance, to [34, 53, 1, 42, 25, 6]. More recently, the terminology of fully well-balanced schemes was introduced to emphasize the ability of some schemes to exactly restore all linear or nonlinear steady solutions, see e.g. [14, 10, 5, 39, 17, 18, 8]. The present work concerns the derivation of a fully well-balanced scheme for the Euler equations with a gravitational source term. To that end, we extend techniques which were successfully applied in the context of fully well-balanced schemes for the shallow water equations, all the while making sure that the resulting scheme is entropy-stable.

To address the three issues (a) – (b) – (c), the paper is organized as follows. In the next section, we introduce the Euler system under consideration, and we recall the entropy inequalities satisfied by the weak solutions. We also present the nonlinear steady solutions with a non-vanishing velocity. Section 3 is then devoted to the derivation of an approximate Riemann solver which must mimic, in a sense to be prescribed, the exact Riemann solution associated to the Euler equations with gravitational source terms. This approximate Riemann solver is defined on the one hand by the integral consistency condition stated by Harten, Lax and van Leer in [33]. On the other hand, the approximate Riemann solver must be at rest as soon as a steady solution is considered. This condition is a natural way to suitably define the discretization of the source term [5, 39, 26, 7]. Next, we impose the entropy integral consistency condition, stated in [33], to recover the expected discrete entropy inequalities. Equipped with the approximate Riemann solver, we exhibit the associated Godunov-type scheme, and we show that the required properties are satisfied: admissibility of the approximate solutions, entropy stability, and fully well-balanced property. In Section 4, we adopt a recent technique to increase the order of accuracy of a numerical scheme while conserving the fully well-balanced property [4]. In Section 5, we exhibit several numerical experiments to illustrate the relevance of the numerical procedure designed here. A short conclusion is given in the last section.

## 2 The model equations

This section introduces, in Section 2.1, the Euler equations under a gravitational potential. Its steady solutions are then derived, firstly in Section 2.2 with a nonzero velocity and secondly in Section 2.3 with a vanishing velocity, as the limit of moving steady solutions when the velocity tends to zero.

### 2.1 The Euler equations with gravity

We consider the compressible Euler equations, equipped with a gravitational source term in a one-dimensional setting, governed by

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = -\rho \partial_x \varphi, \\ \partial_t E + \partial_x((E + p)u) = -\rho u \partial_x \varphi, \end{cases} \quad (2.1)$$

where  $\rho > 0$  denotes the density,  $u \in \mathbb{R}$  the velocity and  $E > 0$  the total energy density. Moreover,  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a given time-independent continuous gravitational potential. The system is closed by the ideal gas equation of state, which defines the pressure  $p > 0$  by

$$p = (\gamma - 1) \left( E - \frac{1}{2} \rho u^2 \right), \quad (2.2)$$

where  $\gamma > 1$  denotes the adiabatic index. To shorten notation, we rewrite system (2.1) in the following compact form

$$\partial_t W + \partial_x(F(W)) = S(W), \quad (2.3)$$

where the state vector  $W$ , flux function  $F$  and source term  $S$  are respectively defined by

$$W = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad F(W) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{pmatrix}, \quad \text{and} \quad S(W) = \begin{pmatrix} 0 \\ -\rho \partial_x \varphi \\ -\rho u \partial_x \varphi \end{pmatrix}.$$

In (2.1), we require the density  $\rho$  and pressure  $p$  to be positive. Thus, the state vector  $W$  belongs to the set of admissible states  $\Omega$ , defined by

$$\Omega = \{W \in \mathbb{R}^3 \text{ such that } \rho > 0 \text{ and } p > 0\}. \quad (2.4)$$

To reflect the influence of the gravitational source term on the wave structure of the Euler equations, we augment system (2.1) by the additional equation  $\partial_t \varphi = 0$ . The resulting system is hyperbolic with the eigenvalues

$$\lambda_{\pm} = u \pm c, \quad \lambda_u = u, \quad \lambda_0 = 0,$$

where  $c$  denotes the speed of sound, given for the ideal gas law by

$$c = \sqrt{\frac{\gamma p}{\rho}}.$$

This wave structure is represented in the left panel of [Figure 1](#). Note that, compared to the homogeneous system, which is obtained by setting the gravitational source terms in (2.1) to zero, there is a zero eigenvalue associated to the gravitational potential. This leads to a non-ordered wave structure, since  $\lambda_{\pm}$  and  $\lambda_u$  can be positive or negative, depending on the flow. This has a direct consequence on the construction of the Riemann solver, see e.g. [\[21\]](#) where a relaxation model is developed to circumvent the appearance of a zero wave, or [\[50\]](#) where six different cases of wave order configurations had to be taken into account.

Moreover, admissible entropy weak solutions to system (2.1) satisfy the entropy inequality

$$\partial_t(\rho\eta(s)) + \partial_x(\rho u\eta(s)) \leq 0, \quad (2.5)$$

where the specific entropy  $s$  is defined by

$$s = -\log\left(\frac{p}{\rho^\gamma}\right). \quad (2.6)$$

After [\[45, 21\]](#), the entropy function  $W \mapsto \rho\eta(s)$  is convex as soon as  $\eta$  is any smooth function satisfying

$$\eta'(s) \geq 0 \quad \text{and} \quad \gamma\eta''(s) + \eta'(s) \geq 0.$$

However, after [\[45, 23\]](#), the entropy stability of (2.1) is ensured by merely adopting entropies defined by smooth convex functions  $\eta$  of the state variables such that

$$\eta'(s) \geq 0 \quad \text{and} \quad \eta''(s) \geq 0. \quad (2.7)$$

Also, note that the gravitational source term does not interfere with the definition of the entropy given above. However, the presence of gravity in system (2.1) gives rise to non-trivial steady solutions, which play an important role in many applications. Therefore, [Sections 2.2](#) and [2.3](#) are devoted to a derivation of the steady solutions of system (2.1).

## 2.2 Moving equilibria

Time-invariant solutions of (2.1) with a non-zero velocity  $u \neq 0$ , the so-called moving equilibria, are governed by the following system

$$\partial_x(\rho u) = 0, \quad (2.8a)$$

$$\partial_x(\rho u^2 + p) = -\rho \partial_x \varphi, \quad (2.8b)$$

$$\partial_x((E + p)u) = -\rho u \partial_x \varphi. \quad (2.8c)$$

Assuming smooth enough steady solutions fulfilling (2.8), we obtain from (2.8a) a constant momentum  $q_0 = \rho u \neq 0$ . Substituting and dividing by  $q_0$  in (2.8c), we find that

$$\frac{E + p}{\rho} + \varphi = H_0, \quad (2.9)$$

where  $H_0$  denotes a constant total enthalpy.

Substituting  $\varphi$  from (2.9) into (2.8b), we obtain

$$\partial_x \left( \frac{q_0^2}{\rho} + p \right) = \rho \partial_x \left( \frac{E + p}{\rho} \right) \quad (2.10)$$

Since  $p$  is given by (2.2), we note that

$$E + p = \frac{p}{\gamma - 1} + \frac{1}{2} \frac{q_0^2}{\rho} + p = \frac{\gamma p}{\gamma - 1} + \frac{1}{2} \frac{q_0^2}{\rho}. \quad (2.11)$$

Using the above relation in (2.10) yields

$$-\frac{q_0^2}{\rho^2} \partial_x \rho + \partial_x p = \frac{\gamma}{\gamma - 1} \rho \partial_x \left( \frac{p}{\rho} \right) + \frac{q_0^2}{2} \rho \partial_x \left( \frac{1}{\rho^2} \right).$$

Expanding the derivatives, we obtain

$$-\frac{q_0^2}{\rho^2} \partial_x \rho + \partial_x p = \frac{\gamma}{\gamma - 1} \partial_x p - \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \partial_x \rho - \frac{q_0^2}{\rho^2} \partial_x \rho.$$

Therefore, multiplying the above relation by  $\rho$  and rearranging terms, we get

$$\left( 1 - \frac{\gamma}{\gamma - 1} \right) \rho \partial_x p + \frac{\gamma}{\gamma - 1} p \partial_x \rho = 0.$$

Since  $p$  and  $\rho$  are positive, dividing by  $p\rho$  and multiplying by  $(\gamma - 1)$  leads to

$$-\frac{1}{p} \partial_x p + \gamma \frac{1}{\rho} \partial_x \rho = 0,$$

and we obtain

$$\partial_x (-\log p + \gamma \log \rho) = 0.$$

This immediately leads to

$$\partial_x \left( -\log \left( \frac{p}{\rho^\gamma} \right) \right) = 0,$$

and we recognize the expression of the specific entropy  $s$  defined in (2.6) and thus  $\partial_x s = 0$  holds.

Hence, smooth moving steady solutions obeying the ideal gas law, with  $u \neq 0$ , are necessarily isentropic and are characterized by the constant triplet  $(q_0, H_0, s_0)$  given by

$$\rho u =: q_0, \quad \frac{E + p}{\rho} + \varphi =: H_0, \quad -\log \left( \frac{p}{\rho^\gamma} \right) =: s_0, \quad (2.12)$$

where  $q_0$  is a constant momentum,  $H_0$  is a constant total enthalpy (including the contribution of the potential energy to the total energy), and  $s_0$  a constant specific entropy.

## 2.3 Hydrostatic equilibria

In this section, we shortly recap some properties of hydrostatic steady solutions for the Euler equations with gravity, which are equilibria at rest with  $u = 0$ . Considering time-invariant solutions and setting  $u = 0$  in (2.1) leads to steady solutions characterized by the hydrostatic equation

$$\partial_x p = -\rho \partial_x \varphi. \quad (2.13)$$

Relation (2.13) is the only link between the two unknowns,  $p$  and  $\rho$ . Thus, hydrostatic equilibria are underdetermined and, to obtain unique solutions of (2.13), additional information about the properties of the equilibrium have to be provided. Intensively studied examples are so-called isothermal or isentropic hydrostatic equilibria with constant temperature or constant entropy, respectively. The reader is referred to e.g. [36, 21, 32, 46] for a non-exhaustive list of numerical methods exact on hydrostatic equilibria.

Staying in the framework of the moving steady solutions governed by (2.12), and taking a vanishing velocity  $u \rightarrow 0$ , the resulting hydrostatic equilibrium is necessarily isentropic, and is governed by

$$\rho u = 0, \quad \frac{\kappa \gamma}{\gamma - 1} \rho^{\gamma-1} + \varphi = \hat{H}_0, \quad -\log\left(\frac{p}{\rho^\gamma}\right) =: s_0, \quad (2.14)$$

where  $\kappa = \exp(-s_0)$  is constant. One can easily check that (2.14) indeed satisfies the hydrostatic equation (2.13). Moreover, the usual isentropic hydrostatic steady solution is recovered as a solution of (2.14), given by

$$u = 0, \quad p = \kappa \rho^\gamma, \quad \rho = \left(\frac{\gamma - 1}{\gamma \kappa} (\hat{H}_0 - \varphi)\right)^{\frac{1}{\gamma-1}}. \quad (2.15)$$

In the following, the objective is to derive a Godunov-type finite volume (FV) scheme based on an approximate Riemann solver (A-RS), which is able to preserve moving equilibria and isentropic hydrostatic equilibria up to machine precision, generates admissible solutions in the sense of (2.4), and fulfills all entropy inequalities (2.5) with (2.7).

## 3 The numerical scheme

We start by briefly recalling the principle of Godunov-type FV schemes based on approximate Riemann solvers. The space domain is discretized, for  $i \in \mathbb{Z}$ , in uniform cells  $C_i = (x_{i-1/2}, x_{i+1/2})$ , of center  $x_i$  and of size  $\Delta x$ . The time domain is discretized with time steps of varying size  $\Delta t$ . We seek a way to evolve in time an approximation of the solution  $W$  to the Euler equations (2.3), defined by

$$W_i^n \simeq \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W(x, t^n) dx.$$

To that end, we develop an approximate Riemann solver (A-RS). We first state, in Section 3.1, the structure of our A-RS, as well as the general principle of A-RSs. Then, basic consistency properties are established in Section 3.2, and we define the time evolution of the cell averages in Section 3.3, up to the final choice of the A-RS. The remainder of this section is then devoted to the derivation of the remaining unknowns in the A-RS to achieve the properties given in Section 3.4.

### 3.1 The approximate Riemann solver

Let us define an A-RS, denoted by  $\widetilde{W}$ , whose role is to approximate the solution of the Riemann problems occurring at each cell interface  $x_{i+1/2}$  between cells  $C_i$  and  $C_{i+1}$ . Since this solution is self-similar, the A-RS depends on the variable  $x/t$ , as well as on the two states  $W_i^n$  and  $W_{i+1}^n$  left and right of the interface. For instance, the approximate Riemann solution at interface  $x_{i+1/2}$  is given by  $\widetilde{W}((x - x_{i+1/2})/t; W_i^n, W_{i+1}^n)$ .

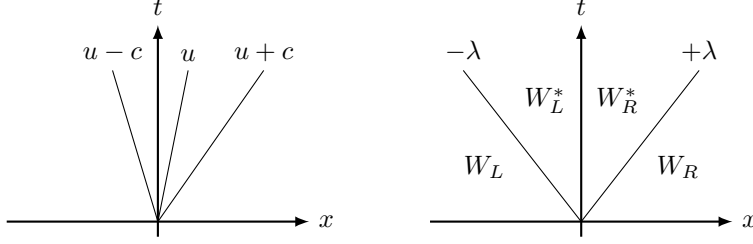


Figure 1: Left panel: One possible wave configuration, with  $u > 0$ , of the Euler equations with gravity (2.1). Right panel: Wave structure of the approximate Riemann solver.

In this work, we select an A-RS made of two intermediate states, as represented in the right panel of Figure 1. Its expression, for given left and right states  $W_L$  and  $W_R$ , reads

$$\widetilde{W}\left(\frac{x}{t}; W_L, W_R\right) = \begin{cases} W_L & \text{if } x < -\lambda t, \\ W_L^* & \text{if } -\lambda t < x < 0, \\ W_R^* & \text{if } 0 < x < \lambda t, \\ W_R & \text{if } x > \lambda t, \end{cases} \quad (3.1)$$

where the components of the vectors  $W_L^*$  and  $W_R^*$  are denoted by

$$W_L^* = \begin{pmatrix} \rho_L^* \\ q_L^* \\ E_L^* \end{pmatrix} \quad \text{and} \quad W_R^* = \begin{pmatrix} \rho_R^* \\ q_R^* \\ E_R^* \end{pmatrix}.$$

In the A-RS (3.1), the approximate wave speed  $\lambda$  is chosen such that the acoustic waves are included in the wave fan of the A-RS, i.e.,  $\lambda \geq |u| + c \geq |\lambda_{\pm}|$ . In this work, we impose that the waves defining the A-RS are symmetric around  $\lambda_0 = 0$ . However, this is just a simplifying assumption. Indeed, in principle, two different values of  $\lambda$  could be chosen, to account for the asymmetric wave configuration of the Euler equations with respect to  $\lambda_0$ , as long as the acoustic waves  $\lambda_{\pm}$  are contained in the A-RS. In practice, we take

$$\lambda = \Lambda \max\left(|u_L| + c(W_L), |u_R| + c(W_R)\right), \quad (3.2)$$

which indeed satisfies  $\lambda \geq |\lambda_{\pm}|$ . The parameter  $\Lambda \geq 1$  is merely used to scale the wave speeds, and will be given in the numerical experiments.

The remainder of this section is devoted to the construction of the two intermediate states  $W_L^*$  and  $W_R^*$ . They will be derived such that the resulting numerical scheme satisfies several key properties, namely consistency, entropy stability (i.e. fulfilling a discrete equivalent of (2.5)), positivity preservation (i.e.  $W_i^n \in \Omega$ ), and well-balancedness. We start with consistency in Section 3.2.

### 3.2 Consistency conditions

Let us first derive conditions on the intermediate states  $W_L^*$  and  $W_R^*$  to ensure the consistency of the FV scheme with the solution of the Euler system. According to [33], the scheme is consistent as soon as the average over a cell of the A-RS (3.1) is equal to that of the exact, self-similar solution to the Riemann problem, denoted by  $W_{\mathcal{R}}$ . Therefore, the following equality must hold

$$\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx. \quad (3.3)$$

Straightforward computations, detailed in [39], then show that the intermediate states must satisfy the following relations:

$$\begin{cases} \rho_L^* + \rho_R^* = 2\rho_{\text{HLL}}, \\ q_L^* + q_R^* = 2q_{\text{HLL}} + \frac{1}{\lambda\Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} (-\rho\partial_x\varphi)_{\mathcal{R}}\left(\frac{x}{t}; W_L, W_R\right) dt dx, \\ E_L^* + E_R^* = 2E_{\text{HLL}} + \frac{1}{\lambda\Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} (-q\partial_x\varphi)_{\mathcal{R}}\left(\frac{x}{t}; W_L, W_R\right) dt dx, \end{cases} \quad (3.4)$$

where  $(-\rho\partial_x\varphi)_{\mathcal{R}}$  and  $(-q\partial_x\varphi)_{\mathcal{R}}$  respectively represent the second and third components of  $S(W_{\mathcal{R}})$ . Moreover, we have introduced in (3.4) the intermediate state  $W_{\text{HLL}}$  of the well-known HLL Riemann solver from Harten, Lax and van Leer [33]. It is given by

$$W_{\text{HLL}} = \begin{pmatrix} \rho_{\text{HLL}} \\ q_{\text{HLL}} \\ E_{\text{HLL}} \end{pmatrix} = \frac{W_L + W_R}{2} - \frac{F(W_R) - F(W_L)}{2\lambda}. \quad (3.5)$$

Next, we introduce two new quantities,  $S^q$  and  $S^E$ , which are approximations of the cell averages of the integrals of the second and third components of the source term, i.e.,

$$\begin{aligned} S^q &\approx \frac{1}{\Delta t} \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} (-\rho\partial_x\varphi)_{\mathcal{R}}\left(\frac{x}{t}; W_L, W_R\right) dt dx, \\ S^E &\approx \frac{1}{\Delta t} \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} (-q\partial_x\varphi)_{\mathcal{R}}\left(\frac{x}{t}; W_L, W_R\right) dt dx, \end{aligned} \quad (3.6)$$

where  $S^q$  and  $S^E$  have to be consistent with  $-\rho\partial_x\varphi$  and  $-q\partial_x\varphi$ , in a sense that will be prescribed later. Then, using relations (3.6) in the consistency condition (3.4) yields the following consistency relations to be satisfied by the intermediate states

$$\begin{cases} \rho_L^* + \rho_R^* = 2\rho_{\text{HLL}}, \\ q_L^* + q_R^* = 2q_{\text{HLL}} + \frac{S^q\Delta x}{\lambda}, \\ E_L^* + E_R^* = 2E_{\text{HLL}} + \frac{S^E\Delta x}{\lambda}. \end{cases} \quad (3.7)$$

To simplify computations in the following steps, let us introduce  $\widehat{W}$  and  $\delta W$ , defined such that

$$\widehat{W} = \frac{W_L^* + W_R^*}{2} \quad \text{and} \quad \delta W = \frac{W_R^* - W_L^*}{2}. \quad (3.8)$$

We highlight that this corresponds to rewriting the intermediate states in terms of the unknowns  $\widehat{W}$  and  $\delta W$ . Namely, we have  $W_L^* = \widehat{W} - \delta W$  and  $W_R^* = \widehat{W} + \delta W$ . Equipped with this notation, (3.7) immediately gives the following expressions for the components of  $\widehat{W}$ :

$$\begin{cases} \widehat{\rho} = \rho_{\text{HLL}}, \\ \widehat{q} = q_{\text{HLL}} + \frac{S^q\Delta x}{2\lambda}, \\ \widehat{E} = E_{\text{HLL}} + \frac{S^E\Delta x}{2\lambda}. \end{cases} \quad (3.9)$$

We are thus left with the determination of the three components of  $\delta W$  and of the two source term averages  $S^q$  and  $S^E$ , i.e. five unknowns in total. However, whatever the expressions of these unknowns, the scheme (3.12) is constructed to satisfy (3.3), and is therefore consistent by definition.

The remainder of Section 3 is devoted to the derivation of these five unknowns, ensuring positivity preservation, entropy stability and well-balancedness. However, first, we explain in Section 3.3 how to use the A-RS (3.1) to derive the time update of the numerical scheme, and precisely define in Section 3.4 the properties to be satisfied by the scheme, and thus by the intermediate states.



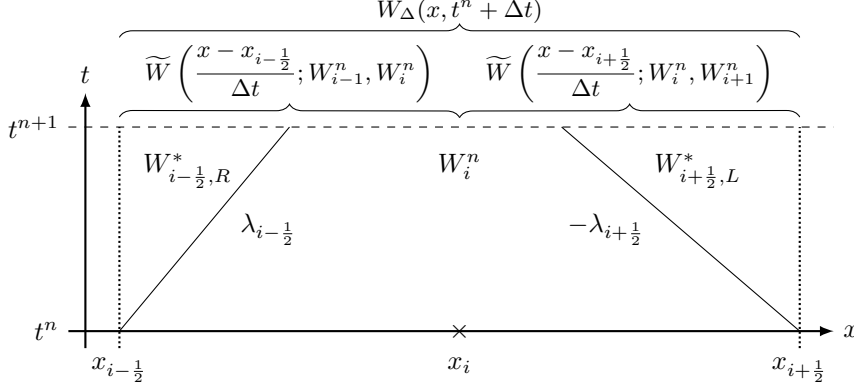


Figure 2: Juxtaposition of subsequent A-RSs defining  $W_\Delta$  using definition (3.14) of the intermediate states  $W_{i-1/2,R}^*$  and  $W_{i+1/2,L}^*$ .

### 3.3 The Godunov-type scheme finite volume scheme

We start by defining the cell-wise juxtaposition  $W_\Delta$  of the approximate Riemann solver (3.1), by

$$\forall i \in \mathbb{Z}, \forall t \in (0, \Delta t], \forall x \in (x_i, x_{i+1}), W_\Delta(x, t^n + t) = \widetilde{W}\left(\frac{x - x_{i+1/2}}{t}; W_i^n, W_{i+1}^n\right).$$

This definition is illustrated on Figure 2. The time update of the Godunov-type scheme is then given by integrating the juxtaposition  $W_\Delta$  of Riemann solvers at time  $t^{n+1} = t^n + \Delta t$  over each cell  $C_i$ , yielding

$$W_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} W_\Delta(x, t^{n+1}) dx. \quad (3.10)$$

For stability purposes, we impose the following CFL (Courant-Friedrichs-Lewy) condition on  $\Delta t$ , see [33]:

$$\Delta t \leq \frac{1}{2} \frac{\Delta x}{\max_i \lambda_{i+1/2}}. \quad (3.11)$$

To properly compute the integral in (3.10), we note that  $W_L^*$  and  $W_R^*$  are given as functions of the left and right states  $W_L$  and  $W_R$ , as well as of the corresponding left and right gravitational potentials  $\varphi_L$  and  $\varphi_R$ . Since the structure of the A-RS (3.1) comprises a stationary wave with velocity 0, the scheme, see e.g. [39], is given by

$$W_i^{n+1} = W_i^n - \lambda \frac{\Delta t}{\Delta x} \left[ W_L^*(W_i^n, W_{i+1}^n, \varphi_i, \varphi_{i+1}) - W_R^*(W_{i-1}^n, W_i^n, \varphi_{i-1}, \varphi_i) \right]. \quad (3.12)$$

We emphasize that the scheme (3.12) can also be rewritten in the standard conservation form

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}(W_i^n, W_{i+1}^n) - \mathcal{F}(W_{i-1}^n, W_i^n)) + \frac{\Delta t}{2} (\mathcal{S}_{i-1/2}^n + \mathcal{S}_{i+1/2}^n), \quad (3.13)$$

where  $\mathcal{F}(W_i^n, W_{i+1}^n)$  is the numerical flux at interface  $x_{i+1/2}$ , and where  $\mathcal{S}_{i+1/2}^n$  is the numerical source term at interface  $x_{i+1/2}$ . The expressions of  $\mathcal{F}(W_i^n, W_{i+1}^n)$  and  $\mathcal{S}_{i+1/2}^n$  are obtained thanks to straightforward computations presented in e.g. [33]. Setting

$$W_{i+1/2,L}^* = W_L^*(W_{i-1}^n, W_i^n, \varphi_{i-1}, \varphi_i) \quad \text{and} \quad W_{i+1/2,R}^* = W_R^*(W_{i-1}^n, W_i^n, \varphi_{i-1}, \varphi_i), \quad (3.14)$$

we obtain

$$\mathcal{F}(W_i^n, W_{i+1}^n) = \frac{F(W_i^n) + F(W_{i+1}^n)}{2} - \frac{\lambda_{i+1/2}}{2} (W_{i+1/2,L}^* - W_i^n) + \frac{\lambda_{i+1/2}}{2} (W_{i+1/2,R}^* - W_{i+1}^n).$$

The numerical source term is given by

$$\mathcal{S}_{i+\frac{1}{2}}^n = \begin{pmatrix} 0 \\ (S^q)_{i+\frac{1}{2}}^n \\ (S^E)_{i+\frac{1}{2}}^n \end{pmatrix},$$

where  $(S^q)_{i+\frac{1}{2}}^n$  and  $(S^E)_{i+\frac{1}{2}}^n$  denote the source term approximations (3.6) evaluated at the interface  $x_{i+\frac{1}{2}}$ .

By (3.13), or equivalently (3.12), we have defined the time update of the consistent Godunov-type finite volume scheme, up to the definition of the intermediate states  $W_L^*$  and  $W_R^*$  and source term approximations  $S^q$  and  $S^E$ . Before giving their derivation, we detail in Section 3.4 the key properties the scheme must satisfy.

### 3.4 Definitions and conditions to be satisfied by the A-RS

The objective of this section is to state important definitions and give theoretical results regarding the well-balanced property, positivity and entropy stability. They are fundamental for the derivation of the unknowns in the A-RS, which will be carried out in Sections 3.5 and 3.6.

#### 3.4.1 Well-balancedness

We start with the definition of a well-balanced scheme. We emphasize that we are interested in developing a so-called *fully well-balanced* scheme, which exactly preserves the moving steady solutions described in Section 2.2 and isentropic hydrostatic equilibria as consequence.

To properly introduce our definition of well-balancedness, we first need to introduce the notion of Interface Steady Solution (ISS). Indeed, from (2.12), we know that a solution is steady if the momentum  $q$ , the total enthalpy  $H$  and the specific entropy  $s$  are constant. This leads us to the following definition of an ISS.

**Definition 3.1** (Interface Steady Solution (ISS)). A pair  $(W_L, W_R)$  of admissible states is said to be an Interface Steady Solution (ISS) if, and only if,

$$q_L = q_R, \quad H(W_L, \varphi_L) = H(W_R, \varphi_R), \quad \text{and} \quad s(W_L) = s(W_R).$$

Equipped with the definition of the ISS, we can now properly define a well-balanced scheme.

**Definition 3.2** (Well-balancedness). Scheme (3.12) is said to be well-balanced if

$$\forall i \in \mathbb{Z}, (W_i^n, W_{i+1}^n) \text{ is an ISS} \quad \implies \quad \forall i \in \mathbb{Z}, W_i^{n+1} = W_i^n.$$

**Lemma 3.3.** Let  $(W_L, W_R)$  be an ISS. A sufficient condition for well-balancedness is that  $W_L^* = W_L$  and  $W_R^* = W_R$ .

*Proof.* Assume that  $(W_i^n, W_{i+1}^n)$  and  $(W_{i-1}^n, W_i^n)$  are ISSs. We need to show that  $W_i^{n+1} = W_i^n$  in order for the scheme to preserve the steady solution. If the intermediate states satisfy  $W_L^* = W_L$  and  $W_R^* = W_R$ , then  $W_L^*(W_i^n, W_{i+1}^n, \varphi_i, \varphi_{i+1}) = W_i^n$  and  $W_R^*(W_{i-1}^n, W_i^n, \varphi_{i-1}, \varphi_i) = W_i^n$ . Plugging these relations into (3.12) immediately yields  $W_i^{n+1} = W_i^n$ , which concludes the proof.  $\square$

According to Lemma 3.3, and recalling the definitions (3.8) of  $\widehat{W}$  and  $\delta W$ , it is sufficient that the following conditions hold

$$(W_L, W_R) \text{ is an ISS} \quad \implies \quad \widehat{W} = \frac{W_L + W_R}{2} \quad \text{and} \quad \delta W = \frac{W_R - W_L}{2}. \quad (3.15)$$

Put in other words,  $(W_L^*, W_R^*)$  should also be an ISS, i.e.,

$$q_L^* = q_R^*, \quad H(W_L^*, \varphi_L) = H(W_R^*, \varphi_R) \quad \text{and} \quad s(W_L^*) = s(W_R^*). \quad (3.16)$$

Conditions (3.16) lead for instance to a single momentum component for both intermediate states, which we denote by  $q^* := q_L^* = q_R^*$ . Moreover, they define a single intermediate entropy

$$s^* := s(W_L^*) = s(W_R^*). \quad (3.17)$$

Conditions (3.15) and (3.16) will be instrumental when determining the expressions of  $\delta W$  and  $S^q$ ,  $S^E$  in Sections 3.5 and 3.6, respectively.

### 3.4.2 Entropy stability

The second crucial property that we wish to satisfy is entropy stability, defined below.

**Definition 3.4** (Entropy stability). The scheme (3.12) is said to be entropy-stable if, for all smooth functions  $\eta$  satisfying (2.7) and for all  $i \in \mathbb{Z}$ , the states  $W_i^n$  and  $W_i^{n+1}$  fulfil the discrete entropy inequality

$$\rho_i^{n+1} \eta(s_i^{n+1}) \leq \rho_i^n \eta(s_i^n) - \frac{\Delta t}{\Delta x} \left( (\rho \eta(s) u)_{i+1/2}^n - (\rho \eta(s) u)_{i-1/2}^n \right). \quad (3.18)$$

In order to reach this important stability property and since we are considering Godunov-type schemes, we here adopt the well-known integral entropy consistency stated in [33]. As a consequence, the scheme (3.12) will be entropy-preserving by satisfying a discrete entropy inequality (3.18) as soon as the A-RS, given by  $\widetilde{W}$  in the form (3.1), verifies

$$\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \widetilde{(\rho \eta(s))} \left( \frac{x}{\Delta t}, W_L, W_R \right) dx \leq \frac{1}{2} (\rho_L \eta(s_L) + \rho_R \eta(s_R)) - \frac{\Delta t}{\Delta x} (\rho_R \eta(s_R) u_R + \rho_L \eta(s_L) u_L), \quad (3.19)$$

where, with abuse of notation, we have defined  $s_L = s(W_L)$  and  $s_R = s(W_R)$ . According to (3.1), we readily show that the integral in (3.19) satisfies

$$\begin{aligned} \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \widetilde{(\rho \eta(s))} \left( \frac{x}{\Delta t}, W_L, W_R \right) dx &= \left( \frac{1}{2} - \lambda \frac{\Delta t}{\Delta x} \right) \rho_L \eta(s_L) + \lambda \frac{\Delta t}{\Delta x} \rho_L^* \eta(s(W_L^*)) \\ &\quad + \lambda \frac{\Delta t}{\Delta x} \rho_R^* \eta(s(W_R^*)) + \left( \frac{1}{2} - \lambda \frac{\Delta t}{\Delta x} \right) \rho_R \eta(s_R), \end{aligned}$$

so that, by involving  $s(W_L^*) = s(W_R^*) =: s^*$  imposed in (3.17), (3.19) is immediately rewritten as

$$\frac{1}{2} (\rho_L^* + \rho_R^*) \eta(s^*) \leq \frac{1}{2} (\rho_L \eta(s_L) + \rho_R \eta(s_R)) - \frac{1}{2\lambda} (\rho_R \eta(s_R) u_R + \rho_L \eta(s_L) u_L).$$

For the sake of simplicity in forthcoming notation, in addition to (3.5), we define

$$(\rho s)_{\text{HLL}} = \frac{1}{2} (\rho_L s_L + \rho_R s_R) - \frac{1}{2\lambda} (\rho_R s_R u_R - \rho_L s_L u_L), \quad (3.20)$$

$$(\rho \eta(s))_{\text{HLL}} = \frac{1}{2} (\rho_L \eta(s_L) + \rho_R \eta(s_R)) - \frac{1}{2\lambda} (\rho_R \eta(s_R) u_R - \rho_L \eta(s_L) u_L). \quad (3.21)$$

As a consequence of this notation, and because (3.7) implies that  $\rho_L^* + \rho_R^* = 2\rho_{\text{HLL}}$ , the integral entropy consistency condition (3.19) to be satisfied by the A-RS now reads

$$\rho_{\text{HLL}} \eta(s^*) \leq (\rho \eta(s))_{\text{HLL}}. \quad (3.22)$$

In order to select a suitable definition of  $s^*$ , we now state one of our main results.

**Theorem 3.5.** *Let  $W_L \in \Omega$  and  $W_R \in \Omega$  be two given states, and let us adopt a definition of  $\lambda$  such that  $\rho_{\text{HLL}} > 0$ . Then, for all convex smooth functions  $\eta$ , we have*

$$\eta \left( \frac{(\rho s)_{\text{HLL}}}{\rho_{\text{HLL}}} \right) \leq \frac{(\rho \eta(s))_{\text{HLL}}}{\rho_{\text{HLL}}}. \quad (3.23)$$

*Proof.* The establishment of this important result relies on the introduction of a standard HLL scheme for an auxiliary system given by

$$\partial_t \rho + \partial_x (\rho u) = 0, \quad (3.24a)$$

$$\partial_t (\rho u) + \partial_x (\rho u^2 + p) = 0, \quad (3.24b)$$

$$\partial_t (\rho s) + \partial_x (\rho s u), \quad (3.24c)$$

where the pressure law is given by  $s = -\log(p/\rho^\gamma)$  as in (2.6). We remark that the weak solutions of the above system satisfy, for all smooth functions  $g$ ,

$$\partial_t(\rho g(s)) + \partial_x(\rho g(s)u). \quad (3.25)$$

We introduce  $U_{\mathcal{R}}(x/t; W_L, W_R)$  the exact Riemann solution of (3.24), and we consider an A-RS, denoted by  $\tilde{U}(x/t; W_L, W_R)$ , made of a single intermediate state  $(\bar{\rho}_{\text{HLL}}, \bar{\rho}u_{\text{HLL}}, \bar{\rho}s_{\text{HLL}})$ . Once again, by considering the integral consistency condition, here given by

$$\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \tilde{U}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} U_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx,$$

we obtain after straightforward computations

$$\begin{aligned} \bar{\rho}_{\text{HLL}} &= \frac{1}{2}(\rho_L + \rho_R) - \frac{1}{2\lambda}(\rho_R u_R - \rho_L u_L), \\ \bar{\rho}s_{\text{HLL}} &= \frac{1}{2}(\rho_L s_L + \rho_R s_R) - \frac{1}{2\lambda}(\rho_R s_R u_R - \rho_L s_L u_L). \end{aligned}$$

According to notation (3.5) and (3.20), we remark that

$$\bar{\rho}_{\text{HLL}} = \rho_{\text{HLL}} \quad \text{and} \quad \bar{\rho}s_{\text{HLL}} = (\rho s)_{\text{HLL}}. \quad (3.26)$$

Next, we integrate the equation (3.24c), governing  $\rho s$ , over a space-time slab to get

$$\int_0^{\Delta t} \int_{-\lambda\Delta t}^{\lambda\Delta t} (\partial_t(\rho s) + \partial_x(\rho s u)) dx dt = 0.$$

Computing the integral, we obtain

$$\bar{\rho}s_{\text{HLL}} = \frac{1}{2\lambda\Delta t} \int_{-\lambda\Delta t}^{\lambda\Delta t} (\rho s)_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx,$$

where  $(\rho s)_{\mathcal{R}}$  is the last component of  $U_{\mathcal{R}}$ . Then, we have

$$\frac{\bar{\rho}s_{\text{HLL}}}{\bar{\rho}_{\text{HLL}}} = \frac{1}{2\lambda\Delta t} \int_{-\lambda\Delta t}^{\lambda\Delta t} s_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) \frac{\rho_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx}{\bar{\rho}_{\text{HLL}}},$$

where we have set  $s_{\mathcal{R}} = (\rho s)_{\mathcal{R}}/\rho_{\mathcal{R}}$ . Now, remark that the measure

$$\frac{\rho_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx}{2\lambda\Delta t \bar{\rho}_{\text{HLL}}}$$

is a probability measure over  $(-\lambda\Delta t, \lambda\Delta t)$ . Hence, for all convex real functions  $\eta$ , Jensen's inequality gives

$$\eta\left(\frac{\bar{\rho}s_{\text{HLL}}}{\bar{\rho}_{\text{HLL}}}\right) \leq \frac{1}{2\lambda\Delta t} \int_{-\lambda\Delta t}^{\lambda\Delta t} \eta\left(s_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right)\right) \frac{\rho_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx}{\bar{\rho}_{\text{HLL}}}. \quad (3.27)$$

Next, since  $U_{\mathcal{R}}$  defines an exact solution of (3.24), from (3.25) we get, with  $g = \eta$ ,

$$\int_0^{\Delta t} \int_{-\lambda\Delta t}^{\lambda\Delta t} (\partial_t(\rho\eta(s)) + \partial_x(\rho\eta(s)u)) dx dt = 0.$$

We thus easily obtain

$$\frac{1}{2\lambda\Delta t} \int_{-\lambda\Delta t}^{\lambda\Delta t} (\rho\eta(s))_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx = \frac{1}{2}(\rho_L\eta(s)_L + \rho_R\eta(s)_R) - \frac{1}{2\lambda}(\rho_R\eta(s)_R u_R - \rho_L\eta(s)_L u_L).$$

Arguing notation (3.21), we have

$$\frac{1}{2\lambda\Delta t} \int_{-\lambda\Delta t}^{\lambda\Delta t} (\rho\eta(s))_{\mathcal{R}} \left( \frac{x}{\Delta t}; W_L, W_R \right) dx = (\rho\eta(s))_{\text{HLL}}$$

so that the relation (3.27) now reads

$$\eta \left( \frac{\bar{\rho}s_{\text{HLL}}}{\bar{\rho}_{\text{HLL}}} \right) \leq \frac{(\rho\eta(s))_{\text{HLL}}}{\bar{\rho}_{\text{HLL}}}.$$

Arguing the notation equivalence (3.26), the estimate (3.23) is established, and the proof is thus completed.  $\square$

Now, arguing **Theorem 3.5**, we remark that the required integral entropy consistency condition (3.22) is immediately satisfied as soon as

$$\rho_{\text{HLL}} s^* = (\rho s)_{\text{HLL}}, \quad (3.28)$$

which defines the intermediate entropy  $s^*$ .

### 3.4.3 Positivity preservation

After defining entropy stability, we are now able to precisely state the third required property, positivity preservation, and to give additional conditions on the intermediate states. Indeed, since  $\Omega$  given by (2.4) is a non-convex set, entropy stability is crucial for a part of the positivity proof.

**Definition 3.6** (Positivity preservation). Let  $W_i^n \in \Omega$  for all  $i \in \mathbb{Z}$  and  $n \geq 0$ . The scheme (3.12) is said to be positivity-preserving if, for all  $i \in \mathbb{Z}$ ,  $W_i^{n+1} \in \Omega$ , where  $\Omega$  is the set of admissible states defined in (2.4).

We first derive a sufficient condition for the positivity of the density. To that end, recall that  $\rho_{\text{HLL}} > 0$  under suitable conditions on  $\lambda$ . Indeed, we have

$$\rho_{\text{HLL}} = \frac{\rho_L}{2\lambda}(\lambda + u_L) + \frac{\rho_R}{2\lambda}(\lambda - u_R).$$

As a consequence, with  $\rho_L > 0$  and  $\rho_R > 0$ , the required positivity of  $\rho_{\text{HLL}}$  is satisfied as soon as  $\lambda > \max(|u_L|, |u_R|)$ , which is the case here since  $\lambda$  is given by (3.2). Next, a way to ensure positivity preservation on  $\rho_i^{n+1}$  is to impose that  $\rho_L^* > 0$  and  $\rho_R^* > 0$ . Indeed, in this case, all densities in the Riemann fan are positive, which implies the positivity of the density at time  $t^{n+1}$ . Finally, recall that  $\rho_L^* = \rho_{\text{HLL}} - \delta\rho$  and  $\rho_R^* = \rho_{\text{HLL}} + \delta\rho$ . Therefore, a sufficient condition for the positivity of density is

$$|\delta\rho| < \rho_{\text{HLL}}. \quad (3.29)$$

Now, with  $\rho_L^* > 0$  and  $\rho_R^* > 0$ , under assumption (3.29), we may also exhibit a natural condition to get a positive updated pressure  $p(W_i^{n+1})$ . In fact, we argue the discrete entropy inequality to address this issue.

**Lemma 3.7** (Positivity of the pressure). Let  $W_L \in \Omega$  and  $W_R \in \Omega$  two given states, and let us adopt a definition of  $\lambda$  such that  $\rho_{\text{HLL}} > 0$ . Moreover, let  $\rho_L^* > 0$  and  $\rho_R^* > 0$ , and consider  $s^*$  given by (3.28). Then, for all  $i \in \mathbb{Z}$ ,  $p^{i+1} > 0$ .

*Proof.* Since  $W \mapsto \rho\eta(s)$  is convex, by applying Jensen's inequality to the integral form of  $W_i^{n+1}$  given by (3.10), we obtain

$$\rho_i^{n+1} \eta(s(W_i^{n+1})) \leq \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho_{\Delta}(x, t^{n+1}) \eta(s_{\Delta}(x, t^{n+1})) dx,$$

where we have set  $s_{\Delta} = s(W_{\Delta})$ . Using the definition (2.6) of  $s$ , we have

$$\eta \left( -\log \left( \frac{p(W_i^{n+1})}{\rho_i^{n+1}} \right) \right) \leq \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \eta \left( -\log \left( \frac{p(W_{\Delta}(x, t^{n+1}))}{(\rho_{\Delta}(x, t^{n+1}))^{\gamma}} \right) \right) \frac{\rho_{\Delta}(x, t^{n+1}) dx}{\Delta x \rho_i^{n+1}}. \quad (3.30)$$

Moreover, we remark that, since  $\rho_i^{n+1} > 0$ , the measure defined by

$$\frac{\rho_\Delta(\cdot, t^{n+1})dx}{\Delta x \rho_i^{n+1}} = \frac{\rho_\Delta(\cdot, t^{n+1})dx}{\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho_\Delta(x, t^{n+1})dx}$$

is a probability measure over  $(x_{i-1/2}, x_{i+1/2})$ . Now, let us introduce the function

$$\begin{aligned} \Xi &: (0, +\infty) \rightarrow \mathbb{R} \\ z &\mapsto \eta(-\log(z)). \end{aligned}$$

Equipped with the properties (2.7) of  $\eta$ , it is easy to show that  $\Xi$  is a monotonically decreasing and convex function. Therefore, it is invertible, and its inverse  $\Xi^{-1}$  is monotonically decreasing and concave. Applying  $\Xi^{-1}$  to (3.30) yields

$$\frac{p(W_i^{n+1})}{\rho_i^{n+1}} \geq \Xi^{-1} \left( \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Xi \left( \frac{p(W_\Delta(x, t^{n+1}))}{(\rho_\Delta(x, t^{n+1}))^\gamma} \right) \frac{\rho_\Delta(x, t^{n+1})dx}{\Delta x \rho_i^{n+1}} \right).$$

Since  $\Xi^{-1}$  is a concave function, applying Jensen's inequality to the right-hand side of the above inequality, we obtain

$$\frac{p(W_i^{n+1})}{(\rho_i^{n+1})^\gamma} \geq \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{p(W_\Delta(x, t^{n+1}))}{(\rho_\Delta(x, t^{n+1}))^\gamma} \frac{\rho_\Delta(x, t^{n+1})}{\Delta x \rho_i^{n+1}} dx.$$

Since  $W_\Delta(x, t^{n+1})$  is a juxtaposition of A-RSs defined by (3.1), because of the positivity of  $\rho_i^{n+1}$  and  $\rho_\Delta(x, t^{n+1})$  given by both  $\rho_L^* > 0$  and  $\rho_R^* > 0$ , it results that  $p(W_i^{n+1}) > 0$  as soon as

$$p(W_L^*) > 0 \quad \text{and} \quad p(W_R^*) > 0.$$

Finally, according to the definition (2.6) of  $s$ , the pressures of the intermediate states satisfy  $p(W_L^*) = (\rho_L^*)^\gamma \exp(-s^*)$  and  $p(W_R^*) = (\rho_R^*)^\gamma \exp(-s^*)$ . Thus, the above positivity condition always holds, and  $p(W_i^{n+1})$  is therefore positive.  $\square$

Equipped with these definitions and conditions (3.15) – (3.16) – (3.28) – (3.29), we now have all the ingredients we need to determine  $\delta W$  in Section 3.5, and  $S^q, S^E$  in Section 3.6.

### 3.5 Determination of $\delta W$

The goal of this section is to use the definitions and conditions established in Section 3.4 to derive an expression of the three yet unknown components of  $\delta W$ , such that  $\delta W = [W]/2$  as soon as  $(W_L, W_R)$  is an ISS, where we have denoted by  $[X] := X_R - X_L$  the jump of some quantity  $X$ . Moreover, for stability purposes, we wish to recover the HLL solver in the absence of a gravitational source term. Therefore, we also impose that, if  $\varphi_L = \varphi_R$ , then  $\delta W$  must vanish.

We start with the momentum component  $\delta q$ . Note that, according to (3.16), we set  $q_L^* = q_R^*$ . Therefore,  $\delta q = 0$  is a suitable choice.

Now, we turn to the density component  $\delta \rho$ . Recall that, if  $(W_L, W_R)$  is an ISS, then  $H(W_L, \varphi_L) = H(W_R, \varphi_R)$ , that is to say

$$\frac{E_L + p_L}{\rho_L} + \varphi_L = \frac{E_R + p_R}{\rho_R} + \varphi_R. \quad (3.31)$$

Defining the enthalpy

$$h = \frac{E + p}{\rho}, \quad (3.32)$$

and setting  $h_L = h(W_L)$  and  $h_R = h(W_R)$ , (3.31) rewrites as

$$[h] = -[\varphi]. \quad (3.33)$$

In other words, if  $[h] \neq 0$ , then

$$y := 1 + \frac{[\varphi]}{[h]} = 0. \quad (3.34)$$

The function  $y \mapsto \delta\rho(y)$  must then fulfil the following properties:

$$\delta\rho(0) = [\rho]/2, \quad (\text{i})$$

$$\delta\rho(1) = 0, \quad (\text{ii})$$

$$|\delta\rho| < \rho_{\text{HLL}}, \quad (\text{iii})$$

$$\lim_{y \rightarrow \pm\infty} \delta\rho(y) = 0. \quad (\text{iv})$$

Property (i) corresponds to the ISS case, and ensures that  $\delta\rho$  satisfies (3.15). Property (ii) corresponds to the no-gravity case, since  $y = 1$  as soon as  $[\varphi] = 0$ . In this case, we wish  $\delta\rho$  to vanish, so that  $\rho_L^* = \rho_R^* = \rho_{\text{HLL}}$ . Property (iii) is the required condition (3.29) for positivity preservation. Property (iv) ensures that  $\delta\rho$  is well-defined and goes to 0 when  $[h]$  vanishes. To determine the final expression of  $\delta\rho$  satisfying properties (i) – (iv), we first introduce a relevant function  $\psi$ .

**Lemma 3.8.** *Let  $\psi$  be a smooth function given by*

$$\psi(y) = \cos\left(\frac{\pi}{2}y\right) \exp(-2y^2). \quad (3.35)$$

*Then  $\psi$  satisfies the following properties:*

$$\psi(0) = 1, \quad (\psi\text{-i})$$

$$\psi(1) = 0, \quad (\psi\text{-ii})$$

$$|\psi| \leq 1, \quad (\psi\text{-iii})$$

$$\lim_{y \rightarrow \pm\infty} \psi(y) = 0, \quad (\psi\text{-iv})$$

$$\psi(1+y) = \mathcal{O}(y) \quad \text{as } y \rightarrow 0. \quad (\psi\text{-v})$$

*Proof.* Properties (ψ-i) and (ψ-ii) are straightforward to verify. For illustration, we provide in Figure 3 a graphical representation of  $\psi$ . Regarding property (ψ-iii), we note that for all  $y \in \mathbb{R}$ ,

$$|\psi(y)| = \left| \cos\left(\frac{\pi}{2}y\right) \exp(-2y^2) \right| \leq \exp(-2y^2) \leq 1.$$

Property (ψ-iv) holds since the cosine function is bounded and  $y \mapsto e^{-2y^2}$  goes to 0 as  $y$  goes to  $\pm\infty$ . Finally, to prove property (ψ-v), we compute  $\psi(1+y)$  for  $y \in \mathbb{R}$ . We obtain

$$\psi(1+y) = \cos\left(\frac{\pi}{2} + \frac{\pi}{2}y\right) \exp(-2(1+y)^2) = -\sin\left(\frac{\pi}{2}y\right) \exp(-2(1+y)^2).$$

Therefore,

$$\psi(1+y) \sim -\frac{\pi}{2}e^{-2}y \quad \text{as } y \rightarrow 0,$$

which proves (ψ-v) and concludes the proof.  $\square$

We now have all the ingredients we need to construct  $\delta\rho$  as a function of  $y$ . Note that multiple expressions of such a function  $y \mapsto \delta\rho(y)$  are possible. We state our choice in the next Lemma.

**Lemma 3.9.** *Let  $W_L \in \Omega$  and  $W_R \in \Omega$  two given states, and let  $W_{\text{HLL}}$  be given by (3.5) with  $\lambda$  sufficiently large. Further, let  $y \mapsto \delta\rho(y)$  be a smooth function given by*

$$\delta\rho(y) = \frac{[\rho]}{2} \psi(y),$$

*where  $y$  is defined by (3.34) and  $\psi$  is given by (3.35). Then, conditions (i) – (iv) are satisfied by  $\delta\rho(y)$ .*

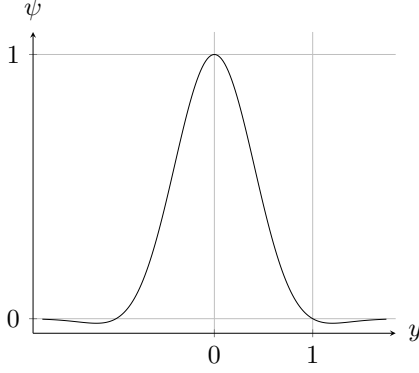


Figure 3: Graph of the function  $\psi$  defined in [Lemma 3.8](#). We observe that  $\psi$  indeed satisfies properties [\(ψ-i\)](#) – [\(ψ-iv\)](#).

*Proof.* First, we note that the properties [\(ψ-i\)](#), [\(ψ-ii\)](#) and [\(ψ-iv\)](#) directly map to properties [\(i\)](#), [\(ii\)](#) and [\(iv\)](#). Second, to prove property [\(iii\)](#), we need to show that  $|\delta\rho| < \rho_{\text{HLL}}$ . To that end, we analyze the extrema of the function  $\delta\rho(y)$ . According to property [\(ψ-iii\)](#) of  $\psi$ , the global extremum of  $\delta\rho(y)$  is  $[\rho]/2$ , reached for  $y = 0$ . Therefore, since  $\rho_L > 0$  and  $\rho_R > 0$ , we immediately obtain

$$|\delta\rho| \leq \frac{|\rho_R - \rho_L|}{2} \leq \frac{\rho_L + \rho_R}{2}.$$

Hence, according to the expression [\(3.5\)](#) of  $\rho_{\text{HLL}}$ , there exists a large enough  $\lambda$  such that  $|\delta\rho| < \rho_{\text{HLL}}$ , and property [\(iii\)](#) is satisfied. The proof is thus concluded.  $\square$

Finally, we turn to the energy component  $\delta E$ . Analogously to the derivation of  $\delta q$  and  $\delta\rho$ , we use the last remaining ISS condition on  $s$  to determine  $\delta E$ . Namely, we wish to impose that  $s(W_L^*) = s(W_R^*)$  as soon as  $(W_L, W_R)$  is an ISS, as prescribed in [\(3.16\)](#). Note that this implies, using the definition [\(2.6\)](#) of  $s$ , that

$$-\log\left(\frac{p(W_L^*)}{(\rho_L^*)^\gamma}\right) = -\log\left(\frac{p(W_R^*)}{(\rho_R^*)^\gamma}\right), \quad \text{thus,} \quad \frac{p(W_L^*)}{(\rho_L^*)^\gamma} = \frac{p(W_R^*)}{(\rho_R^*)^\gamma}.$$

Therefore, using the expression [\(2.2\)](#) of the pressure law, we obtain

$$\frac{1}{(\rho_L^*)^\gamma} \left( E_L^* - \frac{(q_L^*)^2}{2\rho_L^*} \right) = \frac{1}{(\rho_R^*)^\gamma} \left( E_R^* - \frac{(q_R^*)^2}{2\rho_R^*} \right).$$

However, recall that, for a steady solution, the momentum is constant. Hence, we replace  $(q_L^*)^2$  and  $(q_R^*)^2$  in the above relation by a consistent average of  $q^2$ , denoted by  $\tilde{q}^2$ , to be determined later on. We then obtain

$$\frac{1}{(\rho_L^*)^\gamma} \left( E_L^* - \frac{\tilde{q}^2}{2\rho_L^*} \right) = \frac{1}{(\rho_R^*)^\gamma} \left( E_R^* - \frac{\tilde{q}^2}{2\rho_R^*} \right), \quad (3.36)$$

Arguing  $E_L^* = \hat{E} - \delta E$  and  $E_R^* = \hat{E} + \delta E$  leads to the following expression of  $\delta E$

$$\delta E = \frac{1}{(\rho_L^*)^\gamma + (\rho_R^*)^\gamma} \left( (\rho_R^*)^\gamma \left( \hat{E} - \frac{\tilde{q}^2}{2\rho_L^*} \right) - (\rho_L^*)^\gamma \left( \hat{E} - \frac{\tilde{q}^2}{2\rho_R^*} \right) \right). \quad (3.37)$$

Note that this expression of  $\delta E$  satisfies both properties required for well-balancedness, independently of the expression of  $\tilde{q}^2$ . Hence,  $\tilde{q}^2$  is a degree of freedom in the numerical scheme. It turns out that one can determine  $\tilde{q}^2$  such that the scheme is entropy-satisfying, by leveraging condition [\(3.28\)](#) on the intermediate entropy  $s^*$ . Indeed,



recall that we have defined  $s^* = s(W_L^*) = s(W_R^*)$ , and that  $s^*$  is given by  $s^* = (\rho s)_{\text{HLL}}/\rho_{\text{HLL}}$ . By definition of a constant-entropy intermediate state, we obtain

$$-\log\left(\frac{p(W_L^*)}{(\rho_L^*)^\gamma}\right) = s^* = \frac{(\rho s)_{\text{HLL}}}{\rho_{\text{HLL}}}.$$

Therefore under application of the exponential function, we have

$$\frac{1}{(\rho_L^*)^\gamma} \left( \widehat{E} - \delta E - \frac{\widetilde{q}^2}{2\rho_L^*} \right) = \frac{1}{\gamma - 1} \exp\left(-\frac{(\rho s)_{\text{HLL}}}{\rho_{\text{HLL}}}\right).$$

Substituting  $\delta E$  by (3.37) in the above relation leads, after straightforward computations, to

$$\widetilde{q}^2 = \frac{2\rho_L^*\rho_R^*}{\rho_L^* + \rho_R^*} \left( 2\widehat{E} - \exp\left(-\frac{(\rho s)_{\text{HLL}}}{\rho_{\text{HLL}}}\right) \frac{(\rho_L^*)^\gamma + (\rho_R^*)^\gamma}{\gamma - 1} \right).$$

Note that this expression is indeed consistent with a squared momentum, since the expression in brackets is consistent with a kinetic energy.

### 3.6 Determination of $S^q$ and $S^E$

Equipped with the expression of  $\delta W$ , we now turn to the determination of the last two unknowns,  $S^q$  and  $S^E$ . To that end, we leverage the expressions (3.9) of  $\widehat{W}$ , and recall that, if  $(W_L, W_R)$  is an ISS, then  $\widehat{W} = \overline{W}$ , where we have defined  $\overline{X} := (X_L + X_R)/2$  the arithmetic mean of two quantities  $X_L$  and  $X_R$ . Therefore, plugging the expression of  $W_{\text{HLL}}$  in (3.9), we obtain that, if  $(W_L, W_R)$  is an ISS, then  $S^q$  and  $S^E$  must satisfy

$$S^q \Delta x = \left( \frac{q_R^2}{\rho_R} + p_R \right) - \left( \frac{q_L^2}{\rho_L} + p_L \right), \quad (3.38a)$$

$$S^E \Delta x = \left( \frac{q_R}{\rho_R} (E_R + p_R) \right) - \left( \frac{q_L}{\rho_L} (E_L + p_L) \right). \quad (3.38b)$$

Moreover, recall that  $S^q$  and  $S^E$  should be consistent approximations of the source term averages. Therefore, the goal of this section is to find such consistent averages satisfying (3.38).

*Remark 3.10.* Since  $\varphi$  is assumed to be a smooth function, for each interface, the following identities hold:

$$\varphi_R = \varphi_L + \mathcal{O}(\Delta x), \quad \text{i.e.,} \quad [\varphi] = \mathcal{O}(\Delta x).$$

We start with the energy source term  $S^E$ . Its expression and related properties are given in the following lemma.

**Lemma 3.11.** *Let two given states  $W_L \in \Omega$  and  $W_R \in \Omega$ . Further, let*

$$S^E = -\frac{q_L + q_R}{2} \frac{\varphi_R - \varphi_L}{\Delta x}. \quad (3.39)$$

*Then  $S^E$  is consistent with  $-q\partial_x\varphi$  and satisfies the well-balanced condition (3.38b).*

*Proof.* To prove consistency with the continuous source term, we take  $q_L = q(x)$  and  $q_R = q(x + \Delta x)$ , with  $\varphi_L = \varphi(x)$  and  $\varphi_R = \varphi(x + \Delta x)$ . Arguing Remark 3.10, a simple Taylor expansion of  $S^E$  then shows that  $S^E = -q\partial_x\varphi + \mathcal{O}(\Delta x)$ , which proves the consistency of  $S^E$ .

For the well-balancedness, recall that the enthalpy is given in (3.32) and satisfies  $E + p = \rho h$ . So, if  $(W_L, W_R)$  is an ISS with  $q_L = q_R = \bar{q}$ , then  $S^E \Delta x$  must satisfy  $S^E \Delta x = \bar{q}(h_R - h_L)$  after (3.38b). However, from (3.33) we have that  $[h] = -[\varphi]$  if  $(W_L, W_R)$  is an ISS. Therefore,  $S^E$  given by (3.39) satisfies (3.38b) by construction, which concludes the proof.  $\square$

Next, we consider the momentum source term  $S^q$ . Its expression and derivation are more involved and are summarized in the following lemma.

**Lemma 3.12.** *Let two given states  $W_L \in \Omega$  and  $W_R \in \Omega$ . Further, let*

$$S^q = -\frac{2\rho_L\rho_R}{\rho_L + \rho_R} \frac{\varphi_R - \varphi_L}{\Delta x} + \frac{\varepsilon}{\Delta x} \psi \left( 1 + \left( \frac{\varphi_R - \varphi_L}{h_R - h_L} \right)^3 \right), \quad (3.40)$$

where  $\psi$  is given by (3.35) and  $\varepsilon$  is given by

$$\varepsilon = e^{-s} \left( (\rho_R^\gamma - \rho_L^\gamma) - \frac{2\rho_L\rho_R}{\rho_L + \rho_R} \frac{\gamma}{\gamma - 1} (\rho_R^{\gamma-1} - \rho_L^{\gamma-1}) \right), \quad (3.41)$$

with  $e^{-s} = (e^{-s_L} + e^{-s_R})/2$ . Then  $S^q$  is consistent with  $-\rho\partial_x\varphi$  and satisfies the well-balanced condition (3.38a).

*Proof.* We first prove that  $S^q$  is consistent with the continuous source term. In this context, we emphasize that  $(W_L, W_R)$  is not necessarily an ISS, but rather a general pair of states. Performing a Taylor expansion of the first term of  $S^q$ , and recalling Remark 3.10, we immediately see that

$$-\frac{2\rho_L\rho_R}{\rho_L + \rho_R} \frac{\varphi_R - \varphi_L}{\Delta x} \underset{\Delta x \rightarrow 0}{=} -\rho\partial_x\varphi + \mathcal{O}(\Delta x).$$

For the second term of  $S^q$ , we distinguish two cases: either the solution is smooth and  $W_R = W_L + \mathcal{O}(\Delta x)$ , or the solution is not smooth (e.g. in a shock wave) and  $W_R = W_L + \mathcal{O}(1)$ . It turns out that the multiplication by  $\psi$  in (3.40) is crucial to overcome a consistency defect in the non-smooth case. In the first case, performing a Taylor expansion of  $\varepsilon$  given by (3.41) and noting that  $[\varphi]/[h] = \mathcal{O}(1)$  since both  $[\varphi]$  and  $[h]$  are  $\mathcal{O}(\Delta x)$ , we obtain that

$$W_R = W_L + \mathcal{O}(\Delta x) \implies \varepsilon = \mathcal{O}(\Delta x^3) \quad \text{and} \quad \psi \left( 1 + \frac{[\varphi]^3}{[h]^3} \right) = \mathcal{O}(1).$$

In the second case, arguing property ( $\psi$ -v) of  $\psi$  and remarking that  $[\varphi] = \mathcal{O}(\Delta x)$  and  $[h] = \mathcal{O}(1)$ , we get

$$W_R = W_L + \mathcal{O}(1) \implies \varepsilon = \mathcal{O}(1) \quad \text{and} \quad \psi \left( 1 + \frac{[\varphi]^3}{[h]^3} \right) = \mathcal{O} \left( \frac{[\varphi]^3}{[h]^3} \right) = \mathcal{O}(\Delta x^3).$$

As a consequence, in both cases, the second term of  $S^q$  is consistent with 0, with an error of  $\mathcal{O}(\Delta x^2)$ . Therefore, we have obtained that  $S^q$  is indeed consistent with the continuous source term, i.e., that  $S^q = -\rho\partial_x\varphi + \mathcal{O}(\Delta x)$ .

We now turn to the well-balanced property; we have to prove that  $S^q$ , given by (3.40), satisfies (3.38a) as soon as  $(W_L, W_R)$  is an ISS. In this case, we recall that  $[\varphi] = -[h]$ . Therefore, arguing property ( $\psi$ -i) of  $\psi$ , we obtain  $\psi(1 + ([\varphi]/[h])^3) = 1$ . Hence, we have to prove that

$$-\frac{2\rho_L\rho_R}{\rho_L + \rho_R} (\varphi_R - \varphi_L) + \varepsilon = \left( \frac{q_R^2}{\rho_R} + p_R \right) - \left( \frac{q_L^2}{\rho_L} + p_L \right)$$

as soon as  $(W_L, W_R)$  is an ISS and for  $\varepsilon$  given by (3.41). To that end, we first tackle  $\varepsilon$ . Note that, since we are considering an ISS, we have  $s_L = s_R$ . Therefore,

$$e^{-s} = \frac{1}{2} (e^{-s_L} + e^{-s_R}) = \frac{1}{2} \left( \frac{p_L}{\rho_L^\gamma} + \frac{p_R}{\rho_R^\gamma} \right) = \frac{p_L}{\rho_L^\gamma} = \frac{p_R}{\rho_R^\gamma}.$$

Using these identities to distribute  $e^{-s}$ ,  $\varepsilon$  rewrites

$$\varepsilon = (p_R - p_L) - \frac{2\rho_L\rho_R}{\rho_L + \rho_R} \frac{\gamma}{\gamma - 1} \left( \frac{p_R}{\rho_R} - \frac{p_L}{\rho_L} \right),$$

and, from the expression (3.40) of  $S^q$ , we obtain

$$S^q \Delta x = (p_R - p_L) - \frac{2\rho_L\rho_R}{\rho_L + \rho_R} \left( (\varphi_R - \varphi_L) + \frac{\gamma}{\gamma - 1} \left( \frac{p_R}{\rho_R} - \frac{p_L}{\rho_L} \right) \right). \quad (3.42)$$

Now, using the steady relation  $[\varphi] = -[h]$  together with the definition (2.11) of  $h$ , we get

$$\varphi_R - \varphi_L = - \left( \frac{E_R + p_R}{\rho_R} - \frac{E_L + p_L}{\rho_L} \right) = - \frac{\gamma}{\gamma - 1} \left( \frac{p_R}{\rho_R} - \frac{p_L}{\rho_L} \right) - \left( \frac{1}{2} \frac{q_R^2}{\rho_R^2} - \frac{1}{2} \frac{q_L^2}{\rho_L^2} \right).$$

Plugging this relation into (3.42), the term in brackets reduces to

$$(\varphi_R - \varphi_L) + \frac{\gamma}{\gamma - 1} \left( \frac{p_R}{\rho_R} - \frac{p_L}{\rho_L} \right) = - \left( \frac{1}{2} \frac{q_R^2}{\rho_R^2} - \frac{1}{2} \frac{q_L^2}{\rho_L^2} \right).$$

Now, recall that one of the ISS relations is  $q_L = q_R = \bar{q}$ . Consequently, we obtain

$$\frac{2\rho_L\rho_R}{\rho_L + \rho_R} \left( \frac{1}{2} \frac{q_R^2}{\rho_R^2} - \frac{1}{2} \frac{q_L^2}{\rho_L^2} \right) = \bar{q}^2 \frac{2\rho_L\rho_R}{\rho_L + \rho_R} \left( \frac{1}{\rho_R^2} - \frac{1}{\rho_L^2} \right) = \bar{q}^2 \left( \frac{1}{\rho_R} - \frac{1}{\rho_L} \right) = \frac{q_R^2}{\rho_R} - \frac{q_L^2}{\rho_L}.$$

Substituting into (3.42) leads to

$$S^q \Delta x = (p_R - p_L) + \left( \frac{q_R^2}{\rho_R} - \frac{q_L^2}{\rho_L} \right),$$

which is nothing but the well-balancedness condition (3.38b). The proof is thus concluded.  $\square$

### 3.7 Summary of the numerical scheme and main properties

We finally summarize the approximate Riemann solver and state the main properties of the scheme deriving from this A-RS. The A-RS (3.1) is given by

$$\widetilde{W} \left( \frac{x}{t}; W_L, W_R \right) = \begin{cases} W_L & \text{if } x < -\lambda t, \\ W_L^* = \widehat{W} - \delta W & \text{if } -\lambda t < x < 0, \\ W_R^* = \widehat{W} + \delta W & \text{if } 0 < x < \lambda t, \\ W_R & \text{if } x > \lambda t. \end{cases} \quad (3.43)$$

This A-RS is based on the HLL solver, whose intermediate state is given by (3.5), with the wave speed  $\lambda$  given by (3.2). Recall also the pressure law (2.2) defining  $p$ , the expression (2.6) of the entropy  $s$ , and the expression (2.11) of the enthalpy  $h$ . Equipped with these definitions, (3.9) gives the following expression of  $\widehat{W}$ :

$$\begin{cases} \widehat{\rho} = \rho_{\text{HLL}}, \\ \widehat{q} = q_{\text{HLL}} + \frac{S^q \Delta x}{2\lambda}, \\ \widehat{E} = E_{\text{HLL}} + \frac{S^E \Delta x}{2\lambda}, \end{cases} \quad (3.44)$$

where the approximate source terms  $S^q$  and  $S^E$  have been derived in Section 3.6 and are given by

$$\begin{aligned} S^E &= - \frac{q_L + q_R}{2} \frac{\varphi_R - \varphi_L}{\Delta x}, \\ S^q &= - \frac{2\rho_L\rho_R}{\rho_L + \rho_R} \frac{\varphi_R - \varphi_L}{\Delta x} + \frac{\varepsilon}{\Delta x} \psi \left( 1 + \left( \frac{\varphi_R - \varphi_L}{h_R - h_L} \right)^3 \right), \end{aligned} \quad (3.45)$$

where  $\psi(y) = \cos(\frac{\pi}{2}y)e^{-2y^2}$ , and where  $\varepsilon$  satisfies

$$\varepsilon = \frac{1}{2} \left( e^{-s_L} + e^{-s_R} \right) \left( (\rho_R^\gamma - \rho_L^\gamma) - \frac{2\rho_L\rho_R}{\rho_L + \rho_R} \frac{\gamma}{\gamma - 1} \left( \rho_R^{\gamma-1} - \rho_L^{\gamma-1} \right) \right).$$

The components of  $\delta W$  have been derived in [Section 3.5](#), and we have set

$$\begin{cases} \delta\rho = \frac{\rho_R - \rho_L}{2} \psi \left( 1 + \frac{\varphi_R - \varphi_L}{h_R - h_L} \right), \\ \delta q = 0, \\ \delta E = \frac{1}{(\rho_L^*)^\gamma + (\rho_R^*)^\gamma} \left( (\rho_R^*)^\gamma \left( \hat{E} - \frac{\tilde{q}^2}{2\rho_L^*} \right) - (\rho_L^*)^\gamma \left( \hat{E} - \frac{\tilde{q}^2}{2\rho_R^*} \right) \right), \end{cases} \quad (3.46)$$

where  $\tilde{q}^2$  satisfies

$$\tilde{q}^2 = \frac{2\rho_L^* \rho_R^*}{\rho_L^* + \rho_R^*} \left( 2\hat{E} - \exp \left( -\frac{(\rho s)_{\text{HLL}}}{\rho_{\text{HLL}}} \right) \frac{(\rho_L^*)^\gamma + (\rho_R^*)^\gamma}{\gamma - 1} \right).$$

Equipped with these expressions, we are ready to state our main result, summarizing all properties of the resulting scheme.

**Theorem 3.13.** *Let the time step  $\Delta t$  be given by [\(3.11\)](#) and assume that the initial data satisfies  $W_i^0 \in \Omega$  for all  $i \in \mathbb{Z}$ . Then, the numerical scheme [\(3.12\)](#) with the approximate Riemann solver [\(3.43\)](#), where  $\widehat{W}$  is given by [\(3.44\)](#) and  $\delta W$  is given by [\(3.46\)](#), satisfies the following properties:*

1. consistency with the Euler system [\(2.1\)](#);
2. positivity preservation: for all  $n \geq 0$ ,

$$\forall i \in \mathbb{Z}, W_i^n \in \Omega \quad \implies \quad \forall i \in \mathbb{Z}, W_i^{n+1} \in \Omega;$$

3. entropy stability: for all smooth functions  $\eta$  satisfying [\(2.7\)](#), for all  $i \in \mathbb{Z}$ , for all  $n \geq 0$ ,

$$\rho_i^{n+1} \eta(s_i^{n+1}) \leq \rho_i^n \eta(s_i^n) - \frac{\Delta t}{\Delta x} \left( (\rho \eta(s) u)_{i+1/2}^n - (\rho \eta(s) u)_{i-1/2}^n \right);$$

4. well-balancedness:

$$\forall i \in \mathbb{Z}, (W_i^n, W_{i+1}^n) \text{ is an ISS} \quad \implies \quad \forall i \in \mathbb{Z}, W_i^{n+1} = W_i^n.$$

*Proof.* We prove the four properties in order, using the results derived in the previous sections.

1. According to [\[33\]](#), the scheme is consistent as soon as the A-RS satisfies the integral consistency relation [\(3.3\)](#), which holds by construction, as detailed in [Section 3.2](#). Thus, the A-RS is consistent, and therefore the numerical scheme [\(3.12\)](#) is also consistent.
2. We have to prove the positivity of the density and the pressure. A sufficient condition for the positivity of the intermediate density is given by [\(3.29\)](#). It is satisfied due to [Lemma 3.9](#), which yields  $\rho_L^* > 0$  and  $\rho_R^* > 0$  in the A-RS, and thus  $\rho_i^{n+1} > 0$  in the numerical scheme. As a consequence, [Lemma 3.7](#) can be applied, yielding the positivity of the pressure  $p_i^{n+1} > 0$ .
3. After [\[33\]](#), the entropy stability holds as soon as [\(3.19\)](#) is satisfied. However, following [Theorem 3.5](#), this relation is verified if the intermediate entropy of the A-RS is given by [\(3.28\)](#), which is the case for the A-RS [\(3.43\)](#). This proves that the scheme is entropy-stable.
4. A sufficient condition for well-balancedness is given in [Lemma 3.3](#), from which follow the sufficient conditions [\(3.15\)](#) to be satisfied by  $\widehat{W}$  and  $\delta W$  in the intermediate states of the A-RS. As shown in [Sections 3.5](#) and [3.6](#), these conditions are satisfied by construction, and consequently the numerical scheme is fully well-balanced.

Thus, all four properties have been proven, which concludes the proof.  $\square$

## 4 A well-balanced higher order extension

Equipped with the first-order scheme described in [Theorem 3.13](#), we now turn to the construction of a high-order well-balanced extension. Since the first-order scheme does not require solving any non-linear system of equations, we wish to preserve this property for the high-order scheme. Other strategies have been developed over the years to produce high-order well-balanced schemes, but most of them require solving non-linear systems, see e.g. the non-exhaustive list [[13](#), [51](#), [11](#), [28](#)]. To avoid an ill posed problem, i.e. multiple possible solutions associated to solving non-linear systems, we follow the procedure described in detail in [[4](#)] and used e.g. in [[41](#)] for the shallow water equations. Note that by adopting a strategy that circumvents non-linear iterative solvers, the associated computational overhead is avoided.

Here, we briefly outline the procedure, which is based on the observation that, if the solution is steady, the well-balanced scheme is exact, and as such it will be more accurate than any high-order scheme. As a consequence, this procedure consists in computing the error between the solution and a steady solution, and using this error to detect a steady solution and correct the high-order scheme. First, we introduce the uncorrected high-order scheme, see [[22](#)] for further details. The scheme of order  $d + 1, d \geq 1$ , is based on the polynomial reconstruction

$$\widetilde{W}_i^n(x) = W_i^n + \Pi_i^n(x - x_i), \quad (4.1)$$

where  $\Pi_i^n$  is a polynomial of degree  $d$ , defined such that the reconstruction satisfies the following two relations

$$\widetilde{W}_i^n(x) = W(x, t^n) + \mathcal{O}(\Delta x^{d+1}) \quad \text{and} \quad \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{W}_i^n(x) dx = W_i^n.$$

At this point we want to remark that is at times more convenient to reconstruct in primitive variables  $\rho, u, p$ . Equipped with the reconstruction (4.1), interface reconstructions on the cell  $C_i$  are given by

$$\widetilde{W}_{i+\frac{1}{2},-}^n = \widetilde{W}_i^n(x_{i+\frac{1}{2}}) \quad \text{and} \quad \widetilde{W}_{i+\frac{1}{2},+}^n = \widetilde{W}_{i+1}^n(x_{i+\frac{1}{2}}),$$

while the high-order source term satisfies

$$\check{S}_i^n = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(W(x, t^n)) dx + \mathcal{O}(\Delta x^{d+1}). \quad (4.2)$$

The integral in (4.2) is computed with a quadrature rule of order  $d$ . Then, using the interface values  $\widetilde{W}_{i\pm 1/2,\pm}^n$  and the high-order source terms  $\check{S}_i$ , the scheme of order  $d$  reads

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} \left( \mathcal{F}(\widetilde{W}_{i+\frac{1}{2},-}^n, \widetilde{W}_{i+\frac{1}{2},+}^n) - \mathcal{F}(\widetilde{W}_{i-\frac{1}{2},-}^n, \widetilde{W}_{i-\frac{1}{2},+}^n) \right) + \Delta t \check{S}_i^n. \quad (4.3)$$

Note that the scheme (4.3) is high-order accurate, but not well-balanced. Therefore, we employ the indicator from [[4](#)], to detect whether the current solution approximates a steady state. It relies on the verification of the ISS property [Definition 3.1](#) on all cells, thus the quantity

$$\sigma_{i+\frac{1}{2}}^n = \left\| \begin{pmatrix} q_{i+1}^n - q_i^n \\ H(W_{i+1}^n, \varphi_{i+1}^n) - H(W_i^n, \varphi_i^n) \\ s(W_{i+1}^n) - s(W_i^n) \end{pmatrix} \right\|. \quad (4.4)$$

vanishes if, and only if,  $(W_i^n, W_{i+1}^n)$  is an ISS. Thus  $\sigma_{i+1/2}^n$  is well suited as a steady solution detector. To allow for a self adaptive correction, the high-order reconstructed interface values  $\widetilde{W}_{i\pm 1/2,\pm}^n$  and source term discretization  $\check{S}_i^n$  are modified according to a parameter  $\theta_{i+1/2} \in [0, 1]$  yielding the cell average values in case of a steady solution, thus resorting locally to the first-order well-balanced scheme. It is given by

$$\widetilde{W}_{i+\frac{1}{2},\pm}^n = \left( 1 - \theta_{i+\frac{1}{2}}^n \right) W_i^n + \theta_{i+\frac{1}{2}}^n \widetilde{W}_{i+1/2,\pm}^n \quad (4.5)$$

Therein, the indicator is based on the detector (4.4) and is defined by

$$\theta_{i+\frac{1}{2}}^n = \frac{\sigma_{i+\frac{1}{2}}^n}{\sigma_{i+\frac{1}{2}}^n + \left(\frac{\Delta x}{C_{i+\frac{1}{2}}^n}\right)^{d+1}}. \quad (4.6)$$

Note that  $\theta_{i+1/2}^n = 0$  as soon as  $(W_i^n, W_{i+1}^n)$  is an ISS, and otherwise yields  $\theta_{i+1/2}^n = 1 - \mathcal{O}(\Delta x^{d+1})$ . Thus, the modified reconstruction given in (4.5) satisfies

$$\widetilde{W}_{i+\frac{1}{2},-}^n = \begin{cases} W_i^n & \text{if } (W_i^n, W_{i+1}^n) \text{ is an ISS,} \\ \widetilde{W}_i^n(x_{i+\frac{1}{2}}) + \mathcal{O}(\Delta x^{d+1}) & \text{otherwise,} \end{cases}$$

which indeed shows that the scheme will be well-balanced. In (4.6), the quantity  $C_{i+\frac{1}{2}}^n$  is given by  $C_{i+\frac{1}{2}}^0 = 0$  in the first time step  $n = 0$  and

$$C_{i+\frac{1}{2}}^n = C_\theta \frac{1}{2} \left( \frac{\|W_{i+1}^n - W_{i+1}^{n-1}\|}{\Delta t} + \frac{\|W_i^n - W_i^{n-1}\|}{\Delta t} \right) \quad \text{if } n \geq 1. \quad (4.7)$$

The reader is referred to [4] for a discussion regarding the expression of  $C_{i+1/2}^n$ , and to the following section for the choice of the parameter  $C_\theta$  in the numerical experiments.

Similarly to (4.5), the modified high-order well-balanced source term is defined as

$$\check{S}_i^n = \left( 1 - \frac{\theta_{i-\frac{1}{2}}^n + \theta_{i+\frac{1}{2}}^n}{2} \right) \check{S}_i^n + \frac{\theta_{i+\frac{1}{2}}^n \mathcal{S}_{i+\frac{1}{2}}^n + \theta_{i-\frac{1}{2}}^n \mathcal{S}_{i-\frac{1}{2}}^n}{2}. \quad (4.8)$$

Summarizing, using (4.5) and (4.8), the high-order well-balanced scheme reads

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} \left( \mathcal{F}(\widetilde{W}_{i+\frac{1}{2},-}^n, \widetilde{W}_{i+\frac{1}{2},+}^n) - \mathcal{F}(\widetilde{W}_{i-\frac{1}{2},-}^n, \widetilde{W}_{i-\frac{1}{2},+}^n) \right) + \Delta t \check{S}_i^n. \quad (4.9)$$

To prevent spurious oscillations around discontinuities due to the high order reconstruction, the scheme (4.9) is endowed with a slope limiter. Since this is merely modifying the reconstruction operator  $\Pi_i^n$  in (4.1) the well-balanced property is not lost.

## 5 Numerical results

In this section, we perform numerical test cases to validate the theoretical properties of the first, second and third order well-balanced schemes, denoted respectively by  $\mathbb{P}_0$ ,  $\mathbb{P}_1^{\text{WB}}$  and  $\mathbb{P}_2^{\text{WB}}$ . The  $\mathbb{P}_1^{\text{WB}}$  scheme uses the `minmod` limiter, see for instance [49], and the `SSPRK2` time integrator, while the  $\mathbb{P}_2^{\text{WB}}$  scheme makes uses the third order TVD reconstruction from [43] and the `SSPRK3` time integrator. Both Runge-Kutta schemes can be found for instance in [30].

For all test cases, we consider the ideal gas EOS (2.2) with  $\gamma = 1.4$ . Further, we will use the gravitational potentials  $\varphi_1(x) = (x - 0.5)^2/2$  and  $\varphi_2(x) = \sin(x)$ . Unless otherwise mentioned, the parameters  $\Lambda$  in (3.2) and  $C_\theta$  in (4.7) are both set to 1.

### 5.1 Accuracy of the numerical schemes

To verify the experimental order of convergence (EOC) of the numerical schemes, we study an exact solution for the Euler equations with gravity (2.1) taken from [37], which is a variation of the test introduced in [52]. The analytical solution is given by

$$\begin{aligned} \rho(x, t) &= 1 + \frac{1}{5} \sin(k\pi(x - u_0 t)), \\ u(x, t) &= u_0, \\ p(x, t) &= \frac{9}{2} - (x - u_0 t) + \frac{1}{k\pi} \cos(k\pi(x - u_0 t)), \end{aligned} \quad (5.1)$$

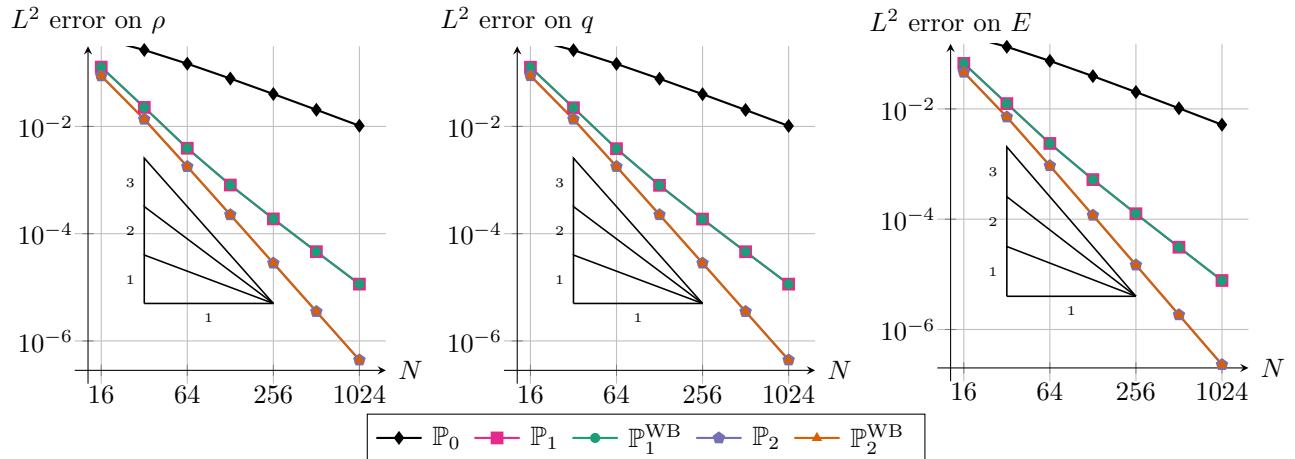


Figure 4: Analytical solution (5.1) with  $\varphi(x) = x$ :  $L^2$  errors and EOC for the first, second and third order schemes, with or without the high-order well-balanced correction.

where  $u_0$  is a constant velocity, taken as  $u_0 = 1$  in the experiment. We take  $k = 5$ , which yields a highly oscillatory solution. This test case is designed for a linear potential  $\varphi(x) = x$ . The simulation is carried out on the computational domain  $\Omega = [0, 2]$  up to a final time of  $t_f = 0.1$ , on seven grids with  $N = 16 \cdot 2^j$  cells, where  $j \in \{0, \dots, 6\}$ . We prescribe exact boundary conditions. In Figure 4, the  $L^2$  errors are depicted with respect to the number of cells. We recover the expected order of convergence for all schemes. Namely, we observe that the high-order well-balanced correction does not negatively affect the EOC of the respective schemes.

## 5.2 Well-balanced test cases

To numerically verify the well-balanced property of the schemes, we compute the numerical solutions of a hydrostatic and a moving equilibrium.

### 5.2.1 Hydrostatic equilibrium

We consider the isentropic hydrostatic atmosphere (2.15) with  $\hat{H}_0 = 1$  and  $s_0 = 1$ , thus it is given by

$$\rho_{\text{hyd}}(x) = \left(1 - \frac{\gamma - 1}{\gamma} \varphi(x)\right)^{1/(\gamma-1)}, \quad p_{\text{hyd}}(x) = \rho(x)^\gamma, \quad u = 0. \quad (5.2)$$

We use the two gravitational potentials  $\varphi_1$  and  $\varphi_2$ . The computational domain  $\Omega = [0, 1]$  is discretized in 50 cells, thus constitutes a very coarse mesh. Exact boundary conditions are prescribed for  $\varphi_1$ , and periodic boundary conditions are prescribed for  $\varphi_2$ . In Tables 1 and 2, the  $L^2$  errors at final time  $t_f = 1$  are compared to a non-well-balanced HLL solver for  $\varphi_1$  and  $\varphi_2$  respectively. All well-balanced schemes are able to preserve the hydrostatic equilibrium at machine precision, whereas the HLL scheme, as expected, has a quite large error. Moreover, due to the non-trivial contribution of  $\varphi_2$  the error using the HLL scheme is much higher. This underlines the necessity of well-balanced schemes.

### 5.2.2 Moving equilibrium

Next, a moving equilibrium is computed, characterized by the triplet  $q_0 = 1$ ,  $s_0 = 0$  and  $H_0 = 5$ . The gravitational potentials  $\varphi_1$  and  $\varphi_2$  are considered. For each  $\varphi$ , the initial conditions for  $\rho$  and  $E$  are obtained by solving the nonlinear system (2.12) employing Newton's method. The computational domain is given by  $\Omega = [0, 1]$ , and is partitioned in 50 cells. Homogeneous exact and periodic boundary conditions are prescribed for  $\varphi_1$  and  $\varphi_2$ , respectively. The  $L^2$  errors at time  $t = 1$  are given in Tables 3 and 4 for  $\varphi_1$  and  $\varphi_2$  respectively. Similarly to the simulation of the hydrostatic atmosphere from Section 5.2.1, all well-balanced schemes are able to preserve the

|        | HLL                   | $\mathbb{P}_0$         | $\mathbb{P}_1^{\text{WB}}$ | $\mathbb{P}_2^{\text{WB}}$ |
|--------|-----------------------|------------------------|----------------------------|----------------------------|
| $\rho$ | $6.390 \cdot 10^{-3}$ | $2.658 \cdot 10^{-15}$ | $2.727 \cdot 10^{-15}$     | $2.436 \cdot 10^{-15}$     |
| $q$    | $5.002 \cdot 10^{-4}$ | $1.154 \cdot 10^{-15}$ | $1.498 \cdot 10^{-15}$     | $1.691 \cdot 10^{-15}$     |
| $E$    | $2.289 \cdot 10^{-3}$ | $7.640 \cdot 10^{-15}$ | $7.640 \cdot 10^{-15}$     | $1.020 \cdot 10^{-14}$     |

Table 1: Well-balanced test case hydrostatic atmosphere with  $\varphi(x) = (x - 0.5)^2/2$ .

|        | HLL                   | $\mathbb{P}_0$         | $\mathbb{P}_1^{\text{WB}}$ | $\mathbb{P}_2^{\text{WB}}$ |
|--------|-----------------------|------------------------|----------------------------|----------------------------|
| $\rho$ | $1.109 \cdot 10^{-1}$ | $3.819 \cdot 10^{-15}$ | $2.620 \cdot 10^{-15}$     | $4.255 \cdot 10^{-15}$     |
| $q$    | $1.898 \cdot 10^{-2}$ | $1.439 \cdot 10^{-15}$ | $1.477 \cdot 10^{-15}$     | $1.522 \cdot 10^{-15}$     |
| $E$    | $7.170 \cdot 10^{-2}$ | $7.536 \cdot 10^{-15}$ | $6.822 \cdot 10^{-15}$     | $7.270 \cdot 10^{-15}$     |

Table 2: Well-balanced test case hydrostatic atmosphere with  $\varphi(x) = \sin(x)$ .

moving equilibrium up to machine precision, whereas the HLL scheme yields quite large errors. This verifies and illustrates the ability of the new well-balanced solvers to capture moving equilibria up to machine precision.

### 5.3 Perturbation of equilibrium states

As the main motivation behind the construction of well-balanced schemes lies in the resolution of small perturbations around equilibria, the next two sets of test cases concern the performance of the novel fully well-balanced solver on such flows. In particular, this allows the use of coarse meshes, as the solution is not polluted by background errors stemming from the resolution of the equilibrium state, which would then require a substantial grid refinement to reduce the truncation error and make the perturbation visible.

In this section, we consider the quadratic potential  $\varphi_1$ , and the computational domain is given by  $\Omega = [0, 1]$ , discretized using 50 cells, with inhomogeneous Dirichlet boundary conditions corresponding to the exact, unperturbed steady solution. For all cases, we set  $C_\theta = 1$  for the  $\mathbb{P}_1^{\text{WB}}$  scheme and  $C_\theta = 0.15$  for the  $\mathbb{P}_2^{\text{WB}}$  scheme.

#### 5.3.1 Perturbations of a hydrostatic equilibrium

As a first example, we consider a perturbation around an isentropic atmosphere (5.2) taken e.g. from [16]. The initial condition is given by

$$\rho(x, 0) = \rho_{\text{hyd}}(x), \quad u(x, 0) = 0, \quad p(x, 0) = p_{\text{hyd}}(x) + A \exp\left(-100 \left(x - \frac{1}{2}\right)^2\right), \quad (5.3)$$

where we set the amplitudes of the perturbations to be  $A = 10^{-4}$  and  $10^{-12}$ , where the latter is of the order of the truncation error of the well-balanced solver. The difference of the numerical solution against the equilibrium  $W_{\text{steady}} - W_{\text{num}}$  obtained with the novel well-balanced solvers at final time  $t = 0.075$  are depicted in Figure 5. The top panels contain the results with an initial perturbation of amplitude  $10^{-4}$ , while the bottom ones display

|        | HLL                   | $\mathbb{P}_0$         | $\mathbb{P}_1^{\text{WB}}$ | $\mathbb{P}_2^{\text{WB}}$ |
|--------|-----------------------|------------------------|----------------------------|----------------------------|
| $\rho$ | $5.569 \cdot 10^{-3}$ | $3.204 \cdot 10^{-15}$ | $2.897 \cdot 10^{-15}$     | $2.907 \cdot 10^{-15}$     |
| $q$    | $1.129 \cdot 10^{-3}$ | $1.916 \cdot 10^{-15}$ | $1.727 \cdot 10^{-15}$     | $1.803 \cdot 10^{-15}$     |
| $E$    | $4.188 \cdot 10^{-3}$ | $7.742 \cdot 10^{-15}$ | $8.970 \cdot 10^{-15}$     | $6.764 \cdot 10^{-15}$     |

Table 3: Well-balanced test case moving equilibrium with  $\varphi(x) = (x - 0.5)^2/2$ .



|        | HLL                    | $\mathbb{P}_0$         | $\mathbb{P}_1^{\text{WB}}$ | $\mathbb{P}_2^{\text{WB}}$ |
|--------|------------------------|------------------------|----------------------------|----------------------------|
| $\rho$ | $8.9233 \cdot 10^{-2}$ | $2.627 \cdot 10^{-15}$ | $3.607 \cdot 10^{-15}$     | $4.246 \cdot 10^{-15}$     |
| $q$    | $2.0153 \cdot 10^{-2}$ | $1.877 \cdot 10^{-15}$ | $1.727 \cdot 10^{-15}$     | $1.922 \cdot 10^{-15}$     |
| $E$    | $6.7186 \cdot 10^{-2}$ | $9.930 \cdot 10^{-15}$ | $9.357 \cdot 10^{-15}$     | $9.607 \cdot 10^{-15}$     |

Table 4: Well-balanced test case moving equilibrium with  $\varphi(x) = \sin(x)$ .

the results with the amplitude reduced to  $10^{-12}$ . In both cases, we observe that the perturbations are clearly visible and there are no spurious artifacts from the background atmosphere.

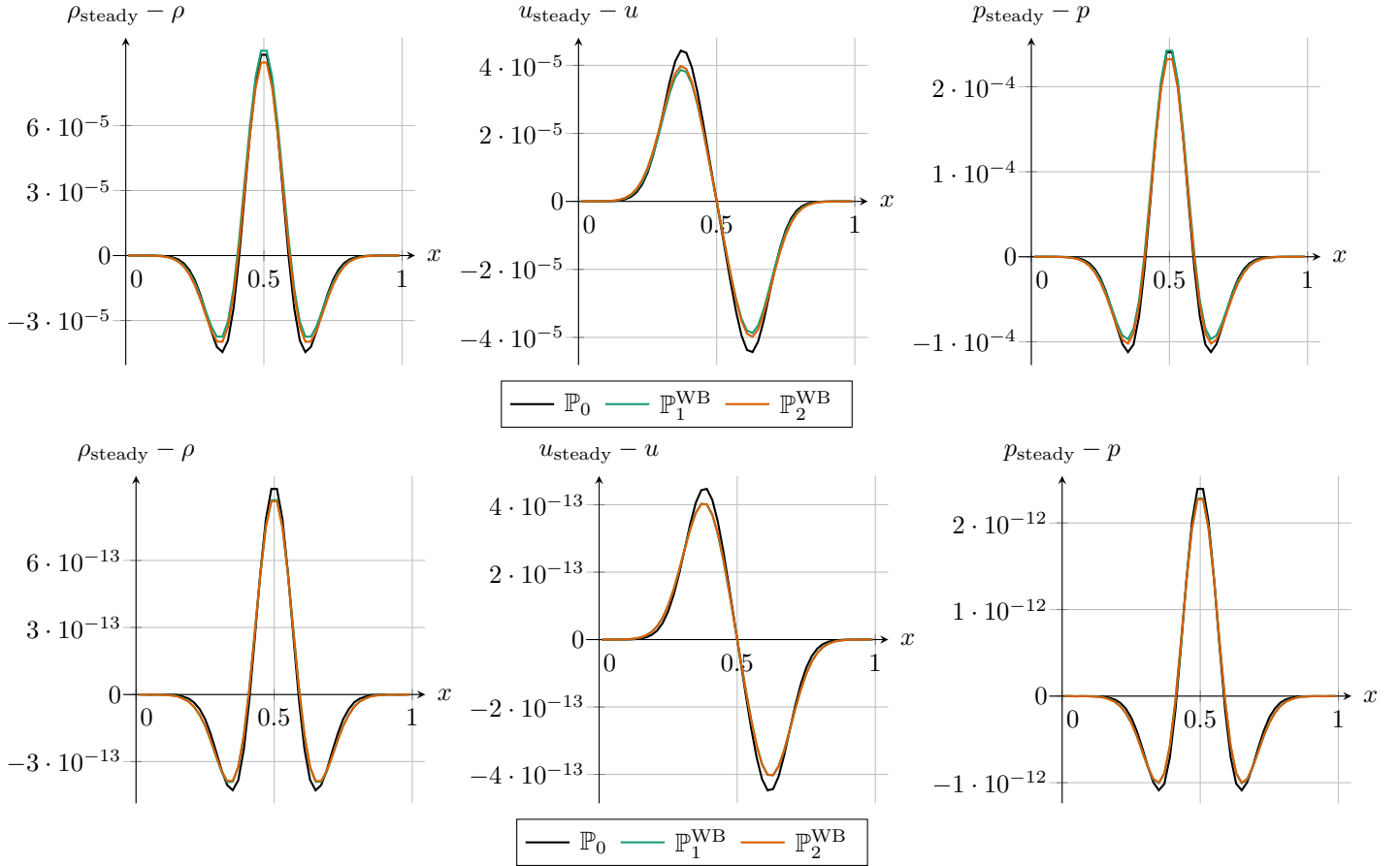


Figure 5: Perturbation of a hydrostatic equilibrium. Top panels: perturbation with initial amplitude  $A = 10^{-4}$ . Bottom: perturbation with initial amplitude  $A = 10^{-12}$ .

### 5.3.2 Perturbations of a moving equilibrium

Next, a perturbation of the moving equilibrium is considered. The initial condition is given by

$$\rho(x, 0) = \rho_0(x), \quad q(x, 0) = q_0, \quad p(x, 0) = p_0(x) + A \exp\left(-100 \left(x - \frac{1}{2}\right)^2\right), \quad (5.4)$$

where the equilibrium states are obtained from the triplet  $(q_0, s_0, H_0)$  already introduced to build the steady solution in Section 5.2.2. The numerical results are given in Figure 6 at the final time  $t = 0.075$ , arranged similarly to Figure 5. The perturbation around the moving equilibrium is once again well-captured and no spurious errors are introduced from the background equilibrium. In comparison to the hydrostatic atmosphere case from Figure 5, the perturbations are skewed towards the right and travel faster. This is due to a non-zero positive background velocity.

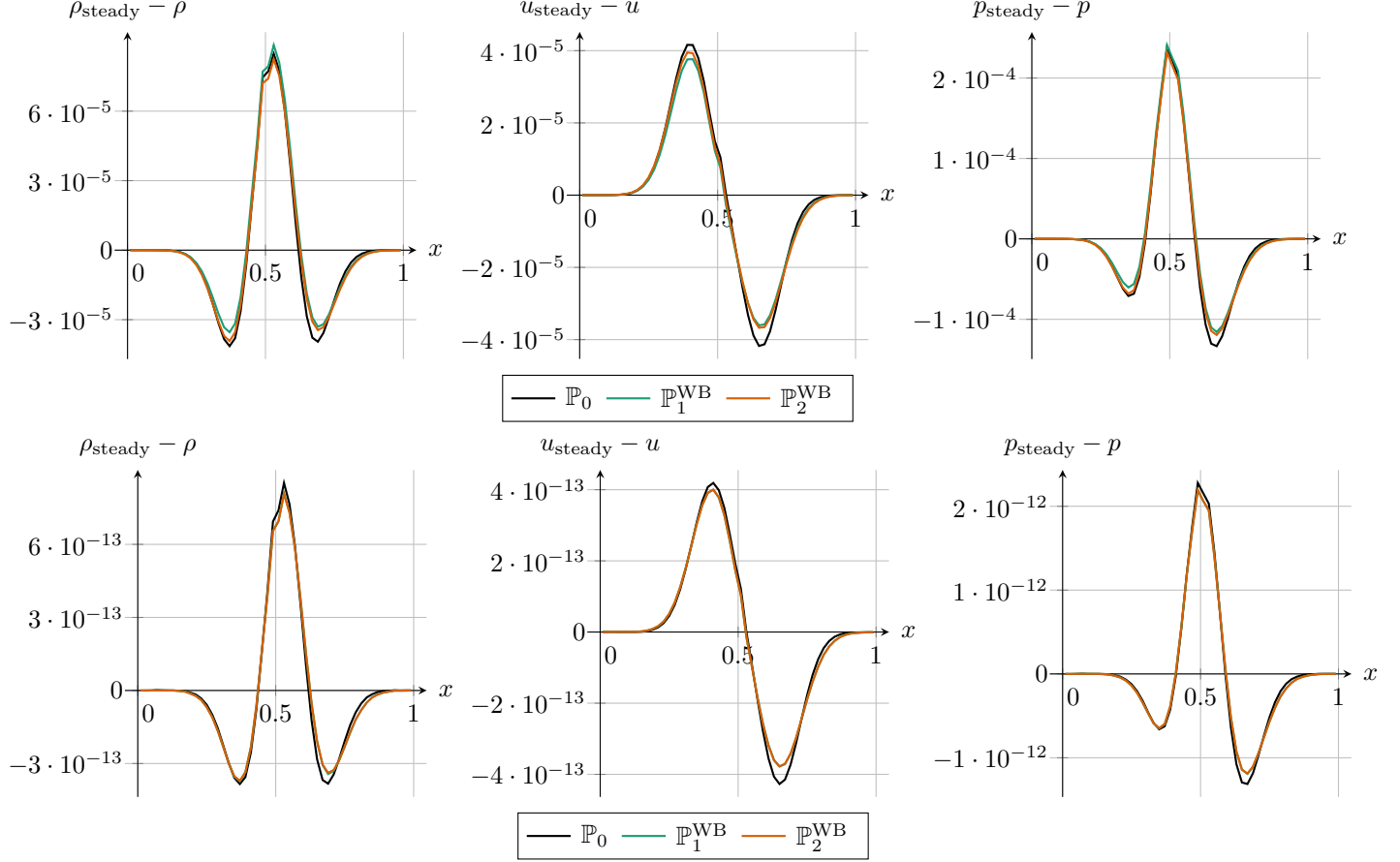


Figure 6: Perturbation of a moving equilibrium. Top panels: perturbation with initial amplitude  $A = 10^{-4}$ . Bottom: perturbation with initial amplitude  $A = 10^{-12}$ .

Another interesting test case concerns a moving steady solution perturbed by a wave stemming from a time-dependent boundary condition. We consider again the moving equilibrium given by  $(q_0, s_0, H_0)$ , with the perturbation applied as a right boundary condition acting only on the velocity, which is given at  $x = 1$  and  $t > 0$  by

$$u(1, t) = 10^{-8} \sin(8\pi t) \quad (5.5)$$

instead of by the unperturbed equilibrium. The computational domain is given by  $[0, 1]$  using 512 cells and the simulation is stopped shortly before the perturbation reaches the opposite boundary, at  $t = 0.72$ . Similar set-ups can be found in [24], motivated by the study of wave propagation in stellar atmospheres. The difference between numerical results and the moving equilibrium are given in Figure 7, where we have set  $\Lambda = 5$  to add some diffusion to the scheme. In this case, the perturbation remains well-resolved. This further verifies the ability of the numerical scheme to resolve small perturbations around hydrostatic and moving equilibria on coarse grids.

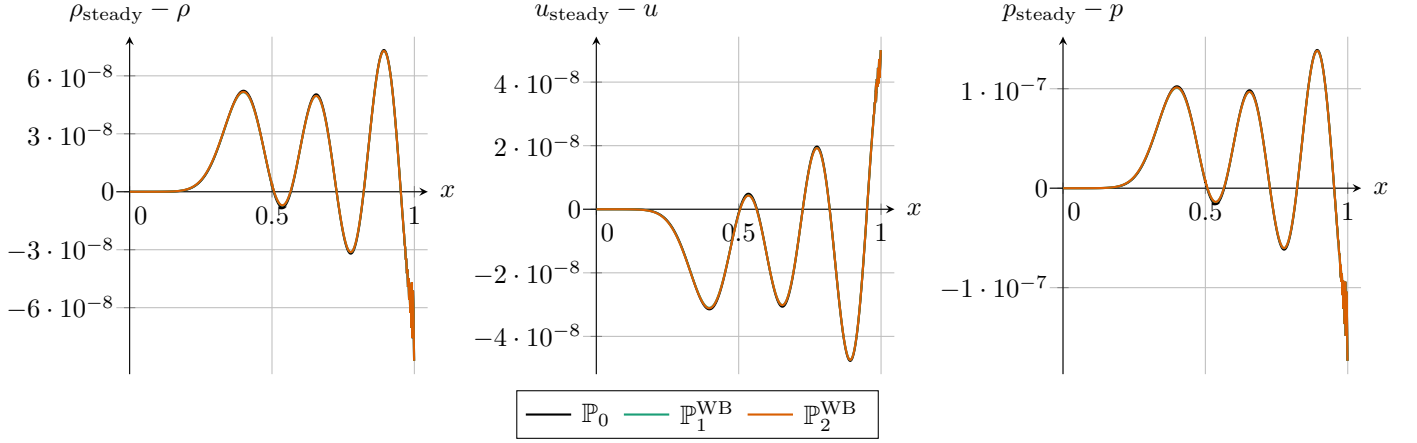


Figure 7: Perturbation of moving equilibrium from the boundary.

## 5.4 Riemann Problems

As a final series of test cases, we consider three Riemann Problems, two classical ones for the Euler equations without a source term, and one in presence of a gravitational field, far from an equilibrium.

In all these cases, the space domain is  $\Omega = [0, 1]$ , and we prescribe homogeneous Neumann boundary conditions. Moreover, the initial condition takes the form of a Riemann problem, i.e.,

$$W(x, 0) = \begin{cases} W_L & \text{if } x < x_0, \\ W_R & \text{if } x \geq x_0, \end{cases}$$

with the jump position  $x_0 = 0.5$ , and where the left and right states are given differently for each problem. In each case, we take a grid consisting of 75 cells.

### 5.4.1 Sod problem

We first consider the standard Sod problem, which is a well-known benchmark for Riemann solvers. The initial condition is given by

$$\rho_L = 1, \quad \rho_R = 0.125, \quad u_L = u_R = 0, \quad p_L = 1, \quad p_R = 0.1. \quad (5.6)$$

The gravitational potential is given by  $\varphi = 0$ . The results are given in [Figure 8](#), in primitive variables  $\rho$ ,  $u$  and  $p$  at time  $t = 0.1644$ . The first order scheme is diffusive, in particular over the contact and shock waves, whereas increasing the order of the new well-balanced scheme increases the resolution of all waves. As is typical for high-order schemes, small oscillations can be observed near the shock wave. However, all schemes are able to correctly determine the shock position and amplitude. The numerical results are compared against an exact reference solution obtained with an exact Riemann solver [\[47\]](#).

### 5.4.2 Double rarefaction

We then turn to a double rarefaction, whose initial data is

$$\rho_L = \rho_R = 1, \quad u_L = -\frac{10}{3}, u_R = \frac{10}{3}, \quad p_L = p_R = 1. \quad (5.7)$$

This is a challenging problem, as it will lead to a near-vacuum state in the middle of the domain. We still take the given gravitational potential  $\varphi = 0$ . The results are displayed in [Figure 8](#) at time  $t = 0.09$ . Despite the fact that  $\rho$  and  $p$  are very close to zero in the middle of the domain, no negative values of  $\rho$  and  $p$  are observed. The numerical results are compared against an exact reference solution obtained with an exact Riemann solver [\[47\]](#).

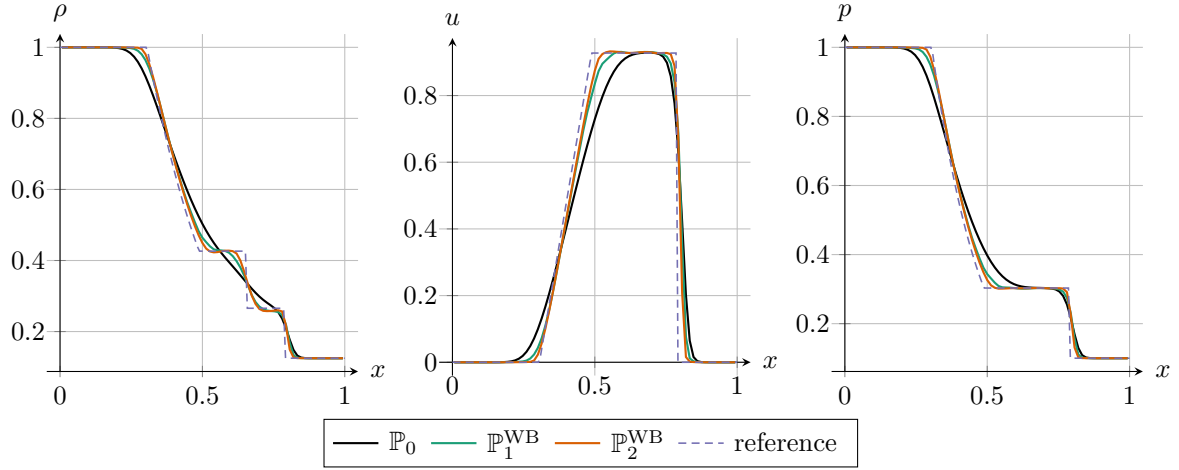


Figure 8: Riemann problem without a gravitational field influence: Sod problem displayed in primitive variables  $\rho$ ,  $u$  and  $p$ , at time  $t = 0.1644$  using 75 cells.

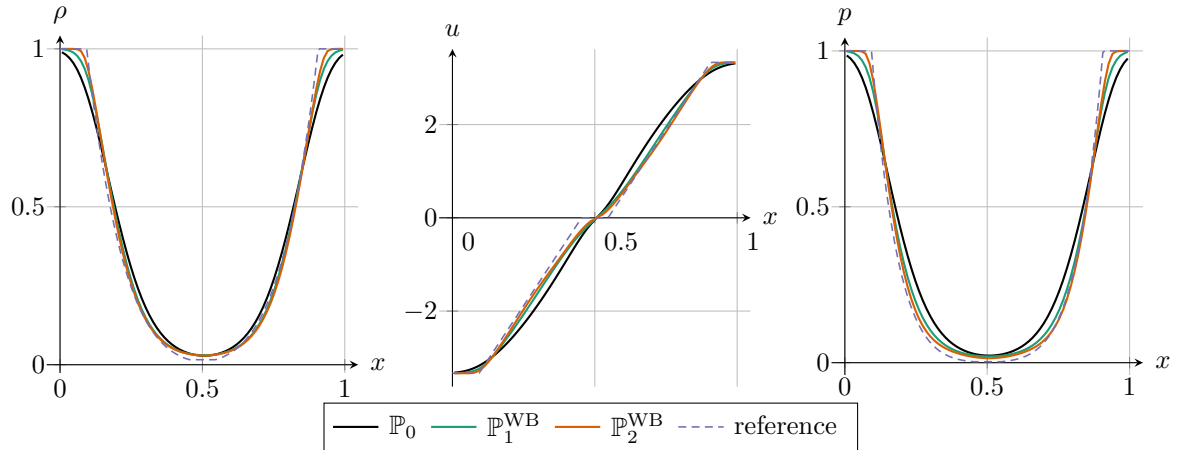


Figure 9: Riemann problem without a gravitational field influence: double rarefaction problem displayed in primitive variables  $\rho$ ,  $u$  and  $p$ , at time  $t = 0.09$  using 75 cells.

### 5.4.3 Riemann problem in a gravitational field

The last Riemann problem is a Sod-like problem in the presence of a gravitational source term, given by  $\varphi_1$ . The initial data is given in steady variables  $q$ ,  $s$  and  $H$ , as follows:

$$q_L = 0.5, q_R = 0, \quad s_L = 0, s_R = -\log(0.75), \quad H_L = 6, H_R = 3. \quad (5.8)$$

The results are displayed in [Figure 10](#) at time  $t = 0.2$ . This time, the equilibrium variables  $q$ ,  $s$  and  $H$  are displayed corresponding to the variables, in which the initial data is given. We observe that the solution generated by the new well-balanced scheme accurately captures the arising shock and rarefaction waves. The numerical results are compared against a reference solution obtained with the classical HLL scheme with a centered discretization of the source terms on 2000 cells.

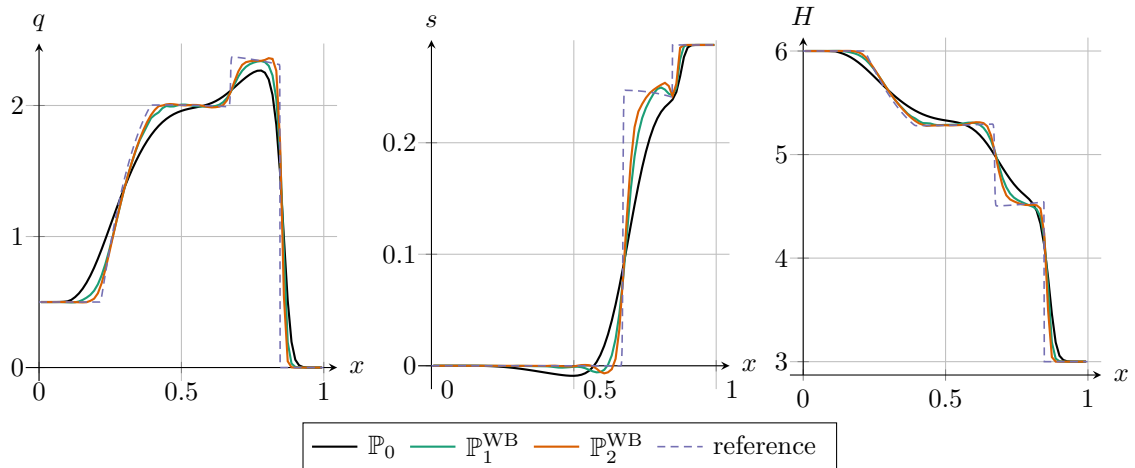


Figure 10: Riemann problem under a gravitational field influence: modified Sod problem displayed in steady variables  $q$ ,  $s$  and  $H$ , at time  $t = 0.2$  using 75 cells.

## 6 Conclusions

In this paper, we derived a new numerical scheme to approximate weak solutions of the Euler equations with a gravitational source term. The presented Godunov-type scheme is based on an approximate Riemann solver composed of two intermediate states. It is constructed such that the scheme is consistent, positivity-preserving, i.e., the positivity of density and pressure is preserved, and satisfies the discrete entropy inequalities. The proof of these properties are summarized in [Theorem 3.13](#) which relies on a main result of this work, given in [Theorem 3.5](#), where we showed that the specific entropy given by the HLL Riemann solver [33] satisfies a decreasing principle. The remaining degrees of freedom in the approximate Riemann solver were exploited, on the one hand, to incorporate the fully well-balanced property in the Riemann solver, and on the other hand, to make the Riemann solver coincide with the well-known HLL Riemann solver in the absence of a gravitational field. Thus, besides the above-mentioned properties, the new numerical scheme provably is fully well-balanced, i.e., the scheme preserves moving equilibria and the associated isentropic hydrostatic equilibria up to machine precision, as proven in [Theorem 3.13](#). Further, we have shown a strategy allowing an extension of the first-order Godunov-type numerical scheme to higher order accuracy, while preserving the fully well-balanced property of the first-order scheme. The performed numerical test cases illustrated the theoretical results presented in this work.

**Acknowledgments.** V. Michel-Dansac extends his thanks to ANR-22-CE25-0017 OptiTrust. The SHARK-FV conference has greatly contributed to this work.

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