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COMPUTATIONS REGARDING THE TORSION HOMOLOGY OF OELJEKLAUS–TOMA MANIFOLDS

DUNG PHUONG PHAN[1], TUAN ANH BUI[2], AND ALEXANDER D. RAHM[1]

Abstract. This article investigates the torsion homology behaviour in towers of Oeljeklaus–Toma (OT) manifolds. This adapts an idea of Silver–Williams from knot theory to OT manifolds and extends it to higher degree homology groups. In the case of surfaces, i.e. Inoue surfaces of type S^0, the torsion grows exponentially in both H_1 and H_2 according to a parameters which already plays a role in Inoue's classical paper. This motivates running example calculations in all homological degrees.

1. Introduction

In order to motivate the computations in this paper, let us for a moment look at an idea from knot theory. Suppose

$$K \hookrightarrow S^3$$

is a tame knot. Then one can put a tubular neighbourhood T around K and the knot complement is the space

$$X_K = S^3 - (\text{interior of } T)\,.$$

This makes sense not just as a topological space, but indeed gives a compact connected 3-manifold. By a standard computation, one always has

$$H_1(X_K, \mathbb{Z}) \simeq \mathbb{Z}, \tag{1}$$

see for example [6, Remark 18.4], independently of the choice of the knot. By the Hurewicz theorem, this means that the quotient of the fundamental group to its abelianization, i.e. the first homology group of X_K, is given by

$$q : \pi_1(X_K, *) \twoheadrightarrow \mathbb{Z}. \tag{2}$$

This map is almost canonical. Really, in both Equation 1 and 2 one just gets infinite cyclic groups canonically, so for identifying them with $\mathbb{Z}$ one needs to choose a generator of this infinite cyclic group, and changing this would change the maps by multiplication with -1.

Consider the subgroups

$$q^{-1}(n\mathbb{Z}) \subseteq \pi_1(X_K, *)$$

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This paper is dedicated to Oliver Braunling.

for integers $n \geq 1$, where q is as in Equation 2. These subgroups are well-defined, since multiplication by -1 preserves them. By covering space theory, each such subgroup corresponds to a finite covering

$$\begin{array}{c} X_{K,n} \\ | \\ X_K \end{array} \tag{3}$$

of degree n. Now, one can study the torsion homology of the spaces $X_{K,n}$, depending on n. Gonzáles-Acuña–Short [5] and Riley [9] have independently proven in the early 1990s that

$$\lim_{n \longrightarrow +\infty} \frac{\log |H_1(X_{K,n}, \mathbb{Z})_{tor}|}{n} = \log M(\Delta_K), \tag{4}$$

where Δ_K denotes the Alexander polynomial of the knot, and $M(-)$ the Mahler measure. So, loosely speaking, this shows that the amount of torsion first homology classes of the spaces $X_{K,n}$ grows in a very controlled way along n. For example, in the case $|M(\Delta_K)| > 1$ one obtains the asymptotic exponential growth

$$|H_1(X_{K,n}, \mathbb{Z})_{tor}| \sim M(\Delta_K)^n \qquad \text{as} \qquad n \longrightarrow +\infty.$$

Note that this is just an asymptotic; for small n this is a bad heuristic. The case $|M(\Delta_K)| > 1$ is fairly common, so one sees that the spaces $X_{K,n}$ for large n will have a tremendous amount of torsion classes in H_1. More qualitatively, one can also say that once the torsion homology is not bounded in n, it must already grow exponentially, and if Lehmer's conjecture is true, there is even a lower bound on the minimal possible exponential growth.

As a brief comment on the literature, we note that many articles instead discuss the first homology of the branched covering spaces $\widehat{X}_{K,n}$ instead (e.g., [5]), but one just has $H_1(X_{K,n}, \mathbb{Z})_{tor} \cong H_1(\widehat{X}_{K,n}, \mathbb{Z})$, so these considerations are just a different viewpoint, but equivalent, see [3, Chapter 8].

In 2002 Daniel Silver and Susan Williams have pointed out in their article [10] that many of the above considerations do not need to be restricted to knot complements. Instead, they point out that whenever one has a connected manifold X with a surjection

$$q : \pi_1(X, *) \twoheadrightarrow \mathbb{Z}$$

imitating Equation 2, one can run the analogous analysis for any such X. One has the corresponding covering spaces, call them X_n, and can study the limit

$$\lim_{n \longrightarrow +\infty} \frac{\log |H_1(X_n, \mathbb{Z})_{tor}|}{n}.$$

This note is about a special type of complex manifolds where this idea can be implemented:

Concretely, the situation envisioned by Silver–Williams is met for Inoue surfaces [7]; and in this note we only look at Inoue surfaces of so-called type S^0 (these are the ones discussed in §2 loc. cit.). These are compact non-Kähler complex surfaces (so instead of dimension 3, we now move to compact 4-dimensional real manifolds with rich extra structure. While the knot complements are frequently hyperbolic, the Inoue surfaces carry at least a locally conformal Kähler metric. This plays no role here though).

Inoue surfaces X are one of the types appearing in Kodaira's classification of minimal compact complex surfaces. They occur in the not fully understood Class

VII_0. They satisfy $H_1(X,\mathbb{Q}) \simeq \mathbb{Q}$, vaguely analogous to Equation 1, and one can set up a surjection

$$q : \pi_1(X,*) \twoheadrightarrow \mathbb{Z}$$

as in Equation 2, again canonical up to multiplication with -1 in $\mathbb{Z}$. As is pointed out in [2], one obtains

$$\lim_{n\longrightarrow+\infty} \frac{\log|H_1(X_n,\mathbb{Z})_{tor}|}{n} = \log M(f), \tag{5}$$

where $M(f)$ is the Mahler measure of a polynomial f which in the setting of Inoue's paper is the minimal polynomial of the matrix he denotes by "M" in [7,§2], or, in the setting of [2] the minimal polynomial of a unit in a certain number field (this translation follows the philosophy of [8]).

This result is entirely in line with the philosophy laid out by Silver and Williams in [10]. This motivates the question to investigate the remaining torsion homology

$$|H_r(X_n,\mathbb{Z})_{tor}|$$

for $r \neq 1$.

Theorem 1. *For Inoue surfaces of type S^0 (resp. Oeljeklaus–Toma manifolds with $r_1, r_2 = 1$), the torsion homology growth satisfies*

$$\log|H_r(X_n,\mathbb{Z})_{tor}| \sim \begin{cases} n\log M(f) & \text{for } r = 1,2 \\ 0 & \text{for } r = 0,3,4 \end{cases}$$

as $n \longrightarrow +\infty$, where $M(f)$ is the Mahler measure of the minimal polynomial as described above in the text. This is > 1, so the orders of the torsion part of both H_1 and H_2 grow exponentially with n.

We refer to §2 for the proof. However, Inoue surfaces of type S^0 admit a generalization to higher dimensions due to Oeljeklaus and Toma [8]. Given a number field K with $r_1 \geq 1$ real places and $r_2 = 1$ complex places, and any torsion-free finite-index subgroup

$$U \subseteq \mathcal{O}_K^{\times,+}$$

of the totally positive units (i.e. the units which are positive under any homomorphism $\mathcal{O}_K^\times \to \mathbb{R}$), they attach a complex manifold

$$X(K,U)$$

which is

- connected compact of complex dimension $r_1 + 1$,
- real dimension $2r_1 + 2$,
- non-Kähler (but carries a locally conformal Kähler metric),
- and whose underlying real manifold is a locally symmetric space for a solvable Lie group.

For any such manifold $X(K,U)$ the fundamental group sits in a canonical exact sequence

$$1 \longrightarrow G \longrightarrow \pi_1(X(K,U),*) \xrightarrow{q} U \longrightarrow 1,$$

see [1,Prop. 6]. In the special case of $r_1 = 1$, Dirichlet's Unit Theorem implies that $\mathcal{O}_K^{\times,+} \simeq \mathbb{Z}$, canonical up to multiplication with -1, and $X(K,U)$ is merely an Inoue surface. So, this is just the case as discussed above. However, for $r_1 = 2$, we have

$$\mathcal{O}_K^{\times,+} \simeq \mathbb{Z}^2,$$

canonical up to the action of $\mathrm{GL}_2(\mathbb{Z})$.

A good source of examples stems from fourth roots. Consider the number field

$$K = \mathbb{Q}(\sqrt[4]{p})$$

for a prime p. The polynomial $X^4 - p$ is irreducible by the Eisenstein criterion, it has two real roots $\pm\sqrt[4]{p}$ and a pair of complex conjugate roots $\pm i\sqrt[4]{p}$, so that K indeed satisfies $r_1 = 2$ and $r_2 = 1$. Hence, this is a great source for OT threefolds.

Let σ_1 and σ_2 be the real embeddings and σ_3 be a complex embedding of K respectively.

The ring of integers can be determined by a general result of Gassert [4,Theorem 1.1]. As soon as p satisfies

$$p^2 \neq p \bmod 4,$$

the ring of integers is

$$\mathcal{O}_K = \mathbb{Z}[\sqrt[4]{p}]$$

(the theorem is stated in Gassert's paper only as claiming that the ring of integers is monogenic, but the proof explicitly shows that the index $[\mathcal{O}_K : \mathbb{Z}[\sqrt[4]{p}]]$, using our notation, is one, so this shows that $\sqrt[4]{p}$ is a concrete generator). The totally multiplicative units of O_K is

$$\mathcal{O}_K^{\times} \simeq \mathbb{Z}\langle u, v\rangle$$

for generating units u, v.

Let

$$i = \min\left\{k \in \mathbb{N}^* : \sigma_1(u^k) > 0, \quad \sigma_2(u^k) > 0\right\},$$
$$j = \min\left\{k \in \mathbb{N}^* : \sigma_1(v^k) > 0, \quad \sigma_2(v^k) > 0\right\},$$

then $O_K^{\times,+} = \mathbb{Z} < u^i, v^j >$.

For each $m, n \geq 1$, define the monomorphism $\varphi :< u^{im}, v^{jn} >\to Aut(O_K)$, $x \mapsto \varphi(x)$ such that $\varphi(x)(y) = xy$, for all $y \in O_K$. With this monomorphism, the semidirect product $O_K \rtimes_\varphi \mathbb{Z} < u^{im}, v^{jn} >$ is defined with the product

$$(h_1, u^{im_1}v^{jn_1}).(h_2, u^{im_2}v^{jn_2}) = (h_1 + u^{im_1}v^{jn_1}h_2, u^{i(m_1+m_2)}v^{j(n_1+n_2)}),$$

for all $h_1, h_2 \in O_K$, $u^{im_1}v^{jn_1}, u^{im_2}v^{jn_2} \in \mathbb{Z} < u^{im}, v^{jn} >$.

The action

$$* : (O_K \rtimes_\varphi \mathbb{Z} < u^{im}, v^{jn} >) \times (\mathbb{H}^2 \times \mathbb{C}) \to (\mathbb{H}^2 \times \mathbb{C})$$
$$((h, k), (z_1, z_2, z_3)) \mapsto (\sigma_1(h) + \sigma_1(k)z_1, \sigma_2(h) + \sigma_2(k)z_2, \sigma_3(h) + \sigma_3(k)z_3)$$

is a properly discontinuous action of $O_K \rtimes_\varphi < u^{im}, v^{jn} >$ on $\mathbb{H}^2 \times \mathbb{C}$. Then

$$X_{m,n} = X(K, \mathbb{Z} < u^{im}, v^{jn} >) := (\mathbb{H}^2 \times \mathbb{C})/(O_K \rtimes_\varphi \mathbb{Z} < u^{im}, v^{jn} >)$$

is itself an Oeljeklaus–Toma threefold (or "OT threefold").

In §3, we describe an algorithm to find the homology of an OT manifold $X_{m,n}$.

Next, a detailed algorithm for the case $p = 2$ will be described in §4, and a GAP program which implements this algorithm will be distributed online [11].

Proposition 2. *Our algorithm produces the following results for the r-th degree homology of $X_{m,n}$ when $p = 2$:*

m	n	r=1	r=2	r=3	r=4	r=5	r=6
1	1	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus \mathbb{Z}$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
2	1	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/3)^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
3	1	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus \mathbb{Z}$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
1	2	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus \mathbb{Z}$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
2	2	$(\mathbb{Z}/2)^4 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/16 \oplus \mathbb{Z}$				
3	2	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus \mathbb{Z}$				
1	3	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus \mathbb{Z}$				
2	3	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/9)^2 \oplus (\mathbb{Z}/43)^2 \oplus \mathbb{Z}$				
3	3	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^5 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}$				

Finally, in Section 5, we determine explicitly the first homology of some classes of the manifolds $X_{m,n}$ in the case $K = \mathbb{Q}(\sqrt[4]{2})$ and obtain the following tow propositions.

Proposition 3. *For Oeljeklaus–Toma manifolds with $r_1 = 2$, $r_2 = 1$ when $p = 2$, the first homology is of isomorphism type*

$$H_1(X_{m,1}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{for} \quad m \nmid 6 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/14\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{for} \quad m \mid 6. \end{cases}$$

Proposition 4. *For Oeljeklaus–Toma manifolds with $r_1 = 2$, $r_2 = 1$ when $p = 2$, the first homology is of isomorphism type $H_1(X_{m,n}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ for all $n \in \mathbb{N}^*$ when m is odd.*

For the proof of Propositions 3 and 4, see Section 5.

Proposition 5. *Our computational results on the isomorphism type of the first homology of some $X_{m,n}$ are the following:*

m	n= 1	n=2	n=3	n=4	n=5
1	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$
2	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$
3	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$
4	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/8)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$
5	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$
6	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/7)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}^2$
7	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$
8	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/3)^4 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/8)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$
9	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$
10	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$
11	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}^2$
12	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/7)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/8)^2 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}^2$

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2. Proof of Theorem 1

Let $X_n = X(K, \mathbb{Z} < v^n >)$. As OT manifold X_n of dimension 4 is oriented closed, according to Poincaré duality,

$$H^k(X) \cong H_{4-k}(X), 0 \leq k \leq 4.$$

With $k = 3$, $H^3(X) \cong H_1(X)$, so the torsion of $H^3(X)$ is isomorphic to the torsion of $H_1(X)$. On the other hand, by the Universal Coefficient Theorem,

$$H^1(X_n) \cong Torsion(H_0(X_n)) \oplus [H_1(X_n)/Torsion(H_1(X_n))].$$

Since $H_0(X_n) \cong \mathbb{Z}$,

$$H_3(X_n) \cong H^1(X_n) \cong [H_1(X_n)/Torsion(H_1(X_n))].$$

This implies

$$Torsion(H_3(X_n)) \cong Torsion(H^1(X_n)) = 0.$$

In the case of $H^2(X_n)$, again with the Universal Coefficient Theorem,

$$H^2(X_n) \cong Torsion(H_1(X_n)) \oplus [H_2(X_n)/Torsion(H_2(X_n))].$$

This implies

$$Torsion(H_2(X_n)) \cong Torsion(H^2(X_n)) \cong Torsion(H_1(X_n)).$$

Usage of Equation (5), stated above, completes the proof of the theorem.

3. An algorithm computing the homology of $X_{m,n}$

3.1. **A resolution of $\mathbb{Z}$ over $\mathbb{Z}^2$.** Consider the topological space $\mathbb{R}^2$ and the multiplicative group

$$G =< x, y |\ x \text{ and } y \text{ commute} >,$$

with the generators acting via

$$(a, b) \mapsto x(a, b) := (a + 1, b) \text{ and } (a, b) \mapsto y(a, b) := (a, b + 1),$$

for all $(a, b) \in \mathbb{R}^2$. Then the group G acts on the space $\mathbb{R}^2$ by the translation $G \times \mathbb{R}^2 : (g, u) \mapsto gu := g(u)$.
With this action, G forms a CW-structure on $\mathbb{R}^2$, and this CW-structure induces a CW-structure on the quotient space $\mathbb{R}^2/G$. Let $C_k(\mathbb{R}^2/G)$ be the free abelian group generated by all k-cells in $\mathbb{R}^2/G$, then the cellular chain complex

$$0 \to C_2(\mathbb{R}^2/G) \xrightarrow{\partial_2} C_1(\mathbb{R}^2/G) \xrightarrow{\partial_1} C_0(\mathbb{R}^2/G) \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

is a resolution of $\mathbb{Z}$ over $\mathbb{Z}^2$ with the contracting homotopies

- $h_1 : C_1 \to C_2$ with

$$h_1(x^m y^n e_1^1) = 0;$$

$$h_1(x^m y^n e_2^1) = sign(m) \left(\sum_{i=\overline{m}}^{m-1-\overline{m}} x^i \right) y^n e_{12}^2.$$

- $h_0 : C_0 \to C_1$ with

$$h_0(x^m y^n e^0) = sign(m) \left(\sum_{i=\overline{m}}^{m-1-\overline{m}} x^i \right) y^n e_1^1 + sign(n) \left(\sum_{j=\overline{n}}^{n-1-\overline{n}} y^j \right) e_2^1.$$

In the above formulas,

$$\overline{m} = \frac{m-|m|)}{2}, \quad \overline{n} = \frac{n-|n|)}{2},$$
$$e_1^1 = \{(a,0) : 0 < a < 1\}, \quad e_2^1 = \{(0,b) : 0 < b < 1\},$$
$$e_{12}^2 = e_1^1 \times e_2^1.$$

3.2. **A resolution of $\mathbb{Z}$ over $\mathbb{Z}^4$.** Consider the topological space $\mathbb{R}^4$ and the multiplicative group

$$G =< x,\ y,\ z,\ t \mid x,\ y,\ z \text{ and } t \text{ commute} >,$$

with the generators acting via

$$(a,b,c,d) \mapsto x(a,b,c,d) := (a+1,b,c,d),$$
$$(a,b,c,d) \mapsto y(a,b,c,d) := (a,b+1,c,d),$$
$$(a,b,c,d) \mapsto z(a,b,c,d) := (a,b,c+1,d),$$
$$(a,b,c,d) \mapsto t(a,b,c,d) := (a,b,c,d+1),$$

for all $(a,b,c,d) \in \mathbb{R}^4$. Then the group G acts on the space $\mathbb{R}^4$ by the translation $G \times \mathbb{R}^4 : (g,u) \mapsto g(u)$.
With this action, G forms a CW-structure on $\mathbb{R}^4$ and this CW-structure induces a CW-structure on the quotient space $\mathbb{R}^4/G$. Let $C_k(\mathbb{R}^4/G)$ be the free abelian group generated by all k-cells in $\mathbb{R}^4/G$, then the cellular chain complex

$$0 \to C_4(\mathbb{R}^4/G) \overset{\partial_4}{\to} C_3(\mathbb{R}^4/G) \overset{\partial_3}{\to} C_2(\mathbb{R}^4/G) \overset{\partial_2}{\to} C_1(\mathbb{R}^4/G) \overset{\partial_1}{\to} C_0(\mathbb{R}^4/G) \overset{\varepsilon}{\to} \mathbb{Z} \to 0,$$

is a resolution of $\mathbb{Z}$ over $\mathbb{Z}^4$ with the contracting homotopies

- $h_3 : C_3 \to C_4$ with

$$h_3(x^m y^n z^p t^q e_{123}^3) = h_3(x^m y^n z^p t^q e_{124}^3) = h_3(x^m y^n z^p t^q e_{134}^3) = 0,$$
$$h_3(x^m y^n z^p t^q e_{234}^3) = sign(m) \left(\sum_{i=\overline{m}}^{m-1-\overline{m}} x^i \right) y^n z^p t^q e_{1234}^4.$$

- $h_2 : C_2 \to C_3$, with

$$h_2(x^m y^n z^p t^q e_{12}^2) = h_2(x^m y^n z^p t^q e_{13}^2) = h_2(x^m y^n z^p t^q e_{14}^2) = 0,$$
$$h_2(x^m y^n z^p t^q e_{23}^2) = sign(m) \left(\sum_{i=\overline{m}}^{m-1-\overline{m}} x^i \right) y^n z^p t^q e_{123}^3,$$
$$h_2(x^m y^n z^p t^q e_{24}^2) = sign(m) \left(\sum_{i=\overline{m}}^{m-1-\overline{m}} x^i \right) y^n z^p t^q e_{124}^3,$$
$$h_2(x^m y^n z^p t^q e_{34}^2) = sign(m) \left(\sum_{i=\overline{m}}^{m-1-\overline{m}} x^i \right) y^n z^p t^q e_{134}^3 + sign(n) \left(\sum_{j=\overline{n}}^{n-1-\overline{n}} y^j \right) z^p t^q e_{234}^3.$$

- $h_1 : C_1 \to C_2$, with

$$h_1(x^m y^n z^p t^q e_1^1) = 0,$$

$$h_1(x^m y^n z^p t^q e_2^1) = sign(m) \left(\sum_{i=\overline{m}}^{m-1-\overline{m}} x^i \right) y^n z^p t^q e_{12}^2,$$

$$h_1(x^m y^n z^p t^q e_3^1) = sign(m) \left(\sum_{i=\overline{m}}^{m-1-\overline{m}} x^i \right) y^n z^p t^q e_{13}^2 + sign(n) \left(\sum_{j=\overline{n}}^{n-1-\overline{n}} y^j \right) z^p t^q e_{23}^2,$$

$$h_1(x^m y^n z^p t^q e_4^1) = sign(m) \left(\sum_{i=\overline{m}}^{m-1-\overline{m}} x^i \right) y^n z^p t^q e_{14}^2 + sign(n) \left(\sum_{j=\overline{n}}^{n-1-\overline{n}} y^j \right) z^p t^q e_{24}^2$$
$$+ sign(p) \left(\sum_{k=\overline{p}}^{p-1-\overline{p}} z^k \right) t^q e_{34}^2.$$

- $h_0 : C_0 \to C_1$ with

$$h_0(x^m y^n z^p t^q e^0) = sign(m) \left(\sum_{i=\overline{m}}^{m-1-\overline{m}} x^i \right) y^n z^p t^q e_1^1 + sign(n) \left(\sum_{j=\overline{n}}^{n-1-\overline{n}} y^j \right) z^p t^q e_2^1$$
$$+ sign(p) \left(\sum_{k=\overline{p}}^{p-1-\overline{p}} z^k \right) t^q e_3^1 + sign(q) \left(\sum_{l=\overline{q}}^{q-1-\overline{q}} t^l \right) e_4^1.$$

In the above formulas,

$$\overline{m} = \frac{m-|m|}{2}, \quad \overline{n} = \frac{n-|n|}{2}, \quad \overline{p} = \frac{p-|p|}{2}, \quad \overline{q} = \frac{q-|q|}{2},$$
$$e_1^1 = \{(a,0,0,0) : 0 < a < 1\}, \quad e_2^1 = \{(0,b,0,0) : 0 < b < 1\},$$
$$e_3^1 = \{(0,0,c,0) : 0 < c < 1\}, \quad e_4^1 = \{(0,0,0,d) : 0 < d < 1\},$$
$$e_{ij}^2 = e_i^1 \times e_j^1, \quad i = \overline{1,4}, \quad j = \overline{i+1,4},$$
$$e_{ijk}^3 = e_i^1 \times e_j^1 \times e_k^1, \quad i = \overline{1,4}, \quad j = \overline{i+1,4}, \quad k = \overline{j+1,4},$$
$$e_{1234}^4 = e_1^1 \times e_2^1 \times e_3^1 \times e_4^1.$$

3.3. **The isomorphism from $O_K \rtimes_\varphi \mathbb{Z} < u^{im}, v^{jn} >$ to $\mathbb{Z}^4 \rtimes_{\overline{\varphi}} \mathbb{Z}^2$.** The multiplicative group $\mathbb{Z} < u^{im}, v^{jn} >$ is isomorphic to the additive group $\mathbb{Z}^2$ by the isomorphism $\alpha : (u^{im})^p (v^{jn})^q \mapsto (p,q)$. The additive group of the ring of integers

$$O_K = \left\{ a + bw + cw^2 + dw^3 / a,b,c,d \in \mathbb{Z}, w^4 = p \right\}$$

of K is isomorphic to $\mathbb{Z}^4$ by the isomorphism $\psi : a + bw + cw^2 + dw^3 \mapsto (a,b,c,d)$. Using α and ψ, we define $\overline{\varphi} : \mathbb{Z}^2 \to Aut(\mathbb{Z}^4)$, $(s,t) \mapsto \overline{\varphi}(s,t)$, where

$$\overline{\varphi}(s,t)(y) = \psi(\varphi(\alpha^{-1}(s,t)))\psi^{-1}(y), \quad \text{for all} \quad y \in \mathbb{Z}^4.$$

In detail, for $y = (a,b,c,d) \in \mathbb{Z}^4$,

$$\varphi(\alpha^{-1}(s,t))\psi^{-1}(y) = \begin{pmatrix} 1 & w & w^2 & w^3 \end{pmatrix} M^{ims} N^{jnt} y^T$$

where M and N are the matrices defined by

$$u\begin{pmatrix}1 & w & w^2 & w^3\end{pmatrix} = \begin{pmatrix}1 & w & w^2 & w^3\end{pmatrix} M,$$
$$v\begin{pmatrix}1 & w & w^2 & w^3\end{pmatrix} = \begin{pmatrix}1 & w & w^2 & w^3\end{pmatrix} N$$

respectively. Finally, $\overline{\varphi}(s,t)(y) = \psi(\varphi(\alpha^{-1}(s,t)))\psi^{-1}(y) = (A_{s,t}, B_{s,t}, C_{s,t}, D_{s,t})$ with

$$\begin{pmatrix}A_{s,t}\\ B_{s,t}\\ C_{s,t}\\ D_{s,t}\end{pmatrix} = M^{ims}N^{jnt}y^T$$

This homomorphism helps us to form the semidirect product $\mathbb{Z}^4 \rtimes_{\overline{\varphi}} \mathbb{Z}^2$ which is isomorphic to $O_K \rtimes_{\varphi} \mathbb{Z} < u^{im}, v^{jn} >$.
Since two groups which are isomorphic have the same homology groups, to find the homology groups of $O_K \rtimes_{\varphi} \mathbb{Z} < u^{im}, v^{jn} >$, we just need to find the homology groups of $\mathbb{Z}^4 \rtimes_{\overline{\varphi}} \mathbb{Z}^2$. The following is the algorithm to find the homologies of $\mathbb{Z}^4 \rtimes_{\overline{\varphi}} \mathbb{Z}^2$.

3.4. **An algorithm for computing the homology of $\mathbb{Z}^4 \rtimes_{\overline{\varphi}} \mathbb{Z}^2$.**

Step 1. Construct $\overline{\varphi} : \mathbb{Z}^2 \to Aut(\mathbb{Z}^4)$, $(s,t) \mapsto \overline{\varphi}(s,t)$, with $\overline{\varphi}(s,t) : (a,b,c,d) \mapsto (A_{s,t}, B_{s,t}, C_{s,t}, D_{s,t})$, where $A_{s,t}, B_{s,t}, C_{s,t}, D_{s,t}$ are defined by the following formula:

$$\begin{pmatrix}A_{s,t}\\ B_{s,t}\\ C_{s,t}\\ D_{s,t}\end{pmatrix} = M^{ims}N^{jnt}\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix}.$$

Then construct the semidirect product $E = \mathbb{Z}^4 \rtimes_{\overline{\varphi}} \mathbb{Z}^2$ with this $\overline{\varphi}$.

Step 2. The projection $p : E \to \mathbb{Z}^2$, $(a,b,c,d,e,f) \mapsto (e,f)$, for all $(a,b,c,d,e,f) \in E$ has $\ker(p) = \mathbb{Z}^4$. So E is a group extension of its normal subgroup $\mathbb{Z}^4$ and the quotient group $\mathbb{Z}^2$. Let R and S be the free resolution of $\mathbb{Z}$ over $\mathbb{Z}^4$ and $\mathbb{Z}^2$ respectively. The twisted tensor product of these resolutions, which we obtain using C.T.C Wall's method, gives us a free resolution for the semidirect product E.

Step 3. Compute the desired homology groups using the above resolution.

4. An explicit example when $p = 2$

In this case, the field K is $\mathbb{Q}(\sqrt[4]{2})$. Two generators of the free part of $O_K^{\times}$ are $u = w^2 - 1$, $v = w + 1$, with $w = \sqrt[4]{2}$, and $O_K^{\times} = \mathbb{Z} < u, v >$.
Since $\sigma_1(u) > 0$, $\sigma_2(u) > 0$, $\sigma_1(v) > 0$, $\sigma_2(v) < 0$, $O_K^{\times,+} = \mathbb{Z} < u, v^2 >$.

We observe that

$$u\begin{pmatrix}1 & w & w^2 & w^3\end{pmatrix} = \begin{pmatrix}1 & w & w^2 & w^3\end{pmatrix} M,$$

$$v\begin{pmatrix}1 & w & w^2 & w^3\end{pmatrix} = \begin{pmatrix}1 & w & w^2 & w^3\end{pmatrix} N,$$

with

$$M = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

So, $O_K \rtimes_\varphi \mathbb{Z} < u^m, v^{2n} >$ is isomorphic to $\mathbb{Z}^4 \rtimes_{\overline{\varphi}} \mathbb{Z}^2$ by the isomorphism $\overline{\varphi}(s,t)(y) = \psi(\varphi(\alpha^{-1}(s,t)))\psi^{-1}(y) = (A_{s,t}, B_{s,t}, C_{s,t}, D_{s,t})$ with

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \\ C_{s,t} \\ D_{s,t} \end{pmatrix} = M^{ms} N^{2nt} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

5. Proof of Propositions 3 and 4

Let us now provide the proofs of Propositions 3 and 4 stated in the Introduction.

Proposition 3. *For Oeljeklaus–Toma manifolds with $r_1 = 2$, $r_2 = 1$ when $p = 2$, the first homology is of isomorphism type*

$$H_1(X_{m,1}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{for} \quad m \nmid 6 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/14\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{for} \quad m \mid 6. \end{cases}$$

Proof.

$$\mathbb{Z}^4 =< f_1, f_2, f_3, f_4 / f_i f_j = f_j f_i, i = \overline{1,4}, j = \overline{1,4} >=< X/R >$$
$$\mathbb{Z}^2 =< u_1, u_2 / u_1 u_2 = u_2 u_1 >=< Y/S >$$

$$E = \mathbb{Z}^4 \rtimes_{\overline{\varphi}} \mathbb{Z}^2 =< X, Y/R, S, yxy^{-1} = \overline{\varphi}(y)(x), \forall x \in X, \forall y \in Y >$$

where with $y = (s,t)$,

$$\overline{\varphi}(y)(x) = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}^{ms} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^{2t} x.$$

We can see $H_1(E) \cong E/[E,E]$.
To find $[E,E]$, we consider the system of equations

$$u_i f_j u_i^{-1} = \varphi(u_i)(f_j) = f_1^{a_{ij}} f_2^{b_{ij}} f_3^{c_{ij}} f_4^{d_{ij}}, \quad i = \overline{1,2}, \quad j = \overline{1,4},$$

from which we can imply a system of equations of equivalence classes

$$\overline{f_1}^{a_{ij} - e_{1j}} \overline{f_2}^{b_{ij} - e_{2j}} \overline{f_3}^{c_{ij} - e_{3j}} \overline{f_4}^{d_{ij} - e_{4j}} = \overline{1}, \quad i = \overline{1,2}, \quad j = \overline{1,4},$$

where $e_{kj} = \begin{cases} 1 & \text{if} \quad k = j \\ 0 & \text{if} \quad k \neq j. \end{cases}$
Picking up the power of $\{\overline{f_i}\}$ in each equation, we obtain a matrix of order 8×4.

In the case m is arbitrary and $n = 1$, the matrix has the form

$$\begin{pmatrix} a_m - 1 & 0 & b_m & 0 \\ 0 & a_m - 1 & 0 & b_m \\ 2b_m & 0 & a_m - 1 & 0 \\ 0 & 2b_m & 0 & a_m - 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 2 \\ 4 & 2 & 0 & 0 \end{pmatrix}$$

where

$$a_m = \frac{1}{2}(-1+\sqrt{2})^m + \frac{1}{2}(-1-\sqrt{2})^m,$$

$$b_m = \frac{1}{4}(-1+\sqrt{2})^{m+1} + \frac{1}{4}(-1-\sqrt{2})^{m+1} + \frac{1}{4}(-1+\sqrt{2})^m + \frac{1}{4}(-1-\sqrt{2})^m.$$

Applying the row transformations, we change this matrix into the matrix (*) as below

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & a_m + 4b_m - 1 \\ 0 & 0 & 0 & 2(a_m - 1) + b_m \end{pmatrix} \quad (*)$$

Now, when $m = 6k, k \in \mathbb{N}^*$, $gcd(7, a_m + 4bm - 1, 2(a_m - 1) + b_m) = 7$, then the matrix (*) is row equivalent to the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case, $H_1(X_{m,1}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.
On the other hand, when $m \neq 6k, k \in \mathbb{N}^*$,

$$gcd(7, a_m + 4bm - 1, 2(a_m - 1) + b_m) = 1,$$

then the matrix (*) is row equivalent to the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case, $H_1(X_{m,1}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Proposition 4. *For Oeljeklaus–Toma manifolds with $r_1 = 2$, $r_2 = 1$ when $p = 2$, the first homology is of isomorphism type $H_1(X_{m,n}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ for all $n \in \mathbb{N}^*$ when m is odd.*

Proof. Apply an analogous procedure as in the proof of Proposition 2 for the case m is odd and n is arbitrarily, we obtain the matrix of powers of $\{\overline{f_i}\}$ of the form

$$\begin{pmatrix} a_m-1 & 0 & b_m & 0 \\ 0 & a_m-1 & 0 & b_m \\ 2b_m & 0 & a_m-1 & 0 \\ 0 & 2b_m & 0 & a_m-1 \\ 2a_n & 2b_n & c_n & d_n \\ 2d_n & 2a_n & 2b_n & c_n \\ 2c_n & 2d_n & 2a_n & 2b_n \\ 4b_n & 2c_n & 2d_n & 2a_n \end{pmatrix}$$

By applying some row operations $\alpha r_i + r_j \to r_j$, this matrix turns into the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -a_m & 0 \\ 0 & 0 & 0 & -a_m \\ 0 & 0 & -a_nc_m+c_n & -b_nc_m+d_n \\ 0 & 0 & -d_nc_m+2b_n & -a_nc_m+c_n \\ 0 & 0 & -c_nc_m+2a_n & -d_nc_m+2b_n \\ 0 & 0 & -2b_nc_m+2d_n & -c_nc_m+2a_n \end{pmatrix}$$

Let

$$\alpha = gcd(-a_m, -a_nc_m+c_n, -d_nc_m+2b_n, -c_nc_m+2a_n, -2b_nc_m+2d_n),$$
$$\beta = gcd(-am, -b_nc_m+d_n, -a_nc_m+c_n, -d_nc_m+2b_n, -c_nc_m+2a_n),$$

then the homology is of isomorphism type

$$H_1(X_{m,n}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/\alpha\mathbb{Z} \oplus \mathbb{Z}/\beta\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

6. References

[1] O. Braunling, *Oeljeklaus–Toma manifolds and arithmetic invariants*, Math.Z. (2016).
[2] O. Braunling and V. Vuletescu, *Automorphisms of ot manifolds and ray class numbers*, arXiv:1807.11596.
[3] G. Burde and H. Zieschang, *Knots*, second ed., de Gruyter Studies in Mathematics, vol. 5, Walter de Gruyter & Co., Berlin, 2003. MR 1959408
[4] T. Alden Gassert, *A note on the monogeneity of power maps*, Albanian J.Math. 11 (2017), no. 1, 3–12. MR 3659215
[5] Francisco González-Acu˜na and Hamish Short, *Cyclic branched coverings of knots and homology spheres*, Rev. Mat. Univ. Complut. Madrid 4 (1991), no. 1, 97–120. MR 1142552
[6] Marvin J. Greenberg and John R. Harper, *Algebraic topology*, Mathematics Lecture Note Series, vol. 58, Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1981, A first course. MR 643101
[7] M. Inoue, *On surfaces of Class VII0*, Invent. Math. 24 (1974), 269–310. MR 0342734 (49 #7479)
[8] K. Oeljeklaus and M. Toma, *Non-Kähler compact complex manifolds associated to number fields*, Ann. Inst. Fourier (Grenoble) 55 (2005), no. 1, 161–171. MR 2141693 (2006c:32020)

[9] R. Riley, *Growth of order of homology of cyclic branched covers of knots*, Bull. London Math. Soc. 22 (1990), no. 3, 287–297. MR 1041145
[10] Daniel S. Silver and Susan G. Williams, *Torsion numbers of augmented groups with applications to knots and links*, Enseign. Math. (2) 48 (2002), no. 3-4, 317–343. MR 1955605
[11] Dung Ph. Phan, *A program for computing the homology of Oeljeklaus-Toma manifolds in GAP* (2024), `https://github.com/arahm/Oeljeklaus-Toma`