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Dilation Choice Sets, Dulmage-Mendelsohn decomposition and structural controllability

Christian Commault* and Jacob van der Woude

Abstract—One of the two conditions for the controllability of linear structured systems is a rank condition. It can be phrased as the absence of so-called dilations in the directed graph representing the system. Starting from dilations in the graph of a system without control, input vertices and edges can be added such that these dilations are removed. Their presence can be investigated by searching for minimal dilations, and by combining them into larger ones, leading to the recently introduced notion of Dilation Choice Set. However, from a computational point of view this searching for minimal dilations is not efficient as their number may grow rapidly. For this reason, in the current paper, first a fundamental decomposition of the related bipartite graph is recalled. The decomposition, called the Dulmage-Mendelsohn, can be obtained by well-known and efficient methods. Having the decomposition, the dilations can be found and removed in straightforward way, making sure that the rank condition for structural controllability is fulfilled. Using a refined version of the decomposition, this process can even be refined. Finally, the relevance of vertices and edges for the removal of dilations can be characterized.

Index Terms—Generic controllability, linear structured systems, graph theory methods, minimal dilations, Dulmage-Mendelsohn decomposition

I. INTRODUCTION

Since more than a decade, there has been an increasing interest for the controllability issues in networks, cf. [1], [2], [3] and [4]. Considering that each vertex of the network is associated with one state variable, the network is then modeled as a finite dimensional linear dynamical system in the sense of control theory. It appeared then that an old concept of control theory, the notion of structured system, cf. [5], was well fitted for the study of controllability for networks. A structured system is a usual dynamical finite dimensional linear system, defined by its matrices A and B , where the entries of the matrices are either fixed zeros or unknown values, and where the structure of the system is lying in the location of fixed zeros. A graph can be associated with the system, where vertices are representing input and state variables, while edges are associated with nonzero entries of A and B . For more information on structured systems, see the survey papers [6] and [7]. In his seminal work, Lin proved that a structured system (A, B) is generically (*i.e.* for almost any value of the unknown entries) controllable if and only if two conditions are satisfied, cf. [5]. The first condition, usually called input connection condition, states that each state vertex must be reachable from an input vertex through a path in the graph. This condition is very natural, it simply means

that each state must be influenced by at least one input, this condition is also necessary for more general models, cf. [8]. Moreover the condition is simple to check using classical algorithms based on the decomposition of the graph in strongly connected components. The second condition states that the generic rank of the composite matrix $[A, B]$ must be n , being the dimension of the state space. This is why we usually call this condition the rank condition. The rank condition is much less intuitive than the input connection condition and is closely related with the linearity of the model. This condition can be stated in several equivalent forms, in particular in graph terms. One of these alternative formulations asks for the absence of dilations in the graph, where a dilation is a set of state vertices originating from a fewer number of predecessors. Another formulation of the rank condition is obtained with the interpretation of the generic rank of $[A, B]$ as the size of a maximum matching in the graph.

The formulation of the controllability problem by the network community is very specific. It starts with a network without inputs and asks for a minimum number of inputs and the way they act on the state vertices in order to insure controllability of the controlled system. In simple words, one starts from a given structured matrix A and looks for a B such that the pair (A, B) is controllable. The minimum number of necessary inputs was given in [3] and the minimum number of dedicated inputs (*i.e.* inputs impacting a unique vertex) received a nice algorithmic solution in [9], [10] and [11]. See also [12] for an alternate and more general approach based on Linear Programming.

The objective of this paper is to go deeper into the rank condition in order to get more precise information on the input additions fulfilling the condition.

The connection between the rank condition and dilations appeared first in [5]. Dilations easily reveal which nodes may be selected as driver inputs in order to obtain structural controllability. However, because of the potentially large amount of dilations, and their possible overlapping, the selection may be not as small as possible. Recently, in [13] the authors analysed finely the structure of the set of dilations and introduced the appealing notion of Dilation Choice set (DCS) in which overlapping dilations are taken into account. They proved that DCS's allow the characterization of (smallest sets of) driver inputs for structural controllability, and give tools for the analysis of the set of all (smallest) possible solutions. For further details, see [13].

The drawback of DCS's is that they are constructed from minimal dilations, which can only be obtained by inspection, and therefore no efficient algorithm can be found to characterize DCS's. The motivation for the investigations leading

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to the current paper was to show the relations between DCS's and a well-known decomposition of bipartite graphs called the Dulmage-Mendelsohn decomposition (DM decomposition), cf. [14] and [15].

The main contributions of this paper are the following

- We prove that DCS's correspond with the connected components of the maximal inconsistent part of the DM decomposition of the bipartite graph representing the structure of the underlying structured system without controls.
- This implies that these DCS's can be obtained with a simple algorithm which has the complexity of a maximum matching algorithm. Then DCS's can be computed in polynomial time.
- The correspondence between DCS's and the DM decomposition suggests a refined partition of the inconsistent components of the DM decomposition.
- We show that the classification of vertices and edges, which was introduced in [13], but was only partially characterized through DCS's, can fully be characterized using the DM decomposition.
- More generally, we hope to convince the reader of the crucial importance of the DM decomposition for dealing with problems related to the rank condition for structural controllability.

The outline of this paper is as follows. In Section II we introduce linear structured systems and the controllability of such systems. In Section III the bipartite graphs are introduced together with the concept of (maximum) matching. The decomposition, called the Dulmage-Mendelsohn decomposition, is described in Section IV. With its properties, the decomposition provides the possibility to find the minimal number of input vertices to be added, and also the locations where to add these input vertices. Some of the known results are recalled and are illustrated by means of an example. In Section V the notion of dilation, introduced for directed graphs, is translated into the language of bipartite graphs and their decompositions. The notion of DCS is introduced, together with the location of where to find these DCS's in terms of parts of the decomposition of the bipartite graph. Section VI contains a refinement of the minimal and maximal inconsistent part of the Dulmage-Mendelsohn decomposition. Also the consequences are included of where to add inputs vertices and edges in order to fulfil the rank condition for structural controllability. In Section VII the classification of vertices and edges is recalled from [13] and fully characterized. In Section VIII some of the computational aspects of the method in this paper are highlighted. Section IX contains some conclusions. The literature consulted for this paper is listed in the Reference section.

II. LINEAR STRUCTURED SYSTEMS AND STRUCTURAL CONTROLLABILITY

A. The structural controllability theorem

We consider a standard dynamical linear system, described by

$$\Sigma: \dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, and $u(t) \in \mathbb{R}^m$ is the input signal. Further, A and B are real matrices of appropriate dimensions. System (1) is called a linear structured system if each of the entries of the composite matrix $J = [A, B]$ is either a fixed zero, or has an unknown value (possibly zero). The entries which are not fixed zeros are usually, with some misuse of terminology, called nonzeros, cf. [6]. Clearly, the structure of the system is defined by the zero/nonzero patterns in matrices A and B .

For structured linear systems, one can study generic properties, *i.e.* properties which are true for almost any value of the nonzeros. More precisely, a property of Σ is generic if the set of values of the nonzero entries for which the property is not true is a proper algebraic variety in the space of possible values of these nonzeros, cf. [15]. A matrix Q with such zero/nonzero entries is called a structured matrix and its generic rank is denoted by $\text{g-rank}(Q)$.

A directed graph $\mathcal{G}(\Sigma) = (V; W)$ can be associated with the structured system Σ of type (1), with

- the vertex set $V = X \cup U$, where X and U are the state and input vertex sets, given by $\{x_1, x_2, \dots, x_n\}$ and $\{u_1, u_2, \dots, u_m\}$, respectively,
- the edge set $W = \{(x_j, x_i) | a_{ij} \neq 0\} \cup \{(u_j, x_i) | b_{ij} \neq 0\}$, where (x_j, x_i) denotes an edge from vertex x_j to vertex x_i and $a_{ij} \neq 0$ means that the (i, j) -th entry of the matrix A is a nonzero, and likewise, (u_j, x_i) denotes an edge from vertex u_j to vertex x_i and $b_{ij} \neq 0$ means that the (i, j) -th entry of the matrix B is a nonzero.

Occasionally, we will deal with systems without input, *i.e.* without matrix B , and with only matrix A . The associated directed graph will then be denoted $\mathcal{G}(A)$. Clearly, the sets V and X then coincide and may be used interchangeably.

A path in $\mathcal{G}(\Sigma)$ from a vertex v_0 to a vertex v_q is a sequence of directed edges $(v_0, v_1), (v_1, v_2), \dots, (v_{q-1}, v_q)$, such that $v_t \in V$ for $t = 0, 1, \dots, q$, and $(v_{t-1}, v_t) \in W$ for $t = 1, 2, \dots, q$. The path is then said to *cover* the vertices v_0, v_1, \dots, v_q . The vertex v_0 is called the begin vertex of the path, and v_q its end vertex. The path is called *closed* if $v_0 = v_q$, *i.e.* its begin and end vertex coincide. If $v_0 \in U$ and $v_q \in X$, the path is called an *input-state path*.

The system Σ is said to be *input-connected* if for any state vertex $v \in X$, there exists an input-state path with end vertex v . An input-state path which does not visit the same vertex twice is called a *stem*. A *cycle* is a closed path which does not visit the same vertex twice (except for the begin/end vertex).

Two vertices $v_i, v_j \in V$ are said to be strongly connected to each other if there is a path from v_i to v_j , and from v_j to v_i . The relation "is strongly connect to" is an equivalence relation and partitions the vertex set V into equivalence classes of vertices that are strongly connected to each other. A maximally strongly connected subgraph is a subgraph made up of a collection of such vertices that is as large as possible, and together with all edges between them. Maximally strongly connected components (or simply strongly connected components) can be ordered in an acyclic way such that there may be edges from one component to another, but not the other way around. Corresponding notions can be introduced for non directed graphs. Paths will become walks and strongly

connected components just connected components. Connected components can be associated with a directed graph if we do not take into account the direction of edges.

Let v_i and v_j be two vertices in V , then v_j is said to be a *predecessor* of v_i if $(v_j, v_i) \in W$. Note that in the context of this paper only vertices in X can have predecessors. A subset $L \subset X$ is called a *dilation*, if $|\mathcal{P}(L)| < |L|$, where $\mathcal{P}(L)$ is the set of predecessors of vertices in L , $|L|$ denotes the cardinality of L , i.e. the number of elements of L , and likewise for $|\mathcal{P}(L)|$.

Structural (or generic) controllability, i.e. controllability for almost any value of the nonzeros, was introduced by Lin, who proved the following result, cf. [5]

Theorem 1: Let Σ be the linear structured system defined by (1) with associated graph $\mathcal{G}(\Sigma)$. The system is structurally controllable if and only if

- 1) The system Σ is input-connected and,
- 2) a) $\text{g-rank}[A, B]=n$, or
b) $\mathcal{G}(\Sigma)$ has no dilation.

In the following, the conditions 1 and 2 of Theorem 1 will be referred to as the input connection condition and the rank condition, respectively.

The input connection condition is quite natural and simply means that any state vertex can be influenced by some input vertex. Hence, input connection can be seen as a ‘global’ property. The rank condition is a bit more subtle and is tightly related with the linear nature of the model. The rank condition has a more ‘local’ character in the sense that it implies that each state vertex can be individually controlled by means of its predecessor state and/or input vertices.

Example 1: In the rest of the paper we will illustrate all the concepts by an example of a structured dynamical system without input, defined by the following $n \times n$ matrix A , where $n = 9$ and a $*$ stands for a nonzero entry.

$$A = \begin{bmatrix} 0 & 0 & * & * & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 \\ * & * & 0 & 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & * & 0 \end{bmatrix}. \quad (2)$$

The system is represented by the directed graph $\mathcal{G}(A)$ in Figure 1. The example will be used as a running example in the remainder of the paper.

This system is clearly not controllable since having no input it is not input connected. Moreover, notice that the set $L = \{x_4, x_8\}$ is a dilation since $\mathcal{P}(L) = \{x_5\}$ and therefore $|\mathcal{P}(L)| < |L|$.

III. BIPARTITE GRAPH, MAXIMUM MATCHING AND THE RANK CONDITION

We will recall here a characterization of the rank condition of Theorem 1, i.e. $\text{g-rank}[A, B] = n$. This generic rank will be computed using a bipartite graph associated with the system

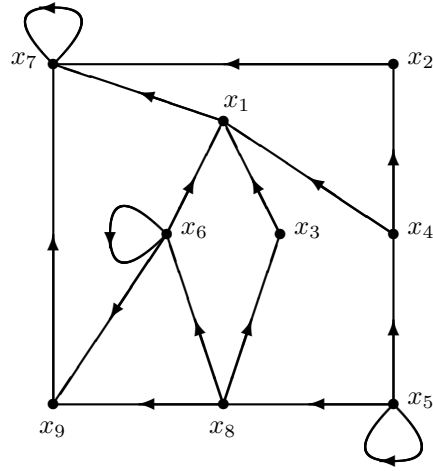


Fig. 1. Digraph $\mathcal{G}(A)$ of Example 1.

Σ of type (1). The bipartite graph will be denoted by $\mathcal{B}(\Sigma)$ and is defined as $\mathcal{B}(\Sigma) = (V^+, V^-; W')$, with

- two disjoint vertex sets $V^+ = X^+ \cup U$ and $V^- = X^-$, where $X^+ = \{x_1^+, \dots, x_n^+\}$ is the first set of state vertices, $X^- = \{x_1^-, \dots, x_n^-\}$ is the second set of state vertices, and $U = \{u_1, \dots, u_m\}$ is the set of input vertices. Notice that here we have split each state vertex x_i of $\mathcal{G}(\Sigma)$ into two vertices x_i^+ and x_i^- in $\mathcal{B}(\Sigma)$,
- the edge set $W' = \{(x_j^+, x_i^-) | a_{ij} \neq 0\} \cup \{(u_j, x_i^-) | b_{ij} \neq 0\}$, where (x_j^+, x_i^-) denotes an edge from vertex x_j^+ to vertex x_i^- and $a_{ij} \neq 0$ means that the (i, j) -th entry of the matrix A is a nonzero, and likewise, (u_j, x_i^-) denotes an edge from vertex u_j to vertex x_i^- and $b_{ij} \neq 0$ means that the (i, j) -th entry of the matrix B is a nonzero.

Similarly to the directed graph before, when the system has no input, the graph $\mathcal{B}(\Sigma)$ will be denoted $\mathcal{B}(A)$. Then the sets V^+ and X^+ coincide and may be used interchangeably, and so do V^- and X^- .

Notice that the representations of a structured linear system Σ of type (1) by the directed graph $\mathcal{G}(\Sigma)$ and the bipartite graph $\mathcal{B}(\Sigma)$ are completely equivalent. It is straightforward to commute from one representation to the other.

A *matching* in a bipartite graph $\mathcal{B} = (V^+, V^-; W')$ is an edge set $M \subseteq W'$ such that the edges in M have no common vertex. Vertices that are covered by (or are incident with) edges in M are said to be *matched* by the matching. The remaining vertices are said to be *unmatched*, i.e. they are not covered by any edge in M . The cardinality of a matching, i.e. the number of edges it consists of, is also called its size. A matching M is called *maximum* if its cardinality is maximal among the cardinalities of all possible matchings. The maximum matching problem is the problem of finding such a matching of maximal cardinality. This maximum matching problem is a standard problem of combinatorics and a lot of efficient algorithms have been proposed to solve it, cf. [16]. For Example 1, the bipartite graph together with a maximum matching in red are represented in Figure 2.

First we will highlight the use of a maximum matching for structural controllability. We recall the following proposition, cf. [17].

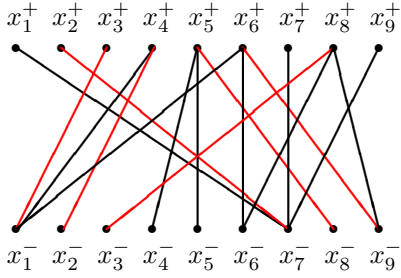


Fig. 2. Bipartite graph $\mathcal{B}(A)$ of Example 1 with a maximum matching.

Proposition 1: Let Σ be the linear structured system defined by (1) with associated bipartite graph $\mathcal{B}(\Sigma)$. The generic rank of $[A, B]$ is equal to the size of a maximum matching in $\mathcal{B}(\Sigma)$. In particular, $\text{g-rank}[A, B] = n$ if and only if there exists a size n matching in $\mathcal{B}(\Sigma)$.

Some remarks follow from this result.

Remark 1: By Proposition 1, the rank condition of Theorem 1 can be equivalently replaced by the existence of a size n matching in the bipartite graph $\mathcal{B}(\Sigma)$.

Remark 2: In the literature, the rank condition of Theorem 1 is sometimes formulated in terms of the graph $\mathcal{G}(\Sigma)$. Indeed, it can be shown that the rank condition is equivalent to the covering of all state vertices by a set of disjoint stems and cycles.

Remark 3: In network theory, a fundamental problem is the following: given a network where each vertex is associated with a state variable, find a minimum number of external inputs, and the way they have to impact the vertices, such that the whole system is controllable. In terms of structured systems the previous problem amounts to: given the A matrix, find a minimum number of inputs and a matrix B , such that the pair (A, B) is controllable.

For this problem, an important issue is the *rank defect* d_r , which is defined as $d_r = n - \text{g-rank}(A)$.

From [3], if a maximum matching of $\mathcal{B}(A)$ has size $n - d_r$, a matching of size n can be obtained by adding d_r inputs with an edge from one input to an unmatched state vertex in the given maximum matching. Some edges starting from these inputs may be necessary to fulfill the input connection condition of Theorem 1. Finally, the minimum number of inputs for controllability is $\max(d_r, 1)$.

Other approaches allow each input to only impact one state vertex, they are then called dedicated inputs. Their minimum number can then be obtained through some weighted maximum matching algorithm, cf. [9], [10] and [11].

For Example 1, it appears, that since $n = 9$ and the size of a maximum matching is six, then $d_r = 3$. Three inputs are then necessary to fulfill the rank condition. Adding three inputs acting on the unmatched vertices $\{x_4, x_5, x_6\}$ will imply with the corresponding B matrix, $\text{g-rank}[A, B] = 9$. Moreover, an input acting on x_5 induces input connection, therefore three inputs acting on vertices $\{x_4, x_5, x_6\}$ insure structural controllability.

The tight relation between the size of a maximum matching in a bipartite graph and the rank of the corresponding matrix

is a fundamental tool for the characterization of the rank condition in structural controllability. However, when starting with the bipartite graph of the matrix A , and looking for new inputs and a matrix B for reaching the rank condition, we are faced with the problem of the non unicity of the maximum matching in $\mathcal{B}(A)$. The Dulmage-Mendelsohn decomposition of a bipartite graph which will be introduced in the next section is a tool to overcome the problem of non unicity of a maximum matching.

IV. DULMAGE-MENDELSON DECOMPOSITION OF A BIPARTITE GRAPH

A useful tool to parameterize all the maximum matchings in a bipartite graph is the Dulmage-Mendelsohn (DM) decomposition, which will be presented now.

Given a bipartite graph $\mathcal{B} = (V^+, V^-; W')$, the DM decomposition allows to decompose \mathcal{B} into a uniquely defined family of bipartite subgraphs $\mathcal{B}_i = (V_i^+, V_i^-; W_i')$, $i = 0, 1, \dots, r, \infty$, called the DM components, where $\{V_0^+, V_1^+, \dots, V_r^+, V_\infty^+\}$ is a partition of V^+ in disjoint subsets, and likewise for V^- and W' . This partition has the fundamental property that any maximum matching on \mathcal{B} can be decomposed into maximum matchings on the DM components.

A. Finding the Dulmage-Mendelsohn decomposition

Consider a general bipartite graph $\mathcal{B} = (V^+, V^-; W')$ together with a maximum matching $M \subseteq W'$. Vertices that are covered by the edges in M are said to be matched by the matching. The remaining vertices are said to be unmatched, i.e. they are not covered by any edge in M .

1) *Auxiliary graph based on maximum matching in bipartite graph:* To find the DM decomposition, a so-called auxiliary graph is used. This auxiliary graph, denoted \mathcal{B}^M , can be seen as a directed graph with vertex set $V^+ \cup V^-$, and edge set $W' \cup M^\leftarrow$, i.e. $\mathcal{B}^M = (V^+ \cup V^-; W' \cup M^\leftarrow)$. Here V^+ and V^- are the two vertex sets of \mathcal{B} , W' is the edge set of \mathcal{B} , and M^\leftarrow is the edge set consisting of all edges of M , however with their direction reversed. Hence, $M^\leftarrow = \{(v, v') \in V^- \times V^+ | (v', v) \in M\}$. Clearly, the edges of W' are directed from V^+ to V^- , whereas the edges of M^\leftarrow are directed from V^- to V^+ . In the following, the edges of W' will occasionally be referred to as black edges and the edges in M^\leftarrow as red edges. Further, the notions of path and path length in \mathcal{B}^M will be as usual. Hence, a path in \mathcal{B}^M consists of an alternating sequence of black and red edges.

A consequence of the fact that M is a maximum matching is that in \mathcal{B}^M there are no paths from unmatched vertices in V^+ to unmatched vertices in V^- . Indeed, when existing, such a path would consist of an odd number of edges that alternatively are coloured black and red, starting and ending with a black edge. Note that the red edges correspond to the matching M . However, using an alternating path argument, these red edges can be replaced by the black edges in the path, creating a new matching with a size one larger than the size of M . Hence, the matching M would not have been a maximum matching.

2) *Decomposition based on maximum matching*: Now assume that a maximum matching M is given, and denote the set of all unmatched vertices in V^+ by S^+ . Then, all vertices in $V^+ \setminus S^+$ are matched.

Next consider all paths in \mathcal{B}^M that start in S^+ . Collect all vertices on these paths in the set V_0 , and write $V_0^+ = V_0 \cap V^+$, $V_0^- = V_0 \cap V^-$. Note that none of the vertices in V_0^- is unmatched because M is maximum, *i.e.* they are all matched. See also the previous paragraph. Further, any vertex of $V_0^+ \setminus S^+$ is matched because it can only be reached from S^+ by a path with a final red edge. Hence, the number of matched vertices in V_0^+ and V_0^- is the same, and the two sets of matched vertices are connected by means of a matching of size $|V_0^-|$. Further, note that by definition there are no edges from V_0^+ to $V^- \setminus V_0^-$. However, there may be edges from $V^+ \setminus V_0^+$ to V_0^- . Denote $\mathcal{B}_0 = (V_0^+, V_0^-; W'_0)$, where $W'_0 = \{(v, v') \in W' | v \in V_0^+, v' \in V_0^-\}$. The bipartite subgraph \mathcal{B}_0 is called the minimal inconsistent part of the DM decomposition.

Now denote the set of all unmatched vertices in V^- by S^- . Then all vertices of $V^- \setminus S^-$ are matched and consider all paths in \mathcal{B}^M that end in S^- . Collect all the vertices on these paths in the set V_∞ and write $V_\infty^+ = V_\infty \cap V^+$, $V_\infty^- = V_\infty \cap V^-$. Denote $\mathcal{B}_\infty = (V_\infty^+, V_\infty^-; W'_\infty)$, where $W'_\infty = \{(v, v') | v \in V_\infty^+, v' \in V_\infty^-\}$. The bipartite subgraph \mathcal{B}_∞ is called the maximal inconsistent part of the DM decomposition. Then the properties of \mathcal{B}_∞ can be obtained in a way completely dual to the way they are obtained for \mathcal{B}_0 .

Next recall that the vertices in S^+ and S^- are unmatched in V^+ and V^- , respectively. Therefore, as indicated above, there are no paths from S^+ to S^- . So, it follows that $V_0 \cap V_\infty = \emptyset$. Hence, it makes sense to introduce $V_f = V \setminus (V_0 \cup V_\infty)$, with 'f' for 'finite', and write $V_f^+ = V_f \cap V^+$ and $V_f^- = V_f \cap V^-$. Note that $V_f^+ \subseteq V^+ \setminus S^+$ and $V_f^- \subseteq V^- \setminus S^-$, from which it follows that all vertices in V_f^+ and V_f^- are matched. Since the matched vertices in V_0^+ and V_0^- are connected by means of a matching of size $|V_0^-|$, and the matched vertices in V_∞^+ and V_∞^- are connected by means of a matching of size $|V_\infty^+|$, it follows that the vertices in V_f^+ and V_f^- are connected by means of a matching of size $|V_f^+| = |V_f^-|$. Denote $\mathcal{B}_f = (V_f^+, V_f^-; W'_f)$, where $W'_f = \{(v, v') | v \in V_f^+, v' \in V_f^-\}$.

Recall that there are no edges from V_0^+ to $V^- \setminus V_0^-$, and no edges from $V^+ \setminus V_0^+$ to V_∞^- . Observe further that $\{V_0^+, V_f^+, V_\infty^+\}$ forms partition of V^+ and $\{V_0^-, V_f^-, V_\infty^-\}$ forms partition of V^- . Hence, it follows that there are no edges from V_0^+ to V_f^- , from V_0^+ to V_∞^- and from V_f^+ to V_∞^- .

In \mathcal{B}_f , consisting of all vertices in V_f and edges between these vertices, every vertex is covered by a cycle of even length, consisting of alternating black and red edges. The directed subgraph can be decomposed in, say r , maximally strongly connected components that can be ordered in an acyclic way. The latter implies that the bipartite subgraph $\mathcal{B}_f = (V_f^+, V_f^-; W'_f)$ can be decomposed in r bipartite subgraphs $\mathcal{B}_i = (V_i^+, V_i^-; W'_i)$, $i = 1, \dots, r$, where each subgraph contains a matching of size $|V_i^+| = |V_i^-|$, $i = 1, \dots, r$, and where there are no edges from V_i^+ to V_j^- for all $i, j \in \{1, \dots, r\}$ with $i < j$. The bipartite subgraphs

$\mathcal{B}_i = (V_i^+, V_i^-; W'_i)$, $i = 1, \dots, r$ are called the consistent parts the DM decomposition.

B. Application to Example 1

For Example 1, starting with the maximum matching $\{(x_2^+, x_7^-), (x_3^+, x_1^-), (x_4^+, x_2^-), (x_5^+, x_8^-), (x_6^+, x_9^-), (x_8^+, x_3^-)\}$, the decomposition of Subsection IV-A, boils down to the following.

- $S^+ = \{x_1^+, x_7^+, x_9^+\}$, $V_0 = \{x_1^+, x_7^-, x_2^+, x_7^-, x_9^+\}$, yielding $V_0^+ = \{x_1^+, x_2^+, x_7^+, x_9^+\}$ and $V_0^- = \{x_7^-\}$. This gives $\mathcal{B}_0 = (V_0^+, V_0^-; W'_0)$ with V_0^+ and V_0^- as indicated, and with $W'_0 = \{(x_1^+, x_7^-), (x_2^+, x_7^-), (x_7^+, x_7^-), (x_9^+, x_7^-)\}$.
- $S^- = \{x_4^-, x_5^-, x_6^-\}$, $V_\infty = \{x_4^-, x_5^+, x_8^-, x_5^-, x_6^-, x_6^+, x_9^-, x_8^+, x_3^-\}$, yielding $V_\infty^+ = \{x_5^+, x_6^+, x_8^+\}$ and $V_\infty^- = \{x_3^-, x_4^-, x_5^-, x_6^-, x_8^-, x_9^-\}$. This gives $\mathcal{B}_\infty = (V_\infty^+, V_\infty^-; W'_\infty)$ with V_∞^+ and V_∞^- as indicated, and with $W'_\infty = \{(x_5^+, x_4^-), (x_5^+, x_5^-), (x_5^+, x_8^-), (x_6^+, x_6^-), (x_6^+, x_9^-), (x_8^+, x_3^-), (x_8^+, x_6^-), (x_8^+, x_9^-)\}$.
- $V_f = \{x_1^-, x_2^-, x_3^+, x_4^+\}$, yielding $V_f^+ = \{x_3^+, x_4^+\}$ and $V_f^- = \{x_1^-, x_2^-\}$.
- The vertices x_1^- and x_3^+ with the two edges between them in \mathcal{B}^M , a black and a red one, form a strongly connected component. The same applies to the vertices x_2^- and x_4^+ . Note that it is possible in \mathcal{B}^M to go from the component with $\{x_4^+, x_2^-\}$ to the the component with $\{x_3^+, x_1^-\}$, but not the other way around. Hence, there are two distinct strongly connected components in \mathcal{B}^M , implying two distinct bipartite subgraphs, namely $\mathcal{B}_1 = (V_1^+, V_1^-; W'_1)$ with $V_1^+ = \{x_3^+\}$, $V_1^- = \{x_1^-\}$ and $W'_1 = \{(x_3^+, x_1^-)\}$, and $\mathcal{B}_2 = (V_2^+, V_2^-; W'_2)$ with $V_2^+ = \{x_4^+\}$, $V_2^- = \{x_2^-\}$ and $W'_2 = \{(x_4^+, x_2^-)\}$.

The resulting Dulmage-Mendelsohn decomposition can be displayed as in Figure 3. Note that in Figure 3, the bipartite subgraph \mathcal{B}_∞ can be further subdivided in \mathcal{B}_∞^1 and \mathcal{B}_∞^2 . This observation will be formalized in Section VI. Also the decomposition nicely illustrates the property that there are no edges from V_i^+ to V_j^- for all $i, j \in \{0, 1, 2, \infty\}$ with $i < j$.

C. Main properties of the DM decomposition

In the following Proposition, inspired by Theorem 2.2.22 of [15], we summarize the main properties of the DM decomposition.

Proposition 2: Let be given a general bipartite graph $\mathcal{B} = (V^+, V^-; W')$ and its DM decomposition, with $\mathcal{B}_i = (V_i^+, V_i^-; W'_i)$, $i = 0, 1, \dots, r, \infty$, being its DM components. One has the following properties.

- 1) A maximum matching on \mathcal{B} is a union of maximum matchings on the DM components \mathcal{B}_i , $i = 0, 1, \dots, r, \infty$.
- 2) A vertex v in V_0^- (or in V_i^+ , V_i^- , $i = 1, \dots, r$, or in V_∞^+) is covered by any maximum matching on \mathcal{B} .
- 3) A vertex v in V^+ belongs to V_0^+ if and only if there exists a maximum matching on \mathcal{B} that does not cover v .
- 4) A vertex v in V^- belongs to V_∞^- if and only if there exists a maximum matching on \mathcal{B} that does not cover v .
- 5) There is no edge (v^+, v^-) from a vertex v^+ in V_i^+ to a vertex v^- in V_j^- , for any $i, j \in \{0, 1, \dots, r, \infty\}$ with $i < j$.

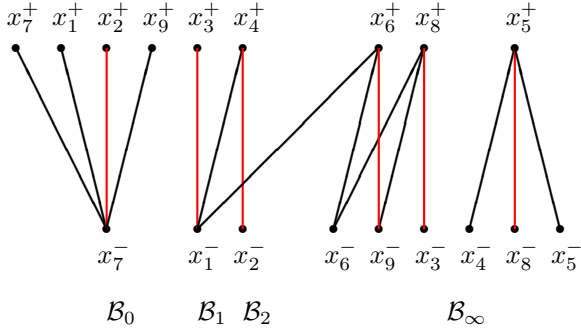


Fig. 3. DM decomposition of the bipartite graph $\mathcal{B}(A)$ of Example 1.

- 6) Any edge in a DM component belongs to some maximum matching on \mathcal{B} .

The DM decomposition is useful in a number of problems of applied mathematics. It allows in particular to get a block triangular form of the associated matrix by permutation of rows and columns. In the same spirit, it allows to decompose a set of algebraic equations in over-determined, just-determined and under-determined sets of equations, cf. [18].

D. DM decomposition and structural controllability

The rank condition for structural controllability can be expressed using only the maximal inconsistent part of the DM decomposition of $\mathcal{B}(\Sigma)$, cf. [19], [20], [21].

In the same vein, it is possible to be more precise than the main result of [3], concerning the location of the steering vertices to fulfil the rank condition in structural controllability.

Proposition 3: Consider a linear structured system defined by the matrix A , with bipartite graph $\mathcal{B}(A)$, and the corresponding DM decomposition. Assume that $\text{g-rank}(A) = n - d_r$ with $d_r > 0$. To fulfil the rank condition of structural controllability, one must add d_r input vertices u_1, \dots, u_{d_r} , connected with d_r state vertices $x_{i_1}, \dots, x_{i_{d_r}}$, where x_{i_k} is such that $x_{i_k}^-$ is an unmatched vertex of $V_\infty^-(A)$, for some maximum matching in \mathcal{B}_∞ .

Proof of Proposition 3: From [3], the steering vertices are unmatched vertices by some maximum matching in $\mathcal{G}(A)$, and therefore are in V^- of $\mathcal{B}(A)$. Moreover, from the decomposition of a maximum matching on the components of the DM decomposition and the fact that the unmatched vertices of V^- are located in V_∞^- , the result follows. \triangle

It can be concluded from Figure 3, that we here have $V_\infty^+ = \{x_5^+, x_6^+, x_8^+\}$ and $V_\infty^- = \{x_3^-, x_4^-, x_5^-, x_6^-, x_8^-, x_9^-\}$. The rank defect, $d_r = n - (\text{g-rank}[A, B]) = |V_\infty^-| - |V_\infty^+|$, is equal to 3. Moreover, adding a new input will result in a larger rank if and only if this input acts on a vertex in $\{x_3, x_4, x_5, x_6, x_8, x_9\}$.

V. DILATIONS AND DM DECOMPOSITION

In this section, for a given system without inputs, defined by the structured matrix A , we show some important connections between recently introduced DCS's of $\mathcal{G}(A)$ and the DM decomposition of the bipartite graph $\mathcal{B}(A)$.

Recall that the subset $L \subset X$ is a dilation if $|\mathcal{P}(L)| < |L|$, where $\mathcal{P}(L)$ is the set of predecessors of vertices in L . A

dilation D is said to be minimal if no proper subset of D is a dilation. Note that the total number of minimal dilations is finite and that these dilations therefore can be enumerated. The following important definition was given in [13] (in slightly modified form).

Definition 1: Denote the minimal dilations of a graph as D_1, D_2, \dots, D_N . A Dilation Choice Set (DCS) is a largest union of minimal dilations such that each one of them overlaps with at least one other minimal dilation in the set. More precisely, the set $D = D_{i_1} \cup D_{i_2} \cup \dots \cup D_{i_q}$ is a DCS if for each $i \in \{i_1, i_2, \dots, i_q\}$, there exists a $j \in \{i_1, i_2, \dots, i_q\}$, $i \neq j$, with $D_i \cap D_j \neq \emptyset$, and q as large as possible.

From the bipartite graph $\mathcal{B}(A)$ of a structured system without inputs, and its DM decomposition as presented in section IV, define the sets of vertices $X_\infty = \{x_i \in X | x_i^- \in V_\infty^-\}$ and $\bar{X}_\infty = X / X_\infty$.

Proposition 4: Let D of cardinality ν be a dilation such that $D \cap \bar{X}_\infty \neq \emptyset$ and has cardinality μ . Then $D \cap X_\infty$ is a dilation of cardinality $\nu - \mu$.

Proof of Proposition 4: From the properties of the DM decomposition, for any $x_i \in D \cap \bar{X}_\infty$, vertex x_i^- is matched by any maximum matching of the bipartite graph. Therefore, the μ vertices in $D \cap \bar{X}_\infty$ have at least μ predecessors in V^+ / V_∞^+ . From point 5 of Proposition 2 it follows that vertices in $D \cap X_\infty$ only have predecessors in V_∞^+ . Since the total number of predecessors of dilation D is less than ν , the number of predecessors of $D \cap X_\infty$ is less than $\nu - \mu$, and consequently $D \cap X_\infty$ is a dilation. \triangle

Notice incidentally that the previous result shows that any dilation has a non empty intersection with X_∞ .

Proposition 4 has an important consequence for DCS's.

Corollary 1: Minimal dilations and DCS's are contained in X_∞ .

Proof of Corollary 1: From Proposition 4, if D is a dilation such that $D \cap \bar{X}_\infty \neq \emptyset$, then $D \cap X_\infty$ is a dilation, proving that D is not a minimal dilation. Therefore, minimal dilations must be contained in X_∞ . DCS's, which are obtained from union of minimal dilations, are also contained in X_∞ . \triangle

This establishes that minimal dilations and DCS's are to be searched only in X_∞ .

Now let us come back to the way V_∞ is constructed in the bipartite graph, see Subsection IV-A. We construct first the auxiliary directed graph \mathcal{B}^M associated with a particular maximal matching M . The graph \mathcal{B}^M is obtained from the initial bipartite graph \mathcal{B} by adding an edge (x_i^-, x_j^+) when the edge (x_j^+, x_i^-) belongs to the maximal matching M .

Let x_i be a vertex such that x_i^- is not matched by M . Now define the elementary set associated with x_i , as the set of vertices x_j such that there exists a path from x_j^- to x_i^- in \mathcal{B}^M and denote this set by $E^M(x_i)$.

In Example 1, choosing the maximum matching $M = \{(x_2^+, x_7^-), (x_3^+, x_1^-), (x_4^+, x_2^-), (x_5^+, x_8^-), (x_6^+, x_9^-), (x_8^+, x_3^-)\}$ leaves the vertices x_4^-, x_5^- and x_6^- unmatched. The corresponding elementary sets are $E^M(x_4) = \{x_4, x_8\}$, $E^M(x_5) = \{x_5, x_8\}$, $E^M(x_6) = \{x_3, x_6, x_9\}$. It can be checked that the elementary sets depend on the chosen maximum matching. For example, if in the maximum

matching we had chosen (x_5^+, x_5^-) instead of (x_5^+, x_8^-) , we would get $E^M(x_4) = \{x_4, x_5\}$.

Proposition 5: The elementary set $E^M(x_i)$ associated with a vertex x_i , where x_i^- is not in a given maximal matching M , is a minimal dilation.

Proof of Proposition 5: Let us consider an elementary set $E^M(x_i)$ associated with a vertex x_i , where x_i^- is not in a given maximal matching M and let us first prove that it is a dilation. For any $x_j \neq x_i$ in $E^M(x_i)$, the first edge in the path from x_j^- to x_i^- in the auxilliary graph \mathcal{B}^M is $(x_j^-, x_{k_j}^+)$ where $(x_{k_j}^+, x_j^-)$ belongs to the maximum matching M . Then the set $E^M(x_i)$ has at least $l-1$ predecessors if the set $E^M(x_i)$ has cardinality l . Assume that some x_j in $E^M(x_i)$ has a predecessor x_q which does not belong to this set of cardinality $l-1$. From the properties of the DM decomposition, there exists an edge $(x_q^+, x_{i_q}^-)$ in M , where $x_{i_q}^- \notin E^M(x_i)$. Then there exists a path from $x_{i_q}^-$ to x_i^- in the auxilliary graph \mathcal{B}^M , which is in contradiction with the definition of $E^M(x_i)$. Therefore the set $E^M(x_i)$ of cardinality l has a set of predecessors of cardinality $l-1$ and is then a dilation.

If $E^M(x_i)$ is not minimal it must contain a set of vertices $E' \subset E^M(x_i)$ which is a minimal dilation. Notice first that if $x_i \notin E'$, E' is not a dilation because for each $x_j \in E'$ there exists x_{k_j} such that $(x_{k_j}^+, x_j^-) \subset M$ and then the cardinality of the set of predecessors of E' is at least as large as the cardinality of E' . Now consider a minimal dilation E' , where $x_i \in E'$. Let x_k be such that $x_k \in E^M(x_i)$ but $x_k \notin E'$. By definition of $E^M(x_i)$, there is a path from x_k^- to x_i^- in the auxilliary graph \mathcal{B}^M . The first edge of this path is $(x_k^-, x_{k_q}^+)$ where $(x_{k_q}^+, x_k^-) \in M$. Moreover, this path will intersect E' in a first vertex which cannot belong to V_∞^+ . This is because there would then exist an edge $(x_{k_l}^-, x_l^+)$, where $(x_l^+, x_{k_l}^-)$, $x_{k_l} \notin E'$, is in M . But there also exists an edge $(x_l^+, x_{k_l}^-)$ of M with $x_{k_l}^-$ in E' which is a contradiction. Then this path will intersect E' in a first vertex x_p^- . Each vertex in E' , except for x_i , has a dedicated predecessor defined by M , but x_p has an additional predecessor in the path coming from x_k , then the cardinality of the set of predecessors of E' is at least as large as the cardinality of E' and E' is not a dilation. Therefore, $E^M(x_i)$ has no proper subset which is a minimal dilation, i.e. $E^M(x_i)$ is a minimal dilation. \triangle

Remark 4: The maximal inconsistent part can be decomposed into connected n_∞ components as $V_\infty = V_\infty^1 \cup V_\infty^2 \cup \dots, V_\infty^{n_\infty}$. For $i = 1, \dots, n_\infty$, denote $V_\infty^{i-} = V_\infty^i \cup V^-$, $V_\infty^{i+} = V_\infty^i \cup V^+$, and $X_\infty^i = \{x_i \in X | x_i^- \in V_\infty^{i-}\}$. Then there is no edge from $x_{k_i}^+ \in V_\infty^{i+}$ to $x_{k_j}^- \in V_\infty^{j-}$ for $i \neq j$.

In Example 1, there are clearly two connected components defined by $V_\infty^{1+} = \{x_5^+\}$, $V_\infty^{1-} = \{x_4^-, x_5^-, x_8^-\}$ and $V_\infty^{2+} = \{x_6^+, x_8^+\}$, $V_\infty^{2-} = \{x_3^-, x_6^-, x_9^-\}$ and therefore $X_\infty^1 = \{x_4, x_5, x_8\}$ and $X_\infty^2 = \{x_3, x_6, x_9\}$.

With the decomposition of the maximal inconsistent part in connected components, we can now refine the result of Corollary 1.

Proposition 6: A minimal dilation and a DCS belong to a particular set X_∞^i .

Proof of Proposition 6: Consider a minimal dilation D which can be decomposed as $D = D_1 \cup D_2, \dots, \cup D_k$ where

$D_i = D \cap X_\infty^i$ for $i = 1, \dots, n_\infty$. The sets of predecessors for each D_i are disjoint from the definition of the X_∞^i 's. D being a dilation, the total number of predecessors of D is less than the cardinality of D . This implies that there is at least a D_i such that the number of predecessors of D_i is less than the cardinality of D_i . Then this D_i is a dilation which induces that D is a minimal dilation only if $D_j = \emptyset$ for all $j \neq i$, i.e. D is contained in X_∞^i . Then any minimal dilation is contained in some X_∞^i . A DCS being an union of overlapping minimal dilations, since two minimal dilations belonging to two different X_∞^i 's cannot overlap, from Definition 1, a DCS's is also contained in some X_∞^i . \triangle

The previous results can be summarized in a simple correspondence between DCS's and the connected parts of \mathcal{B}_∞ .

Theorem 2: The sets of vertices X_∞^i , for $i = 1, \dots, n_\infty$, corresponding with the connected components of the maximal inconsistent part of the DM decomposition of the bipartite graph $\mathcal{B}(A)$, are the Dilation Choice Sets of the graph $\mathcal{G}(A)$.

Proof of Theorem 2: Consider a maximum matching M in the bipartite graph $\mathcal{B}(A)$ and the corresponding auxilliary graph \mathcal{B}^M . Let $x_l^- \in V_\infty^{i-}$ be a vertex not matched by M . The associated elementary set $E^M(x_l)$, from Proposition 5 is a minimal dilation and from Proposition 6 is contained in X_∞^i . By construction of the V_∞ part of the DM decomposition, the union of the elementary sets associated with unmatched vertices of V_∞^{i-} contains all the vertices of X_∞^i . Then, X_∞^i is an union of minimal dilations which do not overlap with minimal dilations of X_∞^j for $j \neq i$, therefore X_∞^i is a DCS. \triangle

VI. REVISITING THE DM DECOMPOSITION

The concepts introduced in Section V suggest that the DM decomposition can be further refined. The n_∞ connected components of the maximal inconsistent part in Remark 4 allow to refine the decomposition of the bipartite graph. In a dual way, the minimal inconsistent part of the DM decomposition can be partitioned into connected n_0 components.

Therefore, we can write $\mathcal{B}_\infty = \mathcal{B}_\infty^1 \cup \dots \cup \mathcal{B}_\infty^{n_\infty}$, $\mathcal{B}_0 = \mathcal{B}_0^1 \cup \dots \cup \mathcal{B}_0^{n_0}$ and decompose the sets V_∞^+ , V_∞^- , V_0^+ and V_0^- accordingly.

Then Proposition 2 can be generalized as follows.

Proposition 7: Let $\mathcal{B} = (V^+, V^-; W')$ be a bipartite graph and its DM decomposition with $\mathcal{B}_i = (V_i^+, V_i^-; W'_i)$, $i = 1, \dots, r$, $\mathcal{B}_j^i = (V_\infty^{j+}, V_\infty^{j-}; W'^{ij})$, $j = 1, \dots, n_\infty$, and $\mathcal{B}_0^k = (V_0^{k+}, V_0^{k-}; W_0'^k)$, $k = 1, \dots, n_0$, its DM components. One has the following properties:

- 1) A maximum matching on \mathcal{B} is a union of maximum matchings on the DM components \mathcal{B}_i , $i = 1, \dots, r$, $\mathcal{B}_j^i = (V_\infty^{j+}, V_\infty^{j-}; W'^{ij})$, $j = 1, \dots, n_\infty$, and $\mathcal{B}_0^k = (V_0^{k+}, V_0^{k-}; W_0'^k)$, $k = 1, \dots, n_0$.
- 2) A vertex v in V_0^- (or in V_i^+ , V_i^- , $i = 1, \dots, r$, or in V_∞^+) is covered by any maximum matching on \mathcal{B} .
- 3) A vertex v in V^+ belongs to V_0^+ if and only if there exists a maximum matching on \mathcal{B} that does not cover v .
- 4) A vertex v in V^- belongs to V_∞^- if and only if there exists a maximum matching on \mathcal{B} that does not cover v .

- 5) There is no edge (v^+, v^-) from a vertex v^+ in V_i^+ to a vertex v^- in V_j^- , for any $i, j \in \{0, 1, \dots, r, \infty\}$ with $i < j$.
- 6) There is no edge (v^+, v^-) from a vertex v^+ in V_0^{i+} to a vertex v^- in V_0^{j-} , for any $i, j \in \{1, \dots, n_0\}$ with $i \neq j$.
- 7) There is no edge (v^+, v^-) from a vertex v^+ in V_∞^{i+} to a vertex v^- in V_∞^{j-} , for any $i, j \in \{1, \dots, n_\infty\}$ with $i \neq j$.
- 8) There is no edge (v_k^+, v_l^-) if $v_k^+ \in V_0^{i+}$, $v_l^- \in V_0^{j-}$ and $i \neq j$ for $i, j \in \{1, \dots, n_0\}$. Any edge in a DM component belongs to some maximum matching on \mathcal{B} .

Proof of Proposition 7: Properties 6 and 7 follow from the decomposition of the inconsistent parts into connected components. Similarly, Property 1 results from the decomposition of a maximum matching of an inconsistent part into its connected parts. \triangle

In Example 1, The vertices of V_∞ in the subset $\{x_6^+, x_8^+, x_3^-, x_6^-, x_9^-\}$ are connected to each other by means of an undirected path. The same applies to the vertices of V_∞ in the subset $\{x_3^+, x_4^-, x_5^-, x_8^-\}$, and to all the vertices of V_0 . Then these observations lead to the bipartite subgraphs $\mathcal{B}_0^1 = \mathcal{B}_0$, and $\mathcal{B}_\infty^1 = (V_\infty^{1+}, V_\infty^{1-}; W_\infty^{1+})$ with $V_\infty^{1+} = \{x_6^+, x_8^+\}$, $V_\infty^{1-} = \{x_3^-, x_6^-, x_9^-\}$, $W_\infty^{1+} = \{(x_6^+, x_6^-), (x_6^+, x_9^-), (x_8^+, x_3^-), (x_8^+, x_6^-), (x_8^+, x_9^-)\}$, and $\mathcal{B}_\infty^2 = (V_\infty^{2+}, V_\infty^{2-}; W_\infty^{2+})$ with $V_\infty^{2+} = \{x_5^+\}$, $V_\infty^{2-} = \{x_4^-, x_5^-, x_8^-\}$, $W_\infty^{2+} = \{(x_5^+, x_4^-), (x_5^+, x_5^-), (x_5^+, x_8^-)\}$.

This refined DM decomposition allows also to refine Proposition 3 in the following way.

Proposition 8: Consider a linear structured dynamical system defined by the matrix A , with bipartite graph $\mathcal{B}(A)$ and the corresponding refined DM decomposition. Assume that $\text{g-rank}(A) = n - d_r$ with $d_r > 0$. Denote $l_i = |V_\infty^{i-}| - |V_\infty^{i+}|$ for the i th connected part of the maximal inconsistent component, yielding $d_r = \sum_{i=1}^{n_\infty} l_i$. To fulfil the rank condition of structural controllability, one must add l_i inputs u_1, \dots, u_{l_i} connected with l_i state vertices $x_{i_1}, \dots, x_{i_{l_i}}$, where x_{i_k} is such that $x_{i_k}^-$ is an unmatched vertex of $V_\infty^{i-}(A)$ for any connected part $V_\infty^i(A)$.

Proof of Proposition 8: This Proposition makes more precise Proposition 3. The proof follows the same lines as in [20] and [21] and taking into account the partition of the maximal inconsistent component in connected parts. \triangle

From Proposition 8, to satisfy the rank condition in Example 1, two vertices must be controlled in $X_\infty^1 = \{x_4, x_5, x_8\}$, and one vertex must be controlled in $X_\infty^2(A) = \{x_3, x_6, x_9\}$. It can be further noticed that here any two vertices in X_∞^1 and any vertex in X_∞^2 would do the job, but this is not true in general.

VII. VERTEX AND EDGE CLASSIFICATION FROM THE DM DECOMPOSITION

From Theorem 2, the DCS's are indeed obtained from a refined partition of the maximal inconsistent part of the DM decomposition of the bipartite graph $\mathcal{B}(A)$. Hence, the properties of DCS's given in [13], can directly result from the DM decomposition.

In [13], the authors propose a classification of vertices and edges of the graph relative to their importance in maximum matchings.

- *Critical vertices* (resp. *critical edges*), which belong to any maximal matching,
- *Redundant vertices* (resp. *redundant edges*), which belong to no maximal matching,
- *Intermittent vertices* (resp. *intermittent edges*), which belong to some maximal matching, but not to all maximal matchings.

The classification of vertices has been given in [23] and [22] and in terms of the DM decomposition in [20] and [21].

A. Edge classification from the DM decomposition

In this subsection, we prove that the full characterization of edges simply follows from the DM decomposition of $\mathcal{B}(A)$.

Proposition 9: Consider a linear structured dynamical system defined by the matrix A , with bipartite graph $\mathcal{B}(A)$ and the corresponding DM decomposition. The classification of edges is characterized as follows:

- An edge (x_k, x_l) is a critical edge if and only if there exists a consistent component \mathcal{B}_i of the DM decomposition such that $V_i^+ = \{x_k^+\}$ and $V_i^- = \{x_l^-\}$.
- An edge (x_k, x_l) is a redundant edge if and only if there exist two components \mathcal{B}_i and \mathcal{B}_j of the DM decomposition such that $x_k^+ \in V_i^+$ and $x_l^- \in V_j^-$.
- An edge (x_k, x_l) is an intermittent edge if it is neither critical nor redundant.

Proof of Proposition 9:

- Assume that the edge (x_k^+, x_l^-) is such that there exists a consistent component \mathcal{B}_i of the DM decomposition with $V_i^+ = \{x_k^+\}$ and $V_i^- = \{x_l^-\}$. From point 1 of Proposition 2, the edge (x_k^+, x_l^-) belongs to any maximum matching.

Conversely, from point 1 of Proposition 2 the edge (x_k^+, x_l^-) can be critical only if it belongs to a DM component. Moreover, if this component contains more than one edge, there exists at least one edge e in the same DM component which has a common vertex with (x_k^+, x_l^-) . Then, from point 6 of Proposition 2, there would exist a maximum matching containing e , and therefore not containing (x_k^+, x_l^-) , which would be a contradiction.

- Assume that the edge (x_k^+, x_l^-) is such that $x_k^+ \in V_i^+$ and $x_l^- \in V_j^-$, where V_i and V_j correspond with two different components \mathcal{B}_i and \mathcal{B}_j of the DM decomposition. A maximum matching, being decomposed in the components of the DM decomposition, as in point 1 of Proposition 2, implies then that the edge belongs to no maximum matching, and therefore is redundant.

Conversely, since, by point 6 of Proposition 2, any edge in a DM component belongs to a maximum matching, only edges from one DM component to another one may be redundant.

- The third point follows from the definition of an intermittent edge. \triangle

In Example 1, from [20] vertices x_1, x_2, x_7 are critical, there is no redundant vertex and vertices $x_3, x_4, x_5, x_6, x_8, x_9$ are intermittent.

From Proposition 9, edges (x_3, x_1) and (x_4, x_2) are critical while, for instance, (x_6, x_6) is intermittent and (x_6, x_1) is redundant. \triangle

It is important to notice that the characterization of redundant and critical (and then intermittent) edges involves all the components of the DM decomposition. This implies that a complete characterization of the edges using only DCS's is not possible, because the DCS's are only related to the \mathcal{B}_∞ component.

B. Driver-disrupting intermittent edges

To further precise the importance of the intermittent edges in matching degeneracy, the authors of [13] define the notion of driver-disrupting intermittent edge. A driver-disrupting intermittent edge is an edge e such that after removing e , some but not all control configurations still achieve structural controllability. In other words, removing e restricts the possibilities to achieve structural controllability.

Proposition 10: Consider a linear structured dynamical system defined by the matrix A , with bipartite graph $\mathcal{B}(A)$ and the corresponding DM decomposition. An edge (x_i, x_j) is a driver-disrupting intermittent edge if $(x_i^+, x_j^-) \in \mathcal{B}_\infty$ and x_i is the unique predecessor of x_j .

Proof of Proposition 10: Consider an edge (x_i, x_j) such that $(x_i^+, x_j^-) \in \mathcal{B}_\infty$ and x_i is the unique predecessor of x_j . From Proposition 9, (x_i, x_j) is an intermittent edge. Let M be a maximum matching of \mathcal{B} containing (x_i^+, x_j^-) . The control configuration associated with the maximum matching M is composed of the d_r vertices unmatched by M in X_∞ . If (x_i^+, x_j^-) is deleted, no other maximum matching of the same size can match x_j^- since (x_i^+, x_j^-) is the unique edge ending in x_j^- . Then, as an unmatched vertex, x_j^- belongs to any control configuration, the previous control configuration associated with M is no longer valid. Deleting (x_i, x_j) invalidates a control configuration, hence (x_i, x_j) is a driver-disrupting intermittent edge. \triangle

In Example 1, Proposition 10 shows that edges (x_5, x_4) , (x_5, x_5) , (x_5, x_8) and (x_8, x_3) are driver-disrupting intermittent edges.

Proposition 10 provides a sufficient condition for a given edge to be driver-disrupting which is more precise than Theorem 5 in [13]. However, it can be seen that the condition of Proposition 10 is not necessary. The control configuration $\{x_9, x_4, x_5\}$ associated with the maximum matching $\{(x_2^+, x_7^-), (x_3^+, x_1^-), (x_4^+, x_2^-), (x_6^+, x_6^-), (x_8^+, x_3^-), (x_5^+, x_8^-)\}$ is no longer a control configuration if the intermittent edge (x_6^+, x_6^-) is deleted, which proves that (x_6^+, x_6^-) is driver disrupting although vertex x_6 has two predecessors.

VIII. COMPLEXITY ASPECTS

In [13], the definition of minimal dilations and DCS's are quite natural and lead to important properties for structural controllability. The main drawback of this approach lies in the fact that finding DCS's assumes the possibility to enumerate all

the minimal dilations, which is certainly hard in a reasonable time for large scale systems.

In contrast, the characterization of DCS's through Theorem 2 only needs the computation the DM decomposition of the bipartite graph. The DM decomposition is available from many graph packages and the complexity of computing the DM decomposition is the same as the complexity of finding a maximum matching in the bipartite graph, cf. [15].

When the DM decomposition is obtained, it only remains to decompose the maximal inconsistent component into connected components to get all the DCS's. The DCS's are then obtained in polynomial time.

For more details on complexity, cf. [24], and [16] is a classical reference for the complexity of the maximum matching problem while [9] reviews the complexity of recent algorithms for solving this problem.

IX. CONCLUSIONS

In this paper we were inspired by the notion of Dilation Choice Set (DCS) introduced in [13]. The sets are used in the context of the rank condition of structural controllability of structured systems. The systems can equivalently be represented by directed and bipartite graphs. In [13] the DCS's are determined from minimal dilations by inspection, which does not yield an efficient method to check the rank condition. In this paper we proved that the computation of DCS's can be avoided by decomposing the bipartite graph of the structured system. The decomposition used is the so-called Dulmage-Mendelsohn decomposition. It is based on a maximum matching in the associated bipartite graph and can be computed by efficient methods. Once obtained the decomposition can be used efficiently for the controllability rank test, as the DCS's are located in so-called connected parts of the maximal inconsistent part of the decomposition.

The notions developed in this paper may be important for the analysis of controllability modifications resulting from deletion [25] or addition [26] of edges.

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