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# The property $\mathbf{v} \perp \nabla \times \mathbf{v}$ is true, in general; application to Navier-Stokes equations

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## Abstract

The orthogonality of a vector and its curl, not established within the framework of the continuous medium hypothesis in three dimensions of space, is discussed from the Helmholtz-Hodge theorem where any real vector can be decomposed into a component divergence-free and another curl-free. The cross product of a real vector and its curl shows that the result can be considered as a quaternion whose set forms an algebra over the field of reals. For it to be a vector in three-dimensional space, it is necessary to introduce a compatibility condition where the vector and its curl are orthogonal. The application to the Navier-Stokes equations is straightforward, the non-linear terms expressing inertia in fluid flows can be regarded as rotations in real three-dimensional space. A simple example illustrates the close link between the Helmholtz-Hodge theorem and the profound meaning of inertia.

## Keywords

Navier-Stokes equations; Quaternions; Discrete Mechanics; Helmholtz-Hodge Decomposition; Conservation of Acceleration;

## 1 Introduction

The  $\nabla \times \mathbf{V}$  vector in three dimensions of space is a complex notion in mechanics when it is defined within the framework of the continuous medium hypothesis. The fundamental theorem of calculus, which transforms the volume integral of the velocity of rotation of a vector into a surface integral relating to the tangential component of this velocity, is formally exact when its formulation is integral. The local form obtained by passing to the limit becomes more difficult to conceive and even to apply. The result is a difficulty inherent in the continuous medium hypothesis, where certain vectors have different interpretations in two or three dimensions of space.

In particular, the quantity  $\mathcal{L} = \mathbf{V} \times \nabla \times \mathbf{V}$  is *a priori* a vector of three-dimensional space. While in two dimensions the property  $\mathbf{V} \perp \nabla \times \mathbf{V}$  is well accepted in the continuous medium hypothesis, its three-dimensional version has not been established. The aim is to prove that this property is also true in dimension three.

The vector  $-\mathcal{L}$ , known as Lamb's vector, is a major component of the equations of fluid mechanics and in particular of the Navier-Stokes equations [1, 2, 3, 4]. Inertia is an essential element for understanding turbulent flows, which are the seat of intense rotational motions entangled in complex unsteady vortices. Rotation, and in particular the rotational part of inertia, plays a decisive role in turbulence, and it is therefore important to understand the true nature of the Lamb vector.

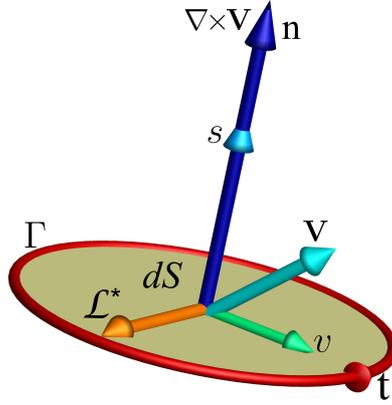
Rotations in three-dimensional space can be derived from Euler angles and rotation matrices, but an efficient method [5] is to use the quaternionic algebra developed by W.R. Hamilton [6]. An element  $\mathbf{V}$  in  $\mathbb{R}^3$  is identified with the quaternion  $(q_0, q_1, q_2, q_3)$  which can be written  $\mathbf{q} = (q_0, \mathbf{V})$  where  $\mathbf{V}$  is called the vector part of  $\mathbf{q}$ ; the latter is a pure imaginary when  $q_0 = 0$ . The formal link between the Helmholtz-Hodge decomposition, the quaternion and the Navier-Stokes equations has been conceptualised by a number of authors, including J. Cantarella [7] or E. Postnikov [8].

The Helmholtz-Hodge decomposition is the starting point for the demonstration that invalidates the Lamb vector as the inertia term of the Navier-Stokes equations in three dimensions of space. The corresponding theorem states that any vector  $\mathbb{R}^3$  is uniquely written as the sum of a component with zero divergence, another curl-free, and possibly a third harmonic. The fact that a triplet does not satisfy this theorem, i.e. the fact that none of these forms correspond to it, determines whether this quantity is a vector of  $\mathbb{R}^3$  or not.

## 2 Physical meaning of $\mathcal{L} = \mathbf{V} \times \nabla \times \mathbf{V}$

In continuum mechanics, a vector  $\mathbf{V}$  and its curl  $\nabla \times \mathbf{V}$ , two vectors of  $\mathbb{R}^3$ , are not orthogonal in general, i.e.  $\mathbf{V} \cdot \nabla \times \mathbf{V} \neq 0$ . The question is whether  $\mathcal{L}$  is an element of the  $E$  vector space of  $\mathbb{R}^3$ .

To answer this question, let's consider the scheme in Figure 1.



**Figure 1.** Representation of the vector  $\mathcal{L}^* = \mathbf{V} \times \nabla \times \mathbf{V}$  in the general case

The velocity field is any non-zero vector in the space  $\mathbb{R}^3$ . Let's consider a surface element  $dS$  of sufficiently small area to treat it as a planar surface bounded by the contour  $\Gamma$ . The curl is calculated as the circulation of this vector along the  $\Gamma$  contour,

$$\nabla \times \mathbf{V} = \frac{1}{[dS]} \int_{\Gamma} \mathbf{V} \cdot \mathbf{t} \, dl, \quad (1)$$

where the quantity  $\mathbf{t}$  is the unit vector of the  $\Gamma$  contour and  $\mathbf{n} = \nabla \times \mathbf{V} / \|\nabla \times \mathbf{V}\|$  is the oriented normal of the  $dS$  surface.

The projection of  $\mathbf{V}$  onto the plane  $dS$  is called  $\mathbf{v} = \mathbf{V} \cdot \mathbf{n}$  has the same curl as the space velocity,

$$\nabla \times \mathbf{v} = \nabla \times \mathbf{V}, \quad (2)$$

the vector  $\mathcal{L} = \mathbf{V} \times \nabla \times \mathbf{V}$  is therefore orthogonal to both  $\mathbf{V}$  and  $\nabla \times \mathbf{V}$ , and is therefore a vector of the plane  $dS$ .

Two limiting cases may arise,

- where  $\mathbf{V}$  is collinear with  $\nabla \times \mathbf{V}$ , both vectors being non-zero. Under these conditions, their cross product is a pure scalar,  $s = \mathbf{V} \cdot \nabla \times \mathbf{V}$  carried by  $\mathbf{n}$  which is contradictory with the fact that  $\mathcal{L}$  is a vector of the plane  $dS$  ;
- and the one where the two vectors are orthogonal,  $\mathcal{L}^* = \mathbf{V} \times \nabla \times \mathbf{V}$ , a situation where  $\mathcal{L}^*$  is a true vector of the  $dS$  plane.

Thus, the quantity  $\mathcal{L}$  is not a true vector of  $\mathbb{R}^3$  but a quaternion defined by,

$$\mathcal{L} = \mathbf{V} \times \nabla \times \mathbf{V} = (s, \mathcal{L}^*), \quad (3)$$

where  $s$  is the real part of the quaternion and  $\mathcal{L}^*$  is its imaginary part, also known as the pure quaternion. It is this latter part that best represents an element of the Euclidean vector space on  $\mathbb{R}^3$ . Taking the  $\mathbf{v}$  component of  $\mathcal{L}$  projected onto the  $d\mathcal{S}$  plane, we obtain,

$$\mathcal{L}^* = \mathbf{v} \times \nabla \times \mathbf{v}, \quad (4)$$

such that  $\mathbf{v} \perp \nabla \times \mathbf{v}$ .

Furthermore, the squared norm of  $\mathcal{L}$  must be calculated from the Lagrange identity,

$$\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2, \quad (5)$$

which becomes,

$$\|\mathcal{L}\|^2 = \|\mathbf{V} \times \nabla \times \mathbf{V}\|^2 = \|\mathbf{V}\|^2 \|\nabla \times \mathbf{V}\|^2 - (\mathbf{V} \cdot \nabla \times \mathbf{V})^2. \quad (6)$$

While  $\nabla \times \mathbf{v} = \nabla \times \mathbf{V}$ , this identity is also satisfied when we use the projection  $\mathbf{v}$  of  $\mathbf{V}$  onto the planar surface  $d\mathcal{S}$  defined by  $\nabla \times \mathbf{V}$  is used,

$$\|\mathcal{L}\|^2 = \|\mathbf{v}\|^2 \|\nabla \times \mathbf{v}\|^2 = \|\mathcal{L}^*\|^2. \quad (7)$$

The conclusion drawn from this geometric analysis is a condition  $\mathbf{V} \cdot \nabla \times \mathbf{V} = 0$  to be imposed on the calculation of the quantity  $\mathcal{L}$ , a condition directly guaranteed if we use the velocity component  $\mathbf{v}$  of the surface  $d\mathcal{S}$ .

### 3 Intrinsic property $\mathbf{v} \perp \nabla \times \mathbf{v}$

#### 3.1 Statement

Let  $V(\Omega)$  be the infinite vector space of all smooth vector fields in  $\Omega = \mathbb{R}^3$ , over which we use the inner product  $L^2$ . For any element  $\mathbf{V} \in (L^2(\Omega))^3$ ,

**Theorem:** *The cross product of a vector  $\mathbf{V}$  and its curl  $\nabla \times \mathbf{V}$ , two vectors of  $\mathbb{R}^3$  is itself in  $\mathbb{R}^3$  iff,*

$$\mathbf{V} \perp \nabla \times \mathbf{V}, \quad (8)$$

or else  $\mathbf{V} \cdot \nabla \times \mathbf{V} = 0$ .

#### 3.2 Proof

- Let us first consider the Helmholtz-Hodge decomposition and more precisely the Helmholtz-Hodge theorem. Consider a smooth vector field  $\mathbf{V} : \Omega \rightarrow \mathbb{R}^n$ , where  $\Omega \subseteq \mathbb{R}^n$  (for  $n = 2, 3$ ), then

$$\mathbf{V} = -\nabla\phi + \nabla \times \boldsymbol{\psi} + \mathcal{H}, \quad (9)$$

with  $\nabla \times (\nabla\phi) = 0$ ,  $\nabla \cdot (\nabla \times \boldsymbol{\psi}) = 0$ ,  $\nabla \cdot \mathcal{H} = 0$  and  $\nabla \times \mathcal{H} = 0$ .

The quantities  $\phi$  and  $\boldsymbol{\psi}$  are respectively the scalar and vector potentials of the vector  $\mathbf{V}$  and  $\mathcal{H}$  is a harmonic component. For infinite space  $\Omega = \mathbb{R}^n$  where  $\mathbf{V}$  goes to zero at

infinity, no harmonic flow can exist and thus the decomposition is unique. For a bounded domain  $\Omega \subset \mathbb{R}^n$  the harmonic part is not zero and the solution is not unique [9]; however, appropriate boundary conditions make it possible to obtain a unique solution. Potential theory allows us to write that a scalar potential, for example, is the sum of two others,  $\phi = \phi_i + \phi_h$  with  $\nabla^2 \phi_i = f$  where  $f$  is a source term and  $\nabla^2 \phi_h = 0$ . This last harmonic part is completely and uniquely defined by the boundary conditions. Discrete mechanics makes it possible to extend Galilean invariance to rotational motions in order to exclude all uniform flows from the law of motion linked to the harmonic part.

- Let us then form the quantity  $\mathcal{L} = \mathbf{V} \times \nabla \times \mathbf{V}$ , *a priori* a vector of  $\mathbb{R}^3$  and calculate its divergence and its curl,

$$\begin{cases} \nabla \cdot \mathcal{L} = \mathbf{V} \cdot \nabla (\nabla \cdot \mathbf{V}) + (\nabla \cdot \mathbf{V})^2 - 2 I_2 \neq 0, & (10a) \\ \nabla \times \mathcal{L} = \mathbf{V} \cdot \nabla (\nabla \times \mathbf{V}) - \nabla \times \mathbf{V} \cdot \nabla \mathbf{V} \neq 0, & (10b) \end{cases}$$

where the scalar  $I_2$  is the second invariant of the tensor  $\nabla \mathbf{V}$ ;  $I_2$  is not zero, even when the motion is incompressible  $\nabla \cdot \mathbf{V} = 0$ . Like the other terms of the identities (10), this term is non-zero if  $\mathbf{V}$  or  $\nabla \times \mathbf{V}$  are not themselves non-zero. Since the two terms of a Helmholtz-Hodge decomposition,  $-\nabla \phi$  and  $\nabla \times \boldsymbol{\psi}$  are orthogonal, they can only be equal to a constant under the assumption of a continuous medium. But a vector field can be a gradient field of a scalar function or the curl of another vector function, provided the appropriate rotations in three-dimensional space are considered.

- Using the Helmholtz-Hodge decomposition in  $\mathbb{R}^n$ , we can study how the vector  $\mathcal{L} = \nabla \phi + \nabla \times \boldsymbol{\psi}$  decomposes as a function of space dimension  $n$ ,
  - in 2D,  $\mathbf{V} \cdot \nabla (\nabla \times \mathbf{V})$  and  $\nabla \times \mathbf{V} \cdot \nabla \mathbf{V}$  are equal to zero because  $\mathbf{V}$  and  $\nabla \mathbf{V}$  are in the plane considered.

$$\begin{cases} \nabla \cdot \mathcal{L} = \mathbf{V} \cdot \nabla (\nabla \cdot \mathbf{V}) + (\nabla \cdot \mathbf{V})^2 - 2 I_2 \neq 0, & (11a) \\ \nabla \times \mathcal{L} = 0, & (11b) \end{cases}$$

$\mathcal{L} = \nabla \phi$  is a pure gradient in  $\mathbb{R}^2$  because the harmonic part  $\mathcal{H} = 0$  for an unbounded domain. If the vector  $\nabla \times \mathbf{V}$  is carried by the normal  $\mathbf{n}$  to a planar surface, then the velocity  $\mathbf{V}$  is in that plane.

- in 3D, we form the scalar and vector Poisson equations, whose right-hand sides are the divergence and curl of  $\mathcal{L}$ ,

$$\begin{cases} \nabla^2 \phi = -\mathbf{V} \cdot \nabla (\nabla \cdot \mathbf{V}) - (\nabla \cdot \mathbf{V})^2 + 2 I_2, & (12a) \\ \nabla^2 \boldsymbol{\psi} = -\mathbf{V} \cdot \nabla (\nabla \times \mathbf{V}) - \nabla \times \mathbf{V} \cdot \nabla \mathbf{V}, & (12b) \end{cases}$$

The quantity  $\mathcal{L}$  is neither a pure gradient, nor a pure curl, nor, *a fortiori*, a harmonic function. Nor do we know whether this quantity is decomposable in  $\mathbb{R}^3$ .

- To find out whether the Helmholtz-Hodge decomposition of  $\mathcal{L}$  is possible in  $\mathbb{R}^3$ , we can consider its meaning in a larger space, that of complexes. In 1843, W.R. Hamilton extended the notion of a complex number by encompassing real numbers and complex numbers in three dimensions of space into a single entity called a quaternion. The set of quaternions

$\mathbb{H}$  defines an algebra over the field of reals  $\mathbb{R}$  generated by three elements  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  that satisfy  $\mathbf{i}^2 + \mathbf{j}^2 + \mathbf{k}^2 = \mathbf{ijk} = -1$  known as quaternionic relations. Each quaternion is written in a unique way,

$$\mathbf{q} = q_0 + \mathbf{i} q_1 + \mathbf{j} q_2 + \mathbf{k} q_3 = [q_0, q_1, q_2, q_3] \quad (13)$$

which allows any vector to be represented in the form  $(ix + jy + kz)$  close to that defined by an orthonormal Cartesian coordinate system.

We can express the multiplication of quaternions (Hamilton's product) in the modern language of the cross product and the scalar product of vectors. Instead of the rules  $\mathbf{i}^2 + \mathbf{j}^2 + \mathbf{k}^2 = \mathbf{ijk} = -1$ , we have the rule for multiplying two vectors  $\mathbf{u}$  and  $\mathbf{w}$ ,

$$\mathbf{uw} = -\mathbf{u} \cdot \mathbf{w} + \mathbf{u} \times \mathbf{w} \quad (14)$$

where  $\mathbf{uw}$  is the resulting quaternion,  $\mathbf{u} \times \mathbf{w}$  is the cross product (a vector) and  $\mathbf{u} \cdot \mathbf{w}$  is the scalar product (a number).

The product of two quaternions whose respective vectors are orthogonal is equivalent to the cross product of these two vectors,

$$[0, \mathbf{u}] \cdot [0, \mathbf{w}] = [0, \mathbf{u} \times \mathbf{w}] \iff \mathbf{u} \cdot \mathbf{w} = 0, \quad \mathbf{u}, \mathbf{w} \in \mathbb{R}^3, \quad (15)$$

Consider, for example, the unit vector  $\mathbf{t} = \mathbf{V}/\|\mathbf{V}\|$  and another unit vector  $\mathbf{n}$  any vector orthogonal to  $\mathbf{V}$  such that  $\mathbf{t} \cdot \mathbf{n} = 0$ . The quaternion  $\mathbf{q} = \mathbf{Vt} = (\mathbf{V} \cdot \mathbf{t}, 0)$  is a scalar quaternion and  $\mathbf{q} = \mathbf{Vn} = (0, \mathbf{V} \times \mathbf{t})$  is a vector quaternion. Similarly, the quantity  $\mathbf{q} = \mathbf{VV} = (|\mathbf{V}|^2, 0)$  is a scalar quaternion.

Now consider the Hamilton product such that  $\mathbf{u} = \mathbf{V}$  and  $\mathbf{w} = \nabla \times \mathbf{V}$ ,

$$\mathbf{q} = \mathbf{V}\nabla \times \mathbf{V} = (\mathbf{V} \cdot \nabla \times \mathbf{V}, \mathbf{V} \times \nabla \times \mathbf{V}), \quad (16)$$

where the last term is none other than  $-\mathcal{L} = \mathbf{V} \times \nabla \times \mathbf{V}$ .

- In a vector space, once the basis has been chosen, a vector can be identified in the space as a triplet. If the numbers chosen are real, a space can be identified as  $\mathbb{R}^3$ . A scalar, scalar quaternion or real quaternion is a quaternion whose vector part is equal to 0 and a pure quaternion, or vector quaternion, is a quaternion whose scalar part is equal to  $q_0 = 0$ . A standard 3D vector can easily be stored in a pure quaternion; pure quaternions can then be used to rotate vectors or transform vector coordinates between different rotated reference frames.

So, if  $\mathcal{L}$  is to be defined in the space  $\mathbb{R}^3$ , the scalar  $q_0$  must be set to zero,

$$\mathbf{V} \cdot \nabla \times \mathbf{V} = 0, \quad (17)$$

and the other terms of the quaternion then define rigid rotations in space. The constrained vector  $\mathcal{L}$  (17) becomes a triplet in which only each direction orthogonal to  $\nabla \times \mathbf{V}$  has physical meaning. As Euler's rotation theorem states, "any reorientation of a rigid body is equivalent to a single rotation about an axis". This rotation in three-dimensional space can be achieved in a number of ways, using Euler angles and rotation matrices, but the most accurate and robust way is to use quaternions.

- Either  $\mathbf{V} \perp \nabla \times \mathbf{V}$  the constraint which gives a vector sense to  $\mathcal{L}$ . This constraint then appears as a compatibility condition to be satisfied for any vector  $\mathbf{V}$  of  $\mathbb{R}^3$ . In this case,  $\mathcal{L}$  takes its maximum value when the two vectors  $\mathbf{V}$  and  $\nabla \times \mathbf{V}$  are orthogonal and it is proportional to  $|\mathbf{V}|^2$ .

- By posing,

$$\mathbf{n} = \frac{\nabla \times \mathbf{V}}{\|\nabla \times \mathbf{V}\|}, \quad (18)$$

and  $\mathcal{L}^* = \mathcal{L}(\mathbf{V}) \mid \mathbf{V} \cdot \nabla \times \mathbf{V} = 0$ , a vector of  $\mathbb{R}^3$ , the Helmholtz-Hodge theorem leads to a form of  $\mathcal{L}^*$ ,

$$\mathcal{L}^* = \nabla \times \left( \frac{|\mathbf{V}|^2}{2} \mathbf{n} \right), \quad (19)$$

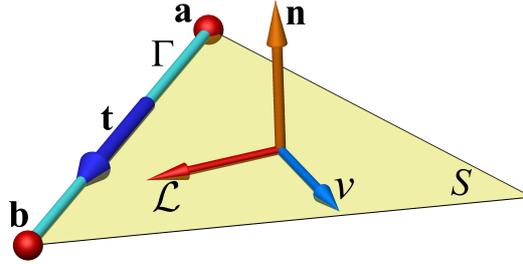
where the vector potential is equal to  $\psi = |\mathbf{V}|^2/2 \mathbf{n}$ .

### 3.3 Application to inertia of Navier-Stokes equations

The inertia of the Navier-Stokes equations can be written in several equivalent ways, two of which are as follows,

$$\mathbf{V} \cdot \nabla \mathbf{V} \equiv \nabla \left( \frac{|\mathbf{V}|^2}{2} \right) - \mathbf{V} \times \nabla \times \mathbf{V} \quad (20)$$

where the last term is none other than the Lamb vector,  $\mathcal{L} = -\mathbf{V} \times \nabla \times \mathbf{V}$ . Its representation in  $\mathbb{R}^2$  is illustrated in Figure 2.



**Figure 2.** Discrete representation of the Lamb's vector in two dimension of space

In two-dimensional space, the plane  $dS$  has a well-defined normal  $\mathbf{n}$  and the velocity vector  $\mathbf{V}$  lies in this plane. The Lamb vector is orthogonal to both  $\mathbf{n}$  and  $\mathbf{V}$ , and is a vector of the surface  $dS$ . Under these conditions, the Lamb vector can only be the gradient of a scalar function,

$$\mathcal{L}|_{2D} = -\nabla\phi. \quad (21)$$

Indeed, the constraint  $\mathbf{V} \cdot \nabla \times \mathbf{V} = 0$  is respected in two dimensions and the Lamb vector can be decomposed into two terms, one to divergence-free and another curl-free, but the only possible choice here is that of a gradient. The inertia is then presented as the sum of two gradients, a result in contradiction with the fact that a vector  $\mathbb{R}^3$  is always written as the sum of an irrotational component and another solenoidal component.

On the other hand, in  $\mathbb{R}^3$  the constraint  $\mathbf{V} \cdot \nabla \times \mathbf{V} = 0$  is not respected *a priori* and it is not possible to grant  $\mathcal{L}$  the status of a vector in order to decompose it into a gradient or a curl. Let's consider the divergence and the curl of the material derivative of velocity,

$$\left\{ \begin{array}{l} \nabla \cdot \left( \frac{d\mathbf{V}}{dt} \right) \equiv \frac{\partial}{\partial t} \nabla \cdot \mathbf{V} + \nabla^2 \phi_i + \mathbf{V} \cdot \nabla (\nabla \cdot \mathbf{V}) + (\nabla \cdot \mathbf{V})^2 - 2 I_2, \\ \nabla \times \left( \frac{d\mathbf{V}}{dt} \right) \equiv \frac{\partial}{\partial t} \nabla \times \mathbf{V} + \mathbf{V} \cdot \nabla (\nabla \times \mathbf{V}) - \nabla \times \mathbf{V} \cdot \nabla \mathbf{V}. \end{array} \right. \quad (22a)$$

$$\left\{ \begin{array}{l} \nabla \cdot \left( \frac{d\mathbf{V}}{dt} \right) \equiv \frac{\partial}{\partial t} \nabla \cdot \mathbf{V} + \nabla^2 \phi_i + \mathbf{V} \cdot \nabla (\nabla \cdot \mathbf{V}) + (\nabla \cdot \mathbf{V})^2 - 2 I_2, \\ \nabla \times \left( \frac{d\mathbf{V}}{dt} \right) \equiv \frac{\partial}{\partial t} \nabla \times \mathbf{V} + \mathbf{V} \cdot \nabla (\nabla \times \mathbf{V}) - \nabla \times \mathbf{V} \cdot \nabla \mathbf{V}. \end{array} \right. \quad (22b)$$

By posing  $\boldsymbol{\omega} = \nabla \times \mathbf{V}$  the curl of the velocity, these expressions become,

$$\begin{cases} 3D : \nabla \times \left( \frac{d\mathbf{V}}{dt} \right) \equiv \frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{V} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{V}, & (23a) \\ 2D : \nabla \times \left( \frac{d\mathbf{V}}{dt} \right) \equiv \frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{V} \cdot \nabla \boldsymbol{\omega}, & (23b) \end{cases}$$

and, as  $\mathbf{V}$  is any vector in a three-dimensional space, the term  $\boldsymbol{\omega} \cdot \nabla \mathbf{V}$  is non-zero.

In summary, the inertia term in the Navier-Stokes equations is not a vector of  $\mathbb{R}^3$ .

### 3.4 The consequences

Some consequences of the theorem (8) limited to the field of fluid mechanics can be given, for example:

- **Vortex formulation**

The compatibility condition  $\mathbf{V} \cdot \nabla \times \mathbf{V} = 0$  allows us to recover the form of the vortex advection-diffusion equation,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{V} \cdot \nabla \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega}, \quad (24)$$

whatever the size of the space under consideration.

- **Navier-Stokes existence and smoothness**

The problem of the existence and regularity of the solutions of the Navier-Stokes equations in  $\mathbb{R}^3$  is posed in these terms: prove the statement "In three dimensions of space and time, given an initial velocity field, there exists a velocity vector and a scalar pressure field, which are both smooth and globally defined, that solve the Navier-Stokes equations."

In two-dimensional space, O. Ladyzhenskaya published a proof in 1969 showing that the solutions are smooth and globally defined.

The theorem (8) makes it possible to show that  $\mathbf{V} \cdot \nabla \mathbf{V}$  is not a vector of  $\mathbb{R}^3$  except to consider the compatibility condition  $\mathbf{V} \cdot \nabla \times \mathbf{V} = 0$ , to be added to the system,

$$\begin{cases} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V}, & (25a) \end{cases}$$

$$\begin{cases} \nabla \cdot \mathbf{V} = 0, & (25b) \end{cases}$$

$$\begin{cases} \mathbf{V} \cdot \nabla \times \mathbf{V} = 0. & (25c) \end{cases}$$

The rigid rotations defined by the quaternion  $\mathbf{V} \cdot \nabla \mathbf{V}$  under the condition  $q_0 = 0$  make it possible to apply the result of O. Ladyzhenskaya's result to each of the three planes orthogonal to each direction of the  $\mathbb{R}^3$  reference frame. Assuming that the theorem (8) is correct, the initial statement of the problem no longer makes sense.

- **Turbulence in fluids**

The non-linearities contained in the  $\mathbf{V} \cdot \nabla \mathbf{V}$  inertia term of the Navier-Stokes equations imply energy exchanges between vortices of different sizes. The divergence-free and curl-free components are at the heart of the phenomenology of turbulent flows.

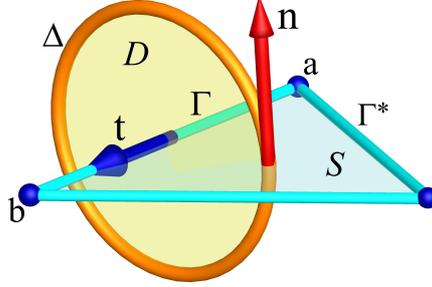
But it is likely that other areas of physics will be influenced by this result.

## 4 Discrete Mechanics

Discrete mechanics takes a completely different view of space [10]. Indeed, the condition,

$$\mathbf{v} \perp \nabla \times \mathbf{v}, \quad (26)$$

is guaranteed by construction.



**Figure 3.** Maxwell local frame of reference

The reference frame of classical mechanics  $\mathbb{R}^3$  is replaced by Maxwell's reference frame  $\mathbb{M}$  of Figure 3. This is composed of two geometric structures, (i) the primal structure is a segment  $\Gamma$  of length  $dh = [a, b]$ , called the discrete horizon oriented by a unit vector  $\mathbf{t}$  and (ii) the dual structure  $\Delta$  oriented by the unit vector  $\mathbf{n}$ . As the segments of the  $\Gamma^*$  family are coplanar, the unit vectors are orthogonal by construction  $\mathbf{t} \cdot \mathbf{n} = 0$ . The primitive surface  $\mathcal{S}$  and the dual surface  $\mathcal{D}$  are also planar. The acceleration  $\boldsymbol{\gamma}$  and the velocity  $\mathbf{v}$  are carried by the segment  $\Gamma$  while the scalar potential  $\phi$  is associated with the vertices of the primal structure and the vector potential  $\boldsymbol{\psi}$  is located on the facets  $\mathcal{S}$ . The discrete properties  $\nabla \times (\nabla \phi) = 0$  and  $\nabla \cdot (\nabla \otimes \boldsymbol{\psi}) = 0$  mimic those of the continuous medium. The law of discrete mechanics written in the form  $\boldsymbol{\gamma} = -\nabla \phi + \nabla \otimes \boldsymbol{\psi}$  of a Helmholtz-Hodge decomposition translates the conservation of acceleration, an energy per unit of mass and length. Reference [10] gives details of these concepts.

The notion of a vector of a global three-dimensional space no longer exists; the vector itself is replaced by a scalar associated with an oriented segment; the velocity of space will remain indeterminate; only its components on  $\Gamma$  make it possible to calculate an oriented curl in a unique way from the circulation of the velocity on the primal contour  $\Gamma^*$ . The notion of a change of reference frame is abandoned, each local Maxwell reference frame is connected to its neighbours by their common vertices and the interactions between local reference frames are causal.

The material derivative of classical mechanics is replaced by the expression,

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{|\mathbf{v}|^2}{2} \right) - \nabla \otimes \left( \frac{|\mathbf{v}|^2}{2} \mathbf{n} \right) = \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\kappa}, \quad (27)$$

where  $\boldsymbol{\kappa}$  represents the curvature of the Bernoulli potential  $\phi = |\mathbf{v}|^2/2$ .

With these definitions, it is possible to calculate the divergence and the primal curl of the material derivative,

$$\left\{ \begin{array}{l} \nabla \cdot \left( \frac{d\mathbf{v}}{dt} \right) \equiv \frac{\partial}{\partial t} \nabla \cdot \mathbf{v} + \nabla^2 \left( \frac{|\mathbf{v}|^2}{2} \right), \end{array} \right. \quad (28a)$$

$$\left\{ \begin{array}{l} \nabla \times \left( \frac{d\mathbf{v}}{dt} \right) \equiv \frac{\partial}{\partial t} \nabla \times \mathbf{v} + \nabla^2 \left( \frac{|\mathbf{v}|^2}{2} \mathbf{n} \right), \end{array} \right. \quad (28b)$$

where, using the vorticity  $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ ,

$$\nabla \times \left( \frac{d\mathbf{v}}{dt} \right) \equiv \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times \nabla \otimes \left( \frac{|\mathbf{v}|^2}{2} \mathbf{n} \right) = \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times \boldsymbol{\mathcal{I}}, \quad (29)$$

where  $\mathcal{I} = \nabla \otimes (|\mathbf{v}|^2/2 \mathbf{n})$  whatever the dimension of the space, 3D or 2D. Including viscous effects, we find

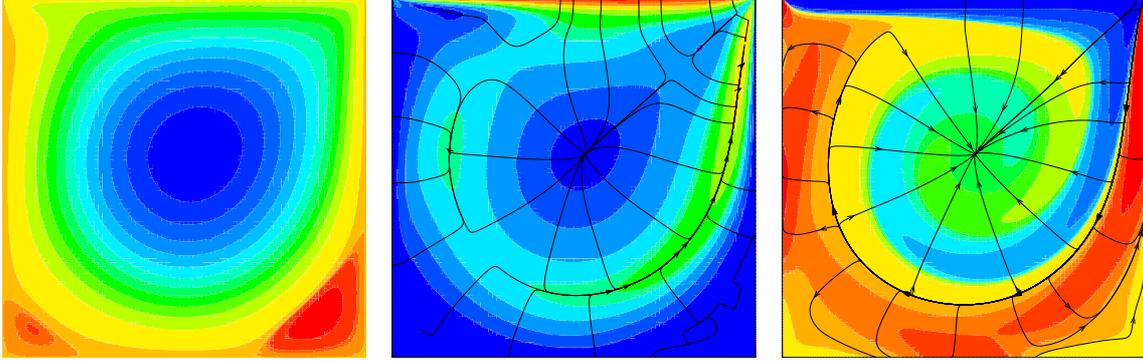
$$\frac{\partial \omega}{\partial t} + \nabla \times \mathcal{I} = -\nabla \times \mathcal{V}, \quad (30)$$

an advection-diffusion equation where  $\mathcal{V} = \nabla \otimes (\nu \nabla \times \mathbf{v})$ .

## 5 Steady solution of the Navier-Stokes equations and the law of discrete mechanics

The simulation results obtained from the Navier-Stokes equations and the discrete formulation are strictly identical, with a few exceptions.

Let's consider the case of a steady incompressible two-dimensional flow generated by the displacement of the upper wall of a cavity containing a viscous Newtonian fluid. The Reynolds number chosen is equal to  $Re = 1000$  but other choices would lead to the same conclusions. Figure 4 shows the results in the same configuration; the velocity divergence is zero at machine accuracy in both cases.



**Figure 4.** Lid driven cavity at  $Re = 1000$ . Left, streamlines in CM and DM, center, in DM, the field  $(|\mathbf{v}|^2/2) \mathbf{e}_z$  and the trajectories of  $\nabla (|\mathbf{v}|^2/2)$  and on the right, in CM, the field  $\nabla \times \mathbf{V}$  and the trajectories of  $\mathcal{L} = -\mathbf{V} \times \nabla \times \mathbf{V}$ .

The image on the left shows the streamlines, which are identical for both formulations. The one in the centre shows the  $(|\mathbf{v}|^2/2) \mathbf{e}_z$  also identical to that obtained from the Navier-Stokes equations; as expected, the trajectories of this field correspond to a series of sources denoting the origin of the gradient of the inertia term. The curl of velocity  $\nabla \times \mathbf{V}$  is shown in the image on the right and the trajectories of the Lamb vector  $\mathcal{L} = -\mathbf{V} \times \nabla \times \mathbf{V}$  are superimposed on it.

We observe:

- that  $\mathbf{V}$  and  $\mathcal{L}$  are necessarily in the same plane and that  $\nabla \times \mathbf{V}$  is orthogonal to the plane of the flow ; This characteristic is found in very many flows simulated by the incompressible Navier-Stokes equations;
- that the fields  $\nabla \times \mathbf{V}$  and  $\mathcal{L} = -\mathbf{V} \times \nabla \times \mathbf{V}$  are orthogonal which suggests that the Lamb vector  $\mathcal{L}$  is the gradient of a scalar,  $\mathcal{L} = \nabla \phi$ ;
- as the results are identical for both formulations, we are led to identify their interaction terms. It should be noted that although the effects are identical, the formulations differ substantially, the first being a curl in discrete mechanics and the second a gradient for

Navier-Stokes; in particular, the structure of Maxwell's reference frame does not allow us to identify the terms by their components, but we can write,

$$-\nabla \otimes (|\mathbf{v}|^2/2 \mathbf{n}) = P(-\mathbf{V} \times \nabla \times \mathbf{V}) = P(\nabla(|\mathbf{v}|^2/2)), \quad (31)$$

where  $P$  is rotation matrix,

$$P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (32)$$

By calculating the global curl on the square cavity as the sum of the curl on each facet  $k$  of the total surface, we obtain a value equal to

$$\nabla \times \mathbf{v}|_{2D} = \frac{1}{[\Sigma]} \sum_k \int_S \mathbf{v} \cdot \mathbf{t} \, dl = -1, \quad (33)$$

a value equal to that obtained from the circulation on the boundaries of the  $\Sigma$  domain. This two-dimensional flow is strictly the same as that calculated on a  $[0, 1]^3$  cubic cavity with sliding side walls for the same Reynolds number, the global curl in volume calculated as the sum over each cell  $j$  of the volume  $\Omega$  is equal to

$$\nabla \times \mathbf{V}|_{3D} = \frac{1}{[\Omega]} \sum_j \int_{\omega} \nabla \times \mathbf{V} \cdot \mathbf{n} \, ds = -1, \quad (34)$$

because  $\mathbf{n} = \pm 1$  for each of the two vertical faces where  $\omega$  is the elementary volume of each cell. The contribution corresponding to the upper face is equal to  $-\mathbf{V} \times \mathbf{N} = 1 \mathbf{e}_z$  in  $y = 1$ .

The solution of the Navier-Stokes equations remains acceptable for two-dimensional flows even if the inertia term is composed of two gradients. In 3D-spaces the use of the  $\mathbf{V} \cdot \nabla \mathbf{V}$  inertia term introduces uncertainty into the solutions for turbulent, unsteady and strongly rotational flows.

## 6 Conclusion

The inertia of the Navier-Stokes equations does not project into  $\mathbb{R}^3$ . In two dimensions of space, the Lamb vector is the gradient of a scalar and the solutions obtained in the plane are correct even if the inertial term is composed of two gradients of different functions. The non-inertial flows then correspond to two equal and opposite terms. In three dimensions of space, the inertial term is a quaternion which, *a priori*, cannot be decomposed into either a gradient of a scalar potential or a curl of a vector potential. The solution of the Navier-Stokes equations in  $\mathbb{R}^3$  becomes uncertain if the compatibility condition  $\mathbf{V} \cdot \boldsymbol{\omega} = 0$  is not satisfied.

## References

- [1] L. Landau, E. Lifchitz, Fluid Mechanics, Pergamon Press, London, 1959.
- [2] A. Tsinober, On one property of Lamb vector in isotropic turbulent flow, Physics of Fluids A: Fluid Dynamics 2 (4) (1990) 484–486. doi:10.1063/1.857747.
- [3] C. Lee, Z. Ran, Lamb vector in isotropic turbulence, Wuli Xuebao/Acta Physica Sinica 64 (02 2015). doi:10.7498/aps.64.034702.

- [4] C. Hamman, J. Klewick, R. Kirby, On the Lamb vector divergence in Navier-Stokes flows, *Journal Fluid Mechanics* 610 (2008) 261–284. doi:10.1017/S0022112008002760.
- [5] J. B. Kuipers, *Quaternions and Rotation Sequences A Primer with Applications to Orbits, Aerospace and Virtual Reality*, Princeton University Press, 1999, pp. 127–143. doi:10.1515/9780691211701.
- [6] W. R. Hamilton, *Elements of Quaternions*, Vol. I & II, Chelsea Publishing, New York, 1999.
- [7] J. Cantarella, D. DeTurck, H. Gluck, Vector calculus and the topology of domains in 3-space, *The American Mathematical Monthly* 109 (5) (2002) 409–442.
- [8] E. Postnikov, O. Stepanova, A variant of the pure quaternionic representation of the Navier-Stokes equation, *Advances in Applied Clifford Algebras* 15 (2005) 67–69. doi:10.1007/s00006-005-0004-1.
- [9] H. Bhatia, V. Pascucci, P. Bremer, The natural helmholtz-hodge decomposition for open-boundary flow analysis, *IEEE Transactions on Visualization and Computer Graphics* 11 (2014). doi:10.1109/TVCG.2014.2312012.
- [10] J.-P. Caltagirone, An alternative to the Navier-Stokes equation based on the conservation of acceleration, *Journal of Fluid Mechanics* 978 (2024) A21. doi:10.1017/jfm.2023.1017.