Prescribed exponential stabilization of a one-layer neural network with delayed feedback

Cyprien Tamekue, Islam Boussaada, Karim Liviu Trabelsi

PRESCRIBED EXPONENTIAL STABILIZATION OF A ONE-LAYER NEURAL NETWORK WITH DELAYED FEEDBACK

CYPRIEN TAMEKUE, ISLAM BOUSSAADA, AND KARIM TRABELSI

ABSTRACT. This paper provides control-oriented delay-based modeling of a one-layer neural network of Hopfield-type subject to an external input designed as delayed feedback. The specificity of such a model is that it makes the considered neuron less susceptible to seizure caused by its inherent dynamic instability. This modeling exploits a recently set partial pole placement for linear functional differential equations, which relies on the coexistence of real spectral values, allowing the explicit prescription of the closed-loop solution's exponential decay. The proposed framework improves some pioneering and scarce results from the literature on the characterization of the exact solution's exponential decay when a simple real spectral value exists. Indeed, it improves neural stability when the inherent dynamic is stable and provides insights into the design of a one-layer neural network that can be stabilized exponentially with delayed feedback and with a prescribed decay rate regardless of whether the inherent neuron dynamic is stable or unstable.

Keywords. Neural networks, Time-delay controller, Neutral equations, PD-controller, Coexistent-real-roots-induced-dominancy, Partial pole placement.
MSCcodes. 34K20, 34K60, 37N25,92B20, 93D15.

1. Introduction

Neural networks exhibit complex dynamics crucial for their biological or artificial functionality. The stability of these systems is essential, as instabilities can lead to dysfunctional behaviours such as seizures in biological systems [DDJB21, DEW+23, YBM+15]. In this paper, we consider the continuous-time Hopfield equation, a model that simulates the state dynamics of a biological neuron or serves as a basic unit within an artificial neural network. The following encapsulates the dynamics [Hop84],

$$\dot{y}(t) = -\nu y(t) + \mu \tanh(y(t)) + I(t) \tag{1.1}$$

where $\dot{y}(t)$ denotes the rate of change of the neuron's state at time t, ν is a positive parameter reflecting the natural decay rate or leakage of the neuron's membrane potential towards its resting state, μ is a positive parameter that scales the influence of the activation function, and $I(t)$ is an external input.

While model (1.1) simplifies the complexity inherent in the neural dynamics of a single cell, it captures essential dynamics related to neuronal excitability modulated by intrinsic properties such as decay and excitation rates when $I(t) = 0$: Under conditions where $\nu \geq \mu$, the zero equilibrium is stable, transitioning from exponential stability ($\nu > \mu$) to asymptotic stability ($\mu = \nu$). Conversely, when $\nu < \mu$, the system exhibits hyperexcitability characteristic of seizure-like activities [Sta06] with the zero equilibrium becoming unstable and the emergence of two additional stable equilibria.

To address the challenges posed by hyperexcitability and to prevent the destabilizing transitions that can lead to seizure-like states, we introduce a feedback control mechanism

$$I(t) = -k_p y(t-\tau) - k_d \dot{y}(t-\tau) \tag{1.2}$$

where $k_p > 0$ and $k_d > 0$ are the proportional and derivative gains, respectively, and $\tau > 0$ represents a synaptic delay. Here, k_p and k_d mimic the biological feedback processes, such as those mediated by inhibitory neurotransmitters that dynamically modulate neuronal excitability based on recent activity patterns [GKNP14].

By implementing a delay in the proportional-derivative (PD) controller, the resulting model

$$\dot{y}(t) = -\nu y(t) + \mu \tanh(y(t)) - k_p y(t-\tau) - k_d \dot{y}(t-\tau) \tag{1.3}$$

reflects the time-dependent nature of neuronal interactions, where the effects of inhibitory actions are not instantaneous but occur with a physiological lag, characteristic of synaptic transmission delays relevant in brain activities [PJ19, SD19]. For further insights into the use of time delays in modeling biological systems, see, for instance, [BBL93, Gop13, Rua06]. In particular, it is commonly accepted to include time delays in modeling sensory and motoric neural pathways due to the time lag one observes in communication. See, for instance, [Ste09, Kua93], and [WR99] for two delays in neural network modeling.

This paper explores the effectiveness of delayed PD controllers (1.2) in stabilizing the equation (1.1) characterized by $\nu < \mu$. Our main goal is to establish a theoretical basis to guide the practical design of a one-layer neural network system that is less susceptible to seizures caused by inherent dynamic instability.

We achieve this general purpose through the linearization technique. We are therefore interested in studying the exponential stability of the equation,

$$\dot{y}(t) = -(\nu - \mu) y(t) - k_p y(t-\tau) - k_d \dot{y}(t-\tau). \tag{1.4}$$

In the proposed framework, we adjust the control parameters — specifically the gains k_p and k_d, and the delay τ — to ensure that the solutions of equation (1.4) are not only asymptotically stable but also exhibit a guaranteed rate of exponential decay. Observe that the global exponential stability of (1.4) is equivalent to the local exponential stability of the zero equilibrium to (1.3).

The equation (1.4) has been widely discussed in the literature regarding its asymptotic and exponential stability [LYH00, HL13, Fri01, BMN22, SBN23]. Sufficient delay-independent conditions for stability have been presented. In [LYH00, Example 1, page 26], the authors proved the global uniform asymptotic stability of equation (1.4) with constant real coefficients (not dependent upon the delay τ) satisfying $\nu - \mu > 0$, $|k_d| < 1$, and $|k_p| < (\nu - \mu)\sqrt{1 - k_d^2}$ using a Lyapunov functional and a linear matrix inequality (LMI). It's worth noting that asymptotic analysis of (1.4) was already considered in [HL13, Chapter 9, Section 9.8, page 294] when $k_p = 0$, $\nu - \mu > 0$, and $|k_d| < 1$, see also [Fri14, Chapter 3, Section 3.3.4, page 69]. Unfortunately, to the best of the authors' knowledge, no known result using a time-domain approach based on a Lyapunov functional and a LMI can be applied to study the asymptotic stability properties of equation (1.4) when $\nu - \mu \leq 0$.

In the frequency domain, the problem reduces to the analysis of the distribution of the roots of the corresponding characteristic function, which is an entire function called *characteristic quasipolynomial*. Interestingly, the corresponding characteristic quasipolynomial is shared with the problem of boundary PI control of the transport equation as studied in [CT15, SBN23], see also [BBNT23] for further insights on the corresponding quasipolynomial. For a deeper discussion of the spectral properties and related stability analysis and control approaches of (1.4), refer to [MN14, Bri15]. Via Laplace's transform, the asymptotic stability of equation (1.4) is reduced to study the location of the spectrum of the quasipolynomial function

$$\Delta_0(s) = s + \nu - \mu + e^{-\tau s}(k_d s + k_p), \qquad s \in \mathbb{C}. \tag{1.5}$$

As a matter of fact, to characterize the exponential stability of solutions of the linearization of (1.4), it amounts to the location of the spectral abscissa of Δ_0, which is a challenging problem in all generality. However, recent works have highlighted a particularly interesting spectral property, called *multiplicity-induced-dominancy (MID)*, which consists in conditions on the system's parameters under which a multiple spectral value corresponds to the spectral abscissa [BNEA⁺20, BMN22, MBN21]. In particular, it appears that a characteristic root of maximal multiplicity (i.e., equal to the degree of the corresponding quasipolynomial) necessarily defines the spectral abscissa of the system, this property occurs in general in *generic* quasipolynomial and is called GMID. However, in the case of *intermediate* multiplicities, that is, multiplicities which are less than the quasipolynomial's degree, the IMID occurs (the largest assigned root corresponds to the spectral abscissa) under some additional conditions, see for instance [BBNT23]. Since these works, the case of the assignment of a characteristic root with maximal multiplicity was recently addressed and thoroughly characterized in [MBN21] (generic retarded case) and in [BMN22] (unifying retarded and neutral cases) for LTI DDEs including a *single delay*. It is essential to point out that the multiplicity of a given spectral value itself is not essential. Still, its connection with the dominancy of this root is a meaningful tool for control synthesis. Namely, it is shown that, under appropriate conditions, the coexistence of exactly the maximal number of distinct negative zeros of quasipolynomial of reduced degree guarantees the exponential stability of the zero solution of the corresponding time-delay system, a property called *Coexisting Real Roots Induced Dominancy (CRRID)*, see for instance [ABBN18, BBN20, SBN23, SBNB24]. These properties opened an interesting perspective in control through the so-called *partial pole placement* method, that is, imposing the multiplicity or the coexistence of simple real characteristic root of the closed-loop system by an appropriate choice of the controller gains guarantees the exponential stability of the closed-loop system with a prescribed decay rate.

The main contributions of this paper are threefold: First, it refines recent results on the CRRID property for the first-order neutral functional differential equations, providing further insights into the qualitative properties of the corresponding quasipolynomials. In particular, in comparison to the result of [SBN23], it gives a more straightforward proof for the GCRRID[1] (G refers to *generic*) to hold for an arbitrary distribution of the real roots as well as the necessary and sufficient conditions for the ICRRID (I refers to *intermediate*) to hold, i.e., when two real roots are assigned. The latter is provided in the two complementary cases: no further real roots exist, and an unintentional third real root coexists.

Secondly, it sheds some light on the intriguing properties of the quasipolynomial satisfying the CRRID and the benefits offered by the CRRID property in terms of exponential decay certification. In fact, in the single-delay case, when the Frasson-Verduyn Lunel's [FVL03] sufficient condition for the dominancy of a simple spectral value is not met, the CRRID remains valid. Furthermore, the CRRID extends some recent results employing time-domain approaches such

[1]The GCRRID occurs when the maximal number of real roots is achieved.

as the ones relying on Lyapunov functional and linear matrix inequalities thanks to the resulting sharp information on the spectrum location. As a matter of fact, when three coexistent spectral values are equidistributed, we exhaustively characterize the remaining spectrum distribution, extending the result obtained in [BMN22] in the GMID case. Nevertheless, when these roots are not equidistributed, we show that the remaining spectrum is asymptotic to an appropriate vertical line in the complex plane. Also, while the negativity of the spectral abscissa is assumed in [SBN23], in our contribution, the negativity of such a spectral abscissa is characterized in the space of the system's parameters. Thanks to the CRRID setting, and inspired from [HL13, Chapter 1, Theorem 6.2], we establish an explicit and more straightforward exponential estimate of the closed-loop system's solution compared to that of [HL13][Chapter 1, Theorem 7.6, page 32] thanks to the special structure of the corresponding quasipolynomial.

Lastly, these findings are interpreted and exploited in modeling a one-layer neural network of Hopfield-type, which is less susceptible to seizure caused by inherent dynamic instability.

The remainder of the paper is organized as follows: Section 2 provides the reader with some prerequisites in the complex analysis used to prove the main results and ends by the problem settings. In Section 3, we present the main results and the corresponding proofs. Section 4 exploits and translates the main results into exponentially stable one-layer neural network modeling.

2. Problem settings and prerequisites

Throughout the following, our focus is on studying the asymptotic behaviour of solutions of the general scalar neutral functional differential equation (NFDE) in Hale's form

$$\frac{d}{dt}(y(t) + \alpha y(t-\tau)) = -ay(t) - \beta y(t-\tau), \tag{NDE}$$

with corresponding initial condition $y(0) = y_0 \in C^0([-\tau, 0])$. To this equation corresponds the characteristic quasipolynomial function given by

$$\Delta(s) = s + a + e^{-\tau s}(\alpha s + \beta) \quad (s \in \mathbb{C}), \tag{2.1}$$

where $(a, \alpha, \beta) \in \mathbb{R}^3$ and $\tau > 0$. It is known that the degree of the quasipolynomial Δ—the sum of the degrees of the involved polynomials plus the number of delays—is equal to three. Moreover, thanks to the Pólya-Szegö bound [PS72, Problem 206.2, page 144], the degree of Δ is a sharp bound for the number of real roots counting multiplicities of the quasipolynomial Δ.

In the next section, we study the spectral properties of the quasipolynomial function Δ, focusing on characterizing its rightmost root. Consider a complex value $s_0 \in \mathbb{C}$ such that $\Delta(s_0) = 0$. We say that s_0 is a *dominant* (respectively, *strictly dominant*) root of Δ if the following holds:

$$\forall s \in \mathbb{C}\backslash\{s_0\}, \qquad \Delta(s) = 0 \quad \Longrightarrow \quad \Re(s) \leq \Re(s_0) \quad (\text{respectively } \Re(s) < \Re(s_0)). \tag{2.2}$$

Despite the challenging question of characterizing the spectral abscissa of (NDE), Frasson-Verduyn Lunel in [FVL03, Lemma B1] provides a test for determining the simplicity and dominance of real spectral values in the multi-delay scalar neutral equations, offering fundamental insights. Notice that such a characterization is closely related to the exponential/asymptotic behavior of the equation (NDE).

According to Frasson-Verduyn Lunel's lemma when restricted to the single-delay case, a real root s_0 of Δ is simple and dominant if $V(s_0) < 1$ where V is a specific functional construct derived from Δ. In other words, the condition is as follows; see, [FVL03, Lemma 5.1].

Lemma 2.1. *Suppose that there exists a real zero s_0 of Δ. If $V(s_0) < 1$, then s_0 is a real simple dominant zero of Δ. Here,*

$$V(s_0) = (|\alpha|(1 + |s_0|\tau) + |\beta|\tau)e^{-s_0\tau},$$

where $(\alpha, \beta) \in \mathbb{R}^2$ are real coefficients appearing in (2.1) and $\tau > 0$.

Frasson-Verduyn Lunel's lemma is relevant, however restrictive, its limitation shall be explored later in Section 3.3. This study uses the CRRID framework, which appears to be broad and flexible, with a sharply located spectrum distribution. Our analysis shows that the CRRID approach ensures that the largest real root is the rightmost and dominant, even when Frasson-Verduyn Lunel's sufficient condition is not satisfied. When the real spectral values assigned to Δ are equidistributed, we also determine the threshold value beyond which the largest one maintains dominancy while no longer satisfying Frasson-Verduyn Lunel's sufficient condition for dominance. This expanded application of the CRRID approach demonstrates its usefulness in providing an effective framework for analyzing the spectrum of quasipolynomials.

Before starting the study of the localisation of the quasipolynomial Δ roots, let us state the following lemma the proof of which can be found in [PS12, Problem 77, page 46].

Lemma 2.2 (Descartes' rule of signs)**.** *Let a_1, a_2, a_3, λ_1, λ_2, λ_3 be real constants, such that $\lambda_1 < \lambda_2 < \lambda_3$. Denote by Z the number of real zeros of the entire function*

$$\mathcal{F}(x) = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + a_3 e^{\lambda_3 x} \tag{2.3}$$

and by C, the number of sign changes in the sequence of numbers a_1, a_2, a_3. Then, $C - Z$ is a non-negative even integer.

One also has the following important result, which extends the [SBN23, Corollary 2]; it includes the case where we only assign two real spectral values to Δ.

Lemma 2.3. *Let Δ be the quasipolynomial defined by (2.1) with real coefficients. Let $\eta \in \{0, 1\}$, if Δ admits exactly $3 - \eta$ real roots, then any root $x + i\omega \in \mathbb{C}$ of Δ with $\omega \neq 0$ satisfies*

$$|\omega| \geq (2 - \eta)\pi/\tau. \tag{2.4}$$

Proof. Recall from [PS72, Problem 206.2, page 144] that if $M_{\alpha,\beta}$ denotes the number of roots of Δ contained in the horizontal strip $\{s \in \mathbb{C} \mid \alpha \leq \Im(s) \leq \beta\}$ counting multiplicities, then the following bound holds

$$\frac{\tau(\beta - \alpha)}{2\pi} - 3 \leq M_{\alpha,\beta} \leq \frac{\tau(\beta - \alpha)}{2\pi} + 3. \tag{2.5}$$

To complete the proof of the lemma, we argue by contradiction. Assume that Δ admits $3-\eta$ real roots, and let $x+i\omega \in \mathbb{C}$ be a root of Δ with $\omega \neq 0$ and $|\omega| < (2-\eta)\pi/\tau$. Then there exists $\varepsilon > 0$ such that $|\omega| < (2 - \eta)\pi/\tau - \varepsilon$. Since Δ has real coefficients, the complex conjugate of ω is also a root of Δ, both belonging to the horizontal strip $\{s \in \mathbb{C} \mid -\frac{(2-\eta)\pi}{\tau} + \varepsilon \leq \Im(s) \leq \frac{(2-\eta)\pi}{\tau} - \varepsilon\}$. It follows that Δ admits at least $5 - \eta$ roots in this strip, which is inconsistent, since by the Pólya-Szegö bound (2.5), the number of zero in this strip satisfies

$$M_{-\frac{(2-\eta)\pi}{\tau}+\varepsilon, \frac{(2-\eta)\pi}{\tau}-\varepsilon} \leq 5 - \eta - \frac{\varepsilon\tau}{\pi} < 5 - \eta. \tag{2.6}$$

□

Let us investigate the coexistence of - non-necessarily equidistributed - three real roots for the quasipolynomial Δ. Due to the linearity of Δ with respect to its coefficients a, α and β, one reduces the system $\Delta(s_1) = \Delta(s_2) = \Delta(s_3) = 0$ to the linear system

$$A_{\tau,3}(s_1, s_2, s_3)X = B, \tag{2.7}$$

where $B = -(s_1 e^{\tau s_1}, s_2 e^{\tau s_2}, s_3 e^{\tau s_3})^t$, $X = (\alpha, \beta, a)^t$ and

$$A_{\tau,3}(s_1, s_2, s_3) = \begin{bmatrix} s_1 & 1 & e^{\tau s_1} \\ s_2 & 1 & e^{\tau s_2} \\ s_3 & 1 & e^{\tau s_3} \end{bmatrix}. \tag{2.8}$$

Using [BBN20, Theorem 2], one immediately obtains that the determinant of the structured functional Vandermonde-type matrix $A_{\tau,3}(s_1, s_2, s_3)$ is given by

$$D_{\tau,3}(s_1, s_2, s_3) = \tau^2 (s_1 - s_2)(s_1 - s_3)(s_2 - s_3) F_{-\tau,2}(s_1, s_2, s_3) \tag{2.9}$$

where $F_{-\tau,2}(s_1, s_2, s_3)$ is defined by

$$F_{-\tau,2}(s_1, s_2, s_3) := \int_0^1 \int_0^1 (1 - t_1) e^{\tau(t_1 s_1 + (1-t_1)(t_2 s_2 + (1-t_2) s_3))} dt_1 dt_2 > 0. \tag{2.10}$$

By integrating (2.10), one may carefully check that

$$F_{-\tau,2}(s_1, s_2, s_3) = \frac{e^{\tau s_3}(s_1 - s_2) + e^{\tau s_2}(s_3 - s_1) + e^{\tau s_1}(s_2 - s_3)}{\tau^2 (s_1 - s_3)(s_2 - s_3)(s_1 - s_2)}. \tag{2.11}$$

For distinct real spectral values $s_3 < s_2 < s_1$, one deduces from (2.9) and (2.10) that for every $\tau > 0$, $D_{\tau,3}(s_1, s_2, s_3) > 0$ holds. It follows that (2.7) is a Cramer system, and one can immediately compute the coefficients α, β and a using the Cramer formula. More precisely, one has the following.

Lemma 2.4. *For a fixed $\tau > 0$, the quasipolynomial Δ admits three distinct real spectral values s_3, s_2 and s_1 if, and only if, the coefficients α, β and a are respectively given by*

$$\beta(\tau) = \frac{1}{D_{\tau,3}(s_1, s_2, s_3)} \det \begin{bmatrix} s_1 & -s_1 e^{\tau s_1} & e^{\tau s_1} \\ s_2 & -s_2 e^{\tau s_2} & e^{\tau s_2} \\ s_3 & -s_3 e^{\tau s_3} & e^{\tau s_3} \end{bmatrix}. \tag{2.12}$$

$$\alpha(\tau) = \frac{1}{D_{\tau,3}(s_1, s_2, s_3)} \det \begin{bmatrix} -s_1 e^{\tau s_1} & 1 & e^{\tau s_1} \\ -s_2 e^{\tau s_2} & 1 & e^{\tau s_2} \\ -s_3 e^{\tau s_3} & 1 & e^{\tau s_3} \end{bmatrix} = \frac{F_{-\tau,2}(s_1 + s_2, s_1 + s_3, s_2 + s_3)}{F_{-\tau,2}(s_1, s_2, s_3)}. \tag{2.13}$$

$$a(\tau) = \frac{1}{D_{\tau,3}(s_1, s_2, s_3)} \det \begin{bmatrix} s_1 & 1 & -s_1 e^{\tau s_1} \\ s_2 & 1 & -s_2 e^{\tau s_2} \\ s_3 & 1 & -s_3 e^{\tau s_3} \end{bmatrix} = -s_1 - \frac{F_{-\tau,1}(s_2, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)} \tag{2.14}$$

$$= -s_2 - \frac{F_{-\tau,1}(s_1, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)}. \tag{2.15}$$

Here, one has

$$F_{-\tau,1}(u, v) := \int_0^1 e^{\tau(tu + (1-t)v)} dt = \frac{e^{\tau u} - e^{\tau v}}{\tau(u - v)} > 0 \quad (\forall u, v \in \mathbb{R}). \tag{2.16}$$

Proof. Relation (2.12) and the first identities in relations (2.13)-(2.15) follow directly by applying the Cramer formulas to the Cramer system (2.7). To obtain the second ones in (2.13)-(2.15), one may use (2.9), (2.11), (2.16) and the following.

$$\det\begin{bmatrix}-s_1e^{\tau s_1} & 1 & e^{\tau s_1}\\ -s_2e^{\tau s_2} & 1 & e^{\tau s_2}\\ -s_3e^{\tau s_3} & 1 & e^{\tau s_3}\end{bmatrix} = e^{\tau(s_1+s_2)}(s_1-s_2)+e^{\tau(s_1+s_3)}(s_3-s_1)+e^{\tau(s_2+s_3)}(s_2-s_3)$$

$$(2.17)\qquad = \tau^2(s_1-s_2)(s_1-s_3)(s_2-s_3)F_{-\tau,2}(s_1+s_2,s_1+s_3,s_2+s_3),$$

and

$$\begin{aligned}\det\begin{bmatrix}s_1 & 1 & -s_1e^{\tau s_1}\\ s_2 & 1 & -s_2e^{\tau s_2}\\ s_3 & 1 & -s_3e^{\tau s_3}\end{bmatrix} &= e^{\tau s_3}(s_2-s_1)s_3+e^{\tau s_2}(s_1-s_3)s_2+e^{\tau s_1}(s_3-s_2)s_1\\ &= -s_1D_{\tau,3}(s_1,s_2,s_3)-(e^{\tau s_2}-e^{\tau s_3})(s_1-s_2)(s_1-s_3)\\ &= -s_2D_{\tau,3}(s_1,s_2,s_3)-(e^{\tau s_1}-e^{\tau s_3})(s_1-s_2)(s_2-s_3)\end{aligned}$$

hence completing the proof of the lemma. □

The next lemma is a key ingredient in simplifying the proofs of our main results in Section 3.1.

Lemma 2.5. *Assume that the quasipolynomial Δ has three real spectral values $s_3 < s_2 < s_1$. Then, the following holds*

$$(2.18)\qquad \begin{aligned} &0<\alpha<1.\\ &\alpha < e^{\tau x} \qquad (\forall x \geq s_1).\end{aligned}$$

Proof. It follows from (2.13) and (2.10) that $\alpha > 0$. Then, one has

$$(2.19)\qquad 1-\alpha = \frac{F_{-\tau,2}(s_1,s_2,s_3)-F_{-\tau,2}(s_1+s_2,s_2+s_3,s_1+s_3)}{F_{-\tau,2}(s_2,s_3,s_1)} > 0$$

since for every $(t_1,t_2)\in[0,1]^2$, the following holds

(2.20)
$$t_1s_1-(1-t_1)(t_2s_2-(1-t_2)s_3)-t_1(s_1+s_2)-(1-t_1)(t_2(s_2+s_3)-(1-t_2)(s_1+s_3)) = \\ -t_1s_2-t_2(1-t_1)s_3-(1-t_1)(1-t_2)s_1 > 0.$$

Now, let $x \geq s_1$, then one has

$$(2.21)\qquad e^{\tau x}-\alpha = \frac{e^{\tau x}F_{-\tau,2}(s_2,s_3,s_1)-F_{-\tau,2}(s_1+s_2,s_2+s_3,s_1+s_3)}{F_{-\tau,2}(s_2,s_3,s_1)} > 0,$$

since $F_{-\tau,2}(s_1,s_2,s_3) = F_{-\tau,2}(s_2,s_3,s_1)$ by [BBN20, Lemma 4] and the fact that for every $(t_1,t_2)\in[0,1]^2$, one has

(2.22)
$$x+t_1s_2+(1-t_1)(t_2s_3+(1-t_2)s_1))-t_1(s_1+s_2)-(1-t_1)(t_2(s_2+s_3)-(1-t_2)(s_1+s_3))) = \\ x-t_1s_1-t_2(1-t_1)s_2-(1-t_1)(1-t_2)s_3 > \\ (1-t_1)s_1-t_2(1-t_1)s_1-(1-t_1)(1-t_2)s_1 = 0$$

completing the proof of the lemma. □

Remark 2.6. Note, from Lemmas 2.4 and 2.5, that, if $s_1 < 0$ then for every $\tau > 0$, one has $b(\tau) = -\alpha s_1 + \zeta e^{\tau s_1} > 0$ and $a(\tau)$ may change sign. Here,

$$\zeta := \zeta(\tau) = \frac{F_{-\tau,1}(s_2, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)} > 0. \tag{2.23}$$

3. Main results

In this section, we establish our main results. Section 3.1 discusses the GCRRID property for the quasipolynomial Δ and derives a new and simpler proof for its validity. In Section 3.2, the ICRRID property is investigated. Next, Section 3.3 provides an example of a simple dominant spectral value violating the sufficient condition established by Frasson-Verduyn Lunel, thereby emphasizing the non-necessary nature of the latter condition. Additionally, we will present a simpler proof of the exponential estimates for solutions of equation (NDE) in Section 3.4.

3.1. Assigning three distinct real spectral values.

This section discusses the GCRRID property for the quasipolynomial Δ. This property involves assigning the maximal number of distinct real spectral values $s_3 < s_2 < s_1$ to Δ and proving that s_1, the rightmost, is the dominant root. We shall then use this result to establish necessary and sufficient conditions to ensure that s_1 is negative, which is essential to the exponential stability of equation (NDE). Finally, we shall fully characterize the remaining spectrum of Δ by assigning three equidistributed real spectral values.

The first main result of this section provides a comprehensive proof of [SBN23, Theorem 5].

Theorem 3.1 (Dominancy of a real root)**.** *Assume that Δ admits three real spectral values $s_3 < s_2 < s_1$. Then, the spectral value s_1 is a strictly dominant root of Δ.*

Proof. Fix $\tau > 0$. It follows from Lemma 2.4 that $a + s_1 < 0$ and $\alpha > 0$. We argue by contradiction. Assume that there exists $s_0 := x + i\omega \in \mathbb{C}$ such that $\Delta(s_0) = 0$ and $x \geq s_1$. In particular, $\omega \neq 0$, since we have already assigned three real spectral values $s_3 < s_2 < s_1$ to Δ. From $\Delta(s_1) = 0$, one deduces that $\beta = -\alpha s_1 - e^{\tau s_1}(a + s_1)$. It follows that $\Delta(s_0) = 0$ if, and only if,

$$e^{\tau s_0} s_0 + e^{\tau s_0} a + \alpha s_0 = \alpha s_1 + e^{\tau s_1}(a + s_1). \tag{3.1}$$

By taking the real and imaginary parts of both sides in (3.1), one gets

$$\begin{aligned} (a + x)\cos(\tau\omega) - \omega \sin(\tau\omega) &= e^{-\tau(x - s_1)}(a + s_1) - \alpha(x - s_1)e^{-\tau x} \\ (a + x)\sin(\tau\omega) + \omega\cos(\tau\omega) &= -\alpha\omega e^{-\tau x}. \end{aligned} \tag{3.2}$$

By squaring each equality in (3.2) and adding them, one obtains

$$\omega^2 = \frac{(e^{\tau s_1}(a + s_1) - \alpha(x - s_1))^2 - (a + x)^2 e^{2\tau x}}{e^{2\tau x} - \alpha^2} \tag{3.3}$$

which is well defined for every $x \geq s_1$ by Lemma 2.5. Let us prove that ω^2 given by (3.3) satisfies $\omega^2 < 1/\tau^2$. To do so, define the function

$$\chi(x) = (e^{\tau s_1}(a + s_1) - \alpha(x - s_1))^2 - (a + x)^2 e^{2\tau x} - \tau^{-2}(e^{2\tau x} - \alpha^2) \quad (\forall x \geq s_1). \tag{3.4}$$

We want to prove that $\chi(x) < 0$ for every $x \geq s_1$. On the one hand, one has

$$\chi(s_1) = -(e^{2\tau s_1} - \alpha^2)\tau^{-2} < 0, \qquad \lim_{x \to \infty} \chi(x) = -\infty \tag{3.5}$$

owing to Lemma 2.5. On the other hand, function χ is infinitely derivable on $[s_1, \infty)$, so that

$$\chi''(x) = -2e^{2\tau x}\left(2\left(\tau(a+x)+1\right)^2 + e^{-2\tau x}(e^{2\tau x}-\alpha^2)\right) < 0 \quad (\forall x \geq s_1) \tag{3.6}$$

since the two terms between the big brackets of $\chi''(x)$ are positive by Lemma 2.5. One also has

(3.7)

$$\chi'(x) = -2\alpha((a+s_1)e^{\tau s_1} - \alpha(x-s_1)) - 2\tau(a+x)^2 e^{2\tau x} - 2(a+x)e^{2\tau x} - 2\tau^{-1}e^{2\tau x} \quad (\forall x \geq s_1)$$

$$\chi'(s_1) = -2\tau\left((a+s_1)e^{\tau s_1} + \frac{\alpha + e^{\tau s_1}}{2\tau}\right)^2 - \frac{(e^{\tau s_1}-\alpha)(\alpha+3e^{\tau s_1})}{2\tau} < 0, \qquad \lim_{x\to\infty}\chi'(x) = -\infty.$$

Here, $\chi'(s_1) < 0$, as the sum of two negative terms by Lemma 2.5. It follows from (3.6) and (3.7) that

$$\chi'(x) < 0 \quad (\forall x \geq s_1). \tag{3.8}$$

Finally, by combining (3.8) and (3.5), one gets that

$$\chi(x) < 0 \quad (\forall x \geq s_1). \tag{3.9}$$

The latter is equivalent to $\omega^2 < \tau^{-2}$ with ω^2 defined in (3.3). So, we have shown that if $x + i\omega$ is a root of Δ with $x \geq s_1$ and $\omega \neq 0$, then $|\omega| < \tau^{-1}$. The latter is inconsistent since one has necessarily $|\omega| \geq 2\pi/\tau$ owing to Lemma 2.3. □

Remark 3.2. It is worth noting that the key property on α required in the proof of Theorem 3.1 is that $e^{2\tau x} - \alpha^2 \geq 0$ for all $x \geq s_1$ which is satisfied owing to Lemma 2.5.

The second main result provides the necessary and sufficient conditions on the delay τ and the coefficient a to guarantee that the dominant root s_1 is negative when three spectral values $s_3 < s_2 < s_1$ are formally assigned to Δ.

Theorem 3.3 (Negativity of the dominant root)**.** *Assume that Δ admits three real spectral values $s_3 < s_2 < s_1$. Then, $s_1 < 0$ if, and only if, there exists a unique $\tau_* > 0$ such that $a(\tau_*) = 0$, where*

$$a(\tau)\begin{cases} = 0 & \text{if} \quad \tau = \tau_*, \\ < 0 & \text{if} \quad \tau < \tau_*, \\ > 0 & \text{if} \quad \tau > \tau_*. \end{cases} \tag{3.10}$$

In particular, $s_1 < 0$ satisfies

$$s_1\begin{cases} = -\zeta(\tau) & \text{if} \quad \tau = \tau_*, \\ > -\zeta(\tau) & \text{if} \quad \tau < \tau_*, \\ < -\zeta(\tau) & \text{if} \quad \tau > \tau_*, \end{cases} \qquad \text{where} \quad \zeta(\tau) = \frac{F_{-\tau,1}(s_2,s_3)}{\tau F_{-\tau,2}(s_1,s_2,s_3)} > 0. \tag{3.11}$$

Proof. Let us assume that $s_1 < 0$. Since $\tau \in (0,\ \infty) \mapsto a(\tau)$ is continuous with respect to τ, one may use the intermediate value theorem. Combining (2.14) and (2.11), one gets

$$a(\tau) = \frac{e^{-\tau(s_1-s_3)}(s_1-s_2)s_3 + e^{-\tau(s_1-s_2)}(s_3-s_1)s_2 + (s_2-s_3)s_1}{e^{-\tau(s_1-s_3)}(s_2-s_1) + e^{-\tau(s_1-s_2)}(s_1-s_3) + (s_3-s_2)}. \tag{3.12}$$

Consequently, $a(\tau) \to -s_1 > 0$ when $\tau \to \infty$. Next, definitions (2.10) and (2.16), and the continuity of $F_{-\tau,1}$ and $F_{-\tau,2}$ with respect to τ entail that $a(\tau) \to -\infty$ as $\tau \to 0$. It follows that there exists at least one $\tau_* > 0$ such that $a(\tau_*) = 0$. To show that τ_* is unique, one applies Lemma 2.2. We observe that $a(\tau) = 0$ if, and only if, its numerator vanishes at τ. Now, let

$$F(\tau) = e^{\tau s_3}(s_1 - s_2)s_3 + e^{\tau s_2}(s_3 - s_1)s_2 + e^{\tau s_1}(s_2 - s_3)s_1 \tag{3.13}$$

be the numerator of $e^{\tau s_1}a(\tau)$. Then, F is analytic in τ, and one has from $s_3 < s_2 < s_1$ that $(s_1 - s_2)s_3 < 0$, $(s_3 - s_1)s_2 > 0$ and $(s_2 - s_3)s_1 < 0$. Let C denote the number of sign changes in the sequence of real numbers $(s_1 - s_2)s_3$, $(s_3 - s_1)s_2$, and $(s_2 - s_3)s_1$, then $C = 2$. Similarly, if Z represents the number of real zeros of the entire function F, then $C - Z = 2 - Z \geq 0$ and therefore $Z = 2$ by Lemma 2.2. Since $F(0) = 0$ and $F(\tau_*) = 0$, the uniqueness of $\tau_* > 0$ follows.

Conversely, assume the existence of a unique $\tau_* > 0$ such that $a(\tau_*) = 0$. One immediately infers from (2.10), (2.14) and (2.16) that

$$s_1 = -\frac{F_{-\tau_*,1}(s_2, s_3)}{\tau_* F_{-\tau_*,2}(s_1, s_2, s_3)} < 0. \tag{3.14}$$ □

Remark 3.4. Observe that Theorem 3.3 states that exponential stability of equation (NDE) may be achieved with a prescribed decay rate, even though $a \leq 0$. Time-domain techniques based on a Lyapunov functional and LMI do not cover this. See, for instance, [LYH00, HL13, Fri01].

In the case of equidistributed real spectral values $s_3 < s_2 < s_1$, the delay $\tau_* > 0$ that enables the design of $s_1 < 0$ may be explicitly computed, as is stated hereafter.

Corollary 3.5. *Assume that the quasipolynomial Δ admits three equidistributed real spectral values $s_k = s_1 - (k-1)d$, for $d > 0$. Then, $s_1 < 0$ if, and only if, the delay $\tau_* > 0$ in Theorem 3.1 reads*

$$\tau_* = \frac{1}{d}\ln\left(\frac{s_3}{s_1}\right) > 0. \tag{3.15}$$

In the specific case where a single real spectral value $s_3 = s_2 = s_1$ is assigned to Δ, the GMID is demonstrated in [BMN22]. Furthermore, the analysis fully characterizes the remaining spectrum of Δ as

$$s = s_1 + \frac{\omega}{\tau}i, \quad \text{where} \quad \tan\left(\frac{\omega}{2}\right) = \frac{\omega}{2}. \tag{3.16}$$

In the same fashion, assuming that the quasipolynomial Δ has three equidistributed real spectral values, we also fully characterize the remaining spectrum of Δ.

Theorem 3.6. *Assume that the quasipolynomial Δ admits three equidistributed real spectral values $s_k = s_1 - (k-1)d$, for $d > 0$ and $k = 1, 2, 3$. Then, the remaining spectrum of Δ reads*

$$s = s_2 + \frac{\omega}{\tau}i, \quad \textit{where} \quad \tan\left(\frac{\omega}{2}\right) = \frac{\omega}{\xi(d)}, \qquad \xi(d) = \tau d \coth\left(\frac{\tau d}{2}\right). \tag{3.17}$$

Here coth *is the cotangent hyperbolic function.*

Proof. First of all, direct computations yield

$$\alpha = e^{\tau(s_1 - d)} = e^{\tau s_2}, \quad a = -s_2 - \eta, \quad \beta = e^{\tau s_2}(-s_2 + \eta), \quad \eta := \eta(d) = d\coth\left(\frac{\tau d}{2}\right) \tag{3.18}$$

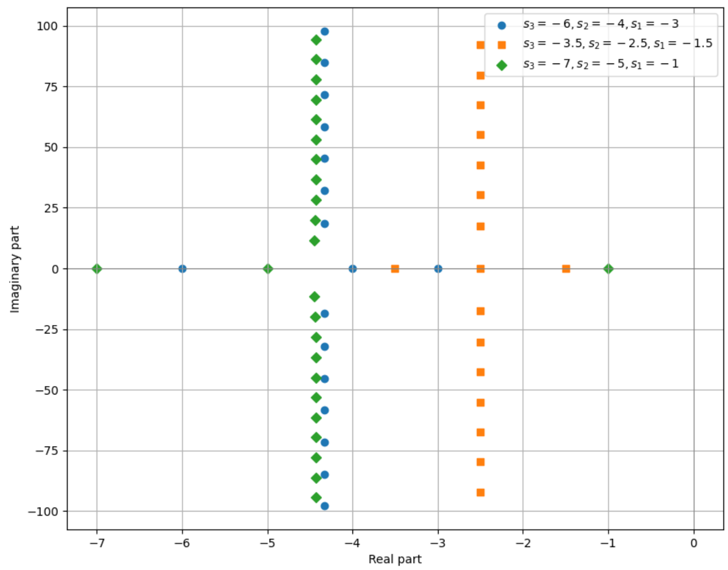


FIGURE 1. Spectrum of the quasipolynomial Δ for various parameters. This is obtained by assigning three real spectral values $s_3 < s_2 < s_1$ to Δ and computing the system parameters a, β, α and τ in each case.

owing to (2.13), (2.15) and (2.12). Let $s_0 = x_0 + i\omega_0$ be a complex spectral value of Δ. Then, $x_0 < s_1$ and $\omega_0 \neq 0$, in particular. From $\Delta(s_0) = 0$ and (3.18), one obtains

$$\frac{x_0 - s_2 - \eta + i\omega_0}{x_0 - s_2 + \eta + i\omega_0} = -e^{-\tau(x_0 - s_2)} e^{-i\tau\omega_0} \tag{3.19}$$

which holds for every $|\omega_0| \geq 2\pi/\tau$ by Lemma 2.3. Taking the magnitude of the above identity and letting $|\omega_0| \to \infty$, one finds

$$1 = e^{\tau(s_2 - x_0)} \qquad (\forall \tau > 0). \tag{3.20}$$

It follows that $x_0 = s_2$ and that the rest of the spectrum is either located on the vertical line $\Re(s) = s_2$ or forms a chain asymptotic to $\Re(s) = s_2$. From $\Delta(s) = (s - s_2) - \eta + e^{-\tau(s - s_2)}(s - s_2 + \eta)$, we can derive $\Delta(2s_2 - \overline{s_0}) = -e^{\tau(\overline{s_0} - s_2)}\Delta(\overline{s_0})$. Since Δ has real coefficients, the complex number $\overline{s_0}$ is a spectral value of Δ if, and only if, $2s_2 - \overline{s_0}$, the reflection of s_0 across the vertical line $\Re(s) = s_2$, is a spectral value of Δ. As a result, the remaining spectrum of Δ exists along the vertical line $\Re(s) = s_2$. Finally, by substituting $s = s_2 + i\omega_0$ into (3.19), one gets

$$\frac{i\omega_0 - \eta}{i\omega_0 + \eta} = -e^{-i\tau\omega_0} \quad \Longleftrightarrow \quad i\frac{\omega_0}{\eta} = \frac{1 - e^{-i\tau\omega_0}}{1 + e^{-i\tau\omega_0}} = i\tan\left(\frac{\tau\omega_0}{2}\right) \tag{3.21}$$

which leads to the desired result via the change of variables $\omega = \tau\omega_0$. □

Remark 3.7. Since $\xi(d) \to 2$ as $d \to 0$, Theorem 3.6 confirms the intuitive concept that the GMID is the limiting case of the GCRRID.

We refer the reader to Figure 1, where we depicted the spectrum of quasipolynomial Δ.

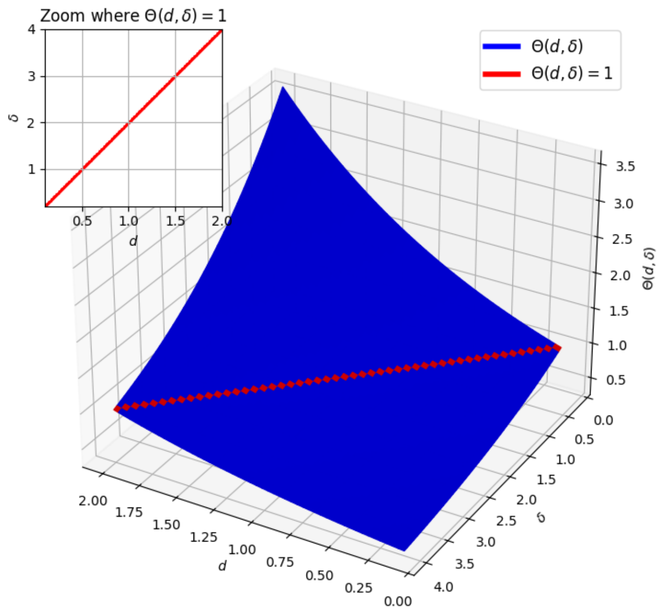


FIGURE 2. Plot of the function $\Theta(d,\delta)$ defined in (3.25). The *red* straight line characterises points (d,δ) where $\Theta(d,\delta)=1$. We magnified the *red* curve in the inset on the top-left.

Remark 3.8. In Theorem 3.6, it is proved that in the equidistributed case, the remaining spectrum of Δ lies on the vertical line $\Re(s) = s_2$. However, it should be noted that if the real spectral values $s_3 < s_2 < s_1$ are not equidistributed, the remaining spectrum of Δ does not perfectly align on a vertical line, though this may not be immediately obvious visually (see Figure 1 for green and blue spectrum distributions).

The proof of the following theorem is presented in A.

Theorem 3.9. *Assume that the quasipolynomial Δ admits three non-equidistributed real spectral values $s_3 < s_2 < s_1$. The remaining spectrum of Δ forms a chain asymptotic to the vertical line*

$$\Re(s) = s_2 + \frac{1}{\tau}\ln(\theta(\tau,d,\delta)), \quad \textit{with} \quad \theta(\tau,d,\delta) = \frac{e^{\tau d}(d(e^{\tau\delta}-1)-\delta(e^{\tau d}-1))}{de^{\tau d}+e^{\tau\delta}(e^{\tau d}(\delta-d)-\delta)}. \tag{3.22}$$

Here $d := s_1 - s_2$ and $\delta := s_1 - s_3$. Moreover, there exists a complex spectral value s_0 such that

$$\Delta(s_0) = 0 \quad \Longrightarrow \quad \Re(s_0) \neq s_2 + \frac{1}{\tau}\ln(\theta(\tau,d,\tau)). \tag{3.23}$$

Remark 3.10. From the definition of $\theta(\tau,d,\delta)$ given by (3.22), one can immediately check that

$$\theta(\tau,d,\delta) \to 1 \quad \text{as} \quad \delta \to 2d \tag{3.24}$$

indicating that we retrieve Theorem 3.6 in the equidistributed scenario. Additionally, we notice that θ seems to rely on three distinct parameters $\tau > 0$, $d > 0$, and $\delta > d$. However, upon closer examination, it is obvious that it depends solely on the products τd and $\tau\delta$ since we have

$$\Theta(\tau d,\tau\delta) := \theta\left(\tau,\frac{d}{\tau},\frac{\delta}{\tau}\right) = \frac{e^d(d(e^\delta-1)-\delta(e^d-1))}{de^d+e^\delta(e^d(\delta-d)-\delta)}. \tag{3.25}$$

We have plotted $\Theta(d,\delta)$ (i.e., $\tau=1$) in Figure 2, illustrating that $\Theta(d,\delta)=1$ only when $\delta=2d$.

3.2. **Assigning two distinct real spectral values.** This section discusses the ICRRID property of the quasipolynomial Δ. The aim is to assign two real spectral values $s_2<s_1$ to the quasipolynomial Δ and to find necessary and sufficient conditions on the coefficients of Δ guaranteeing that s_1 is a dominant root of Δ.

Inspired from [SBN23, Theorem 6], we find these conditions in terms of the quotient $(a+s_1)/(s_1-s_2)$. The challenge in this task relies on the scenario where a third real spectral value x exists in the spectrum of Δ in which case the coefficients a, α, and β are given in terms of x, s_2, s_1 and τ as in Lemma 2.4. Moreover, the greatest between s_2, s_1 and x is the dominant root by Theorem 3.1. Since one wants s_1 to be the dominant root of Δ, we are looking for the necessary and sufficient conditions on $(a+s_1)/(s_1-s_2)$ ensuring that $x<s_2<s_1$ or $s_2<x<s_1$. On the one hand, one deduces from (2.14) that

$$\frac{a+s_1}{s_1-s_2} = -\frac{(s_1-x)(e^{\tau s_2}-e^{\tau x})}{e^{\tau x}(s_1-s_2)+e^{\tau s_2}(x-s_1)+e^{\tau s_1}(s_2-x)}, \qquad (\forall x\in\mathbb{R}\backslash\{s_1,s_2\}) \tag{3.26}$$

which is negative owing to (2.10) and (2.11). On the other hand, introduce the function

$$\varphi(x) = \frac{(s_1-x)(e^{\tau s_2}-e^{\tau x})}{e^{\tau x}(s_1-s_2)+e^{\tau s_2}(x-s_1)+e^{\tau s_1}(s_2-x)}, \qquad (\forall x\in\mathbb{R}\backslash\{s_1,s_2\}). \tag{3.27}$$

Then, $\varphi(x)>0$ for every $x<s_2<s_1$ or $s_2<x<s_1$ or $x>s_1$, and satisfies

$$\lim_{x\to-\infty}\varphi(x) = \frac{1}{e^{\tau(s_1-s_2)}-1}, \qquad \lim_{x\to s_2}\varphi(x) = \frac{\tau(s_1-s_2)}{e^{\tau(s_1-s_2)}-(1+\tau(s_1-s_2))} \tag{3.28}$$

$$\lim_{x\to s_1}\varphi(x) = \frac{e^{\tau(s_1-s_2)}-1}{e^{\tau(s_1-s_2)}(\tau(s_1-s_2)-1)+1} \tag{3.29}$$

$$\begin{aligned}\varphi'(x) &= \frac{(e^{\tau(s_1+s_2)}-e^{\tau(s_1+x)}+e^{2\tau x}-e^{\tau(s_2+x)})(s_1-s_2)+\tau(s_1-x)(s_2-x)(e^{\tau(s_2+x)}-e^{\tau(s_1+x)})}{[e^{x\tau}(-s_1+s_2)+e^{s_2\tau}(s_1-x)+e^{s_1\tau}(-s_2+x)]^2}\\ &= \frac{\tau^2(s_2-x)(s_1-x)(s_1-s_2)(F_{-\tau,1}(x,s_1)F_{-\tau,1}(x,s_2)-e^xF_{-\tau,1}(s_1,s_2))}{[e^{x\tau}(-s_1+s_2)+e^{s_2\tau}(s_1-x)+e^{s_1\tau}(-s_2+x)]^2}\end{aligned}$$

thanks to the integral representation (2.16). Using [BBN20, Lemma 7], one finds that for all $\tau>0$ and $(u,v,w)\in\mathbb{R}^3$, the following holds

$$F_{-\tau,1}(u,v)-F_{-\tau,1}(u,w) = \tau(v-w)F_{-\tau,2}(u,v,w) > 0 \quad\Longleftrightarrow\quad v>w.$$

Therefore, for a fixed $u\in\mathbb{R}$, $v\mapsto F_{-\tau,1}(u,v)$ is a positive and strictly increasing function on $\mathbb{R}$. One deduces that $\varphi'(x)>0$ for every $x<s_2<s_1$ or $s_2<x<s_1$ or $x>s_1$, and that φ is a positive increasing function for $x<s_2<s_1$ or $s_2<x<s_1$ or $x>s_1$. Therefore, if $s_2<s_1$, then

$$\varphi(x) \geq \frac{1}{e^{\tau(s_1-s_2)}-1}, \qquad (\forall x\in\mathbb{R}\backslash\{s_1,s_2\}).$$

We proved the following.

Theorem 3.11. *Assume that Δ admits two real spectral values $s_2 < s_1$. Then, a third real spectral value x coexists in the spectrum of Δ, if and only if*

$$\Lambda := \frac{a+s_1}{s_1-s_2} \leq \frac{-1}{e^{\tau(s_1-s_2)}-1}. \tag{3.30}$$

Furthermore, one has

$$\frac{-\tau(s_1-s_2)}{e^{\tau(s_1-s_2)}-(1+\tau(s_1-s_2))} < \Lambda \leq \frac{-1}{e^{\tau(s_1-s_2)}-1} \iff x < s_2 < s_1, \tag{3.31}$$

$$\frac{1-e^{\tau(s_1-s_2)}}{e^{\tau(s_1-s_2)}(\tau(s_1-s_2)-1)+1} < \Lambda < \frac{-\tau(s_1-s_2)}{e^{\tau(s_1-s_2)}-(1+\tau(s_1-s_2))} \Leftrightarrow s_2 < x < s_1. \tag{3.32}$$

As an immediate consequence of Theorem 3.11, one has the following result.

Corollary 3.12. *Assume that Δ admits two real spectral values $s_2 < s_1$. Then, a third real spectral value x coexists in the spectrum of Δ, and it is the dominant root of Δ if, and only if,*

$$\frac{a+s_1}{s_1-s_2} \leq \frac{1-e^{\tau(s_1-s_2)}}{e^{\tau(s_1-s_2)}(\tau(s_1-s_2)-1)+1}. \tag{3.33}$$

Remark 3.13. It follows from (3.32) and (3.33) that s_1 is the dominant real spectral value of Δ with multiplicity equal to two if

$$\frac{a+s_1}{s_1-s_2} = \frac{1-e^{\tau(s_1-s_2)}}{e^{\tau(s_1-s_2)}(\tau(s_1-s_2)-1)+1}.$$

This equality corresponds to a special configuration where the ICRRID property allows the IMID property to hold.

Let us now investigate the necessary and sufficient conditions on $(a+s_1)/(s_1-s_2)$ guaranteeing the dominance of a simple real spectral value s_1 when exactly two real spectral values $s_2 < s_1$ are assigned to the quasipolynomial Δ. The following interpolation result is of interest when exactly two real spectral values coexist in the spectrum of Δ. The proof is immediate.

Lemma 3.14. *The quasipolynomial Δ admits exactly two distinct real spectral values s_2 and s_1 if, and only if $a \in \mathbb{R}$,*

$$\beta = -\alpha s_1 - e^{\tau s_1}(a+s_1) = -\alpha s_2 - e^{\tau s_2}(a+s_2). \tag{3.34}$$

$$\alpha = \frac{-(a+s_1)e^{\tau s_1}+(a+s_2)e^{\tau s_2}}{s_1-s_2}. \tag{3.35}$$

Remark 3.15. It is an immediate consequence of Theorem 3.11 that, exactly two real spectral values $s_2 < s_1$ coexist in the spectrum of Δ, if and only if, the following inequality holds

$$\frac{-1}{e^{\tau(s_1-s_2)}-1} < \frac{a+s_1}{s_1-s_2}. \tag{3.36}$$

By inspecting the proof of Theorem 3.1 when three real spectral values are assigned to Δ, one can notice that the key point to obtain the desired result relies on the properties of the coefficient α as stated in Remark 3.2. Firstly, equation (3.3) is well-defined if, and only if, $e^{2\tau x} - \alpha^2 \neq 0$ for every $x \geq s_1$. Moreover, relations (3.5), (3.6) and (3.7) are satisfied only if $e^{2\tau x} - \alpha^2 > 0$.

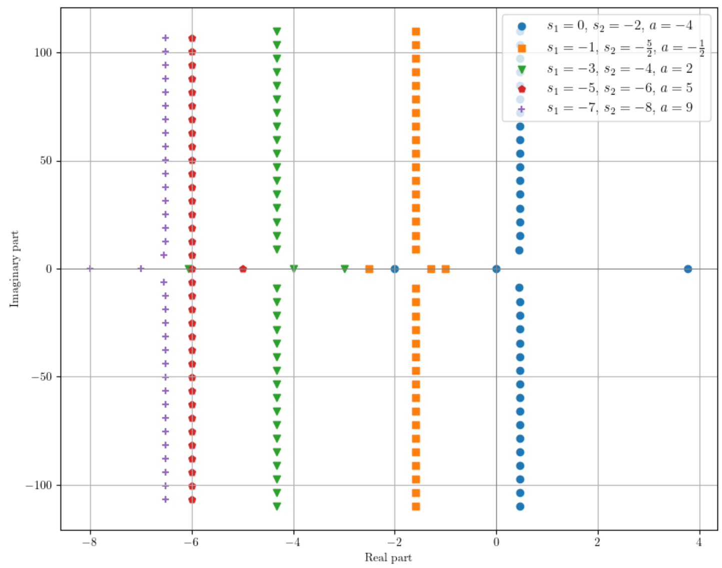


FIGURE 3. Spectrum distributions of Δ obtained when two simple real spectral values $s_2 < s_1$ are assigned to Δ. The delay $\tau = 1$ in all cases. In *blue*, $s_2 = -2$ and $s_1 = 0$ are assigned to Δ, but the spectrum distribution shows that another real root coexists, and it is the dominant root of Δ; in *orange*, $s_2 = -2.5$ and $s_1 = -1$ are assigned to Δ, and s_1 is the strictly dominant root of Δ while another real root coexists in the spectrum between s_2 and s_1; in *green*, $s_2 = -4$ and $s_1 = -3$ are assigned to Δ, and s_1 is the strictly dominant root of Δ while another real root coexists in the spectrum, and it is less than s_2; in *red*, $s_2 = -6$ and $s_1 = -5$ are assigned to Δ, and s_1 is the strictly dominant root of Δ while none real root coexists in the spectrum; finally, in *purple*, $s_2 = -8$ and $s_1 = -7$ are assigned to Δ, and s_1 is not a strictly dominant root of Δ, and none real root coexists in the spectrum.

The next lemma is a key ingredient in proving our second main result.

Lemma 3.16. *Assume that Δ admits exactly two real spectral values $s_2 < s_1$. Then, it holds*

$$\frac{-1}{e^{\tau(s_1-s_2)}-1} < \frac{a+s_1}{s_1-s_2} < 1 \iff -e^{\tau s_1} < \alpha < 0. \tag{3.37}$$

$$\alpha > -e^{\tau x} \qquad (\forall x \geq s_1). \tag{3.38}$$

Moreover, if $s_1 \leq 0$, then $\beta \leq 0$ whenever

$$0 \leq \frac{a+s_1}{s_1-s_2} < 1. \tag{3.39}$$

Proof. Firstly, from (3.35), one gets

$$\alpha = -\frac{a+s_1}{s_1-s_2}(e^{\tau s_1}-e^{\tau s_2}) - e^{\tau s_2}. \tag{3.40}$$

Therefore, the equivalence (3.37) holds owing to

$$-(e^{\tau s_1}-e^{\tau s_2})-e^{\tau s_2}<\alpha<\frac{e^{\tau s_1}-e^{\tau s_2}}{e^{\tau(s_1-s_2)}-1}-e^{\tau s_2} \iff -e^{\tau s_1}<\alpha<0. \tag{3.41}$$

Inequality (3.38) is a consequence of (3.37) and the fact that $x\in\mathbb{R}\mapsto e^x$ is strictly increasing. Finally, if $s_1\leq 0$, then it follows immediately from (3.34) and (3.39) that $\beta\leq 0$. ☐

One can now prove the following main result, which gives necessary and sufficient conditions for the dominance of a real spectral value when Δ admits exactly two real spectral.

Theorem 3.17. *Assume that exactly two real spectral values $s_2<s_1$ coexist in the spectrum of Δ. Then, s_1 is a strictly dominant root of Δ if, and only if, inequalities (3.37) are satisfied.*

Proof. Firstly, it follows from Remark 3.15 that Δ admits exactly two real spectral values if, and only if, the left inequality in (3.37) is verified. Let $s_0:=x+i\omega\in\mathbb{C}$ be such that $\Delta(s_0)=0$. In particular, $\omega\neq 0$. From $\Delta(s_1)=0$, one gets $\beta=-\alpha s_1-e^{\tau s_1}(a+s_1)$, where α and a are given by Lemma 3.14. It follows that $\Delta(s_0)=0$ if, and only if,

$$e^{\tau s_0}s_0+e^{\tau s_0}a+\alpha s_0=\alpha s_1+e^{\tau s_1}(a+s_1). \tag{3.42}$$

By taking the real and imaginary parts of both sides in (3.42), one gets

$$\begin{aligned} &e^{\tau x}((a+x)\cos(\tau\omega)-\omega\sin(\tau\omega))=e^{\tau s_1}(a+s_1)-\alpha(x-s_1)\\ &e^{\tau x}((a+x)\sin(\tau\omega)+\omega\cos(\tau\omega))=-\alpha\omega. \end{aligned} \tag{3.43}$$

By squaring each equality in (3.43) and adding them, one obtains

$$\omega^2=\frac{(e^{\tau s_1}(a+s_1)-\alpha(x-s_1))^2-(a+x)^2e^{2\tau x}}{e^{2\tau x}-\alpha^2}, \tag{3.44}$$

which is well-defined for every $x\geq s_1$ by Lemma 3.16. Therefore, by performing the exact same steps of the proof of dominancy in Theorem 3.1, one gets that $\omega^2<1/\tau^2$, i.e., $|\omega|<1/\tau$. The latter is inconsistent since one has necessarily $|\omega|\geq\pi/\tau$ owing to Lemma 2.3. Hence, s_1 is the dominant root of Δ. Conversely, assume that inequality (3.37) is not satisfied. Then, another real root x coexists in the spectrum of Δ by Theorem 3.11 or s_1 and s_2 are the only real spectral values of Δ, and s_1 is not a strictly dominant root of Δ by Lemma 3.16 and the sufficient part of this theorem. ☐

Remark 3.18. We stress the fact that in the particular case of exactly two real spectral values $s_2<s_1$ coexisting in the spectrum of Δ, and

$$\frac{a+s_1}{s_1-s_2}=1, \tag{3.45}$$

the spectral values $s\in\mathbb{C}$ of Δ are analytically given by $s=s_1+i\frac{2\pi k}{\tau}$, $k\in\mathbb{Z}$, and s_1 is hence a (not strictly) dominant root of Δ. Indeed, considering $a+s_1=s_1-s_2$, and equations (3.34) and (3.35), one finds $\alpha=-e^{\tau s_1}$ and $\beta=s_2e^{\tau s_1}$. Consequently, one has $\Delta(s)=(s-s_2)(1-e^{-\tau(s-s_1)})$ for every $s\in\mathbb{C}$, wherefrom one obtains the desired result.

Remark 3.19. Aside from the necessary and sufficient conditions on the dominancy of the assigned rightmost root s_1, which have already been established in [SBN23, Theorem 6], Theorem 3.11 and Theorem 3.17 provide a complete description of such conditions with respect to the

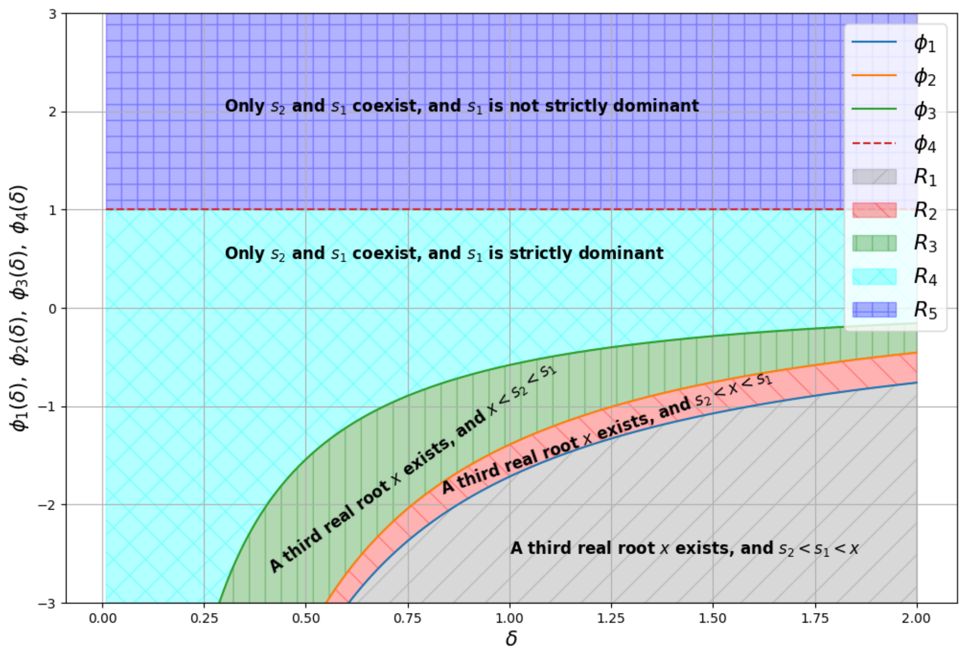


FIGURE 4. Complete characterization of the regions with respect to the values of $(a+s_1)/\delta$ that are necessary and sufficient to ensure the strict dominance or not of the real spectral value s_1 when only two real spectral values $s_2 < s_1$ are assigned to Δ. The figure is depicted when the delay $\tau = 1$ while $\delta := s_1 - s_2$ ranges between 0.01 and 2. The functions used are $\phi_1(\delta) = (1 - e^{\delta})/(e^{\delta}(\delta - 1) + 1)$, $\phi_2(\delta) = -\delta/(e^{\delta} - (1 + \delta))$, $\phi_3(\delta) = -1/(e^{\delta} - 1)$, and $\phi_4(\delta) = 1$. We define the regions R_1 as the range of δ where $\frac{a+s_1}{\delta} \leq \phi_1(\delta)$, R_2 as the range where $\phi_1(\delta) < \frac{a+s_1}{\delta} \leq \phi_2(\delta)$, R_3 as the range where $\phi_2(\delta) < \frac{a+s_1}{\delta} \leq \phi_3(\delta)$, R_4 as the range where $\phi_3(\delta) < \frac{a+s_1}{\delta} < \phi_4(\delta)$, and R_5 as the range where $\frac{a+s_1}{\delta} \geq \phi_4(\delta)$.

number of assigned roots and the potential coexistence of a third real root. We refer to Figure 4 for a full visualization.

3.3. **On Frasson-Verduyn Lunel's sufficient conditions for dominancy and beyond.** This section builds upon Frasson-Verduyn Lunel's seminal work [FVL03, Lemma 5.1], which established a sufficient condition for the dominance of a simple real spectral value of quasipolynomials with multiple delays. While Frasson-Verduyn Lunel's lemma offers a fundamental method for assessing spectral dominance, its applicability is restricted to specific conditions that may only address certain dynamic scenarios encountered in complex systems. Restricted to the single-delay case, when Frasson-Verduyn Lunel's condition is not met, the CRRID property still provides a guarantee for the dominance of a simple real root.

The first main result of this section is the following.

Theorem 3.20. *Let $\tau > 0$. There exist $s_1 \leq 0$ and $d > 0$, such that if Δ admits three equidistributed real spectral values $s_3 = s_1 - 2d$, $s_2 = s_1 - d$ and s_1, then*

$$V(s_1) = (|\alpha|(1 + |s_1|\tau) + |\beta|\tau)e^{-s_1\tau} \tag{3.46}$$

satisfies $V(s_1) \geq 1$ and s_1 is the dominant root of Δ.

Proof. Assume that the quasipolynomial Δ admits three equidistributed real spectral values $s_k = s_1 - (k-1)d$, for some $d > 0$ (that we will choose later on) and $k = 1, 2, 3$ with $s_1 \leq 0$. Necessarily, s_k is simple since Δ cannot admit more than three real roots counting multiplicities [PS72, Problem 206.2, page 144]. Hence, Theorem 3.1 states that s_1 is a strictly dominant root of Δ. Moreover, one deduces from Lemma 2.4 that

$$\alpha = e^{\tau(s_1 - d)}, \qquad \beta = e^{\tau(s_1-d)}\left(-s_1 + d + d\coth\left(\tau d/2\right)\right), \tag{3.47}$$

$$V(s_1) = \left(1 - 2\tau s_1 + \tau d + \tau d \coth\left(\tau d/2\right)\right)e^{-\tau d}. \tag{3.48}$$

Introducing $g(s_1) := V(s_1) - 1$, one finds

$$g(s_1) = e^{-\tau d}\Big(\tau d\coth\left(\tau d/2\right) + \tau d + 1 - e^{\tau d} - 2\tau s_1\Big) \tag{3.49}$$

which is affine in s_1. It follows that $g(s_1) \geq 0$ if, and only if,

$$s_1 \leq (\tau d\coth\left(\tau d/2\right) + \tau d + 1 - e^{\tau d})/2\tau. \tag{3.50}$$

In particular, knowing that Frasson-Verduyn Lunel's sufficient condition is not met the first time when $V(s_1) = 1$, owing to inequality (3.50) the latter is equivalent to ($v := \tau s_1$ and $u := \tau d$)

$$v(u) = (u\coth\left(u/2\right) + u + 1 - e^{u})/2 \qquad (u > 0). \tag{3.51}$$

A straightforward analysis of the function $u \mapsto v(u)$ for $u > 0$ shows that v is strictly decreasing on $(0, \infty)$ and satisfies

$$\lim_{u\to 0} v(u) = 1, \qquad \lim_{u\to\infty} v(u) = -\infty. \tag{3.52}$$

Therefore, there exist $s_1 \leq 0$ and $d > 0$ satisfying (3.50) such that $V(s_1) \geq 1$. ☐

Theorem 3.20 shows that the sufficient condition of Lemma 2.1 is not necessary for the dominance of simple real spectral values when Δ admits three equidistributed real roots.

The function V defined in (3.48) initially seems to depend on three separate parameters: the delay τ, the distance d, and the dominant simple real value s_1. However, upon closer inspection, it becomes apparent that V can be expressed solely in terms of the products τd and τs_1, denoted by u and v, respectively. As a result, the function reduces to

$$W(u, v) := \left(1 - 2v + u + u\coth\left(\frac{u}{2}\right)\right)e^{-u}. \tag{3.53}$$

For visualization, it is sufficient to consider plots of W as a function of $u = \tau d$ and $v = \tau s_1$, treating τ as a positive constant scaling factor. Refer to Figure 5 (image on the *left*).

The second main result of this section is the following.

Theorem 3.21. *Let $\tau > 0$. There exist $s_1 \leq 0$ and $\delta > 0$ such that if Δ admits exactly two simple real spectral values s_1 and $s_2 = s_1 - \delta$, and inequality (3.39) is satisfied, then*

$$Y(s_1) = (|\alpha|(1 + |s_1|\tau) + |\beta|\tau)e^{-s_1\tau} \tag{3.54}$$

satisfies $Y(s_1) \geq 1$ and s_1 is the dominant root of Δ.

Proof. If Δ admits exactly two simple real spectral values $s_2 < s_1$ and inequality (3.39) is satisfied, then s_1 is a strictly dominant root of Δ by Theorem 3.17. Moreover, $\alpha < 0$, and if $s_1 \leq 0$ then $\beta \leq 0$ by Lemma 3.16. If $s_2 = s_1 - \delta$ for some $\delta > 0$ that will be chosen later on, then one has from Lemma 3.14,

$$\alpha = -e^{\tau s_1}\left(\frac{(a+s_1)}{\delta}(1-e^{-\tau\delta}) + e^{-\tau\delta}\right), \qquad \beta = -\alpha s_1 - e^{\tau s_1}(a+s_1),$$

$$\delta Y(s_1) = e^{-\tau\delta}(a+s_1-\delta)(-1+2\tau s_1) + (a+s_1)(1-2\tau s_1+\tau\delta). \tag{3.55}$$

Function Y initially seems to depend on four separate parameters: the coefficient a, τ, δ, and s_1. Upon closer inspection, one finds that Y can be expressed solely in terms of the products τa, $\tau\delta$ and τs_1. Setting $A := \tau a$, $u := \tau\delta$, $v := \tau s_1$, then $A \in \mathbb{R}$, $u > 0$ and $v < 0$. Introducing the functions

$$Z(A,u,v) := Y(s_1) \quad \text{and} \quad h(A,u,v) = Z(A,u,v) - 1 \tag{3.56}$$

one gets

(3.57)

$$uh(A,u,v) = -(2-2e^{-u})v^2 - ((1-e^{-u})(2A-u-1) + ue^{-u})v + (A+e^{-u}-1)u + A(1-e^{-u})$$

which is a second-degree polynomial in v. Let $N(A,u,v)$ be the numerator of $h(A,u,v)$. From $N(A,u,v) = 0$, one finds that its discriminant $D_1(A,u)$ admits two real roots $A_0(u) < 0$ for every $u > 0$ and $A_1(u)$ that can change sign for $u > 0$. One infers that for every $u > 0$, $D_1(A,u) \geq 0$ for $A \leq A_0(u)$ and $A \geq A_1(u)$. For simplicity, assume from now on that $A \geq A_1(u)$. Hence, equation $N(A,u,v) = 0$ admits two real roots

(3.58)

$$v_1 := v_1(A,u) = \frac{-C_b(A,u) - \sqrt{D_1(A,u)}}{2C_a(u)}, \qquad v_2 := v_2(A,u) = \frac{-C_b(A,u) + \sqrt{D_1(A,u)}}{2C_a(u)}$$

where

$$C_a(u) = -2(1-e^{-u}), \qquad C_b(A,u) = -2(1-e^{-u})A + (1+u)(1-e^{-u}) - ue^{-u},$$

$$C_c(A,u) = (1-e^{-u}+u)A - (1-e^{-u})u$$

are the coefficients of $N(A,u,v)$ considered as a second-degree polynomial in v, so that $D_1(A,u) = C_b^2(A,u) - 4C_a(u)C_c(A,u)$. Letting

$$A_2(u) = \frac{(1-e^{-u})u}{1-e^{-u}+u} > 0 \qquad \text{and} \qquad A_3(u) = \frac{(1+u)(1-e^{-u}) - ue^{-u}}{2(1-e^{-u})} > 0 \quad (u > 0) \tag{3.59}$$

one can checks that $A_1(u) \leq A_2(u) \leq A_3(u)$ for all $u > 0$. Moreover, one gets the following sign tab.

A	$-\infty$		$A_2(u)$		$A_3(u)$		$+\infty$
$C_b(A,u)$		$+$		$+$	0	$-$	
$C_c(A,u)$		$-$	0	$+$		$+$	

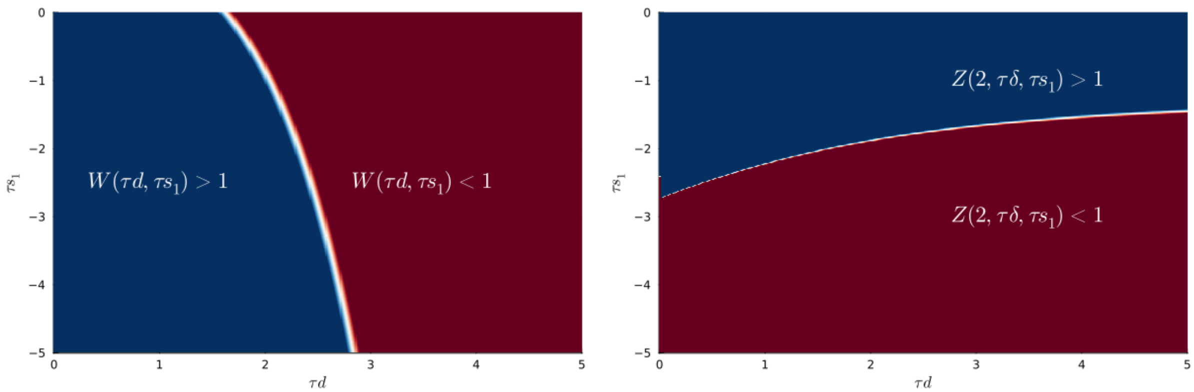


FIGURE 5. Depiction of the functions $W(\tau d, \tau s_1)$ and $Z(2, \tau\delta, \tau s_1)$ defined respectively in (3.53) and (3.64). The *red*-coloured region in the parameters space $(\tau d, \tau s_1)$ corresponds to the region where Frasson-Verduyn Lunel sufficient condition for the dominance of a simple real spectral value is satisfied. The ICRRID conditions extend Frasson-Verduyn Lunel's conditions by the *blue* region.

Hence, for all $A \leq A_2(u)$, one has $v_1 \leq 0$ and $v_2 > 0$. Since $A_1(u) \leq A_2(u)$, and $C_a(u) < 0$ for every $u > 0$, one deduces that for all $A \in [A_1, A_2]$, $u > 0$ and $v \in [v_1, v_2]$, one has $N(A, u, v) \geq 0$. Therefore, for every $s_1 \leq 0$ and every $\delta > 0$, one has $Y(s_1) \geq 1$, whenever

$$A_1(\tau\delta) \leq \tau a \leq A_2(\tau\delta) \qquad \text{and} \qquad v_1(\tau a, \tau\delta) \leq \tau s_1 \leq v_2(\tau a, \tau\delta). \tag{3.60}$$ □

Theorem 3.21 shows that the sufficient condition of Lemma 2.1 is not necessary for the dominance of simple real spectral values when Δ admits exactly two simple real roots.

Remark 3.22. We stress that there are values for s_1 and δ (where $s_1 \leq 0$ and $\delta > 0$) that satisfy (3.60) and still maintain inequality (3.39). From equation (3.60), take, for instance,

$$\tau a = A_2(\tau\delta), \qquad \text{and} \qquad \tau s_1 = v_1(\tau a, \tau\delta). \tag{3.61}$$

One finds, after careful computations, that

$$\frac{a + s_1}{\delta} = \frac{A_2(\tau\delta) + v_1(\tau a, \tau\delta)}{\tau\delta} = \frac{1}{2}\left(1 + \frac{1}{1 - e^{\tau\delta}} + \frac{1}{\tau\delta}\right). \tag{3.62}$$

Finally, as a function of $\tau\delta > 0$, one can immediately check that the third member on the right in (3.62) is bounded between 1/2 and 3/4.

Remark 3.23. When two real roots are assigned, and a third root coexists (see, Theorem 3.11 and Corollary 3.12), one can use the exact same arguments as in the proof of Theorem 3.20 to show that in some parameters regions, the Frasson-Verduyn Lunel's criteria does not apply. Furthermore, when exactly two real spectral values $s_2 < s_1$ coexist in the spectrum of Δ, and

$$\frac{-1}{e^{\tau(s_1 - s_2)} - 1} < \frac{a + s_1}{s_1 - s_2} \leq 0, \tag{3.63}$$

then s_1 is a strictly dominant root of Δ by Theorem 3.17, and we can prove as in Theorem 3.21 that in some parameters regions the Frasson-Verduyn Lunel's criteria does not apply.

In the proof of Theorem 3.21, we introduced the function

$$Z(\tau a,\tau\delta,\tau s_1) := Y(s_1) = \frac{e^{-\tau\delta}(a+s_1-\delta)(-1+2\tau s_1)+(a+s_1)(1-2\tau s_1+\tau\delta)}{\delta}. \tag{3.64}$$

When $a = 2$ (admissible owing to (3.60)), we plotted the heatmap of $Z(2,\tau\delta,\tau s_1)$ in Figure 5 (image on the *right*) showing the regions $(\tau\delta,\tau s_1)$ where $Y(s_1) < 1$, $Y(s_1) > 1$ and $Y(s_1) = 1$.

3.4. **Exponential Estimates.** In [Kha05, Section 6], the author establishes exponential estimates of solutions for time delay systems of neutral-type using quadratic Lyapunov functionals and Lyapunov matrices. Although effective, this method may be computationally involved. Additionally, [HL13][Chapter 1, Theorem 7.6, page 32] also offers an exponential estimate for solutions of a neutral differential equation based on the characteristic quasipolynomial's rightmost root. The proof employs the Cauchy theorem of residues and involves complex analysis arguments, including the periodicity and analyticity of certain functions. However, the explanation may benefit from greater detail to enhance the reader's clarity and ease of understanding.

In the following, we provide an alternative proof of exponential estimates for solutions of (NDE) that integrates the previously outlined spectral analysis of Sections 3.1 and 3.2.

For all $t \geq 0$, $y_t \in C^0([-\tau,0])$ denotes the history function, defined for all $\theta \in [-\tau,0]$ as

$$y_t(\theta) = y(t+\theta), \qquad \text{with} \qquad \|y_t\|_\infty := \sup_{\theta\in[-\tau,0]} |y_t(\theta)|. \tag{3.65}$$

Theorem 3.24. *Let $\tau > 0$. Assume that the quasipolynomial Δ admits three real spectral values $s_3 < s_2 < s_1$. For every $\varepsilon > 0$, there exists a constant $k := k(\varepsilon, s_1, s_2, s_3) \geq 1$ such that the solution $y(t)$ of (NDE) with initial condition $y_0 \in C^0([-\tau,0])$ satisfies*

$$|y(t)| \leq k e^{(s_1+\varepsilon)t}\|y_0\|_\infty \quad (t \geq 0). \tag{3.66}$$

Proof. Applying the Laplace transform to both sides of equation (NDE), one gets

$$\hat{y}(s) = \frac{y_0(0)+\alpha y_0(-\tau)}{\Delta(s)} \quad (s \in \mathbb{C}), \tag{3.67}$$

showing that $\hat{y}$ is an analytic function of s for $\Re(s) > s_1$. Let $\varepsilon > 0$ and set $c_1 := s_1 + \varepsilon$. Then, function y is given by the Bromwich complex contour integral

$$y(t) = \frac{(y_0(0)+\alpha y_0(-\tau))}{2i\pi} \lim_{T\to\infty} \int_{c_1-iT}^{c_1+iT} \frac{e^{st}}{\Delta(s)} ds \quad (t \geq 0). \tag{3.68}$$

Indeed, let $T > 0$ and $c_2 > c_1$, and consider the integration of the function $e^{ts}/\Delta(s)$ over the closed rectangle Γ in the complex plane with vertical boundaries $V_1 := \{c_1 + i\omega \mid -T \leq \omega \leq T\}$ and $V_2 := \{c_2 + i\omega \mid -T \leq \omega \leq T\}$, and horizontal boundaries $H_1 := \{x + iT \mid c_1 \leq x \leq c_2\}$ and $H_2 := \{x - iT \mid c_1 \leq x \leq c_2\}$. Since Δ has no zeroes inside the rectangle Γ, the integral over Γ equals zero. It is then sufficient to show the following.

$$\int_{H_j} \frac{e^{st}}{\Delta(s)} ds \to 0 \qquad \text{as} \qquad T \to \infty \quad (j = 1, 2). \tag{3.69}$$

For $s = x + iT$ with $c_1 \leq x \leq c_2$ and $T > 0$, one has $\Delta(s) = (1+\alpha e^{-\tau s})s + (a + \beta e^{-\tau s})$. Since $1 - \alpha e^{-\tau x} > 0$ thanks to $s_1 < c_1 \leq x \leq c_2$ and Lemma 2.5, one chooses $T_0 > 0$ large enough to have

$$\frac{T}{2}(1-\alpha e^{-\tau x}) \geq |a| + \beta e^{-\tau c_2} \quad (\forall T \geq T_0). \tag{3.70}$$

It follows that

$$|\Delta(s)| \geq T(1-\alpha e^{-\tau x}) - (|a| + \beta e^{-\tau x}) \geq \frac{T}{2}(1-\alpha e^{-\tau x}) \quad (T \geq T_0). \tag{3.71}$$

Therefore, one has

$$\left|\int_{H_1} \frac{e^{st}}{\Delta(s)} ds\right| \leq \frac{2e^{tc_2}}{T} \int_{c_1}^{c_2} \frac{e^{\tau x}}{(e^{\tau x}-\alpha)} dx = \frac{2}{T}\frac{e^{tc_2}}{\tau} \ln\left(\frac{e^{\tau c_2}-\alpha}{e^{\tau c_1}-\alpha}\right) \to 0 \qquad \text{as} \qquad T \to \infty.$$

In the same fashion, one proves that the integral over H_2 tends to zero when $T \to \infty$. It follows that (3.68) defines properly the signal $y(t)$ for every $t \geq 0$. One claims that if $s = c_1 + iT$, then

$$|\Delta(s)| \geq \frac{|T|}{2}(1-\alpha e^{-\tau s_1}), \qquad \text{for all} \quad |T| \geq \frac{4\zeta}{1-\alpha e^{-\tau s_1}} =: T_1 \tag{3.72}$$

where $\zeta > 0$ is defined by (2.23) and $1 - \alpha e^{-\tau s_1} > 0$ owing to Lemma 2.5. Indeed, since $a = -s_1 - \zeta$ and $b = -\alpha s_1 + \zeta e^{\tau s_1}$, one has $\Delta(s) = (1+\alpha e^{-\tau c_1} e^{-i\tau T})(\varepsilon + iT) - \zeta(1 - e^{-\tau\varepsilon} e^{-i\tau T})$. Therefore,

$$|\Delta(s)| \geq |T|(1-\alpha e^{-\tau c_1}) - \zeta(1+e^{-\tau\varepsilon}) \geq |T|(1-\alpha e^{-\tau s_1}) - 2\zeta$$

thanks to the reverse triangle inequality, which completes the proof of the claim. Setting

$$z(t) := \mathcal{L}^{-1}\left\{\frac{1}{\Delta(s)}\right\}(t) = \frac{1}{2i\pi} \lim_{T\to\infty} \int_{c_1-iT}^{c_1+iT} \frac{e^{st}}{\Delta(s)} ds \quad (t \geq 0) \tag{3.73}$$

one immediately observes that for every $s \in \mathbb{C}$ such that $\Re(s) > s_1$, one has

$$\frac{1+\alpha e^{-\tau s}}{\Delta(s)} = \zeta \frac{1 - e^{-\tau(s-s_1)}}{(s-s_1)\Delta(s)} + \frac{1}{s-s_1}$$

so that, taking the inverse Laplace transform and the fact that $\mathcal{L}^{-1}\{1/(s-s_1)\} = e^{s_1 t}$), one obtains

$$z(t) + \alpha z(t-\tau) = \zeta \mathcal{L}^{-1}\left\{\frac{1-e^{-\tau(s-s_1)}}{(s-s_1)\Delta(s)}\right\}(t) + e^{s_1 t}. \tag{3.74}$$

Thanks to (3.72), one has

$$\begin{aligned}
\left|\zeta \mathcal{L}^{-1}\left\{\frac{1-e^{-\tau(s-s_1)}}{(s-s_1)\Delta(s)}\right\}(t)\right| &\leq \frac{\zeta e^{tc_1}}{\pi} \int_{-T_1}^{T_1} \frac{dT}{|\Delta(c_1+iT)|\sqrt{\varepsilon^2+T^2}} + \frac{4\zeta e^{tc_1}}{\pi(1-\alpha e^{-\tau s_1})} \int_{T_1}^{\infty} \frac{dT}{T^2} \\
&= \frac{\zeta e^{tc_1}}{\pi} \int_{-T_1}^{T_1} \frac{dT}{|\Delta(c_1+iT)|\sqrt{\varepsilon^2+T^2}} + \frac{e^{tc_1}}{\pi} \leq k_0 e^{tc_1}
\end{aligned} \tag{3.75}$$

where

$$k_0 := k_0(\varepsilon, s_1, s_2, s_3) = \frac{1}{\pi}\left(1 + 2\zeta T_1 \min_{|T|\leq T_1}\left(|\Delta(c_1+iT)|\sqrt{\varepsilon^2+T^2}\right)\right).$$

Combining (3.74) and (3.75), one obtains

$$|z(t)| - \alpha|z(t-\tau)| \leq |z(t) + \alpha z(t-\tau)| \leq (1+k_0)e^{(s_1+\varepsilon)t} \tag{3.76}$$

which yields

$$|z(t)| \leq (1+k_0)e^{(s_1+\varepsilon)t}\sum_{j=0}^{\infty}\alpha^j e^{-j(s_1+\varepsilon)\tau} = \frac{(1+k_0)}{1-\alpha e^{-\tau(s_1+\varepsilon)}}e^{(s_1+\varepsilon)t} \tag{3.77}$$

since $1-\alpha e^{-\tau(s_1+\varepsilon)} > 0$ owing to Lemma 2.5. As a consequence, the result follows since

$$|y(t)| = |(y_0(0)+\alpha y_0(-\tau))||z(t)| \leq \frac{(1+k_0)(1+\alpha)}{1-\alpha e^{-\tau(s_1+\varepsilon)}}e^{(s_1+\varepsilon)t}\|y_0\|_\infty \tag{3.78}$$

and

$$k(\varepsilon, s_1, s_2, s_3) := (1+k_0)(1+\alpha)/(1-\alpha e^{-\tau(s_1+\varepsilon)}) \geq 1.$$

□

Remark 3.25. In (3.66), it is important to note that the inequality only makes sense when $k \geq 1$. The GCRRID setting derived in Section 3.1 allows to explicitly determine $k \geq 1$, a property which is not explicitly stated in [HL13][Chapter 1, Theorem 7.6, page 32].

Remark 3.26. It is worth noting that in the case where the quasipolynomial Δ has exactly two simple real spectral values $s_2 < s_1$ and that inequality (3.37) is satisfied, the same proof applies due to Lemma 3.16 (since $\alpha < 0$ implies $\alpha < e^{\tau x}$ for every $x \geq s_1$) and Theorem 3.17.

4. Application: Designing an exponentially stable one-layer neural network

Based on the spectral theory developed in the previous sections, this section applies our theoretical insights to the practical design of an exponentially stable one-layer neural network subject to a delayed Proportional-Derivative (PD) controller.

4.1. Implementing delayed PD control for seizure prevention.

Based on the CRRID setting of Section 3.1, and the stability analysis of B, this section specifically looks at how to practically implement a delayed Proportional-Derivative (PD) controller to improve the stability of the HNN when $\nu \geq \mu$ and to stabilize the HNN in situations where $\nu < \mu$. The reason for adding the PD controller is its ability to reduce the natural instabilities that could cause seizure-like patterns in neuronal models.

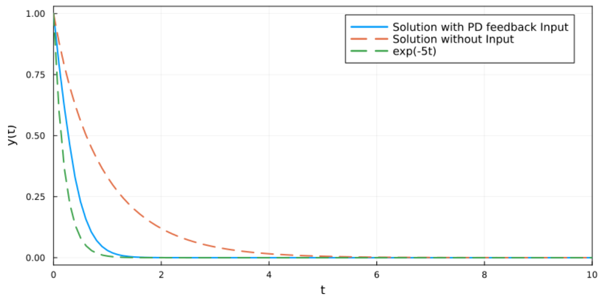
We consider two types of configurations. The first type aims to improve the decay rate to zero or stabilize to zero exponentially—with a prescribed decay—the solutions of equation (1.3) in the case where $\nu > \mu$ or $\nu \leq \mu$. This will be achieved by determining the gain parameters k_p and k_d and the delay $\tau > 0$. The parameters μ and ν are then known in this case. The second type of configuration aims to model a one-layer neural network like (1.3) such that the trivial equilibrium is exponentially stable with a prescribed decay rate, regardless of the sign of $\nu - \mu$. We achieve these tasks locally by studying the asymptotic stability of the linearized equation around the zero equilibrium zero to (1.3).

4.1.1. *Improving the decay rate of an exponentially stable one-layer neural network.*

Consider the one-layer neural network under the form

$$\dot{y}(t) = -2y(t) + \tanh(y(t)) + I(t), \tag{4.1}$$

which is a particular case of (1.3) when $\nu = 2$ and $\mu = 1$. Under no external input, that is, when $I(t) = 0$, it is immediate to obtain that all solutions of (4.1) are globally exponentially stable or equivalently that the trivial equilibrium zero is globally exponentially stable with a decay rate equal to $\nu - \mu = 1$. A classical problem in control theory is choosing the control $I(t)$ in the



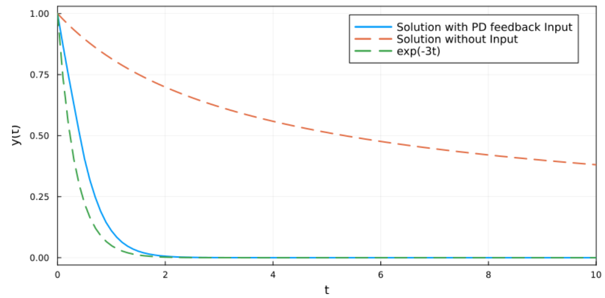


FIGURE 6. Solutions to equations (4.1) (*left*) and (4.3) (*right*) when $I = 0$ and when I is the PD controller (1.2) where τ, k_d and k_p are given by (4.2) (*left*) and (4.4) (*right*) respecively. The initial condition is taken as $y_0 = 1$ for the solution with $I = 0$, and $y_0(t) = 1$, $y_0'(t) = 0$, $t \leq 0$ for the solution with the PD feedback controller.

feedback form to improve (locally at least) the stability properties of (4.1). By letting $I(t) = -k_p y(t-\tau) - k_d \dot{y}(t-\tau)$ and after linearization around zero, the question reduces to studying the localization of the spectrum of the quasipolynomial function $Q_0(s) = s + 1 + e^{-\tau s}(k_d s + k_p)$ which is the particular case of (2.1) where $\alpha = k_d$, $\beta = k_p$ and $a = 1$.

To simplify the control design, assume that we assign three equidistributed real spectral values $s_1 = -5$, $s_2 = -5 - d$ and $s_3 = -5 - 2d$ to Q_0. By letting $d = 1$, owing to Lemma 2.4, one has

$$\tau = \ln(3/2), \qquad k_d = e^{-6\tau}, \qquad k_p = e^{-6\tau}\left(6 + \coth(\tau/2)\right) \tag{4.2}$$

where coth is the cotangent hyperbolic function.

As per the CRRID properties explained in Section 3.1, it can be inferred that parameters in equation (4.2) ensure the local exponential stability of solutions of equation (4.1) by the PD controller $I(t) = -k_p y(t-\tau) - k_d \dot{y}(t-\tau)$, with a decay rate equal to $-5 + \varepsilon$, for a sufficiently small $\varepsilon > 0$ as stated in Theorem 3.24. The reader can refer to Figure 6 (image on the *left*) for a visualization.

4.1.2. *Stabilizing exponentially an asymptotically stable one-layer neural network.* In this section, we will focus on the one-layer neural network equation

$$\dot{y}(t) = -y(t) + \tanh(y(t)) + I(t). \tag{4.3}$$

This equation corresponds to a specific case of the equation (1.3) when $\nu = \mu = 1$. By using a Lyapunov function [Hop84], one can check that all solutions of this equation asymptotically converge to zero when $I(t) = 0$. To make the solutions converge toward zero with a prescribed exponential decay rate, we control this equation with the PD controller (1.2). By linearizing around zero, we can reduce the question to studying the localization of the spectrum of the quasipolynomial $Q_1(s) = s + e^{-\tau s}(k_d s + k_p)$ which is a particular case of (2.1), where $\alpha = k_d$, $\beta = k_p$, and $a = 0$.

Assume that one assigns three non-equidistributed real spectral values $s_3 < s_2 < s_1$ to Q_1, say $s_1 < 0$, $s_2 = s_1 - \delta$ and $s_3 = s_1 - 3\delta$ for some $\delta > 0$. Letting, $s_1 = -2$ and $\delta = 1$, one gets $s_2 = -4$, $s_3 = -6$ and Lemma 2.4 yields

$$\tau = \ln\left((1+\sqrt{5})/2\right), \qquad k_d = -20 + 9\sqrt{5}, \qquad k_p = -66 + 30\sqrt{5}. \tag{4.4}$$

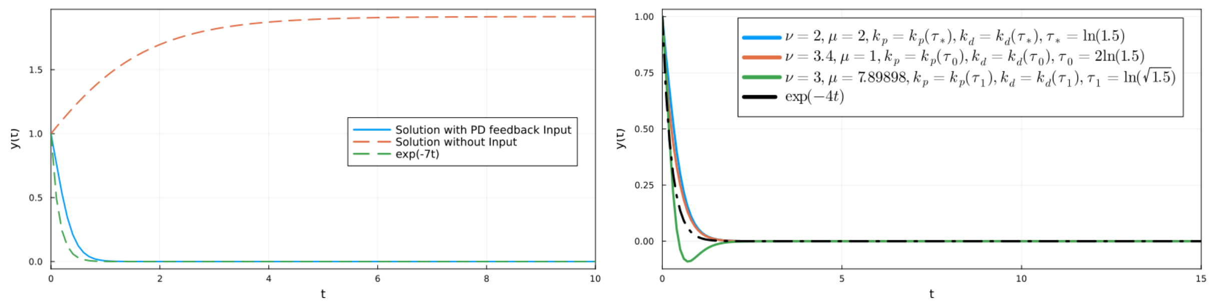


FIGURE 7. On the *left*, solutions to equation (4.5) when $I = 0$ and when I is the PD controller (1.2) where τ, k_d and k_p are given by (4.6). On the *right*, solutions to equations (4.8), (4.9) and (4.10) respectively plotted in *blue*, *orange* and *green*. The initial condition is taken as $y_0 = 1$ for the solution with $I = 0$, and $y_0(t) = 1$, $y_0'(t) = 0$, $t \le 0$ for the solution with the PD feedback controller.

We plotted in Figure 6 (image on the *right*) the solutions of (4.3) when $I(t) = 0$ and when $I(t) = -k_p y(t-\tau) - k_d \dot{y}(t-\tau)$ where $\tau > 0$, $k_d > 0$ and $k_p > 0$ are given by (4.4).

4.1.3. *Stabilizing an unstable one-layer neural network exponentially.* This section focuses on the more interesting scenario where the neural system's natural decay rate is surpassed by the interaction strength so that the neuron's inherent dynamics tend toward instability. More precisely, we consider the one layer-neural network

$$\dot{y}(t) = -y(t) + 2\tanh(y(t)) + I(t), \tag{4.5}$$

which is a particular case of (1.3) when $\nu = 1$ and $\mu = 2$. The linear stability analysis suggests that the trivial equilibrium to (4.5) when $I = 0$ is unstable. To stabilize the neuron activity, we use a PD controller (1.2).

By linearizing the equation, assessing the local asymptotic behavior of solutions to (4.5) is equivalent to studying the spectrum distribution of the quasipolynomial function

$$Q_2(s) = s - 1 + e^{-\tau s}(k_d s + k_p) \quad (s \in \mathbb{C})$$

which is a particular case of (2.1), where $\alpha = k_d$, $\beta = k_p$, and $a = -1$.

Assume that we assign three equidistributed real spectral values $s_1 = -7$, $s_2 = -7 - d$ and $s_3 = -7 - 2d$ to Q_2. By letting $d = 1$, one finds, owing to Lemma 2.4, the following.

$$\tau = \ln(5/4), \qquad k_d = e^{-8\tau}, \qquad k_p = e^{-8\tau}\left(8 + \coth(\tau/2)\right). \tag{4.6}$$

In Figure 7, we depicted the solutions of (4.5) when $I(t) = 0$ and when $I(t) = -k_p y(t-\tau) - k_d \dot{y}(t-\tau)$ where $\tau > 0$, $k_d > 0$ and $k_p > 0$ are given by (4.6).

4.1.4. *Modeling of locally exponentially stable one-layer neural networks.* In the previous sections, we examined the fundamental stability aspects of one-layer neural networks. We specifically looked at how variations in the system's parameters ν and μ, as well as the implementation of a delayed PD controller, influence the system's dynamic behavior.

Building on this analysis, this section aims to provide a comprehensive framework that guides the design and configuration of one-layer neural networks to achieve specific exponential stability criteria. In this scenario, we do not have prior knowledge of the inherent parameters $\nu > 0$ and

$\mu > 0$ in equation (NDE). The only available information is the prescribed decay rate $\gamma > 0$, which is designated to govern the exponential decay of the solution of (NDE) towards zero, regardless of the sign of $\nu - \mu$. Since the local asymptotic behavior of solutions to (NDE) is equivalent to studying the spectrum distribution of the quasipolynomial function Δ_0 defined in (1.5), the following steps allow us to engineer the model parameters.

(1) Firstly, assign three real spectral values $s_3 < s_2 < s_1 := -\gamma$ to Δ_0.

(2) Owing to (2.23) and the interpolation Lemma 2.4, the coefficients of Δ_0 are then given by

(4.7)
$$(\nu - \mu)(\tau) = -s_1 - \zeta(\tau),\ k_d(\tau) = \frac{F_{-\tau,2}(s_1 + s_2 + s_3, s_1 + s_3)}{F_{-\tau,2}(s_1, s_2, s_3)},\ k_p(\tau) = -k_d(\tau)s_1 + \zeta(\tau)e^{\tau s_1},$$

(3) If one wants to design (NDE) with $\nu = \mu$,

(i) Find the unique $\tau_* > 0$ such that $s_1 = -\zeta(\tau_*)$. This equation always admits a positive real solution thanks to Theorem 3.3.

(ii) Compute the corresponding gains $k_p(\tau_*)$ and $k_d(\tau_*)$.

Consequently, the zero equilibrium to (NDE) with $\nu = \mu > 0$, $\tau = \tau_*$, $k_p = k_p(\tau_*)$, and $k_d = k_d(\tau_*)$ is locally exponentially stable according to Theorem 3.24.

(4) If one wants to design (NDE) with $\nu > \mu$,

(i) Find the unique $\tau_* > 0$ such that $s_1 = -\zeta(\tau_*)$.

(ii) Set $\tau_0 > 0$ such that $\tau_0 > \tau_*$ to guarantee that $(\nu - \mu)(\tau_0) > 0$ by Theorem 3.3.

(iii) Compute the corresponding gains $k_p(\tau_0)$ and $k_d(\tau_0)$.

Therefore, the trivial equilibrium zero to (NDE) with $\mu > 0$, $\nu = \mu - s_1 - \zeta(\tau_0)$, $\tau = \tau_0$, $k_p = k_p(\tau_0)$, and $k_d = k_d(\tau_0)$ is locally exponentially stable according to Theorem 3.24.

(5) If one wants to design (NDE) with $\nu < \mu$,

(i) Find the unique $\tau_* > 0$ such that $s_1 = -\zeta(\tau_*)$.

(ii) Set $\tau_1 > 0$ such that $\tau_1 < \tau_*$ to guarantee that $(\nu - \mu)(\tau_1) < 0$ by Theorem 3.3.

(iii) Compute the corresponding gains $k_p(\tau_1)$ and $k_d(\tau_1)$.

It follows that the trivial equilibrium zero to (NDE) with $\nu > 0$, $\mu = \nu + s_1 + \zeta(\tau_1)$, $\tau = \tau_1$, $k_p = k_p(\tau_1)$, and $k_d = k_d(\tau_1)$ is locally exponentially stable thanks to Theorem 3.24.

Abiding by these steps enables the design of a one-layer neural network with a delayed PD controller that prevents seizure-like events regardless of the inherent dynamics.

Consider the practical example consisting of designing (NDE) with either $\nu = \mu$, $\nu > \mu$ and $\mu > \nu$ such that the zero equilibrium is locally exponential stable with the prescribed decay rate $\gamma = 4$. Assigning three equidistributed real spectral values $s_3 = -6$, $s_2 = -5$ and $s_1 = -4$, one finds that the unique $\tau_* > 0$ such that $-4 = \zeta(\tau_*)$ is given by $\tau_* = \ln(3/2)$. Therefore,

$$k_d(\tau_*) = e^{-5\tau_*}, \qquad k_p(\tau_*) = e^{-5\tau_*}\left(5 + \coth\left(\tau_*/2\right)\right).$$

Therefore letting, for instance, $\nu = \mu = 2$, one can design (NDE) in the following fashion

(4.8)
$$\dot{y}(t) = -2y(t) + 2\tanh(y(t)) - k_p(\tau_*)y(t - \tau_*) - k_d(\tau_*)\dot{y}(t - \tau_*).$$

Letting now $\tau_0 > \ln(3/2)$, say $\tau_0 = 2\ln(3/2)$, and $\mu = 1$, one finds

$$\nu = \mu - s_1 - \zeta(\tau_0) = 3.4, \quad k_d(\tau_0) = e^{-5\tau_0}, \qquad k_p(\tau_0) = e^{-5\tau_0}\left(5 + \coth\left(\tau_0/2\right)\right).$$

It follows that one can design (NDE) accordingly

(4.9)
$$\dot{y}(t) = -3.4y(t) + \tanh(y(t)) - k_p(\tau_0)y(t - \tau_0) - k_d(\tau_0)\dot{y}(t - \tau_0).$$

Finally, taking $\tau_1 > 0$ such that $\tau_1 < \tau_*$, say $\tau_1 = \ln(3/2)/2$, and $\nu = 3$, one gets

$$\mu = \nu + s_1 + \zeta(\tau_1) \approx 7.89898, \quad k_d(\tau_1) = e^{-5\tau_1}, \qquad k_p(\tau_1) = e^{-5\tau_1}\left(5 + \coth\left(\tau_1/2\right)\right).$$

Therefore, one can design (NDE) as follows

$$\dot{y}(t) = -3y(t) + 7.89898\tanh(y(t)) - k_p(\tau_1)y(t-\tau_1) - k_d(\tau_1)\dot{y}(t-\tau_1). \tag{4.10}$$

We depicted in Figure 7 solutions to equations (4.8), (4.9) and (4.10).

5. Discussion

This paper demonstrates the use of a delayed Proportional-Derivative (PD) controller in continuous-time modeling of a one-layer Hopfield Neural Network (HNN) to achieve exponential stability in models that are vulnerable to conditions similar to epileptic seizures at the single neuron level. Through a rigorous application of spectral theory analysis, we have developed a methodological framework that improves the stability of neural network models.

We have expanded the spectral theory based on the CRRID property for linear functional differential equations of neutral type to the field of neural dynamics, offering a powerful analytical tool to examine the stability of neural networks based on their spectral properties. This has enabled us to determine the conditions under which the network attains stability, with a primary focus on systems where conventional approaches anticipate instability.

Based on these theoretical insights, integrating a delayed PD controller has shown significant promise in stabilizing the HNN. This approach imitates the natural inhibitory feedback mechanisms within the brain. It offers a biologically inspired method to control and prevent the hyperexcitability that may lead to seizure-like events at the level of a single cell.

It would be beneficial for future research to explore the potential of implementing the delayed PD control strategy and CRRID setting in more complex, multi-layer neural network architectures, which could represent the intricate structures of biological neural systems more accurately. It's important to note that the MID setting developed in [BBNT23, BMN22, MBN21] can address this issue. However, as mentioned in [MBN17], it is widely acknowledged that non-semisimple spectral values are sensitive to minor perturbations due to their splitting mechanism.

The study provides valuable insights into the local stability of the zero equilibrium in the nonlinear model. It is important to recognize that these findings are primarily related to local dynamics. In the future, research should combine the spectral methods used for the linear equation with time-domain approaches based on Lyapunov functionals and linear matrix inequalities. This combined approach would enable a more comprehensive investigation into the global exponential stability of the nonlinear equation with a prescribed decay rate, expanding the applicability and reliability of our findings.

Acknowledgments

The authors wish to thank their colleagues Fazia Bedouhene (LMPA, The Mouloud Mammeri University of Tizi Ouzou, Algeria), Antoine Chaillet, Guilherme Mazanti and Silviu-Iulian Niculescu (L2S, University Paris-Saclay, France), and Timothée Schmoderer (Prisme, University of Orléans, France) for valuable discussions on the CRRID property and references suggestions on time-domain approaches based on Lyapunov functionals.

Appendix A. Proof of Theorem 3.9

We present in this section a proof of Theorem 3.9.

Proof. We start by noticing that direct computations yield

$$\alpha e^{-\tau s_2} = \frac{e^{\tau d}(d(e^{\tau\delta}-1)-\delta(e^{\tau d}-1))}{de^{\tau d}+e^{\tau\delta}(e^{\tau d}(\delta-d)-\delta)} = \theta(\tau,d,\delta) := \theta, \tag{A.1}$$

$$a = s_2 - \frac{\xi}{\tau} \qquad \text{where} \qquad \xi := \xi(\tau,d,\delta) = \frac{de^{\tau d}(e^{\tau d}-1)(\delta-d)\tau}{de^{\tau d}+e^{\tau\delta}(e^{\tau d}(\delta-d)-\delta)}, \tag{A.2}$$

$$\beta = -\alpha s_2 - e^{\tau s_2}(a+s_2) = -\alpha s_2 + \xi e^{\tau s_2}/\tau \tag{A.3}$$

thanks to (2.13), (2.15) and (2.12). Here $d = s_1 - s_2$ and $\delta = s_1 - s_3$. Consider the change of functions

$$\Delta'(z) = \tau\Delta\left(s_2 + \frac{z}{\tau}\right) = z - \xi + e^{-z}(\theta z + \xi) \qquad (z \in \mathbb{C}) \tag{A.4}$$

so that Δ' has three real roots $-\tau(\delta-d)$, 0 and τd. In particular, $\Delta'(0) = 0$ if, and only if, $\Delta(s_2) = 0$, and $z \in \mathbb{C}$ is a spectral value of Δ' if, and only if, $s_2 + z/\tau$ is a spectral value of Δ. Moreover, since s_1 is the dominant root of Δ, then τd is the dominant root of Δ'. Let now $z_0 = x_0 + i\omega_0$ be a complex non-real spectral value of Δ'. Then, $x_0 < \tau d$ and $\omega_0 \neq 0$. In particular, $|\omega_0| \geq 2\pi$ by Lemma 2.3. From $\Delta'(z_0) = 0$, one obtains

$$\frac{z-\xi}{\theta z+\xi} = -e^{-z} \quad \Longrightarrow \quad \sqrt{\frac{(x_0-\xi)^2+\omega_0^2}{(\theta x_0+\xi)^2+\omega_0^2}} = e^{-x_0}. \tag{A.5}$$

Letting $|\omega_0| \to \infty$ in (A.5), one deduces that

$$e^{-x_0} = 1/\theta, \qquad \text{since} \qquad 0 < \theta < e^{\tau(s_1-s_2)} = e^{\tau d}. \tag{A.6}$$

It follows that $x_0 = \ln(\theta)$, and the rest of the spectrum of Δ' is either located on the vertical line $\Re(z) = \ln(\theta)$ or forms a chain asymptotic to $\Re(s) = \ln(\theta)$. Let us show that the former is inconsistent; that is, there exists at least a complex spectral value $z_0 \in \mathbb{C}$ of Δ' such that $\Re(z_0) \neq \ln(\theta)$. Otherwise, for every $z_0 = x_0 + i\omega_0 \in \mathbb{C}$ ($x_0 < \tau d$, $\omega_0 \neq 0$ and $|\omega_0| \geq 2\pi$) $\Delta'(z_0) = 0$ and $x_0 = \ln(\theta)$. Firstly, in all generality, for any complex number $z \in \mathbb{C}$, the reflection $2\ln(\theta) - \overline{z}$ of z across the vertical line $\Re(z) = \ln(\theta)$ satisfies

$$\Delta'(2\ln(\theta) - \overline{z}) = -\overline{z} + \ln(\theta^2) - \xi + e^{\overline{z}}\left(-\frac{\overline{z}}{\theta} + \frac{\ln(\theta^2)}{\theta} + \frac{\xi}{\theta^2}\right). \tag{A.7}$$

Secondly, for $z = z_0 = \ln(\theta) + i\omega_0$ ($\omega_0 \neq 0$ and $|\omega_0| \geq 2\pi$), one has $2\ln(\theta) - \overline{z_0} = z_0$ so that owing to (A.7) and (A.4), one gets

$$z_0 - \xi + e^{\overline{z_0}}\left(\frac{\overline{z_0}}{\theta} + \frac{\xi}{\theta^2}\right) = z_0 - \xi + e^{-z_0}(\theta z_0 + \xi). \tag{A.8}$$

which yields $\overline{z_0} = z_0$. It follows that z_0 is a real spectral value of Δ', which is inconsistent. □

Appendix B. A priori stability analysis

In this section, we study the stability of the continuous-time one-layer Hopfield Neural Network (HNN) with time-independent external inputs. Instead of using linearization to approximate system behavior, we will directly examine the nonlinear characteristics of the model. The analysis will help us understand the conditions under which the system remains stable and will provide a foundation for introducing external control designs.

For the sake of exhaustiveness, we tackle the question in the instance where the constant external input is not identically equal to zero. Then, we consider the following equation

$$\dot{y}(t) = -\nu y(t) + \mu \tanh(y(t)) + I, \tag{B.1}$$

where $\nu > 0$, $\mu > 0$, and $I \in \mathbb{R}$ is time-independent.

The following result is standard, but we provide the proof for the sake of completeness.

Proposition B.1. *For every $I \in \mathbb{R}$, the unique equilibrium $y_* \in \mathbb{R}$ of (B.1) is globally exponentially stable if $0 < \mu < \nu$ and globally asymptotically stable when $\mu = \nu$.*

Proof. The fact that $y_* \in \mathbb{R}$ exists and is unique is a trivial consequence of the intermediate value theorem. Since tanh is globally 1-Lipschitz continuous, the Cauchy-Lipchitz theorem ensures that for every $I \in \mathbb{R}$ and for any $y_0 \in \mathbb{R}$, there exists a unique continuous solution y to equation (B.1) with initial condition $y(0) = y_0$. Set the change of function

$$u(t) = y(t) - y_* \quad (\forall t \geq 0). \tag{B.2}$$

Then, $u(0) = y_0 - y_*$ and u solves the following equation

$$\dot{u}(t) = -\nu u(t) + \mu(\tanh(u(t) + y_*) - \tanh(y_*)), \tag{B.3}$$

since $-\nu y_* + \mu \tanh(y_*) + I = 0$. Duhamel's formula gives us

$$u(t) = e^{-\nu t} u_0 + \mu \int_0^t e^{-\nu(t-s)}(\tanh(u(s) + y_*) - \tanh(y_*))ds \quad (\forall t \geq 0). \tag{B.4}$$

Taking the absolute value of the above identity and Gronwall's lemma yields to

$$e^{\nu t}|u(t)| \leq e^{\mu t}|u_0| \qquad \text{i.e.,} \qquad |y(t) - y_*| \leq e^{-(\nu-\mu)t}|y_0 - y_*| \quad (\forall t \geq 0). \tag{B.5}$$

It follows that the equilibrium y_* (equal to zero when $I = 0$) is globally exponentially stable if $\nu > \mu$.

Let us provide an argument to prove the global asymptotic stability of y_* when $\mu = \nu$ via a Lyapunov function. Define the function

$$V(y) = \nu \int_{y_*}^{y} (r - \tanh(r) - I) \tanh'(r - y_*) dr \quad (y \in \mathbb{R}). \tag{B.6}$$

By letting $g(r) = r - \tanh(r) - I$, function g is derivable on $\mathbb{R}$ and satisfies $g(y_*) = 0$, $g(\pm\infty) = \pm\infty$ and $g'(r) = \tanh^2(r) \geq 0$ for all $r \in \mathbb{R}$. It follows that $g(r) > 0$ for every $r > y_*$ and $g(r) < 0$ for every $r < y_*$, which yields

$$V(y_*) = 0 \qquad \text{and} \qquad V(y) > 0 \quad (\forall y \in \mathbb{R} \backslash \{y_*\}). \tag{B.7}$$

On the other hand, along a solution $y(\cdot)$ of (B.1), one has

$$\frac{dV}{dt}(y(t)) = -\dot{y}(t)^2 \tanh'(y(t) - y_*) \leq 0, \qquad \text{and} \qquad \frac{dV}{dt}(y(t)) = 0 \iff y(t) = y_*. \tag{B.8}$$

Hence, V is a strict Lyapunov function for y_*, and the latter is globally asymptotically stable. $\square$

In the case of $0 < \nu < \mu$, equation (B.1) can have multi equilibria for certain inputs $I \in \mathbb{R}$ and exactly three equilibria $-y_1 < y_* = 0 < y_1$ when $I = 0$. In this case, via linear stability analysis, one proves that $\pm y_1$ are locally exponentially stable, and $y_* = 0$ is unstable.

Consequently, we can assume that $y_* = 0$ ($\tanh(0) = 0$) is the unstable equilibrium. The aim is then to design a suitable delayed PD controller guaranteeing that the trivial equilibrium zero of the closed-loop equation (1.3) is exponentially stable.

Remark B.2. Since equation (B.1) is structurally stable, replacing the tanh with any sigmoid function f satisfying $f(0) = 0$ will lead to the same conclusion.

References

[ABBN18] Souad Amrane, Fazia Bedouhene, Islam Boussaada, and Silviu-Iulian Niculescu. On qualitative properties of low-degree quasipolynomials: further remarks on the spectral abscissa and rightmost-roots assignment. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, 61(4):361–381, 2018.

[BBL93] Anne Beuter, Jacques Bélair, and Christiane Labrie. Feedback and delays in neurological diseases: A modeling study using gynamical systems. *Bulletin of mathematical biology*, 55:525–541, 1993.

[BBN20] Fazia Bedouhene, Islam Boussaada, and Silviu-Iulian Niculescu. Real spectral values coexistence and their effect on the stability of time-delay systems: Vandermonde matrices and exponential decay. *Comptes Rendus. Mathématique*, 358(9-10):1011–1032, 2020.

[BBNT23] Amina Benarab, Islam Boussaada, Silviu-Iulian Niculescu, and Karim L Trabelsi. Multiplicity-induced-dominancy for delay systems: Comprehensive examples in the scalar neutral case. *European Journal of Control*, 74:100835, 2023.

[BMN22] Islam Boussaada, Guilherme Mazanti, and Silviu-Iulian Niculescu. The generic multiplicity-induced-dominancy property from retarded to neutral delay-differential equations: When delay-systems characteristics meet the zeros of kummer functions. *Comptes Rendus. Mathématique*, 360(G4):349–369, 2022.

[BNEA+20] Islam Boussaada, Silviu-Iulian Niculescu, Ali El-Ati, Redamy Pérez-Ramos, and Karim Trabelsi. Multiplicity-induced-dominancy in parametric second-order delay differential equations: Analysis and application in control design. *ESAIM: Control, Optimisation and Calculus of Variations*, 26:57, 2020.

[Bri15] C Briat. Lpv & time-delay systems–analysis, observation, filtering & control. *Advances in Delay and Dynamics*, 3, 2015.

[CT15] Jean Michel Coron and Simona Oana Tamasoiu. Feedback stabilization for a scalar conservation law with PID boundary control. *Chinese Annals of Mathematics, Series B*, 36(5):763–776, 2015.

[DDJB21] Damien Depannemaecker, Alain Destexhe, Viktor Jirsa, and Christophe Bernard. Modeling seizures: From single neurons to networks. *Seizure*, 90:4–8, 2021.

[DEW+23] Damien Depannemaecker, Aitakin Ezzati, Huifang E Wang, Viktor Jirsa, and Christophe Bernard. From phenomenological to biophysical models of seizures. *Neurobiology of Disease*, 182:106131, 2023.

[Fri01] Emilia Fridman. New lyapunov–krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & control letters*, 43(4):309–319, 2001.

[Fri14] Emilia Fridman. *Introduction to time-delay systems: Analysis and control*. Springer, 2014.

[FVL03] Miguel VS Frasson and Sjoerd M Verduyn Lunel. Large time behaviour of linear functional differential equations. *Integral Equations and Operator Theory*, 47:91–121, 2003.

[GKNP14] Wulfram Gerstner, Werner M. Kistler, Richard Naud, and Liam Paninski. *Neuronal Dynamics: From Single Neurons to Networks and Models of Cognition*. Cambridge University Press, 2014.

[Gop13] Kondalsamy Gopalsamy. *Stability and oscillations in delay differential equations of population dynamics*, volume 74. Springer Science & Business Media, 2013.

[HL13] Jack K Hale and Sjoerd M Verduyn Lunel. *Introduction to functional differential equations*, volume 99. Springer Science & Business Media, 2013.

[Hop84] John J Hopfield. Neurons with graded response have collective computational properties like those of two-state neurons. *Proceedings of the national academy of sciences*, 81(10):3088–3092, 1984.

[Kha05] VL Kharitonov. Lyapunov functionals and lyapunov matrices for neutral type time delay systems: a single delay case. *International Journal of Control*, 78(11):783–800, 2005.

[Kua93] Yang Kuang. *Delay differential equations: with applications in population dynamics*. Academic press, 1993.

[LYH00] Chang-Hua Lien, Ker-Wei Yu, and Jer-Guang Hsieh. Stability conditions for a class of neutral systems with multiple time delays. *Journal of Mathematical Analysis and Applications*, 245(1):20–27, 2000.

[MBN17] Wim Michiels, Islam Boussaada, and Silviu-Iulian Niculescu. An explicit formula for the splitting of multiple eigenvalues for nonlinear eigenvalue problems and connections with the linearization for the delay eigenvalue problem. *SIAM Journal on Matrix Analysis and Applications*, 38(2):599–620, 2017.

[MBN21] Guilherme Mazanti, Islam Boussaada, and Silviu-Iulian Niculescu. Multiplicity-induced-dominancy for delay-differential equations of retarded type. *Journal of Differential Equations*, 286:84–118, 2021.

[MN14] Wim Michiels and Silviu-Iulian Niculescu. *Stability, control, and computation for time-delay systems: an eigenvalue-based approach*. SIAM, 2014.

[PJ19] Spase Petkoski and Viktor K Jirsa. Transmission time delays organize the brain network synchronization. *Philosophical Transactions of the Royal Society A*, 377(2153):20180132, 2019.

[PS72] George Pólya and Gabor Szegö. *Problems and theorems in analysis I: Series. Integral. Calculus . Theory of Functions*. Springer-Verlag Berlin Heidelberg GmbH, 1972.

[PS12] George Pólya and Gabor Szegö. *Problems and theorems in analysis II: theory of functions. Zeros. Polynomials. Determinants. Number theory. Geometry*. Springer Science & Business Media, 2012.

[Rua06] S. Ruan. Delay differential equations in single species dynamics. In O. Arino, M.L. Hbid, and E. Ait Dads, editors, *Delay Differential Equations and Applications*, pages 477–517, Dordrecht, 2006. Springer Netherlands.

[SBN23] Timothée Schmoderer, Islam Boussaada, and Silviu-Iulian Niculescu. On boundary control of the transport equation. assigning real spectra & exponential decay. *IEEE Control Systems Letters*, 2023.

[SBNB24] Timothée Schmoderer, Islam Boussaada, Silviu-Iulian Niculescu, and Fazia Bedouhene. Insights on equidistributed real spectral values in second-order delay systems: perspectives in partial pole placement. *Systems & Control Letters*, 185:105728, 2024.

[SD19] Kartik K Sreenivasan and Mark D'Esposito. The what, where and how of delay activity. *Nature reviews neuroscience*, 20(8):466–481, 2019.

[Sta06] Carl E Stafstrom. Epilepsy: a review of selected clinical syndromes and advances in basic science. *Journal of Cerebral Blood Flow & Metabolism*, 26(8):983–1004, 2006.

[Ste09] Gabor Stepan. Delay effects in the human sensory system during balancing. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 367(1891):1195–1212, 2009.

[WR99] Junjie Wei and Shigui Ruan. Stability and bifurcation in a neural network model with two delays. *Physica D: Nonlinear Phenomena*, 130(3-4):255–272, 1999.

[YBM+15] Robert B Yaffe, Philip Borger, Pierre Megevand, David M Groppe, Mark A Kramer, Catherine J Chu, Sabato Santaniello, Christian Meisel, Ashesh D Mehta, and Sridevi V Sarma. Physiology of functional and effective networks in epilepsy. *Clinical Neurophysiology*, 126(2):227–236, 2015.

Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France & Institut Polytechnique des Sciences Avancées (IPSA), 63 boulevard de Brandebourg, 94200 Ivry-sur-Seine, France
Email address: cyprien.tamekue@inria.fr

Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France & Institut Polytechnique des Sciences Avancées (IPSA), 63 boulevard de Brandebourg, 94200 Ivry-sur-Seine, France
Email address: islam.boussaada@centralesupelec.fr

Institut Polytechnique des Sciences Avancées (IPSA), 63 boulevard de Brandebourg, 94200 Ivry-sur-Seine, France
Email address: karim.trabelsi@ipsa.fr