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# PREScribed EXPONENTIAL STABILIZATION OF A ONE-LAYER NEURAL NETWORK WITH DELAYED FEEDBACK: INSIGHTS IN SEIZURE PREVENTION AND NEURAL CONTROL

CYPRIEN TAMEKUE, ISLAM BOUSSAADA, AND KARIM TRABELSI

**ABSTRACT.** This paper provides control-oriented delay-based modelling of a one-layer neural network of Hopfield-type subject to an external input designed as delayed feedback. The specificity of such a model is that it makes the considered neuron less susceptible to seizure caused by its inherent dynamic instability. This modelling exploits a recently set partial pole placement for linear functional differential equations, which relies on the coexistence of real spectral values, allowing the explicit prescription of the closed-loop solution's exponential decay. The proposed framework improves some pioneering and scarce results from the literature on the characterization of the exact solution's exponential decay when a simple real spectral value exists. Indeed, it improves neural stability when the inherent dynamic is stable and provides insights into the design of a one-layer neural network that can be stabilized exponentially with delayed feedback and with a prescribed decay rate regardless of whether the inherent neuron dynamic is stable or unstable.

**Keywords.** Neural networks, Time-delay controller, Neutral equations, PD-controller, Coexistent-real-roots-induced-dominancy, Partial pole placement.

**MSCcodes.** 34K20, 34K60, 92B20, 37N25, 93D15.

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## 1. INTRODUCTION

Neural networks exhibit complex dynamics crucial for their biological or artificial functionality. The stability of these systems is essential, as instabilities can lead to dysfunctional behaviours such as seizures in biological systems [YBM<sup>+</sup>15] or system failures in artificial networks [WY21].

In the study of neurological disorders such as epilepsy, models at the single-neuron level offer profound insights into the cellular mechanisms that support seizure phenomena [DDJB21]. Single neuron-scale models vary significantly in detail, ranging from those that include sub-cellular components like mitochondria [TU19] to those that intricately describe the morphology of dendritic and axonal trees [TGCR14]. Such detailed models often require the neuron’s geometry to be divided into several compartments, which, while biophysically detailed, can lead to complex systems with many differential equations that challenge the complete dynamical analysis.

This paper employs a one-layer continuous-time Hopfield neural network (HNN) [Hop84] to model epileptic seizures at the single neuron level, incorporating a delayed Proportional-Derivative (PD) controller to enhance the model’s stability and responsiveness. The HNN framework offers a more straightforward yet effective way to capture neuronal activity’s fundamental dynamics during seizures, such as high-frequency bursts of action potentials and hypersynchronization of neuronal populations, which are pivotal in initiating and propagating seizures. It reduces the complexity seen in highly detailed models [CUZ<sup>+</sup>09, HH52] while retaining the ability to simulate key biophysical processes. Integrating a delayed PD controller is essential for accurately simulating the regulatory mechanisms that naturally occur within the brain. This controller mimics the effects of inhibitory feedback within neuronal circuits, essential for modulating the excitatory activity that leads to seizures [YBM<sup>+</sup>15]. By implementing delay in the PD controller, our model reflects the time-dependent nature of neuronal interactions, where the effects of inhibitory actions are not instantaneous but occur with a physiological lag, characteristic of synaptic transmission delays relevant in brain activities [PJ19, SD19], for further insights in the use of time-delays in modelling biological systems, see for instance [Gop13, Rua06]. In particular, it is commonly accepted to include time delays in modelling sensory and motoric neural pathways due to the time lag one observes in communication. See, for instance, [Ste09, Kua93, BBL93], and [WR99] for two delays in neural networks modelling.

Consider the continuous Hopfield network, a model that simulates the state dynamics of a biological neuron or serves as a basic unit within an artificial neural network. The following equation encapsulates the governing dynamics of this model [Hop84, BBL93],

$$\dot{y}(t) = -\nu y(t) + \mu \tanh(y(t)) + I(t), \quad (1.1)$$

where  $\dot{y}(t)$  denotes the rate of change of the neuron’s state at time  $t$ ,  $\nu$  is a positive parameter reflecting the natural decay rate or leakage of the neuron’s membrane potential towards its resting state,  $\mu$  is a positive parameter that scales the influence of the activation function  $\tanh$ ,  $\tanh(y(t))$  is the activation function applied to the neuron’s current state, emphasizing real-time processing without the introduction of delay,  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$  represents an external input designed as a Proportional-Derivative (PD) controller, with  $k_p$  and  $k_d$  being the proportional and derivative gains, respectively, and  $\tau > 0$  indicating a delay in the control action to account for the inherent temporal dynamics in neural processing.

The proportional component of the PD controller ( $-k_p$ ) can be compared to inhibitory synaptic strength. Increasing  $k_p$  represents strengthening inhibitory control and stabilizing

the neuron's activity. The derivative component ( $-k_d$ ) helps dampen the rate of change in the neuron's activity, similar to how biological systems use rate-dependent inhibitory mechanisms to prevent excessive firing rates that lead to seizures. We achieve stability with small values for  $k_p$  and  $k_d$ . This smallness guarantees that the inherent dynamics of the system ( $\dot{y}(t) = -\nu y(t) + \mu \tanh(y(t))$ ) are preserved to a greater extent. In particular, this is important for neural networks that aim to mimic biological processes, as it ensures that the control strategy does not overpower the natural behaviours and characteristics of the system.

When a neural system's natural decay rate ( $\nu$ ) is surpassed by the interaction strength ( $\mu$ ), the system tends toward instability. This is a significant theoretical and practical concern, as it resembles the hyperexcitability observed in neuronal populations during epileptic seizures.

This paper explores the effectiveness of delayed PD controllers in stabilizing the equation (1.1) characterized by  $\nu \leq \mu$ . Our main goal is to establish a theoretical basis to guide the practical design of a one-layer neural network system that is less susceptible to seizures caused by inherent dynamic instability. Roughly speaking, our main focus is on parameter identifications, meaning that given  $\gamma < 0$ , how can we engineer the system's parameters  $\nu > 0$ ,  $\mu > 0$ ,  $k_p > 0$ ,  $k_d > 0$  and  $\tau > 0$ , whatever the sign of  $\nu - \mu$  such that the trivial equilibrium to equation (1.1) decays exponentially towards zero with a decay rate  $\gamma + \varepsilon$ , for a sufficiently small  $\varepsilon > 0$ ?

We achieve this general purpose through the linearization technique. We are therefore interested in studying the exponential stability of the equation,

$$\dot{y}(t) = -(\nu - \mu)y(t) - k_p y(t - \tau) - k_d \dot{y}(t - \tau). \quad (1.2)$$

In the proposed framework, we adjust the control parameters - specifically the gains  $k_p$  and  $k_d$ , and the delay  $\tau$  - to ensure that the solutions of equation (1.2) are not only asymptotically stable but also exhibit a guaranteed rate of exponential decay.

The equation (1.2) has been widely discussed in the literature regarding its asymptotic and exponential stability [LYH00, HL13, Fri01, BMN22, SBN23]. Sufficient delay-independent conditions for stability have been presented. In [LYH00, Example 1, page 26], the authors proved the global uniform asymptotic stability of equation (1.2) with constant real coefficients (not dependent upon the delay  $\tau$ ) satisfying  $\nu - \mu > 0$ ,  $|k_d| < 1$ , and  $|k_p| < (\nu - \mu)\sqrt{1 - k_d^2}$  using a Lyapunov functional and a linear matrix inequality (LMI). It's worth noting that asymptotic analysis of (1.2) was already considered in [HL13, Chapter 9, Section 9.8, page 294] when  $k_p = 0$ ,  $\nu - \mu > 0$ , and  $|k_d| < 1$ , see also [Fri14, Chapter 3, Section 3.3.4, page 69]. Unfortunately, to the best of the authors' knowledge, no known result using a time-domain approach based on a Lyapunov functional and a LMI can be applied to study the asymptotic stability properties of equation (1.2) when  $\nu - \mu \leq 0$ .

In the frequency domain, the problem reduces to the analysis of the distribution of the roots of the corresponding characteristic function, which is an entire function called *characteristic quasipolynomial*. Interestingly, the corresponding characteristic quasipolynomial is shared with the problem of boundary PI control of the transport equation as studied in [CT15, SBN23], see also [BBNT23] for further insights on the corresponding quasipolynomial. For a deeper discussion of the spectral properties and related stability analysis and control approaches of (1.2), refer to [MN14, Bri15]. Via Laplace's transform, the asymptotic stability of equation (1.2) is reduced to study the location of the spectrum of the quasipolynomial function

$$\Delta_0(s) = s + \nu - \mu + e^{-\tau s}(k_d s + k_p), \quad s \in \mathbb{C}. \quad (1.3)$$

As a matter of fact, to characterize the exponential stability of solutions of the linearization of (1.2), it amounts to the location of the spectral abscissa of  $\Delta_0$ , which is a challenging problem in all generality. However, recent works have highlighted a particularly interesting spectral property, called *multiplicity-induced-dominancy (MID)*, which consists in conditions on the system's parameters under which a multiple spectral value corresponds to the spectral abscissa [BNEA<sup>+</sup>20, BMN22, MBN21]. In particular, it appears that a characteristic root of maximal multiplicity (i.e., equal to the degree of the corresponding quasipolynomial) necessarily defines the spectral abscissa of the system, this property occurs in general in *generic* quasipolynomial and is called GMID. However, in the case of *intermediate* multiplicities, that is, multiplicities which are less than the quasipolynomial's degree, the IMID occurs (the largest assigned root corresponds to the spectral abscissa) under some additional conditions, see for instance [BBNT23]. Since these works, the case of the assignment of a characteristic root with maximal multiplicity was recently addressed and thoroughly characterized in [MBN21] (generic retarded case) and in [BMN22] (unifying retarded and neutral cases) for LTI DDEs including a *single delay*. It is essential to point out that the multiplicity of a given spectral value itself is not essential. Still, its connection with the dominance of this root is a meaningful tool for control synthesis. Namely, it is shown that, under appropriate conditions, the coexistence of exactly the maximal number of distinct negative zeros of quasipolynomial of reduced degree guarantees the exponential stability of the zero solution of the corresponding time-delay system, a property called *Coexisting Real Roots Induced Dominancy (CRRID)*, see for instance [ABBN18, BBN20, SBN23, SBNB23]. These properties opened an interesting perspective in control through the so-called *partial pole placement* method, that is, imposing the multiplicity or the coexistence of simple real characteristic root of the closed-loop system by an appropriate choice of the controller gains guarantees the exponential stability of the closed-loop system with a prescribed decay rate. For instance, the quasipolynomial (1.3) has been investigated in [SBN23] in the context of studying the boundary control of the transport equation with a proportional-integral (PI) controller, where the CRRID property has been exploited in the prescription of the closed-loop exponential decay.

The contribution of this work is threefold: First, it refines recent results on the CRRID property for the first-order neutral functional differential equations, providing further insights into the qualitative properties of the corresponding quasipolynomials. In particular, in comparison to the result of [SBN23], it gives a more straightforward proof for the GCRRID<sup>1</sup> (G refers to *generic*) to hold for an arbitrary distribution of the real roots as well as the necessary and sufficient conditions for the ICRRID (I refers to *intermediate*) to hold i.e. when two real roots are assigned. The latter is provided in the two complementary cases: no further real roots exist, and an unintentional third real root coexists.

Secondly, it sheds some light on the intriguing properties of the quasipolynomial satisfying the CRRID and the benefits offered by the CRRID property in terms of exponential decay certification. In fact, in the single-delay case, when the Frasson-Verduyn Lunel's [FVL03] sufficient condition for the dominance of a simple spectral value is not met, the CRRID remains valid. Furthermore, the CRRID extends some recent results employing time-domain approaches such as the ones relying on Lyapunov functional and linear matrix inequalities thanks to the resulting sharp information on the spectrum location. As a matter of fact, when three coexistent spectral values are equidistributed, we exhaustively characterize the remaining spectrum distribution,

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<sup>1</sup>The GCRRID occurs when the maximal number of real roots is achieved.

extending the result obtained in [BMN22] in the GMID case. Nevertheless, when these roots are not equidistributed, we show that the remaining spectrum is asymptotic to an appropriate vertical line in the complex plane. Also, while the negativity of the spectral abscissa is assumed in [SBN23], in our contribution, the negativity of such a spectral abscissa is characterized in the space of the system's parameters. Thanks to the CRRID setting, and inspired from [HL13, Chapter 1, Theorem 6.2], we establish an explicit and more straightforward exponential estimate of the closed-loop system's solution compared to that of [HL13][Chapter 1, Theorem 7.6, page 32] thanks to the special structure of the corresponding quasipolynomial.

Lastly, these findings are interpreted and exploited in modelling a one-layer neural network of Hopfield-type, which is less susceptible to seizure caused by inherent dynamic instability.

The remainder of the paper is organized as follows: Section 2 provides the reader with some prerequisites in the complex analysis used to prove the main results and ends by the problem settings. In Section 3, we present the main results and the corresponding proofs. Section 4 exploits and translates the main results into exponentially stable one-layer neural networks of Hopfield-type modelling.

## 2. PROBLEM SETTINGS AND PREREQUISITES

Throughout the following, our focus is on studying the asymptotic behaviour of solutions of the general scalar neutral functional differential equation (NFDE) in Hale's form

$$\frac{d}{dt}(y(t) + \alpha y(t - \tau)) = -ay(t) - \beta y(t - \tau), \quad (\text{NDE})$$

with corresponding initial condition  $y(0) = y_0 \in C^0([-\tau, 0])$ . To this equation corresponds the characteristic quasipolynomial function given by

$$\Delta(s) = s + a + e^{-\tau s}(\alpha s + \beta) \quad (s \in \mathbb{C}), \quad (2.1)$$

where  $(a, \alpha, \beta) \in \mathbb{R}^3$  and  $\tau > 0$ . It is known that the degree of the quasipolynomial  $\Delta$  - the sum of the degrees of the involved polynomials plus the number of delays - is equal to three. Moreover, thanks to the Pólya-Szegő bound [PS72, Problem 206.2, page 144], the degree of  $\Delta$  is a sharp bound for the number of real roots counting multiplicities of the quasipolynomial  $\Delta$ .

Despite the challenging question of characterizing the spectral abscissa of (NDE), Frasson-Verduyn Lunel in [FVL03, Lemma B1] provides a test for determining the simplicity and dominance of real spectral values in the multi-delay scalar neutral equations, offering fundamental insights. Notice that such a characterization is closely related to the exponential/asymptotic behavior of the equation (NDE).

According to Frasson-Verduyn Lunel's lemma when restricted to the single-delay case, a real root  $s_0$  of  $\Delta$  is simple and dominant if  $V(s_0) < 1$  where  $V$  is a specific functional construct derived from  $\Delta$ . In other words, the condition is as follows; see, [FVL03, Lemma 5.1].

**Lemma 2.1.** *Suppose that there exists a real zero  $s_0$  of  $\Delta$ . If  $V(s_0) < 1$ , then  $s_0$  is a real simple dominant zero of  $\Delta$ . Here,*

$$V(s_0) = (|\alpha|(1 + |s_0|\tau) + |\beta|\tau)e^{-s_0\tau},$$

where  $(\alpha, \beta) \in \mathbb{R}^2$  are real coefficients appearing in (2.1) and  $\tau > 0$ .

Frasson-Verduyn Lunel's lemma is relevant, however restrictive, its limitation shall be explored later in Section 3.3. This study uses the CRRID framework which appears to be broad and flexible, with a quite sharp location of the spectrum distribution. Our analysis shows that the CRRID approach ensures that the largest real root is the rightmost and dominant, even when Frasson-Verduyn Lunel's sufficient condition is not satisfied. When the real spectral values assigned to  $\Delta$  are equidistributed, we also determine the threshold value beyond which the largest one maintains dominancy while no longer satisfying Frasson-Verduyn Lunel's sufficient condition for dominancy. This expanded application of the CRRID approach demonstrates its usefulness in providing an effective framework for analyzing the spectrum of quasipolynomials.

Before starting the study of the localisation of the quasipolynomial  $\Delta$  roots, let us state the following lemma the proof of which can be found in [PS12, Problem 77, page 46].

**Lemma 2.2** (Descartes' rule of signs). *Let  $a_1, a_2, a_3, \lambda_1, \lambda_2, \lambda_3$  be real constants, such that  $\lambda_1 < \lambda_2 < \lambda_3$ . Denote by  $Z$  the number of real zeros of the entire function*

$$\mathcal{F}(x) = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + a_3 e^{\lambda_3 x} \quad (2.2)$$

*and by  $C$ , the number of sign changes in the sequence of numbers  $a_1, a_2, a_3$ . Then,  $C - Z$  is a non-negative even integer.*

One also has the following important result which extends the [SBN23, Corollary 2], it includes the case where we only assign two real spectral values to  $\Delta$ .

**Lemma 2.3.** *Let  $\tau > 0$  and  $\Delta$  be the quasipolynomial defined by (2.1) with real coefficients. Let  $\eta \in \{0, 1\}$ , if  $\Delta$  admits exactly  $3 - \eta$  real roots, then any root  $x + i\omega \in \mathbb{C}$  of  $\Delta$  with  $\omega \neq 0$  satisfies*

$$|\omega| \geq \frac{(2 - \eta)\pi}{\tau}. \quad (2.3)$$

*Proof.* Recall from [PS72, Problem 206.2, page 144] that if  $M_{\alpha, \beta}$  denotes the number of roots of  $\Delta$  contained in the horizontal strip  $\{s \in \mathbb{C} \mid \alpha \leq \Im(s) \leq \beta\}$  counting multiplicities, then the following bound holds

$$\frac{\tau(\beta - \alpha)}{2\pi} - 3 \leq M_{\alpha, \beta} \leq \frac{\tau(\beta - \alpha)}{2\pi} + 3. \quad (2.4)$$

To complete the proof of the lemma, we argue by contradiction. Assume that  $\Delta$  admits  $3 - \eta$  real roots, and let  $x + i\omega \in \mathbb{C}$  be a root of  $\Delta$  with  $\omega \neq 0$  and  $|\omega| < (2 - \eta)\pi/\tau$ . Then there exists  $\varepsilon > 0$  such that  $|\omega| < \frac{(2 - \eta)\pi}{\tau} - \varepsilon$ . Since  $\Delta$  has real coefficients, the complex conjugate of  $\omega$  is also a root of  $\Delta$ , both belonging to the horizontal strip  $\{s \in \mathbb{C} \mid -\frac{(2 - \eta)\pi}{\tau} + \varepsilon \leq \Im(s) \leq \frac{(2 - \eta)\pi}{\tau} - \varepsilon\}$ . It follows that  $\Delta$  admits at least  $5 - \eta$  roots in this strip, which is inconsistent, since by the Pólya-Szegő bound (2.4), the number of zero in this strip satisfies

$$M_{-\frac{(2 - \eta)\pi}{\tau} + \varepsilon, \frac{(2 - \eta)\pi}{\tau} - \varepsilon} \leq 5 - \eta - \frac{\varepsilon\tau}{\pi} < 5 - \eta. \quad (2.5)$$

□

In the next section, we study the spectral properties of the quasipolynomial function  $\Delta$ , focusing on characterizing its rightmost root. Consider a complex value  $s_0 \in \mathbb{C}$  such that  $\Delta(s_0) = 0$ . We say that  $s_0$  is a *dominant* (respectively, *strictly dominant*) root of  $\Delta$  if the following holds:

$$\forall s \in \mathbb{C} \setminus \{s_0\}, \quad \Delta(s) = 0 \implies \Re(s) \leq \Re(s_0) \quad (\text{respectively } \Re(s) < \Re(s_0)). \quad (2.6)$$

Let us investigate the coexistence of - non-necessarily equidistributed - three real roots for the quasipolynomial  $\Delta$ . Due to the linearity of  $\Delta$  with respect to its coefficients  $a$ ,  $\alpha$  and  $\beta$ , one reduces the system  $\Delta(s_1) = \Delta(s_2) = \Delta(s_3) = 0$  to the linear system

$$A_{\tau,3}(s_1, s_2, s_3)X = B, \quad (2.7)$$

where  $B = -(s_1 e^{\tau s_1}, s_2 e^{\tau s_2}, s_3 e^{\tau s_3})^t$ ,  $X = (\alpha, \beta, a)^t$  and

$$A_{\tau,3}(s_1, s_2, s_3) = \begin{bmatrix} s_1 & 1 & e^{\tau s_1} \\ s_2 & 1 & e^{\tau s_2} \\ s_3 & 1 & e^{\tau s_3} \end{bmatrix}. \quad (2.8)$$

Using [BBN20, Theorem 2], one immediately obtains that the determinant of the structured functional Vandermonde-type matrix  $A_{\tau,3}(s_1, s_2, s_3)$  is given by

$$D_{\tau,3}(s_1, s_2, s_3) = \tau^2(s_1 - s_2)(s_1 - s_3)(s_2 - s_3)F_{-\tau,2}(s_1, s_2, s_3), \quad (2.9)$$

where  $F_{-\tau,2}(s_1, s_2, s_3)$  is defined by

$$F_{-\tau,2}(s_1, s_2, s_3) := \int_0^1 \int_0^1 (1-t_1)e^{\tau(t_1 s_1 + (1-t_1)(t_2 s_2 + (1-t_2)s_3))} dt_1 dt_2 > 0. \quad (2.10)$$

By integrating (2.10), one may carefully check that

$$F_{-\tau,2}(s_1, s_2, s_3) = \frac{e^{\tau s_3}(s_1 - s_2) + e^{\tau s_2}(s_3 - s_1) + e^{\tau s_1}(s_2 - s_3)}{\tau^2(s_1 - s_3)(s_2 - s_3)(s_1 - s_2)}. \quad (2.11)$$

For distinct real spectral values  $s_3 < s_2 < s_1$ , one deduces from (2.9) and (2.10) that for every  $\tau > 0$ ,  $D_{\tau,3}(s_1, s_2, s_3) > 0$  holds. It follows that (2.7) is a Cramer system, and one can immediately compute the coefficients  $\alpha$ ,  $\beta$  and  $a$  using the Cramer formula. More precisely, one has the following.

**Lemma 2.4.** *For a fixed  $\tau > 0$ , the quasipolynomial  $\Delta$  admits three distinct real spectral values  $s_3$ ,  $s_2$  and  $s_1$  if, and only if, the coefficients  $\alpha$ ,  $\beta$  and  $a$  are respectively given by*

$$\beta(\tau) = \frac{1}{D_{\tau,3}(s_1, s_2, s_3)} \det \begin{bmatrix} s_1 & -s_1 e^{\tau s_1} & e^{\tau s_1} \\ s_2 & -s_2 e^{\tau s_2} & e^{\tau s_2} \\ s_3 & -s_3 e^{\tau s_3} & e^{\tau s_3} \end{bmatrix}, \quad (2.12)$$

$$\alpha(\tau) = \frac{1}{D_{\tau,3}(s_1, s_2, s_3)} \det \begin{bmatrix} -s_1 e^{\tau s_1} & 1 & e^{\tau s_1} \\ -s_2 e^{\tau s_2} & 1 & e^{\tau s_2} \\ -s_3 e^{\tau s_3} & 1 & e^{\tau s_3} \end{bmatrix} = \frac{F_{-\tau,2}(s_1 + s_2, s_1 + s_3, s_2 + s_3)}{F_{-\tau,2}(s_1, s_2, s_3)}, \quad (2.13)$$

$$a(\tau) = \frac{1}{D_{\tau,3}(s_1, s_2, s_3)} \det \begin{bmatrix} s_1 & 1 & -s_1 e^{\tau s_1} \\ s_2 & 1 & -s_2 e^{\tau s_2} \\ s_3 & 1 & -s_3 e^{\tau s_3} \end{bmatrix} = -s_1 - \frac{F_{-\tau,1}(s_2, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)}, \quad (2.14)$$

$$= -s_2 - \frac{F_{-\tau,1}(s_1, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)}, \quad (2.15)$$

where

$$F_{-\tau,1}(u, v) := \int_0^1 e^{\tau(tu + (1-t)v)} dt = \frac{e^{\tau u} - e^{\tau v}}{\tau(u - v)} > 0 \quad (\forall u, v \in \mathbb{R}). \quad (2.16)$$



*Proof.* Relation (2.12) and the first identities in relations (2.13)-(2.15) follow directly by applying the Cramer formulas to the Cramer system (2.7). To obtain the second ones in (2.13)-(2.15), one may use (2.9), (2.11), (2.16) and the following.

$$\begin{aligned} \det \begin{bmatrix} -s_1 e^{\tau s_1} & 1 & e^{\tau s_1} \\ -s_2 e^{\tau s_2} & 1 & e^{\tau s_2} \\ -s_3 e^{\tau s_3} & 1 & e^{\tau s_3} \end{bmatrix} &= e^{\tau(s_1+s_2)}(s_1 - s_2) + e^{\tau(s_1+s_3)}(s_3 - s_1) + e^{\tau(s_2+s_3)}(s_2 - s_3) \\ &= \tau^2(s_1 - s_2)(s_1 - s_3)(s_2 - s_3)F_{-\tau,2}(s_1 + s_2, s_1 + s_3, s_2 + s_3) \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} \det \begin{bmatrix} s_1 & 1 & -s_1 e^{\tau s_1} \\ s_2 & 1 & -s_2 e^{\tau s_2} \\ s_3 & 1 & -s_3 e^{\tau s_3} \end{bmatrix} &= e^{\tau s_3}(s_2 - s_1)s_3 + e^{\tau s_2}(s_1 - s_3)s_2 + e^{\tau s_1}(s_3 - s_2)s_1 \\ &= -s_1 D_{\tau,3}(s_1, s_2, s_3) - (e^{\tau s_2} - e^{\tau s_3})(s_1 - s_2)(s_1 - s_3), \\ &= -s_2 D_{\tau,3}(s_1, s_2, s_3) - (e^{\tau s_1} - e^{\tau s_3})(s_1 - s_2)(s_2 - s_3), \end{aligned}$$

hence completing the proof of the lemma.  $\square$

The next lemma is a key ingredient in simplifying the proofs of our main results in Section 3.1.

**Lemma 2.5.** *Let  $\tau > 0$ . Assume that the quasipolynomial  $\Delta$  has three real spectral values  $s_3 < s_2 < s_1$ . Then, the following holds*

$$0 < \alpha < 1, \quad (2.18)$$

$$\alpha < e^{\tau x} \quad (\forall x \geq s_1). \quad (2.19)$$

*Proof.* It follows from (2.13) and (2.10) that  $\alpha > 0$ . Then, one has

$$1 - \alpha = \frac{F_{-\tau,2}(s_1, s_2, s_3) - F_{-\tau,2}(s_1 + s_2, s_2 + s_3, s_1 + s_3)}{F_{-\tau,2}(s_2, s_3, s_1)} > 0, \quad (2.20)$$

since for every  $(t_1, t_2) \in [0, 1]^2$ , the following holds

$$\begin{aligned} t_1 s_1 - (1 - t_1)(t_2 s_2 - (1 - t_2)s_3) - t_1(s_1 + s_2) - (1 - t_1)(t_2(s_2 + s_3) - (1 - t_2)(s_1 + s_3)) = \\ -t_1 s_2 - t_2(1 - t_1)s_3 - (1 - t_1)(1 - t_2)s_1 > 0. \end{aligned}$$

Now, let  $x \geq s_1$ , then one has

$$e^{\tau x} - \alpha = \frac{e^{\tau x} F_{-\tau,2}(s_2, s_3, s_1) - F_{-\tau,2}(s_1 + s_2, s_2 + s_3, s_1 + s_3)}{F_{-\tau,2}(s_2, s_3, s_1)} > 0, \quad (2.21)$$

since  $F_{-\tau,2}(s_1, s_2, s_3) = F_{-\tau,2}(s_2, s_3, s_1)$  by [BBN20, Lemma 4] and the fact that for every  $(t_1, t_2) \in [0, 1]^2$ , one has

$$\begin{aligned} x + t_1 s_2 + (1 - t_1)(t_2 s_3 + (1 - t_2)s_1) - t_1(s_1 + s_2) - (1 - t_1)(t_2(s_2 + s_3) - (1 - t_2)(s_1 + s_3)) = \\ x - t_1 s_1 - t_2(1 - t_1)s_2 - (1 - t_1)(1 - t_2)s_3 > \\ (1 - t_1)s_1 - t_2(1 - t_1)s_1 - (1 - t_1)(1 - t_2)s_1 = 0. \end{aligned}$$

$\square$

**Remark 2.6.** Note, from Lemmas 2.4 and 2.5, that, if  $s_1 < 0$  then for every  $\tau > 0$ , one has  $b(\tau) = -\alpha s_1 + \zeta e^{\tau s_1} > 0$  and  $a(\tau)$  may change sign. Here,

$$\zeta := \zeta(\tau) = \frac{F_{-\tau,1}(s_2, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)} > 0. \quad (2.22)$$

**Remark 2.7.** It follows from Lemma 2.5, by considering the bound (2.18) in particular, that the coexistence of three real roots for the quasipolynomial  $\Delta$  ensures the strong stability of the difference operator  $D$  [HL13, Section 9.6, page 284], defined for every  $\phi \in C([-\tau, 0])$  by

$$D\phi = \phi(0) + \alpha\phi(-\tau). \quad (2.23)$$

Similarly, Lemma 2.5, and in particular property (2.19), shows that assigning three real spectral values  $s_3 < s_2 < s_1$  to the quasipolynomial  $\Delta$  entails that the real part of spectral values of the difference equation  $y(t) + \alpha y(t - \tau) = 0$  are strictly less than  $s_1$ .

### 3. MAIN RESULTS

In this section, we establish our main results. Section 3.1 discusses the GCRRID property for the quasipolynomial  $\Delta$  and derives a new and simpler proof for its validity. In Section 3.2, the ICRRID property is investigated. Next, Section 3.3 provides an example of a simple dominant spectral value violating the sufficient condition established by Frasson-Verduyn Lunel, thereby emphasizing the non-necessary nature of the latter condition. Additionally, we will present a simpler proof of the exponential estimates for solutions of equation (NDE) in Section 3.4.

**3.1. Assigning three distinct real spectral values.** This section discusses the GCRRID property for the quasipolynomial  $\Delta$ . This property involves assigning the maximal number of distinct real spectral values  $s_3 < s_2 < s_1$  to  $\Delta$  and proving that  $s_1$ , the rightmost, is the dominant root. We shall then use this result to establish necessary and sufficient conditions to ensure that  $s_1$  is negative, which is essential to the exponential stability of equation (NDE). Finally, we shall fully characterize the remaining spectrum of  $\Delta$  by assigning three equidistributed real spectral values.

The first main result of this section provides a more straightforward and comprehensive proof of [SBN23, Theorem 5].

**Theorem 3.1** (Dominancy of a real root). *Assume that  $\Delta$  admits three real spectral values  $s_3 < s_2 < s_1$ . Then, the spectral value  $s_1$  is a strictly dominant root of  $\Delta$ .*

*Proof.* Fix  $\tau > 0$ . It follows from Lemma 2.4 that  $a + s_1 < 0$  and  $\alpha > 0$ . We argue by contradiction. Assume that there exists  $s_0 := x + i\omega \in \mathbb{C}$  such that  $\Delta(s_0) = 0$  and  $x \geq s_1$ . In particular,  $\omega \neq 0$ , since we have already assigned three real spectral values  $s_3 < s_2 < s_1$  to  $\Delta$ . From  $\Delta(s_1) = 0$ , one deduces that  $\beta = -\alpha s_1 - e^{\tau s_1}(a + s_1)$ . It follows that  $\Delta(s_0) = 0$  if, and only if,

$$e^{\tau s_0} s_0 + e^{\tau s_0} a + \alpha s_0 = \alpha s_1 + e^{\tau s_1}(a + s_1). \quad (3.1)$$

By taking the real and imaginary parts of both sides in (3.1), one gets

$$\begin{aligned} (a + x) \cos(\tau\omega) - \omega \sin(\tau\omega) &= e^{-\tau(x-s_1)}(a + s_1) - \alpha(x - s_1)e^{-\tau x}, \\ (a + x) \sin(\tau\omega) + \omega \cos(\tau\omega) &= -\alpha\omega e^{-\tau x}. \end{aligned} \quad (3.2)$$

By squaring each equality in (3.2) and adding them, one obtains

$$\omega^2 = \frac{(e^{\tau s_1}(a + s_1) - \alpha(x - s_1))^2 - (a + x)^2 e^{2\tau x}}{e^{2\tau x} - \alpha^2}, \quad (3.3)$$

which is well defined for every  $x \geq s_1$  by Lemma 2.5. Let us prove that  $\omega^2$  given by (3.3) satisfies  $\omega^2 < 1/\tau^2$ . To do so, define the function

$$\chi(x) = (e^{\tau s_1}(a + s_1) - \alpha(x - s_1))^2 - (a + x)^2 e^{2\tau x} - \tau^{-2}(e^{2\tau x} - \alpha^2) \quad (\forall x \geq s_1). \quad (3.4)$$

We want to prove that  $\chi(x) < 0$  for every  $x \geq s_1$ . On the one hand, one has

$$\chi(s_1) = -(e^{2\tau s_1} - \alpha^2)\tau^{-2} < 0, \quad \lim_{x \rightarrow \infty} \chi(x) = -\infty, \quad (3.5)$$

owing to Lemma 2.5. On the other hand, function  $\chi$  is infinitely derivable on  $[s_1, \infty)$ , so that

$$\chi''(x) = -2e^{2\tau x} \left( 2(\tau(a + x) + 1)^2 + e^{-2\tau x}(e^{2\tau x} - \alpha^2) \right) < 0 \quad (\forall x \geq s_1), \quad (3.6)$$

since the two terms between the big brackets of  $\chi''(x)$  are positive by Lemma 2.5. One also has

$$\begin{aligned} \chi'(x) &= -2\alpha((a + s_1)e^{\tau s_1} - \alpha(x - s_1)) - 2\tau(a + x)^2 e^{2\tau x} - 2(a + x)e^{2\tau x} - 2\tau^{-1}e^{2\tau x} \quad (\forall x \geq s_1), \\ \chi'(s_1) &= -2\tau \left( (a + s_1)e^{\tau s_1} + \frac{\alpha + e^{\tau s_1}}{2\tau} \right)^2 - \frac{(e^{\tau s_1} - \alpha)(\alpha + 3e^{\tau s_1})}{2\tau} < 0, \quad \lim_{x \rightarrow \infty} \chi'(x) = -\infty. \end{aligned} \quad (3.7)$$

Here,  $\chi'(s_1) < 0$ , as the sum of two negative terms by Lemma 2.5. It follows from (3.6) and (3.7) that

$$\chi'(x) < 0 \quad (\forall x \geq s_1). \quad (3.8)$$

Finally, by combining (3.8) and (3.5), one gets that

$$\chi(x) < 0 \quad (\forall x \geq s_1). \quad (3.9)$$

The latter is equivalent to  $\omega^2 < \tau^{-2}$  with  $\omega^2$  defined in (3.3). So, we have shown that if  $x + i\omega$  is a root of  $\Delta$  with  $x \geq s_1$  and  $\omega \neq 0$ , then  $|\omega| < \tau^{-1}$ . The latter is inconsistent since one has necessarily  $|\omega| \geq 2\pi/\tau$  owing to Lemma 2.3.  $\square$

**Remark 3.2.** It is worth noting that the key property on  $\alpha$  required in the proof of Theorem 3.1 is that

$$e^{2\tau x} - \alpha^2 \geq 0 \quad (\forall x \geq s_1). \quad (3.10)$$

which is satisfied owing to Lemma 2.5.

The second main result provides the necessary and sufficient conditions on the delay  $\tau$  and the coefficient  $a$  to guarantee that the dominant root  $s_1$  is negative when three spectral values  $s_3 < s_2 < s_1$  are formally assigned to  $\Delta$ .

**Theorem 3.3** (Negativity of the dominant root). *Assume that  $\Delta$  admits three real spectral values  $s_3 < s_2 < s_1$ . Then, the spectral value  $s_1$  is negative if, and only if, there exists a unique  $\tau_* > 0$  such that  $a(\tau_*) = 0$ , where*

$$a(\tau) \begin{cases} = 0 & \text{if } \tau = \tau_*, \\ < 0 & \text{if } \tau < \tau_*, \\ > 0 & \text{if } \tau > \tau_*. \end{cases} \quad (3.11)$$

Moreover,  $s_1 < 0$  can be designed as follows

$$s_1 \begin{cases} = -\zeta(\tau) & \text{if } \tau = \tau_*, \\ > -\zeta(\tau) & \text{if } \tau < \tau_*, \\ < -\zeta(\tau) & \text{if } \tau > \tau_*, \end{cases} \quad \text{where } \zeta(\tau) = \frac{F_{-\tau,1}(s_2, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)} > 0. \quad (3.12)$$

*Proof.* Let us assume that  $s_1 < 0$ . Since  $\tau \in (0, \infty) \mapsto a(\tau)$  is continuous with respect to  $\tau$ , one may use the intermediate value theorem. Combining (2.14) and (2.11), one gets

$$a(\tau) = \frac{e^{-\tau(s_1-s_3)}(s_1-s_2)s_3 + e^{-\tau(s_1-s_2)}(s_3-s_1)s_2 + (s_2-s_3)s_1}{e^{-\tau(s_1-s_3)}(s_2-s_1) + e^{-\tau(s_1-s_2)}(s_1-s_3) + (s_3-s_2)}. \quad (3.13)$$

Consequently,  $a(\tau) \rightarrow -s_1 > 0$  when  $\tau \rightarrow \infty$ . Next, definitions (2.10) and (2.16), and the continuity of  $F_{-\tau,1}$  and  $F_{-\tau,2}$  with respect to  $\tau$  entail that  $a(\tau) \rightarrow -\infty$  as  $\tau \rightarrow 0$ . It follows that there exists at least one  $\tau_* > 0$  such that  $a(\tau_*) = 0$ . To show that  $\tau_* > 0$  is unique, one applies Lemma 2.2. Firstly, we observe that  $a(\tau) = 0$  if, and only if, its numerator vanishes at  $\tau$ . Now, let

$$F(\tau) = e^{\tau s_3}(s_1-s_2)s_3 + e^{\tau s_2}(s_3-s_1)s_2 + e^{\tau s_1}(s_2-s_3)s_1 \quad (3.14)$$

be the numerator of  $e^{\tau s_1}a(\tau)$ . Then,  $F$  is obviously analytic in  $\tau$ , and one has from  $s_3 < s_2 < s_1$  that  $(s_1-s_2)s_3 < 0$ ,  $(s_3-s_1)s_2 > 0$  and  $(s_2-s_3)s_1 < 0$ . Let  $C$  denote the number of sign changes in the sequence of real numbers  $(s_1-s_2)s_3$ ,  $(s_3-s_1)s_2$ , and  $(s_2-s_3)s_1$ , then  $C = 2$ . Similarly, if  $Z$  represents the number of real zeros of the entire function  $F$ , then  $C - Z = 2 - Z \geq 0$  and  $2 - Z$  is even according to Lemma 2.2, so that  $Z = 2$ . Since  $F(0) = 0$  and  $F(\tau_*) = 0$ , the uniqueness of  $\tau_* > 0$  follows.

Conversely, assume the existence of a unique  $\tau_* > 0$  such that  $a(\tau_*) = 0$ . One immediately infers from (2.10), (2.14) and (2.16) that

$$s_1 = -\frac{F_{-\tau_*,1}(s_2, s_3)}{\tau_* F_{-\tau_*,2}(s_1, s_2, s_3)} < 0, \quad (3.15)$$

completing the proof of the theorem.  $\square$

**Remark 3.4.** Observe that Theorem 3.3 states that exponential stability of equation (NDE) may be achieved with a prescribed decay rate, even though  $a \leq 0$ . Time-domain techniques based on a Lyapunov functional and LMI do not cover this. See, for instance, [LYH00, HL13, Fri01].

In the case of equidistributed real spectral values  $s_3 < s_2 < s_1$ , the delay  $\tau_* > 0$  that enables the design of  $s_1 < 0$  may be explicitly computed, as is stated hereafter.

**Corollary 3.5.** *Assume that the quasipolynomial  $\Delta$  admits three equidistributed real spectral values  $s_k = s_1 - (k-1)d$ , for  $d > 0$  and  $k = 1, 2, 3$ . Then,  $s_1$  is negative if, and only if, the delay  $\tau_* > 0$  in Theorem 3.1 is given by*

$$\tau_* = \frac{1}{d} \ln \left( \frac{s_3}{s_1} \right) > 0. \quad (3.16)$$

*Proof.* Owing to Lemma 3.1,  $s_1$  is negative if, and only if, there exists one, and only one,  $\tau_* > 0$  such that  $a(\tau_*) = 0$ , and

$$s_1 = -\frac{F_{-\tau_*,1}(s_2, s_3)}{\tau_* F_{-\tau_*,2}(s_1, s_2, s_3)}. \quad (3.17)$$

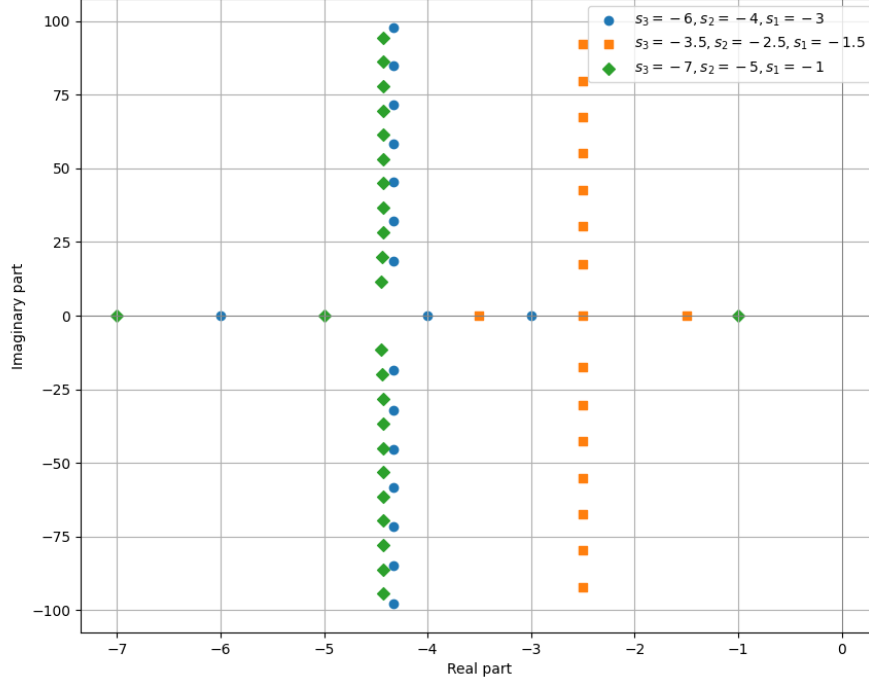


FIGURE 1. Spectrum of the quasipolynomial  $\Delta$  for various parameters. The spectrum distribution is obtained by assigning three real spectral values  $s_3 < s_2 < s_1$  to  $\Delta$  and computing the system parameters  $a$ ,  $\beta$ ,  $\alpha$  and  $\tau$  in each case. For instance, the spectrum distribution in *orange* is obtained by assigning three equidistributed real spectral values  $s_3 = -3.5$ ,  $s_2 = -2.5$  and  $s_1 = -1.5$ . As expected (Theorem 3.6), the remaining spectrum belongs to the vertical line with abscissa  $s_2 = -2.5$ .

By taking now  $s_2 = s_1 - d$  and  $s_3 = s_1 - 2d$  with  $d > 0$ , one finds after careful computations, considering (2.11) and (2.16), that

$$F_{-\tau_*,2}(s_1, s_2, s_3) = \frac{e^{\tau_* s_1} (1 - e^{-\tau_* d})^2}{2\tau_*^2 d^2} \quad \text{and} \quad F_{-\tau_*,1}(s_2, s_3) = \frac{e^{\tau_* s_1} e^{-\tau_* d} (1 - e^{-\tau_* d})}{\tau_* d}. \quad (3.18)$$

By injecting (3.18) into (3.17), one gets

$$s_1 = \frac{-2d}{e^{\tau_* d} - 1}, \quad \text{which is equivalent to} \quad \tau_* = \frac{1}{d} \ln \left( \frac{s_3}{s_1} \right) > 0, \quad (3.19)$$

since  $s_3 < s_1 < 0$ . □

In the specific case where a single real spectral value  $s_3 = s_2 = s_1$  is assigned to  $\Delta$ , the GMID is demonstrated in [BMN22]. Furthermore, the analysis fully characterizes the remaining

spectrum of  $\Delta$  as

$$s = s_1 + \frac{\omega}{\tau}i, \quad \text{where} \quad \tan\left(\frac{\omega}{2}\right) = \frac{\omega}{2}. \quad (3.20)$$

In the same fashion, assuming that the quasipolynomial  $\Delta$  has three equidistributed real spectral values, we also fully characterize the remaining spectrum of  $\Delta$ .

**Theorem 3.6.** *Assume that the quasipolynomial  $\Delta$  admits three equidistributed real spectral values  $s_k = s_1 - (k-1)d$ , for  $d > 0$  and  $k = 1, 2, 3$ . Then, the remaining spectrum of  $\Delta$  is given by*

$$s = s_2 + \frac{\omega}{\tau}i, \quad \text{where} \quad \tan\left(\frac{\omega}{2}\right) = \frac{\omega}{\xi(d)}, \quad \xi(d) = \tau d \coth\left(\frac{\tau d}{2}\right). \quad (3.21)$$

Here  $\coth$  is the cotangent hyperbolic function.

*Proof.* First of all, direct computations yield

$$\alpha = e^{\tau(s_1-d)} = e^{\tau s_2}, \quad a = -s_2 - \eta, \quad \beta = e^{\tau s_2}(-s_2 + \eta), \quad \eta := \eta(d) = d \coth\left(\frac{\tau d}{2}\right), \quad (3.22)$$

owing to (2.13), (2.15) and (2.12). Let  $s_0 = x_0 + i\omega_0$  be a complex spectral value of  $\Delta$ . Then,  $x_0 < s_1$  and  $\omega_0 \neq 0$ , in particular. From  $\Delta(s_0) = 0$  and (3.22), one obtains

$$\frac{s_0 - s_2 - \eta}{s_0 - s_2 + \eta} = -e^{-\tau(s_0 - s_2)}. \quad (3.23)$$

Taking the magnitude of the above identity leads to

$$\sqrt{\frac{(x_0 - s_2 - \eta)^2 + \omega_0^2}{(x_0 - s_2 + \eta)^2 + \omega_0^2}} = e^{\tau(s_2 - x_0)},$$

which holds for every  $|\omega_0| \geq 2\pi/\tau$  by Lemma 2.3. By letting  $|\omega_0| \rightarrow \infty$ , one finds

$$1 = e^{\tau(s_2 - x_0)} \quad (\forall \tau > 0). \quad (3.24)$$

It follows that  $x_0 = s_2$  and that the rest of the spectrum is either located on the vertical line  $\Re(s) = s_2$  or forms a chain asymptotic to  $\Re(s) = s_2$ . Since  $\Delta(s) = (s - s_2) - \eta + e^{-\tau(s - s_2)}(s - s_2 + \eta)$ , we can derive  $\Delta(2s_2 - \bar{s}_0) = -e^{\tau(\bar{s}_0 - s_2)}\Delta(\bar{s}_0)$ . Additionally, since  $\Delta$  has real coefficients, a complex number  $s \in \mathbb{C}$  is a spectral value of  $\Delta$  if, and only if, its complex conjugate  $\bar{s}$  is also a spectral value. Therefore, the complex number  $s_0$  is a spectral value of  $\Delta$  if, and only if,  $2s_2 - \bar{s}_0$ , the reflection of  $s_0$  across the vertical line  $\Re(s) = s_2$ , is a spectral value of  $\Delta$ . As a result, the remaining spectrum of  $\Delta$  exists along the vertical line  $\Re(s) = s_2$ .

Finally, by substituting  $s = s_2 + i\omega_0$  into (3.23), one gets

$$\frac{i\omega_0 - \eta}{i\omega_0 + \eta} = -e^{-i\tau\omega_0} \iff i\frac{\omega_0}{\eta} = \frac{1 - e^{-i\tau\omega_0}}{1 + e^{-i\tau\omega_0}} = i \tan\left(\frac{\tau\omega_0}{2}\right) \quad (3.25)$$

which leads to the desired result via the change of variables  $\omega = \tau\omega_0$ .  $\square$

**Remark 3.7.** Since  $\xi(d) \rightarrow 2$  as  $d \rightarrow 0$ , Theorem 3.6 confirms the intuitive concept that the GMID is the limiting case of the GCRRID.

We refer the reader to Figure 1, where we depicted the spectrum of quasipolynomial  $\Delta$  for various parameters.

**Remark 3.8.** In Theorem 3.6, it is proved that in the equidistributed case, the remaining spectrum of  $\Delta$  lies on the vertical line  $\Re(s) = s_2$ . However, it should be noted that if the real spectral values  $s_3 < s_2 < s_1$  are not equidistributed, the remaining spectrum of  $\Delta$  does not perfectly align on a vertical line, though this may not be immediately obvious visually (see Figure 1 for green and blue spectrum distributions). In fact, there are imperfections in the symmetry of the spectrum with respect to a vertical line. Let us illustrate it in the following.

**Theorem 3.9.** *Let  $\tau > 0$ . Assume that the quasipolynomial  $\Delta$  admits three non-equidistributed real spectral values  $s_3 < s_2 < s_1$ . The remaining spectrum of  $\Delta$  forms a chain asymptotic to the vertical line*

$$\Re(s) = s_2 + \frac{1}{\tau} \ln(\theta(\tau, d, \delta)), \quad \text{with} \quad \theta(\tau, d, \delta) = \frac{e^{\tau d}(d(1 - e^{\tau \delta}) - \delta(1 - e^{\tau d}))}{-de^{\tau d} + e^{\tau \delta}(e^{\tau d}(d - \delta) + \delta)}. \quad (3.26)$$

Here  $d := s_1 - s_2$  and  $\delta := s_1 - s_3$ . Moreover, there exists a complex spectral value  $s_0$  such that

$$\Delta(s_0) = 0 \quad \implies \quad \Re(s_0) \neq s_2 + \frac{1}{\tau} \ln(\theta(\tau, s_1, s_2, s_3)). \quad (3.27)$$

*Proof.* We start by noticing that direct computations yield

$$\alpha e^{-\tau s_2} = \frac{e^{\tau d}(d(e^{\tau \delta} - 1) - \delta(e^{\tau d} - 1))}{de^{\tau d} + e^{\tau \delta}(e^{\tau d}(\delta - d) - \delta)} = \theta(\tau, d, \delta) := \theta, \quad (3.28)$$

$$a = s_2 - \frac{\xi}{\tau} \quad \text{where} \quad \xi := \xi(\tau, d, \delta) = \frac{de^{\tau d}(e^{\tau d} - 1)(\delta - d)\tau}{de^{\tau d} + e^{\tau \delta}(e^{\tau d}(\delta - d) - \delta)}, \quad (3.29)$$

$$\beta = -\alpha s_2 - e^{\tau s_2}(a + s_2) = -\alpha s_2 + \frac{\xi}{\tau} e^{\tau s_2} \quad (3.30)$$

thanks to (2.13), (2.15) and (2.12). Here  $d = s_1 - s_2$  and  $\delta = s_1 - s_3$ . It follows from  $\Delta(s) = a + s + e^{-\tau s}(\alpha s + \beta)$  that

$$\Delta(s) = (s - s_2) - \frac{\xi}{\tau} + e^{-\tau(s-s_2)} \left( \theta(s - s_2) + \frac{\xi}{\tau} \right) \quad (s \in \mathbb{C}). \quad (3.31)$$

Consider the change of functions

$$\Delta'(z) = \tau \Delta \left( s_2 + \frac{z}{\tau} \right) = z - \xi + e^{-z}(\theta z + \xi) \quad (z \in \mathbb{C}) \quad (3.32)$$

so that  $\Delta'$  has three real roots  $-\tau(\delta - d)$ , 0 and  $\tau d$ . In particular,  $\Delta'(0) = 0$  if, and only if,  $\Delta(s_2) = 0$ , and  $z \in \mathbb{C}$  is a spectral value of  $\Delta'$  if, and only if,  $s_2 + z/\tau$  is a spectral value of  $\Delta$ . Moreover, since  $s_1$  is the dominant root of  $\Delta$ , then  $\tau d$  is the dominant root of  $\Delta'$ . Let now  $z_0 = x_0 + i\omega_0$  be a complex non-real spectral value of  $\Delta'$ . Then,  $x_0 < \tau d$  and  $\omega_0 \neq 0$ . In particular,  $|\omega_0| \geq 2\pi$  by Lemma 2.3. From  $\Delta'(z_0) = 0$ , one obtains

$$\frac{z - \xi}{\theta z + \xi} = -e^{-z} \quad \implies \quad \sqrt{\frac{(x_0 - \xi)^2 + \omega_0^2}{(\theta x_0 + \xi)^2 + \omega_0^2}} = e^{-x_0}. \quad (3.33)$$

Letting  $|\omega_0| \rightarrow \infty$  in (3.33), one deduces that

$$e^{-x_0} = \frac{1}{\theta}, \quad \text{since} \quad 0 < \theta < e^{\tau(s_1 - s_2)} = e^{\tau d}. \quad (3.34)$$

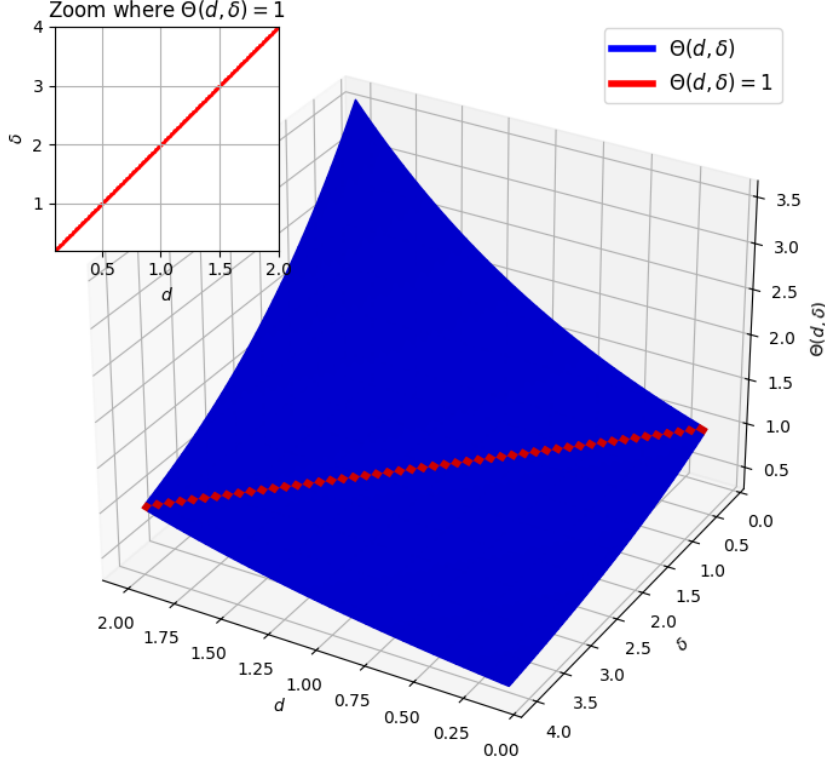


FIGURE 2. Plot of the function  $\Theta(d, \delta)$  defined in (3.38) for  $(d, \delta) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ . The red straight line characterises points  $(d, \delta)$  where  $\Theta(d, \delta) = 1$ . We magnified the red curve in the inset on the top-left, which suggests that it corresponds to the straight line  $\delta = 2d$ .

It follows that  $x_0 = \ln(\theta)$ , and the rest of the spectrum of  $\Delta'$  is either located on the vertical line  $\Re(z) = \ln(\theta)$  or forms a chain asymptotic to  $\Re(s) = \ln(\theta)$ . Let us show that the former is inconsistent; that is, there exists at least a complex spectral value  $z_0 \in \mathbb{C}$  of  $\Delta'$  such that  $\Re(z_0) \neq \ln(\theta)$ . Otherwise, for every  $z_0 = x_0 + i\omega_0 \in \mathbb{C}$  ( $x_0 < \tau d$ ,  $\omega_0 \neq 0$  and  $|\omega_0| \geq 2\pi$ )  $\Delta'(z_0) = 0$  and  $x_0 = \ln(\theta)$ . Firstly, in all generality, for any complex number  $z \in \mathbb{C}$ , the reflection  $2\ln(\theta) - \bar{z}$  of  $z$  across the vertical line  $\Re(z) = \ln(\theta)$  satisfies

$$\Delta'(2\ln(\theta) - \bar{z}) = -\bar{z} + \ln(\theta^2) - \xi + e^{\bar{z}} \left( -\frac{\bar{z}}{\theta} + \frac{\ln(\theta^2)}{\theta} + \frac{\xi}{\theta^2} \right). \quad (3.35)$$

Secondly, for  $z = z_0 = \ln(\theta) + i\omega_0$  ( $\omega_0 \neq 0$  and  $|\omega_0| \geq 2\pi$ ), one has  $2\ln(\theta) - \bar{z}_0 = z_0$  so that owing to (3.35) and (3.32), one gets

$$z_0 - \xi + e^{\bar{z}_0} \left( \frac{\bar{z}_0}{\theta} + \frac{\xi}{\theta^2} \right) = z_0 - \xi + e^{-z_0} (\theta z_0 + \xi). \quad (3.36)$$



which yields  $\overline{z_0} = z_0$ . It follows that  $z_0$  is a real spectral value of  $\Delta'$ , which is inconsistent.  $\square$

**Remark 3.10.** From the definition of  $\theta(\tau, d, \delta)$  given by (3.26), one can immediately check that

$$\theta(\tau, d, \delta) \rightarrow 1 \quad \text{as } \delta \rightarrow 2d \quad (3.37)$$

indicating that we retrieve Theorem 3.6 in the equidistributed scenario. Additionally, we notice that  $\theta$  seems to rely on three distinct parameters  $\tau > 0$ ,  $d > 0$ , and  $\delta > d$ . However, upon closer examination, it is obvious that it depends solely on the products  $\tau d$  and  $\tau \delta$  since we have

$$\Theta(\tau d, \tau \delta) := \theta\left(\tau, \frac{d}{\tau}, \frac{\delta}{\tau}\right) = \frac{e^d(d(1 - e^\delta) - \delta(1 - e^d))}{-de^d + e^\delta(e^d(d - \delta) + \delta)}. \quad (3.38)$$

As a result,  $\tau > 0$  may be treated as a scaling parameter. We have plotted  $\Theta(d, \delta)$  (i.e.,  $\tau = 1$ ) in Figure 2, illustrating that  $\Theta(d, \delta) = 1$  only when  $\delta = 2d$ . This suggests that when three real spectral values coexist in the spectrum of  $\Delta$ , the remaining spectrum exists along a vertical line only if these real spectral values are equidistributed.

While we investigated in this section the case of coexistence of three real spectral values, Section 3.2 will provide necessary and sufficient conditions guaranteeing that the coexistence of two real spectral values  $s_2 < s_1$  for the quasipolynomial  $\Delta$  entails that  $s_1$  is the dominant root.

**3.2. Assigning two distinct real spectral values.** This section discusses the ICRRID property of the quasipolynomial  $\Delta$ . The aim is to assign two real spectral values  $s_2 < s_1$  to the quasipolynomial  $\Delta$  and to find necessary and sufficient conditions on the coefficients of  $\Delta$  guaranteeing that  $s_1$  is a dominant root of  $\Delta$ .

Inspired from [SBN23, Theorem 6], we find these conditions in terms of the quotient  $(a + s_1)/(s_1 - s_2)$ . The challenge in this task relies on the scenario where a third real spectral value  $x$  exists in the spectrum of  $\Delta$  in which case the coefficients  $a$ ,  $\alpha$ , and  $\beta$  are given in terms of  $x$ ,  $s_2$ ,  $s_1$  and  $\tau$  as in Lemma 2.4. Moreover, the greatest between  $s_2$ ,  $s_1$  and  $x$  is the dominant root by Theorem 3.1. Since one wants  $s_1$  to be the dominant root of  $\Delta$ , we are looking for the necessary and sufficient conditions on  $(a + s_1)/(s_1 - s_2)$  ensuring that  $x < s_2 < s_1$  or  $s_2 < x < s_1$ . On the one hand, one deduces from (2.14) that

$$\frac{a + s_1}{s_1 - s_2} = -\frac{(s_1 - x)(e^{\tau s_2} - e^{\tau x})}{e^{\tau x}(s_1 - s_2) + e^{\tau s_2}(x - s_1) + e^{\tau s_1}(s_2 - x)}, \quad (\forall x \in \mathbb{R} \setminus \{s_1, s_2\}) \quad (3.39)$$

which is negative owing to (2.10) and (2.11). On the other hand, introduce the function

$$\varphi(x) = \frac{(s_1 - x)(e^{\tau s_2} - e^{\tau x})}{e^{\tau x}(s_1 - s_2) + e^{\tau s_2}(x - s_1) + e^{\tau s_1}(s_2 - x)}, \quad (\forall x \in \mathbb{R} \setminus \{s_1, s_2\}). \quad (3.40)$$

Then,  $\varphi(x) > 0$  for every  $x < s_2 < s_1$  or  $s_2 < x < s_1$  or  $x > s_1$ , and satisfies

$$\lim_{x \rightarrow -\infty} \varphi(x) = \frac{1}{e^{\tau(s_1 - s_2)} - 1}, \quad \lim_{x \rightarrow s_2} \varphi(x) = \frac{\tau(s_1 - s_2)}{e^{\tau(s_1 - s_2)} - (1 + \tau(s_1 - s_2))}, \quad (3.41)$$

$$\lim_{x \rightarrow s_1} \varphi(x) = \frac{e^{\tau(s_1 - s_2)} - 1}{e^{\tau(s_1 - s_2)}(\tau(s_1 - s_2) - 1) + 1}, \quad (3.42)$$

and

$$\begin{aligned}\varphi'(x) &= \frac{(e^{\tau(s_1+s_2)} - e^{\tau(s_1+x)} + e^{2\tau x} - e^{\tau(s_2+x)})(s_1 - s_2) + \tau(s_1 - x)(s_2 - x)(e^{\tau(s_2+x)} - e^{\tau(s_1+x)})}{[e^{x\tau}(-s_1 + s_2) + e^{s_2\tau}(s_1 - x) + e^{s_1\tau}(-s_2 + x)]^2} \\ &= \frac{\tau^2(s_2 - x)(s_1 - x)(s_1 - s_2)(F_{-\tau,1}(x, s_1)F_{-\tau,1}(x, s_2) - e^x F_{-\tau,1}(s_1, s_2))}{[e^{x\tau}(-s_1 + s_2) + e^{s_2\tau}(s_1 - x) + e^{s_1\tau}(-s_2 + x)]^2},\end{aligned}$$

thanks to the integral representation (2.16). Using [BBN20, Lemma 7], one finds that for all  $\tau > 0$  and  $(u, v, w) \in \mathbb{R}^3$ , the following holds

$$F_{-\tau,1}(u, v) - F_{-\tau,1}(u, w) = \tau(v - w)F_{-\tau,2}(u, v, w) > 0 \iff v > w.$$

Therefore, for a fixed  $u \in \mathbb{R}$ ,  $v \mapsto F_{-\tau,1}(u, v)$  is a positive and strictly increasing function on  $\mathbb{R}$ . One deduces that  $\varphi'(x) > 0$  for every  $x < s_2 < s_1$  or  $s_2 < x < s_1$  or  $x > s_1$ , and that  $\varphi$  is a positive increasing function for  $x < s_2 < s_1$  or  $s_2 < x < s_1$  or  $x > s_1$ . Therefore, if  $s_2 < s_1$ , then

$$\varphi(x) \geq \frac{1}{e^{\tau(s_1-s_2)} - 1}, \quad (\forall x \in \mathbb{R} \setminus \{s_1, s_2\}).$$

We proved the following first main result.

**Theorem 3.11.** *Let  $\tau > 0$ . Assume that  $\Delta$  admits two real spectral values  $s_2 < s_1$ . Then, a third real spectral value  $x$  coexists in the spectrum of  $\Delta$ , if and only if*

$$\frac{a + s_1}{s_1 - s_2} \leq \frac{-1}{e^{\tau(s_1-s_2)} - 1}. \quad (3.43)$$

Furthermore, one has

$$\frac{-\tau(s_1 - s_2)}{e^{\tau(s_1-s_2)} - (1 + \tau(s_1 - s_2))} < \frac{a + s_1}{s_1 - s_2} \leq \frac{-1}{e^{\tau(s_1-s_2)} - 1} \iff x < s_2 < s_1, \quad (3.44)$$

and

$$\frac{1 - e^{\tau(s_1-s_2)}}{e^{\tau(s_1-s_2)}(\tau(s_1 - s_2) - 1) + 1} < \frac{a + s_1}{s_1 - s_2} < \frac{-\tau(s_1 - s_2)}{e^{\tau(s_1-s_2)} - (1 + \tau(s_1 - s_2))} \iff s_2 < x < s_1. \quad (3.45)$$

As an immediate consequence of Theorem 3.11, one has the following result.

**Corollary 3.12.** *Let  $\tau > 0$ . Assume that  $\Delta$  admits two real spectral values  $s_2 < s_1$ . Then, a third real spectral value  $x$  coexists in the spectrum of  $\Delta$ , and it is the dominant root of  $\Delta$  if, and only if,*

$$\frac{a + s_1}{s_1 - s_2} \leq \frac{1 - e^{\tau(s_1-s_2)}}{e^{\tau(s_1-s_2)}(\tau(s_1 - s_2) - 1) + 1}. \quad (3.46)$$

**Remark 3.13.** It follows from (3.45) and (3.46) that  $s_1$  is the dominant real spectral value of  $\Delta$  with multiplicity equal to two if

$$\frac{a + s_1}{s_1 - s_2} = \frac{1 - e^{\tau(s_1-s_2)}}{e^{\tau(s_1-s_2)}(\tau(s_1 - s_2) - 1) + 1}.$$

This equality corresponds to a very special configuration where the ICRRID property allows the IMID property to hold.

Let us now investigate the necessary and sufficient conditions on  $(a+s_1)/(s_1-s_2)$  guaranteeing the dominance of a simple real spectral value  $s_1$  when exactly two real spectral values  $s_2 < s_1$  are assigned to the quasipolynomial  $\Delta$ . The following interpolation result is of interest when exactly two real spectral values coexist in the spectrum of  $\Delta$ .

**Lemma 3.14.** *The quasipolynomial  $\Delta$  admits exactly two distinct real spectral values  $s_2$  and  $s_1$  if, and only if  $a \in \mathbb{R}$ ,*

$$\beta = -\alpha s_1 - e^{\tau s_1}(a + s_1) = -\alpha s_2 - e^{\tau s_2}(a + s_2), \quad (3.47)$$

and

$$\alpha = \frac{-(a + s_1)e^{\tau s_1} + (a + s_2)e^{\tau s_2}}{s_1 - s_2}. \quad (3.48)$$

*Proof.* First, one has  $\Delta(s_1) = \Delta(s_2) = 0$  if, and only if,  $\beta$  is given by (3.47) and

$$\alpha(s_1 - s_2) + (a + s_1)e^{\tau s_1} - (a + s_2)e^{\tau s_2} = 0, \quad (3.49)$$

which yields the desired result.  $\square$

**Remark 3.15.** It is an immediate consequence of Theorem 3.11 that, exactly two real spectral values  $s_2 < s_1$  coexist in the spectrum of  $\Delta$ , if and only if, the following inequality holds

$$\frac{-1}{e^{\tau(s_1-s_2)} - 1} < \frac{a + s_1}{s_1 - s_2}. \quad (3.50)$$

By inspecting the proof of Theorem 3.1 when three real spectral values are assigned to  $\Delta$ , one can notice that the key point to obtain the desired result relies on the properties of the coefficient  $\alpha$  as noticed in Remark 3.2. Firstly, equation (3.3) is well-defined if, and only if,  $e^{2\tau x} - \alpha^2 \neq 0$  for every  $x \geq s_1$ . Moreover, relations (3.5), (3.6) and (3.7) are satisfied only if  $e^{2\tau x} - \alpha^2 > 0$ .

The next lemma is a key ingredient in proving our second main result.

**Lemma 3.16.** *Let  $\tau > 0$ . Assume that exactly two real spectral values  $s_2 < s_1$  coexist in the spectrum of  $\Delta$ . Then, it holds*

$$\frac{-1}{e^{\tau(s_1-s_2)} - 1} < \frac{a + s_1}{s_1 - s_2} < 1, \quad (3.51)$$

if and only if  $\alpha < 0$ , and

$$-e^{\tau x} < \alpha \quad (\forall x \geq s_1). \quad (3.52)$$

Moreover, if  $s_1 \leq 0$ , then  $\beta \leq 0$  whenever

$$0 \leq \frac{a + s_1}{s_1 - s_2} < 1. \quad (3.53)$$

*Proof.* Firstly, Direct computations from equation (3.48) yield

$$\alpha e^{-\tau s_1} = -\frac{a + s_1}{s_1 - s_2} \left(1 - e^{-\tau(s_1-s_2)}\right) - e^{-\tau(s_1-s_2)} < e^{-\tau(s_1-s_2)} - e^{-\tau(s_1-s_2)} = 0, \quad (3.54)$$

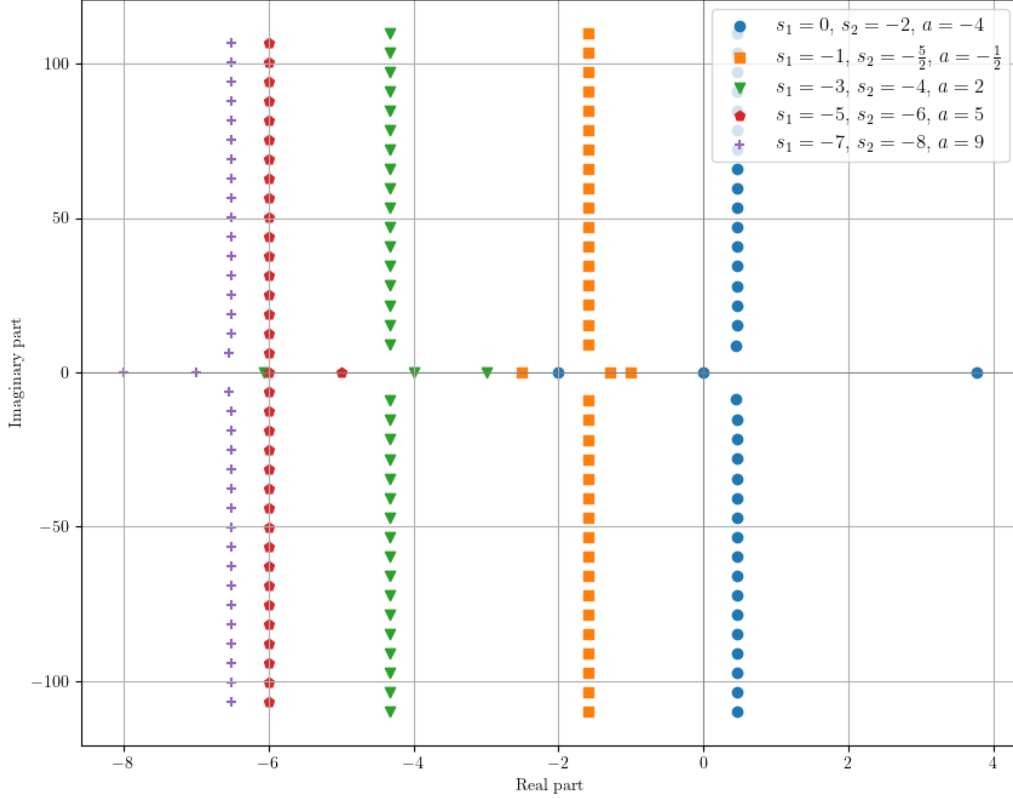


FIGURE 3. The Spectrum distribution of  $\Delta$  obtained when two simple real spectral values  $s_2 < s_1$  are assigned to  $\Delta$ . The delay  $\tau = 1$  in all cases. In *blue*,  $s_2 = -2$  and  $s_1 = 0$  are assigned to  $\Delta$ , but the spectrum distribution shows that another real root coexists, and it is the dominant root of  $\Delta$ ; in *orange*,  $s_2 = -2.5$  and  $s_1 = -1$  are assigned to  $\Delta$ , and  $s_1$  is the strictly dominant root of  $\Delta$  while another real root coexists in the spectrum between  $s_2$  and  $s_1$ ; in *green*,  $s_2 = -4$  and  $s_1 = -3$  are assigned to  $\Delta$ , and  $s_1$  is the strictly dominant root of  $\Delta$  while another real root coexists in the spectrum, and it is less than  $s_2$ ; in *red*,  $s_2 = -6$  and  $s_1 = -5$  are assigned to  $\Delta$ , and  $s_1$  is the strictly dominant root of  $\Delta$  while none real root coexists in the spectrum; finally, in *purple*,  $s_2 = -8$  and  $s_1 = -7$  are assigned to  $\Delta$ , and  $s_1$  is not a strictly dominant root of  $\Delta$ , and none real root coexists in the spectrum.

if, and only if, the first inequality in (3.50) is satisfied. Secondly, one also has for every  $x \geq s_1$ ,

$$\begin{aligned}
 e^{\tau x} + \alpha &= e^{\tau x} + \frac{-(a + s_1)e^{\tau s_1} + (a + s_2)e^{\tau s_2}}{s_1 - s_2} \geq e^{\tau s_1} + \frac{-(a + s_1)e^{\tau s_1} + (a + s_2)e^{\tau s_2}}{s_1 - s_2} \\
 &= -(a + s_2) \frac{e^{\tau s_1} - e^{\tau s_2}}{s_1 - s_2} > 0 \Leftrightarrow a + s_2 < 0.
 \end{aligned}
 \tag{3.55}$$

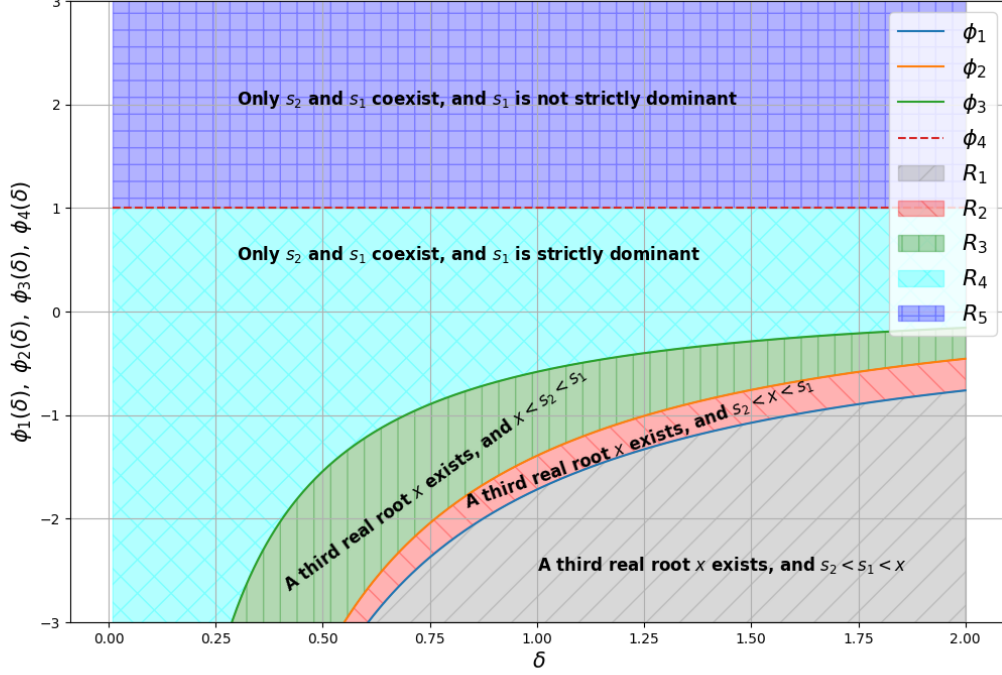


FIGURE 4. Complete characterization of the regions with respect to the values of  $(a+s_1)/\delta$  that are necessary and sufficient to ensure the strict dominance or not of the real spectral value  $s_1$  when only two real spectral values  $s_2 < s_1$  are assigned to  $\Delta$ . The figure is depicted when the delay  $\tau = 1$  while  $\delta := s_1 - s_2$  ranges between 0.01 and 2. The functions used are  $\phi_1(\delta) = (1 - e^\delta)/(e^\delta(\delta - 1) + 1)$ ,  $\phi_2(\delta) = -\delta/(e^\delta - (1 + \delta))$ ,  $\phi_3(\delta) = -1/(e^\delta - 1)$ , and  $\phi_4(\delta) = 1$ . We define the regions  $R_1$  as the range of  $\delta$  where  $\frac{a+s_1}{\delta} \leq \phi_1(\delta)$ ,  $R_2$  as the range where  $\phi_1(\delta) < \frac{a+s_1}{\delta} \leq \phi_2(\delta)$ ,  $R_3$  as the range where  $\phi_2(\delta) < \frac{a+s_1}{\delta} \leq \phi_3(\delta)$ ,  $R_4$  as the range where  $\phi_3(\delta) < \frac{a+s_1}{\delta} < \phi_4(\delta)$ , and  $R_5$  as the range where  $\frac{a+s_1}{\delta} \geq \phi_4(\delta)$ .

It follows that for every  $x \geq s_1$ ,  $e^{\tau x} + \alpha > 0$  if, and only if,

$$\frac{a + s_2}{s_1 - s_2} < 0 \iff \frac{a + s_1}{s_1 - s_2} < 1. \quad (3.56)$$

Finally, if  $s_1 \leq 0$ , then it follows immediately from (3.47) and (3.53) that  $\beta \leq 0$ .  $\square$

One can now prove the following main result, which gives necessary and sufficient conditions for the dominance of a real spectral value when exactly two real spectral values are assigned to the quasipolynomial  $\Delta$ .

**Theorem 3.17.** *Let  $\tau > 0$ . Assume that exactly two real spectral values  $s_2 < s_1$  coexist in the spectrum of  $\Delta$ . Then,  $s_1$  is a strictly dominant root of  $\Delta$  if, and only if, inequalities (3.51) are satisfied.*

*Proof.* Firstly, it follows from Remark 3.15 that exactly two real spectral values coexist in the spectrum of  $\Delta$  if, and only if, the left inequality in (3.51) is verified. Let us prove that inequality (3.51) ensures that  $s_1$  is a dominant root of  $\Delta$ . Let  $s_0 := x + i\omega \in \mathbb{C}$  be such that  $\Delta(s_0) = 0$ . In particular,  $\omega \neq 0$ . From  $\Delta(s_1) = 0$ , one gets  $\beta = -\alpha s_1 - e^{\tau s_1}(a + s_1)$ , where  $\alpha$  and  $a$  are given by Lemma 3.14. It follows that  $\Delta(s_0) = 0$  if, and only if,

$$e^{\tau s_0} s_0 + e^{\tau s_0} a + \alpha s_0 = \alpha s_1 + e^{\tau s_1}(a + s_1). \quad (3.57)$$

By taking the real and imaginary parts of both sides in (3.57), one gets

$$\begin{aligned} e^{\tau x}((a+x)\cos(\tau\omega) - \omega\sin(\tau\omega)) &= e^{\tau s_1}(a+s_1) - \alpha(x-s_1), \\ e^{\tau x}((a+x)\sin(\tau\omega) + \omega\cos(\tau\omega)) &= -\alpha\omega. \end{aligned} \quad (3.58)$$

By squaring each equality in (3.58) and adding them, one obtains

$$\omega^2 = \frac{(e^{\tau s_1}(a+s_1) - \alpha(x-s_1))^2 - (a+x)^2 e^{2\tau x}}{e^{2\tau x} - \alpha^2}, \quad (3.59)$$

which is well-defined for every  $x \geq s_1$  by Lemma 3.16. Therefore, by performing the exact same steps of the proof of dominance in Theorem 3.1, one gets that  $\omega^2 < 1/\tau^2$ , i.e.,  $|\omega| < 1/\tau$ . The latter is inconsistent since one has necessarily  $|\omega| \geq \pi/\tau$  owing to Lemma 2.3. Hence,  $s_1$  is the dominant root of  $\Delta$ . Conversely, assume that inequality (3.51) is not satisfied. If

$$\frac{a+s_1}{s_1-s_2} \leq \frac{-1}{e^{\tau(s_1-s_2)} - 1},$$

then another real root  $x$  coexists in the spectrum of  $\Delta$  by Theorem 3.11. Instead, if

$$\frac{a+s_1}{s_1-s_2} \geq 1,$$

then,  $s_1$  and  $s_2$  are the only real spectral values of  $\Delta$ , and  $s_1$  is not a strictly dominant of  $\Delta$  by Lemma 3.16 and the sufficient part of this theorem. Indeed, the sufficient part of this theorem that we proved holds if, and only if,  $(a+s_1)/(s_1-s_2) < 1$  owing to Lemma 3.16.  $\square$

**Remark 3.18.** We stress the fact that in the particular case of exactly two real spectral values  $s_2 < s_1$  coexisting in the spectrum of  $\Delta$ , and

$$\frac{a+s_1}{s_1-s_2} = 1, \quad (3.60)$$

the spectral values  $s \in \mathbb{C}$  of  $\Delta$  are analytically given by  $s = s_1 + i\frac{2\pi k}{\tau}$ ,  $k \in \mathbb{Z}$ , and  $s_1$  is hence a (not strictly) dominant root of  $\Delta$ . Indeed, considering  $a+s_1 = s_1-s_2$ , and equations (3.47) and (3.48), one finds  $\alpha = -e^{\tau s_1}$  and  $\beta = s_2 e^{\tau s_1}$ . Consequently, one has  $\Delta(s) = (s-s_2)(1-e^{-\tau(s-s_1)})$  for every  $s \in \mathbb{C}$ , wherefrom one obtains the desired result.

**Remark 3.19.** Aside from the necessary and sufficient conditions on the dominance of the assigned rightmost root  $s_1$ , which have already been established in [SBN23, Theorem 6], Theorem 3.11 and Theorem 3.17 provide a complete description of such conditions with respect to the number of assigned roots and the potential coexistence of a third real root. We refer to Figure 4 for a full visualisation.

### 3.3. On Frasson-Verduyn Lunel's sufficient conditions for dominance and beyond.

This section builds upon Frasson-Verduyn Lunel's seminal work [FVL03, Lemma 5.1], which established a sufficient condition for the dominance of a simple real spectral value of quasipolynomials with multiple delays. While Frasson-Verduyn Lunel's lemma offers a fundamental method for assessing spectral dominance, its applicability is restricted to specific conditions that may only address certain dynamic scenarios encountered in complex systems. Restricted to the single-delay case, when Frasson-Verduyn Lunel's condition is not met, the CRRID property still provides a guarantee for the dominance of a simple real root.

The first main result of this section is the following.

**Theorem 3.20.** *Let  $\tau > 0$ . There exist  $s_1 \leq 0$  and  $d > 0$ , such that if  $\Delta$  admits three equidistributed real spectral values  $s_3 = s_1 - 2d$ ,  $s_2 = s_1 - d$  and  $s_1$ , then*

$$V(s_1) = (|\alpha|(1 + |s_1|\tau) + |\beta|\tau)e^{-s_1\tau} \quad (3.61)$$

satisfies  $V(s_1) \geq 1$  and  $s_1$  is the dominant root of  $\Delta$ .

*Proof.* Assume that the quasipolynomial  $\Delta$  admits three equidistributed real spectral values  $s_k = s_1 - (k - 1)d$ , for some  $d > 0$  (that we will choose later on) and  $k = 1, 2, 3$  with  $s_1 \leq 0$ . Necessarily,  $s_k$  is simple since  $\Delta$  cannot admit more than three real roots counting multiplicities [PS72, Problem 206.2, page 144]. Hence, Theorem 3.1 states that  $s_1$  is a strictly dominant root of  $\Delta$ . Moreover, one deduces from Lemma 2.4 that

$$\alpha = e^{\tau(s_1-d)}, \quad \beta = e^{\tau(s_1-d)} \left( -s_1 + d + d \coth \left( \frac{\tau d}{2} \right) \right), \quad (3.62)$$

so that

$$V(s_1) = \left( 1 - 2\tau s_1 + \tau d + \tau d \coth \left( \frac{\tau d}{2} \right) \right) e^{-\tau d}. \quad (3.63)$$

Introducing  $g(s_1) := V(s_1) - 1$ , one finds

$$g(s_1) = e^{-\tau d} \left( \tau d \coth \left( \frac{\tau d}{2} \right) + \tau d + 1 - e^{\tau d} - 2\tau s_1 \right) \quad (3.64)$$

which is affine in  $s_1$ . It follows that  $g(s_1) \geq 0$  if, and only if,

$$s_1 \leq \frac{\tau d \coth \left( \frac{\tau d}{2} \right) + \tau d + 1 - e^{\tau d}}{2\tau}. \quad (3.65)$$

In particular, knowing that Frasson-Verduyn Lunel's sufficient condition is not met the first time when  $V(s_1) = 1$ , owing to inequality (3.65) the latter is equivalent to ( $v := \tau s_1$  and  $u := \tau d$ )

$$v(u) = \frac{u \coth \left( \frac{u}{2} \right) + u + 1 - e^u}{2} \quad (u > 0). \quad (3.66)$$

A straightforward analysis of the function  $u \mapsto v(u)$  for  $u > 0$  shows that  $v$  is strictly decreasing on  $(0, \infty)$  and satisfies

$$\lim_{u \rightarrow 0} v(u) = 1, \quad \lim_{u \rightarrow \infty} v(u) = -\infty. \quad (3.67)$$

Therefore, there exist  $s_1 \leq 0$  and  $d > 0$  satisfying (3.65) such that  $V(s_1) \geq 1$ .  $\square$

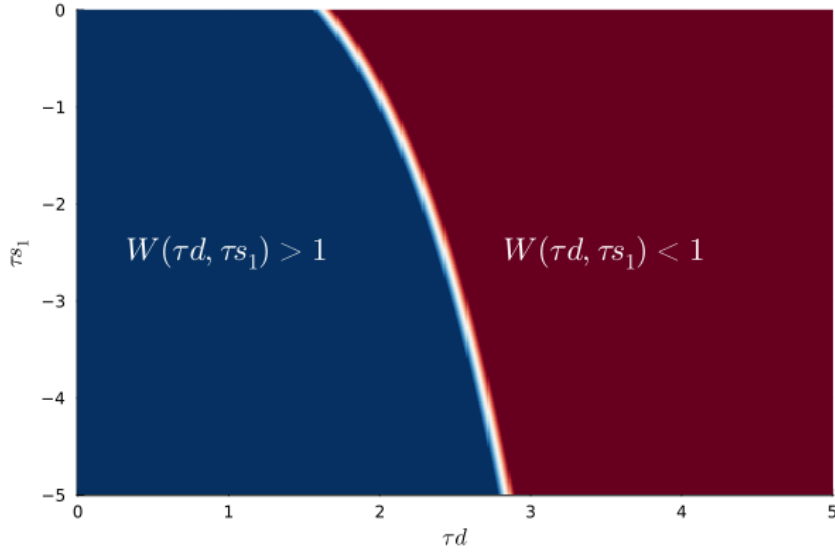


FIGURE 5. For  $(\tau, d, s_1) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_-^*$ , the red coloured region in the parameters space  $(\tau d, \tau s_1)$  corresponds to the region where Frasson-Verduyn Lunel's sufficient condition for the dominance of a simple real spectral value is satisfied, that is  $W(\tau d, \tau s_1) < 1$ . The GCRRID conditions extend Frasson-Verduyn Lunel's conditions by the blue region, that is, for  $W(\tau d, \tau s_1) > 1$ .

Theorem 3.20 shows that Frasson-Verduyn Lunel's sufficient condition is not necessary for the dominance of simple real spectral values when three equidistributed real roots coexist in the spectrum of  $\Delta$ .

The function  $V$  defined in (3.63) initially seems to depend on three separate parameters: the delay  $\tau$ , the distance  $d$ , and the dominant simple real value  $s_1$ . However, upon closer inspection, it becomes apparent that  $V$  can be expressed solely in terms of the products  $\tau d$  and  $\tau s_1$ , denoted by  $u$  and  $v$ , respectively. As a result, the function reduces to

$$W(u, v) := \left(1 - 2v + u + u \coth\left(\frac{u}{2}\right)\right) e^{-u}. \quad (3.68)$$

As a consequence, the behaviour of  $V$  can be fully explored by varying  $u$  and  $v$  without loss of generality. Therefore, for visualization and analysis, it is sufficient to consider plots of  $W$  as a function of  $u = \tau d$  and  $v = \tau s_1$ , treating  $\tau$  as a positive constant scaling factor. Refer to Figure 5 for a visualisation.

The second main result of this section is the following.

**Theorem 3.21.** *Let  $\tau > 0$ . There exist  $s_1 \leq 0$  and  $\delta > 0$  such that if  $\Delta$  admits exactly two simple real spectral values  $s_1$  and  $s_2 = s_1 - \delta$ , and inequality (3.53) is satisfied, then*

$$Y(s_1) = (|\alpha|(1 + |s_1|\tau) + |\beta|\tau)e^{-s_1\tau} \quad (3.69)$$

*satisfies  $Y(s_1) \geq 1$  and  $s_1$  is the dominant root of  $\Delta$ .*

*Proof.* If  $\Delta$  admits exactly two simple real spectral values  $s_2 < s_1$  and inequality (3.53) is satisfied, then  $s_1$  is a strictly dominant root of  $\Delta$  by Theorem 3.17. Moreover,  $\alpha < 0$ , and if



$s_1 \leq 0$  then  $\beta \leq 0$  by Lemma 3.16. If  $s_2 = s_1 - \delta$  for some  $\delta > 0$  that will be chosen later on, then one has from Lemma 3.14,

$$\alpha = -e^{\tau s_1} \left( \frac{(a + s_1)}{\delta} (1 - e^{-\tau\delta}) + e^{-\tau\delta} \right), \quad \beta = -\alpha s_1 - e^{\tau s_1} (a + s_1),$$

so that

$$Y(s_1) = \frac{e^{-\tau\delta}(a + s_1 - \delta)(-1 + 2\tau s_1) + (a + s_1)(1 - 2\tau s_1 + \tau\delta)}{\delta}. \quad (3.70)$$

Function  $Y$  initially seems to depend on four separate parameters: the coefficient  $a$ , the delay  $\tau$ , the distance  $\delta$ , and the dominant simple real value  $s_1$ . However, upon closer inspection, one finds that  $Y$  can be expressed solely in terms of the products  $\tau a$ ,  $\tau\delta$  and  $\tau s_1$ . Setting  $A := \tau a$ ,  $u := \tau\delta$ ,  $v := \tau s_1$ , then  $A \in \mathbb{R}$ ,  $u > 0$  and  $v < 0$ . Introducing the functions

$$Z(A, u, v) := Y(s_1) \quad \text{and} \quad h(A, u, v) = Z(A, u, v) - 1, \quad (3.71)$$

one gets

$$h(A, u, v) = \frac{-(2 - 2e^{-u})v^2 - ((1 - e^{-u})(2A - u - 1) + ue^{-u})v + (A + e^{-u} - 1)u + A(1 - e^{-u})}{u}, \quad (3.72)$$

which is a second-degree polynomial in  $v$ . Let  $N(A, u, v)$  be the numerator of  $h(A, u, v)$ . From  $N(A, u, v) = 0$ , one finds that the discriminant is given by

$$\begin{aligned} D_1(A, u) &= 4(1 - e^{-u})^2 A^2 + 4(1 - e^{-u})(1 - e^{-u} + (1 + 2e^{-u})u)A + (1 - e^{-u})^2 \\ &\quad + u^2(1 - 2e^{-u})^2 + 2u(1 - e^{-u})(2e^{-u} - 3). \end{aligned} \quad (3.73)$$

To study the sign of  $D_1(A, u)$ , a second-degree polynomial in  $A$ , one solves the equation  $D_1(A, u) = 0$ . Therefore, the discriminant of  $D_1(A, u)$  is given by

$$D_2(u) = 128ue^{-u}(1 - e^{-u})^2(e^u + u - 1) > 0, \quad (3.74)$$

since  $e^u + u - 1 > 0$  for every  $u > 0$ . It follows that  $D_1(A, u)$  admits two real roots  $A_0(u) < A_1(u)$  given by

$$A_0 := A_0(u) = \frac{-(1 - e^{-u} + (1 + 2e^{-u})u) - 2\sqrt{2ue^{-u}(e^u + u - 1)}}{2(1 - e^{-u})} < 0 \quad (u > 0), \quad (3.75)$$

$$A_1(u) = \frac{-(1 - e^{-u} + (1 + 2e^{-u})u) + 2\sqrt{2ue^{-u}(e^u + u - 1)}}{2(1 - e^{-u})}. \quad (3.76)$$

One infers that for every  $u > 0$ ,  $D_1(A, u) \geq 0$  for  $A \leq A_0(u)$  and  $A \geq A_1(u)$ . For simplicity, assume from now on that  $A \geq A_1(u)$ . Hence, equation  $N(A, u, v) = 0$  admits two real roots

$$v_1 := v_1(A, u) = \frac{-C_b(A, u) - \sqrt{D_1(A, u)}}{2C_a(u)}, \quad v_2 := v_2(A, u) = \frac{-C_b(A, u) + \sqrt{D_1(A, u)}}{2C_a(u)}, \quad (3.77)$$

where

$$C_a(u) = -2(1 - e^{-u}), \quad C_b(A, u) = -2(1 - e^{-u})A + (1 + u)(1 - e^{-u}) - ue^{-u},$$

and

$$C_c(A, u) = (1 - e^{-u} + u)A - (1 - e^{-u})u,$$

are the coefficients of  $N(A, u, v)$  considered as a second-degree polynomial in  $v$ , so that  $D_1(A, u) = C_b^2(A, u) - 4C_a(u)C_c(A, u)$ . Letting

$$A_2(u) = \frac{(1 - e^{-u})u}{1 - e^{-u} + u} > 0 \quad \text{and} \quad A_3(u) = \frac{(1 + u)(1 - e^{-u}) - ue^{-u}}{2(1 - e^{-u})} > 0 \quad (u > 0), \quad (3.78)$$

one can carefully checks that  $A_1(u) \leq A_2(u) \leq A_3(u)$  for every  $u > 0$ . Moreover, one gets the following sign tab.

$A$	$-\infty$	$A_2(u)$	$A_3(u)$	$+\infty$
$C_b(A, u)$	+	0	+	-
$C_c(A, u)$	-	0	+	+

Consequently, for every  $A \leq A_2(u)$ , one has  $v_1 \leq 0$  ( $v_1 = 0$  if, and only if,  $A = A_2(u)$ ) and  $v_2 > 0$ . Since  $A_1(u) \leq A_2(u)$ , and  $C_a(u) < 0$  for every  $u > 0$ , one deduces that for all  $A \in [A_1, A_2]$ ,  $u > 0$  and  $v \in [v_1, v_2]$ , one has  $N(A, u, v) \geq 0$ . Therefore, for every  $s_1 \leq 0$  and every  $\delta > 0$ , one has  $Y(s_1) \geq 1$ , whenever

$$A_1(\tau\delta) \leq \tau a \leq A_2(\tau\delta) \quad \text{and} \quad v_1(\tau a, \tau\delta) \leq \tau s_1 \leq v_2(\tau a, \tau\delta). \quad (3.79)$$

□

Theorem 3.21 shows that Frasson-Verduyn Lunel's sufficient condition is not necessary for the dominance of simple real spectral values when exactly two simple real roots coexist in the spectrum of  $\Delta$ .

**Remark 3.22.** We want to emphasize that there are values for  $s_1$  and  $\delta$  (where  $s_1 \leq 0$  and  $\delta > 0$ ) that satisfy (3.79) and still maintain inequality (3.53). From equation (3.79), take, for instance,

$$\tau a = A_2(\tau\delta), \quad \text{and} \quad \tau s_1 = v_1(\tau a, \tau\delta). \quad (3.80)$$

One finds, after careful computations, that

$$\frac{a + s_1}{\delta} = \frac{A_2(\tau\delta) + v_1(\tau a, \tau\delta)}{\tau\delta} = \frac{1}{2} \left( 1 + \frac{1}{1 - e^{\tau\delta}} + \frac{1}{\tau\delta} \right). \quad (3.81)$$

Finally, as a function of  $\tau\delta > 0$ , one can immediately check that the third member on the right in (3.81) is bounded between 0 and 1.

**Remark 3.23.** Notice that, when two real roots are assigned, and a third root coexists (see, Theorem 3.11 and Corollary 3.12), one can use the exact same arguments as in the proof of Theorem 3.20 to show that in some parameters regions, the Frasson-Verduyn Lunel's criteria does not apply. Furthermore, when exactly two real spectral values  $s_2 < s_1$  coexist in the spectrum of  $\Delta$ , and

$$\frac{-1}{e^{\tau(s_1 - s_2)} - 1} < \frac{a + s_1}{s_1 - s_2} \leq 0, \quad (3.82)$$

then  $s_1$  is a strictly dominant root of  $\Delta$  by Theorem 3.17, and we can prove as in Theorem 3.21 that in some parameters regions the Frasson-Verduyn Lunel's criteria does not apply.

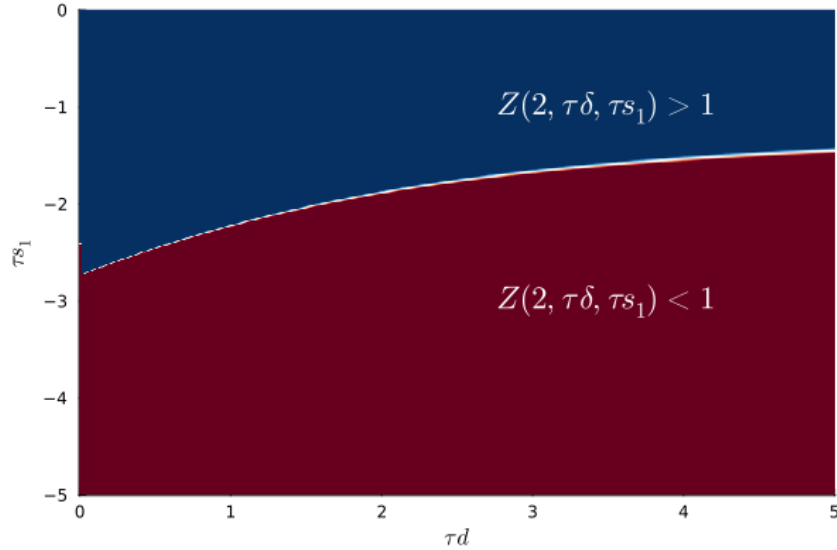


FIGURE 6. An illustration of the function  $Z(\tau a, \tau \delta, \tau s_1)$  defined in (3.83) where  $(\tau, \delta, s_1) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_-^*$  and the parameter  $a = 2$ . The *red*-coloured region in the parameters space  $(\tau \delta, \tau s_1)$  corresponds to the region where Frasson-Verduyn Lunel sufficient condition for the dominance of a simple real spectral value is satisfied that is  $Z(2, \tau \delta, \tau s_1) < 1$ . The ICRRID conditions extend Frasson-Verduyn Lunel's conditions by the *blue* region, that is,  $Z(2, \tau \delta, \tau s_1) > 1$ .

In the proof of Theorem 3.21, we introduced the function

$$Z(\tau a, \tau \delta, \tau s_1) := Y(s_1) = \frac{e^{-\tau \delta}(a + s_1 - \delta)(-1 + 2\tau s_1) + (a + s_1)(1 - 2\tau s_1 + \tau \delta)}{\delta}. \quad (3.83)$$

When  $a = 2$  (admissible owing to (3.79)), we plotted the heatmap of  $Z(2, \tau \delta, \tau s_1)$  in Figure 6 showing the regions  $(\tau \delta, \tau s_1)$  where  $Y(s_1) < 1$ ,  $Y(s_1) > 1$  and  $Y(s_1) = 1$ .

**3.4. Exponential Estimates.** In [Kha05, Section 6], the author establishes exponential estimates of solutions for time delay systems of neutral-type using quadratic Lyapunov functionals and Lyapunov matrices. Although effective, this method may be computationally involved. Additionally, [HL13][Chapter 1, Theorem 7.6, page 32] also offers an exponential estimate for solutions of a neutral differential equation based on the characteristic quasipolynomial's rightmost root. The proof employs the Cauchy theorem of residues and involves complex analysis arguments, including the periodicity and analyticity of certain functions. However, the explanation may benefit from greater detail to enhance the reader's clarity and ease of understanding.

In the following, we provide an alternative proof of exponential estimates for solutions of (NDE) that integrates the previously outlined spectral analysis of Sections 3.1 and 3.2 with the Cauchy theorem of residues.

For all  $t \geq 0$ ,  $y_t \in C^0([-\tau, 0])$  denotes the history function, defined for all  $\theta \in [-\tau, 0]$  as

$$y_t(\theta) = y(t + \theta), \quad \text{with} \quad \|y_t\|_\infty := \sup_{\theta \in [-\tau, 0]} |y_t(\theta)|. \quad (3.84)$$

**Theorem 3.24.** *Let  $\tau > 0$ . Assume that the quasipolynomial  $\Delta$  admits three real spectral values  $s_3 < s_2 < s_1$ . For every  $\varepsilon > 0$ , there exists a constant  $k := k(\varepsilon, s_1, s_2, s_3) \geq 1$  such that the solution  $y(t)$  of (NDE) with initial condition  $y_0 \in C^0([-\tau, 0])$  satisfies*

$$|y(t)| \leq ke^{(s_1+\varepsilon)t} \|y_0\|_\infty \quad (t \geq 0). \quad (3.85)$$

*Proof.* Applying the Laplace transform to both sides of equation (NDE), one gets

$$\hat{y}(s) = \frac{y_0(0) + \alpha y_0(-\tau)}{\Delta(s)} \quad (s \in \mathbb{C}), \quad (3.86)$$

showing that  $\hat{y}$  is an analytic function of  $s$  for  $\Re(s) > s_1$ . Let  $\varepsilon > 0$  and set  $c_1 := s_1 + \varepsilon$ . Then, function  $y$  is given by the Bromwich complex contour integral

$$y(t) = \frac{(y_0(0) + \alpha y_0(-\tau))}{2i\pi} \lim_{T \rightarrow \infty} \int_{c_1 - iT}^{c_1 + iT} \frac{e^{st}}{\Delta(s)} ds \quad (t \geq 0). \quad (3.87)$$

Indeed, let  $T > 0$  and  $c_2 > c_1$ , and consider the integration of the function  $e^{ts}/\Delta(s)$  over the closed rectangle  $\Gamma$  in the complex plane with vertical boundaries  $V_1 := \{c_1 + i\omega \mid -T \leq \omega \leq T\}$  and  $V_2 := \{c_2 + i\omega \mid -T \leq \omega \leq T\}$ , and horizontal boundaries  $H_1 := \{x + iT \mid c_1 \leq x \leq c_2\}$  and  $H_2 := \{x - iT \mid c_1 \leq x \leq c_2\}$ . Since  $\Delta$  has no zeroes inside the rectangle  $\Gamma$ , the integral over  $\Gamma$  equals zero. It is then sufficient to show the following.

$$\int_{H_j} \frac{e^{st}}{\Delta(s)} ds \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty \quad (j = 1, 2). \quad (3.88)$$

For  $s = x + iT$  with  $c_1 \leq x \leq c_2$  and  $T > 0$ , one has  $\Delta(s) = (1 + \alpha e^{-\tau s})s + (a + \beta e^{-\tau s})$ . Since  $1 - \alpha e^{-\tau x} > 0$  thanks to  $s_1 < c_1 \leq x \leq c_2$  and Lemma 2.5 (inequality (2.19)), one chooses  $T_0 > 0$  large enough in order to have

$$\frac{T}{2}(1 - \alpha e^{-\tau x}) \geq |a| + \beta e^{-\tau c_2} \quad (\forall T \geq T_0). \quad (3.89)$$

It follows that

$$|\Delta(s)| \geq T(1 - \alpha e^{-\tau x}) - (|a| + \beta e^{-\tau x}) \geq \frac{T}{2}(1 - \alpha e^{-\tau x}) \quad (T \geq T_0). \quad (3.90)$$

Therefore, one has

$$\left| \int_{H_1} \frac{e^{st}}{\Delta(s)} ds \right| \leq \frac{2e^{tc_2}}{T} \int_{c_1}^{c_2} \frac{e^{\tau x}}{(e^{\tau x} - \alpha)} dx = \frac{2}{T} \frac{e^{tc_2}}{\tau} \ln \left( \frac{e^{\tau c_2} - \alpha}{e^{\tau c_1} - \alpha} \right) \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty.$$

In the same fashion, one proves that the integral over  $H_2$  tends to zero when  $T \rightarrow \infty$ . It follows that (3.87) defines properly the signal  $y(t)$  for every  $t \geq 0$ .

One claims that if  $s = c_1 + iT = s_1 + \varepsilon + iT$ , then

$$|\Delta(s)| \geq \frac{|T|}{2}(1 - \alpha e^{-\tau s_1}), \quad \text{for all} \quad |T| \geq \frac{4\zeta}{1 - \alpha e^{-\tau s_1}} =: T_1, \quad (3.91)$$

where  $\zeta > 0$  is defined by (2.22) and  $1 - \alpha e^{-\tau s_1} > 0$  owing to Lemma 2.5 (inequality (2.19)). Indeed, since  $a = -s_1 - \zeta$  and  $b = -\alpha s_1 + \zeta e^{\tau s_1}$ , one has  $\Delta(s) = (1 + \alpha e^{-\tau c_1} e^{-i\tau T})(\varepsilon + iT) - \zeta(1 - e^{-\tau \varepsilon} e^{-i\tau T})$ . Therefore,

$$|\Delta(s)| \geq |T|(1 - \alpha e^{-\tau c_1}) - \zeta(1 + e^{-\tau \varepsilon}) \geq |T|(1 - \alpha e^{-\tau s_1}) - 2\zeta,$$

thanks to the reverse triangle inequality, which completes the proof of the claim.

Set

$$z(t) := \mathcal{L}^{-1} \left\{ \frac{1}{\Delta(s)} \right\} (t) = \frac{1}{2i\pi} \lim_{T \rightarrow \infty} \int_{c_1 - iT}^{c_1 + iT} \frac{e^{st}}{\Delta(s)} ds \quad (t \geq 0). \quad (3.92)$$

One immediately observes that for every  $s \in \mathbb{C}$  such that  $\Re(s) > s_1$ , one has

$$\frac{1 + \alpha e^{-\tau s}}{\Delta(s)} = \zeta \frac{1 - e^{-\tau(s-s_1)}}{(s-s_1)\Delta(s)} + \frac{1}{s-s_1},$$

so that, taking the inverse Laplace transform and the fact that  $\mathcal{L}^{-1}\{1/(s-s_1)\} = e^{s_1 t}$ , one obtains

$$z(t) + \alpha z(t - \tau) = \zeta \mathcal{L}^{-1} \left\{ \frac{1 - e^{-\tau(s-s_1)}}{(s-s_1)\Delta(s)} \right\} (t) + e^{s_1 t}. \quad (3.93)$$

Thanks to (3.91), one has

$$\begin{aligned} \left| \zeta \mathcal{L}^{-1} \left\{ \frac{1 - e^{-\tau(s-s_1)}}{(s-s_1)\Delta(s)} \right\} (t) \right| &\leq \frac{\zeta e^{tc_1}}{\pi} \int_{-T_1}^{T_1} \frac{dT}{|\Delta(c_1 + iT)|\sqrt{\varepsilon^2 + T^2}} + \frac{4\zeta e^{tc_1}}{\pi(1 - \alpha e^{-\tau s_1})} \int_{T_1}^{\infty} \frac{dT}{T^2} \\ &= \frac{\zeta e^{tc_1}}{\pi} \int_{-T_1}^{T_1} \frac{dT}{|\Delta(c_1 + iT)|\sqrt{\varepsilon^2 + T^2}} + \frac{e^{tc_1}}{\pi} \leq k_0 e^{tc_1}, \end{aligned} \quad (3.94)$$

where

$$k_0 := k_0(\varepsilon, s_1, s_2, s_3) = \frac{1}{\pi} \left( 1 + 2\zeta T_1 \min_{|T| \leq T_1} (|\Delta(c_1 + iT)|\sqrt{\varepsilon^2 + T^2}) \right).$$

Combining (3.93) and (3.94), one obtains

$$|z(t)| - \alpha |z(t - \tau)| \leq |z(t) + \alpha z(t - \tau)| \leq (1 + k_0) e^{(s_1 + \varepsilon)t}, \quad (3.95)$$

which yields

$$|z(t)| \leq (1 + k_0) e^{(s_1 + \varepsilon)t} \sum_{j=0}^{\infty} \alpha^j e^{-j(s_1 + \varepsilon)\tau} = \frac{(1 + k_0)}{1 - \alpha e^{-\tau(s_1 + \varepsilon)}} e^{(s_1 + \varepsilon)t}, \quad (3.96)$$

since  $1 - \alpha e^{-\tau(s_1 + \varepsilon)} > 0$  owing to Lemma 2.5 (inequality (2.19)). As a consequence, the result follows since

$$|y(t)| = |(y_0(0) + \alpha y_0(-\tau))| |z(t)| \leq \frac{(1 + k_0)(1 + \alpha)}{1 - \alpha e^{-\tau(s_1 + \varepsilon)}} e^{(s_1 + \varepsilon)t} \|y_0\|_{\infty} \quad (3.97)$$

and

$$k(\varepsilon, s_1, s_2, s_3) := \frac{(1 + k_0)(1 + \alpha)}{1 - \alpha e^{-\tau(s_1 + \varepsilon)}} \geq 1. \quad \square$$

**Remark 3.25.** In (3.85), it is important to note that the inequality only makes sense when  $k \geq 1$ . The GCRRID setting derived in Section 3.1 allows to explicitly determine  $k \geq 1$ , a property which is not explicitly stated in [HL13][Chapter 1, Theorem 7.6, page 32].

**Remark 3.26.** It is worth noting that in the case where the quasipolynomial  $\Delta$  has exactly two simple real spectral values  $s_2 < s_1$  and that inequality (3.51) is satisfied, the same proof applies due to Lemma 3.16 (since  $\alpha < 0$  implies  $\alpha < e^{\tau x}$  for every  $x \geq s_1$ ) and Theorem 3.17.

## 4. APPLICATION: DESIGNING AN EXPONENTIALLY STABLE ONE-LAYER NEURAL NETWORK

Based on the spectral theory developed in the previous sections, this section applies our theoretical insights to the practical design of a one-layer continuous-time Hopfield Neural Network (HNN). The aim is to prevent mechanisms that support seizure phenomena. Our model uses a delayed Proportional-Derivative (PD) controller, a proven effective strategy in modulating the dynamic instabilities that lead to such neurological events. The PD controller, customized through precise spectral analysis, allows us to explicitly prescribe the closed-loop solution's exponential decay, thereby enhancing the model's stability and responsiveness.

**4.1. A priori stability analysis.** This section will study the stability of the continuous-time one-layer Hopfield Neural Network (HNN) with time-independent external inputs. Instead of using linearization to approximate system behaviour, we will directly examine the nonlinear characteristics of the model. The analysis will help us understand the conditions under which the system remains stable and will provide a foundation for introducing external control designs.

For the sake of exhaustiveness, we tackle the question in the instance where the constant external input is not identically equal to zero. Then, we consider the following equation

$$\dot{y}(t) = -\nu y(t) + \mu \tanh(y(t)) + I, \quad (4.1)$$

where  $\nu > 0$ ,  $\mu > 0$ , and  $I \in \mathbb{R}$  is time-independent.

The following result is standard, but we provide the proof for the sake of completeness.

**Proposition 4.1.** *For every  $I \in \mathbb{R}$ , the unique equilibrium  $y_* \in \mathbb{R}$  of (4.1) is globally exponentially stable if  $0 < \mu < \nu$  and globally asymptotically stable when  $\mu = \nu$ .*

*Proof.* The fact that  $y_* \in \mathbb{R}$  exists and is unique is a trivial consequence of the intermediate value theorem. Since  $\tanh$  is globally 1-Lipschitz continuous, the Cauchy-Lipchitz theorem ensures that for every  $I \in \mathbb{R}$  and for any  $y_0 \in \mathbb{R}$ , there exists a unique continuous solution  $y$  to equation (4.1) with initial condition  $y(0) = y_0$ . Set the change of function

$$u(t) = y(t) - y_* \quad (\forall t \geq 0). \quad (4.2)$$

Then,  $u(0) = y_0 - y_*$  and  $u$  solves the following equation

$$\dot{u}(t) = -\nu u(t) + \mu(\tanh(u(t) + y_*) - \tanh(y_*)), \quad (4.3)$$

since  $-\nu y_* + \mu \tanh(y_*) + I = 0$ . Duhamel's formula gives us

$$u(t) = e^{-\nu t} u_0 + \mu \int_0^t e^{-\nu(t-s)} (\tanh(u(s) + y_*) - \tanh(y_*)) ds \quad (\forall t \geq 0). \quad (4.4)$$

Taking the absolute value of the above identity leads to

$$e^{\nu t} |u(t)| \leq |u_0| + \mu \int_0^t e^{\nu s} |u(s)| ds, \quad (4.5)$$

which, by Gronwall's lemma, implies that

$$e^{\nu t} |u(t)| \leq e^{\mu t} |u_0| \quad \text{i.e.,} \quad |y(t) - y_*| \leq e^{-(\nu-\mu)t} |y_0 - y_*| \quad (\forall t \geq 0). \quad (4.6)$$

It follows that the equilibrium  $y_*$  (equal to zero when  $I = 0$ ) is globally exponentially stable if  $\nu > \mu$ .

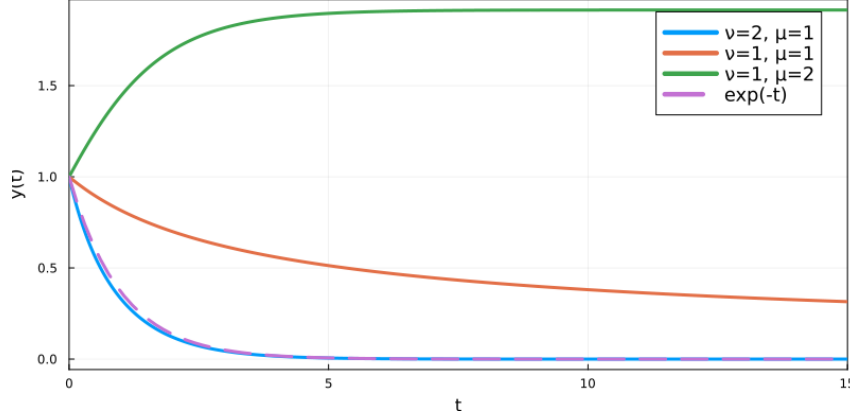


FIGURE 7. Solutions to equation (4.1) corresponding to different values of  $\mu$  and  $\nu$  when  $I = 0$ . The initial condition in each case is taken to be  $y_0 = 1$ .

Let us provide an argument to prove the global asymptotic stability of  $y_*$  when  $\mu = \nu$  via a Lyapunov function. Define the function

$$V(y) = \nu \int_{y_*}^y (r - \tanh(r) - I) \tanh'(r - y_*) dr \quad (y \in \mathbb{R}). \quad (4.7)$$

By letting  $g(r) = r - \tanh(r) - I$ , function  $g$  is derivable on  $\mathbb{R}$  and satisfies  $g(y_*) = 0$ ,  $g(\pm\infty) = \pm\infty$  and  $g'(r) = \tanh^2(r) \geq 0$  for all  $r \in \mathbb{R}$ . It follows that  $g(r) > 0$  for every  $r > y_*$  and  $g(r) < 0$  for every  $r < y_*$ , which yields

$$V(y_*) = 0 \quad \text{and} \quad V(y) > 0 \quad (\forall y \in \mathbb{R} \setminus \{y_*\}). \quad (4.8)$$

On the other hand, along a solution  $y(\cdot)$  of (4.1), one has

$$\frac{dV}{dt}(y(t)) = -\dot{y}(t)^2 \tanh'(y(t) - y_*) \leq 0, \quad \text{and} \quad \frac{dV}{dt}(y(t)) = 0 \iff y(t) = y_*. \quad (4.9)$$

As a result,  $V$  is a strict Lyapunov function for  $y_*$ , and the latter is globally asymptotically stable.  $\square$

In the case of  $0 < \nu < \mu$ , equation (4.1) can have multi equilibria for certain inputs  $I \in \mathbb{R}$  and exactly three equilibria  $-y_1 < y_* = 0 < y_1$  when  $I = 0$ . In this case, via linear stability analysis, one proves that  $\pm y_1$  are locally exponentially stable, and  $y_* = 0$  is unstable.

Consequently, we can assume that  $y_* = 0$  ( $\tanh(0) = 0$ ) is the unstable equilibrium. The aim is then to design a suitable delayed PD controller guaranteeing that the trivial equilibrium zero of the closed-loop equation (1.1) is exponentially stable.

**Remark 4.2.** It follows from the proof of Proposition 4.1 that when  $0 < \mu \leq \nu$  and  $I = 0$ , one has  $y_* = 0$ , and the asymptotic stability of the equilibrium zero is equivalent to say that any solution of (4.1) converges towards zero exponentially if  $\mu < \nu$  and asymptotically when  $\mu = \nu$ ; refer to Figure 7.

**Remark 4.3.** Since equation (4.1) is structurally stable, replacing the  $\tanh$  with any sigmoid function  $f$  satisfying  $f(0) = 0$  will lead to the same conclusion.

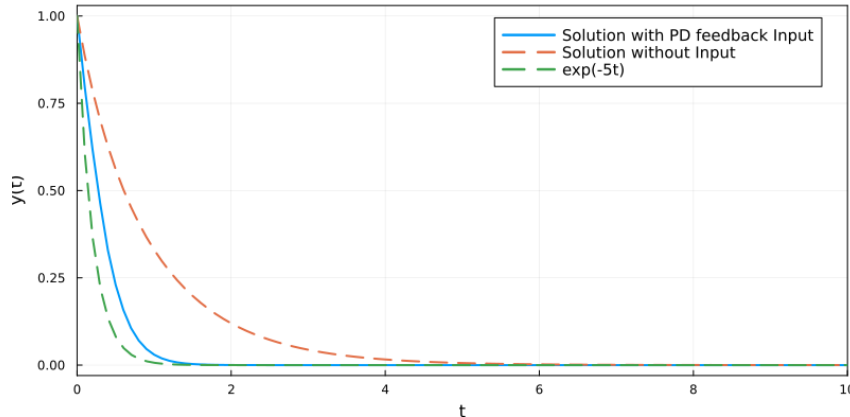


FIGURE 8. Solutions to equation (4.10) when  $I = 0$  and when  $I$  is the PD controller  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$ , where  $\tau > 0$ ,  $k_d > 0$  and  $k_p > 0$  are given by (4.12). We also plotted the function  $t \mapsto \exp(-5t)$ , since  $-5 + \varepsilon$  controls the exponential decay of solutions of the linearized equation (4.10) for a sufficiently small  $\varepsilon > 0$  as stated in Theorem 3.24. The initial condition is taken as  $y_0 = 1$  for the solution with  $I = 0$ , and  $y_0(t) = 1$ ,  $y'_0(t) = 0$ ,  $t \leq 0$  for the solution with the PD feedback input.

Now that we understand the system's inherent stability properties, we will move on to implementing delayed PD control strategies. These strategies are designed to improve the system's dynamic properties and ensure stability, even in the more challenging scenario where  $0 < \nu < \mu$ . The next subsection will explain how these control mechanisms are implemented and their effectiveness in achieving the desired stability outcomes.

**4.2. Implementing delayed PD control for seizure prevention.** Based on the CRRID setting of Section 3.1 and the stability analysis of Section 4.1, this section specifically looks at how to practically implement a delayed Proportional-Derivative (PD) controller to improve the stability of the HNN when  $\nu \geq \mu$  and to stabilize the HNN in situations where  $\nu < \mu$ . The reason for adding the PD controller is its ability to reduce the natural instabilities that could cause seizure-like patterns in neuronal models. It is worth noting that in the case of exclusive access to the delayed Proportional action, that is,  $k_d = 0$ , the analysis reduces to the retarded case where a complete analysis is provided in [BBN20].

We consider two types of configurations. The first type aims to improve the decay rate to zero or stabilize to zero exponentially - with a prescribed decay - the solutions of equation (1.1) in the case where  $\nu > \mu$  or  $\nu \leq \mu$ . This will be achieved by determining the gain parameters  $k_p$  and  $k_d$  and the delay  $\tau > 0$ . The parameters  $\mu$  and  $\nu$  are then known in this case. The second type of configuration aims to model a one-layer neural network like (1.1) such that the trivial equilibrium is exponentially stable with a prescribed decay rate, regardless of the sign of  $\nu - \mu$ . We achieve both of these tasks locally by studying the asymptotic stability of the linearized equation around the trivial equilibrium zero to (1.1).



4.2.1. *Improving the decay rate of an exponentially stable one-layer neural network.* Consider the one-layer neural network under the form

$$\dot{y}(t) = -2y(t) + \tanh(y(t)) + I(t), \quad (4.10)$$

which is a particular case of (1.1) when  $\nu = 2$  and  $\mu = 1$ . Under no external input, that is, when  $I(t) = 0$ , Proposition 4.1 ensures that all solutions of (4.10) are globally exponentially stable or equivalently that the trivial equilibrium zero is globally exponentially stable with a decay rate equal to  $\nu - \mu = 1$ . A classical problem in control theory is choosing the control  $I(t)$  in the feedback form to improve (locally at least) the stability properties of (4.10). By letting  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$  and after linearization around zero, the question reduces to studying the localization of the spectrum of the quasipolynomial function

$$Q_0(s) = s + 1 + e^{-\tau s}(k_d s + k_p) \quad (s \in \mathbb{C}),$$

which is the particular case of (2.1) where  $\alpha = k_d$ ,  $\beta = k_p$  and  $a = 1$ .

To simplify the control design, assume that we assign three equidistributed real spectral values  $s_1 = -5$ ,  $s_2 = -5 - d$  and  $s_3 = -5 - 2d$  to  $Q_0$ . By letting  $d = 1$ , owing to Lemma 2.4 one has

$$1 = 5 - \frac{2}{e^\tau - 1}, \quad k_d = e^{-6\tau}, \quad k_p = e^{-6\tau} \left( 6 + \coth\left(\frac{\tau}{2}\right) \right), \quad (4.11)$$

where  $\coth$  is the cotangent hyperbolic function. By solving the first equation in (4.11), one obtains the following.

$$\tau = \ln\left(\frac{3}{2}\right), \quad k_d = e^{-6\tau}, \quad k_p = e^{-6\tau} \left( 6 + \coth\left(\frac{\tau}{2}\right) \right). \quad (4.12)$$

As per the CRRID properties explained in Section 3.1, it can be inferred that parameters in equation (4.12) ensure the local exponential stability of solutions of equation (4.10) by the PD controller  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$ , with a decay rate equal to  $-5 + \varepsilon$ , for a sufficiently small  $\varepsilon > 0$  as stated in Theorem 3.24. The reader can refer to Figure 8 for a visualization.

4.2.2. *Stabilizing exponentially an asymptotically stable one-layer neural network.* In this section, we will focus on the one-layer neural network equation

$$\dot{y}(t) = -y(t) + \tanh(y(t)) + I(t). \quad (4.13)$$

This equation corresponds to a specific case of the equation (1.1) when  $\nu = \mu = 1$ . According to Proposition 4.1, all solutions of this equation asymptotically converge to zero without external input. See also Figure 7. To make the solutions converge toward zero with a prescribed exponential decay rate, we control this equation with the PD controller  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$ . By linearizing around zero, we can reduce the question to studying the localization of the spectrum of the quasipolynomial function

$$Q_1(s) = s + e^{-\tau s}(k_d s + k_p) \quad (s \in \mathbb{C}),$$

which is a particular case of (2.1), where  $\alpha = k_d$ ,  $\beta = k_p$ , and  $a = 0$ .

Assume that one assigns three non-equidistributed real spectral values  $s_3 < s_2 < s_1$  to  $Q_1$ , say  $s_1 < 0$ ,  $s_2 = s_1 - \delta$  and  $s_3 = s_1 - 3\delta$  for some  $\delta > 0$ . One has

$$F_{-\tau,1}(s_2, s_3) = \frac{e^{\tau s_3}(e^{2\tau\delta} - 1)}{2\tau\delta}, \quad F_{-\tau,2}(s_1, s_2, s_3) = \frac{e^{\tau s_3}(e^{\tau\delta} - 1)^2(1 + 2e^{\tau\delta})}{6\tau^2\delta^2}. \quad (4.14)$$

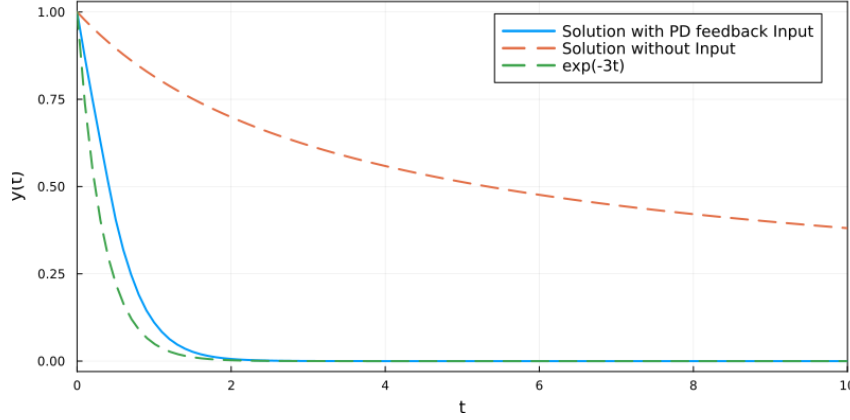


FIGURE 9. Solutions to equation (4.13) when  $I = 0$  and when  $I$  is the PD controller  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$ , where  $\tau > 0$ ,  $k_d > 0$  and  $k_p > 0$  are given by (4.15). We also plotted the function  $t \mapsto \exp(-3t)$ , since  $-3 + \varepsilon$  controls the exponential decay of solutions of the linearised equation (4.10) for a sufficiently small  $\varepsilon > 0$  as stated in Theorem 3.24. The initial condition is taken as  $y_0 = 1$  for the solution with  $I = 0$ , and  $y_0(t) = 1$ ,  $y'_0(t) = 0$ ,  $t \leq 0$  for the solution with the PD feedback input.

Lemma 2.4 states that

$$s_1 = -\frac{3\delta(e^{\tau\delta} + 1)}{(e^{\tau\delta} - 1)(1 + 2e^{\tau\delta})}, \quad k_d = \frac{e^{\tau s_2}(2 + e^{\tau\delta})}{1 + 2e^{\tau\delta}}, \quad k_p = \frac{-s_3 + (s_1 + 3\delta) \cosh(\tau\delta) - 3s_2 \sinh(\tau\delta)}{e^{-\tau s_1}(e^{\tau\delta} - 1)(1 + 2e^{\tau\delta})}.$$

Letting,  $s_1 = -2$  and  $\delta = 1$ , one gets  $s_2 = -4$ ,  $s_3 = -6$  and

$$\tau = \ln\left(\frac{1 + \sqrt{5}}{2}\right), \quad k_d = -20 + 9\sqrt{5}, \quad k_p = -66 + 30\sqrt{5}. \quad (4.15)$$

We plotted in Figure 9 the solutions of (4.13) when  $I(t) = 0$  and when  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$  where  $\tau > 0$ ,  $k_d > 0$  and  $k_p > 0$  are given by (4.15).

**4.2.3. Stabilizing exponentially an unstable one-layer neural network.** This section focuses on the more interesting scenario where the neural system's natural decay rate is surpassed by the interaction strength so that the neuron's inherent dynamics tend toward instability. More precisely, we consider the one layer-neural network

$$\dot{y}(t) = -y(t) + 2 \tanh(y(t)) + I(t), \quad (4.16)$$

which is a particular case of (1.1) when  $\nu = 1$  and  $\mu = 2$ . The linear stability analysis and numerical simulations depicted in Figure 7 suggest that the trivial equilibrium to (4.16) when  $I = 0$  is unstable. To stabilize the neuron activity, we use a PD controller  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$  where the gain  $k_p$  can be compared to inhibitory synaptic strength, the gain  $k_d$  helps dampen the rate of change in the neuron's activity and the delay  $\tau > 0$  accounts for the inherent temporal dynamics of the neuron.

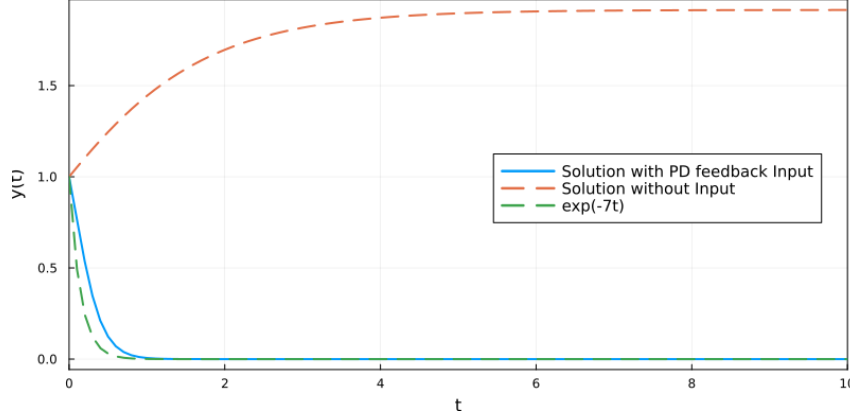


FIGURE 10. Solutions to equation (4.16) when  $I = 0$  and when  $I$  is the PD controller  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$ , where  $\tau > 0$ ,  $k_d > 0$  and  $k_p > 0$  are given by (4.17). We also plotted the function  $t \mapsto \exp(-7t)$ , since  $-7 + \varepsilon$  controls the exponential decay of solutions of the linearized equation (4.16) for a sufficiently small  $\varepsilon > 0$  as stated in Theorem 3.24. The initial condition is taken as  $y_0 = 1$  for the solution with  $I = 0$ , and  $y_0(t) = 1$ ,  $y'_0(t) = 0$ ,  $t \leq 0$  for the solution with the PD feedback input.

By linearizing the equation, assessing the local asymptotic behaviour of solutions to (4.16) is equivalent to studying the spectrum distribution of the quasipolynomial function

$$Q_2(s) = s - 1 + e^{-\tau s}(k_d s + k_p) \quad (s \in \mathbb{C})$$

which is a particular case of (2.1), where  $\alpha = k_d$ ,  $\beta = k_p$ , and  $a = -1$ .

Assume that we assign three equidistributed real spectral values  $s_1 = -7$ ,  $s_2 = -7 - d$  and  $s_3 = -7 - 2d$  to  $Q_2$ . Lemma 2.4 provides the following

$$-1 = 7 - \frac{2d}{e^{\tau d} - 1}, \quad k_d = e^{\tau s_2}, \quad k_p = e^{\tau s_2} \left( -s_2 + d \coth \left( \frac{\tau d}{2} \right) \right).$$

By letting  $d = 1$ , one finds

$$\tau = \ln \left( \frac{5}{4} \right), \quad k_d = e^{-8\tau}, \quad k_p = e^{-8\tau} \left( 8 + \coth \left( \frac{\tau}{2} \right) \right). \quad (4.17)$$

In Figure 9, we depicted the solutions of (4.16) when  $I(t) = 0$  and when  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$  where  $\tau > 0$ ,  $k_d > 0$  and  $k_p > 0$  are given by (4.17).

**4.2.4. Modeling of locally exponentially stable one-layer neural networks.** In the previous sections, we examined the fundamental stability aspects of one-layer neural networks. We specifically looked at how variations in the system's parameters  $\nu$  and  $\mu$ , as well as the implementation of a delayed PD controller, influence the system's dynamic behaviour. Building on this analysis, this section aims to provide a comprehensive framework that guides the design and configuration of one-layer neural networks to achieve specific exponential stability criteria. It involves a detailed exploration of how to adjust the network parameters, specifically the gains of the PD

controller ( $k_p$  and  $k_d$ ) and the delay  $\tau$ , to ensure that the network's equilibrium behaviour aligns with desired stability profiles.

In this scenario, we do not have prior knowledge of the inherent parameters  $\nu > 0$  and  $\mu > 0$  in equation (NDE). The only available information is the prescribed decay rate  $\gamma > 0$ , which is designated to govern the exponential decay of the solution of (NDE) towards zero, regardless of the sign of  $\nu - \mu$ . Since the local asymptotic behaviour of solutions to (NDE) is equivalent to studying the spectrum distribution of the quasipolynomial function  $\Delta_0$  defined in (1.3), the following steps allow us to engineer the model parameters.

- (1) Firstly, assign (one can let them equidistributed to simplify the design) three real spectral values  $s_3 < s_2 < s_1 := -\gamma$  to  $\Delta_0$ .
- (2) Owing to the interpolation Lemma 2.4, the coefficients of  $\Delta_0$  are then given by

$$(\nu - \mu)(\tau) = -s_1 - \zeta(\tau), \quad k_d(\tau) = \frac{F_{-\tau,2}(s_1 + s_2 + s_3, s_1 + s_3)}{F_{-\tau,2}(s_1, s_2, s_3)}, \quad k_p(\tau) = -k_d(\tau)s_1 + \zeta(\tau)e^{\tau s_1}, \quad (4.18)$$

where  $\zeta$  is defined in (2.22).

- (3) If one wants to design (NDE) with  $\nu = \mu$ ,
  - (i) Find the unique  $\tau_* > 0$  such that  $s_1 = -\zeta(\tau_*)$ . This equation always admits a positive real solution thanks to Theorem 3.3.
  - (ii) Compute the corresponding gains  $k_p(\tau_*)$  and  $k_d(\tau_*)$ .

Consequently, the trivial equilibrium zero to (NDE) having parameters  $\nu = \mu$  (equal to any positive real number),  $\tau = \tau_*$ ,  $k_p = k_p(\tau_*)$ , and  $k_d = k_d(\tau_*)$  will be locally exponentially stable with a decay rate  $s_1 + \varepsilon$  for a sufficiently small  $\varepsilon > 0$ , according to Theorem 3.24.
- (4) If one wants to design (NDE) with  $\nu > \mu$ ,
  - (i) Find the unique  $\tau_* > 0$  such that  $s_1 = -\zeta(\tau_*)$ .
  - (ii) Choose some  $\tau_0 > 0$  such that  $\tau_0 > \tau_*$  to guarantee that  $(\nu - \mu)(\tau_0) > 0$  owing to Theorem 3.3.
  - (iii) Compute the corresponding gains  $k_p(\tau_0)$  and  $k_d(\tau_0)$ .

Therefore, the trivial equilibrium zero to (NDE) having parameters  $\mu > 0$ ,  $\nu = \mu - s_1 - \zeta(\tau_0)$ ,  $\tau = \tau_0$ ,  $k_p = k_p(\tau_0)$ , and  $k_d = k_d(\tau_0)$  will be locally exponentially stable with a decay rate  $s_1 + \varepsilon$  for a sufficiently small  $\varepsilon > 0$ , according to Theorem 3.24.
- (5) If one wants to design (NDE) with  $\nu < \mu$ ,
  - (i) Find the unique  $\tau_* > 0$  such that  $s_1 = -\zeta(\tau_*)$ .
  - (ii) Choose some  $\tau_1 > 0$  such that  $\tau_1 < \tau_*$  to guarantee that  $(\nu - \mu)(\tau_1) < 0$  owing to Theorem 3.3.
  - (iii) Compute the corresponding gains  $k_p(\tau_1)$  and  $k_d(\tau_1)$ .

It follows that the trivial equilibrium zero to (NDE) having parameters  $\nu > 0$ ,  $\mu = \nu + s_1 + \zeta(\tau_1)$ ,  $\tau = \tau_1$ ,  $k_p = k_p(\tau_1)$ , and  $k_d = k_d(\tau_1)$  will be locally exponentially stable with a decay rate  $s_1 + \varepsilon$  for a sufficiently small  $\varepsilon > 0$ , thanks to Theorem 3.24.

Abiding by these steps enables the design of a one-layer neural network with a delayed PD controller that prevents seizure and system failure regardless of the inherent dynamics.

Consider the practical example consisting of designing (NDE) with either  $\nu = \mu$ ,  $\nu > \mu$  and  $\mu > \nu$  such that the trivial equilibrium zero is locally exponential stable with the prescribed decay rate  $\gamma = 4$ . Assigning three equidistributed real spectral values  $s_3 = -6$ ,  $s_2 = -5$  and

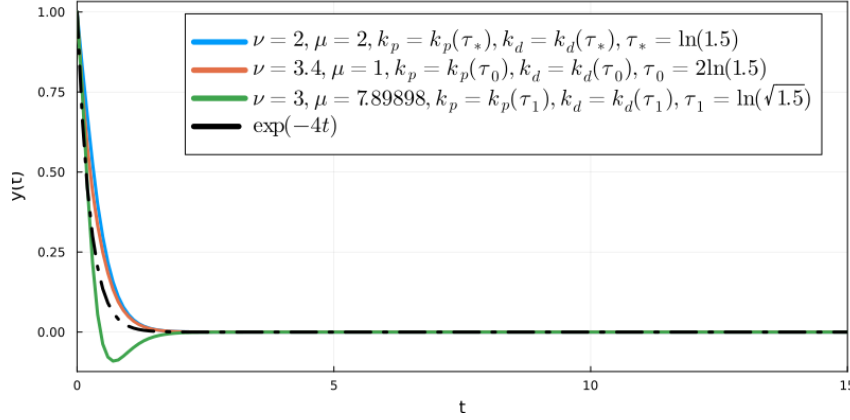


FIGURE 11. Solutions to equations (4.19), (4.20) and (4.21) respectively plotted in *blue*, *orange* and *green*. We also plotted the function  $t \mapsto \exp(-4t)$ , since  $-4 + \varepsilon$  controls the exponential decay of solutions of the linearized equation (4.19), (4.20) and (4.21) for a sufficiently small  $\varepsilon > 0$  as stated in Theorem 3.24. The initial condition is taken as  $y_0(t) = 1$ ,  $y'_0(t) = 0$ ,  $t \leq 0$  in each case.

$s_1 = -4$ , one finds that the unique  $\tau_* > 0$  such that  $-4 = \zeta(\tau_*)$  is given by  $\tau_* = \ln(3/2)$ . One can compute the associated gains

$$k_d(\tau_*) = e^{-5\tau_*}, \quad k_p(\tau_*) = e^{-5\tau_*} \left( 5 + \coth\left(\frac{\tau_*}{2}\right) \right).$$

Therefore letting, for instance,  $\nu = \mu = 2$ , one can design (NDE) in the following fashion

$$\dot{y}(t) = -2y(t) + 2 \tanh(y(t)) - k_p(\tau_*)y(t - \tau_*) - k_d(\tau_*)\dot{y}(t - \tau_*). \quad (4.19)$$

Letting now  $\tau_0 > \ln(3/2)$ , say  $\tau_0 = 2 \ln(3/2)$ , and  $\mu = 1$ , one finds

$$\nu = \mu - s_1 - \zeta(\tau_0) = 3.4, \quad k_d(\tau_0) = e^{-5\tau_0}, \quad k_p(\tau_0) = e^{-5\tau_0} \left( 5 + \coth\left(\frac{\tau_0}{2}\right) \right).$$

It follows that one can design (NDE) accordingly

$$\dot{y}(t) = -3.4y(t) + \tanh(y(t)) - k_p(\tau_0)y(t - \tau_0) - k_d(\tau_0)\dot{y}(t - \tau_0). \quad (4.20)$$

Finally, taking  $\tau_1 > 0$  such that  $\tau_1 < \tau_*$ , say  $\tau_1 = \ln(3/2)/2$ , and  $\nu = 3$ , one gets

$$\mu = \nu + s_1 + \zeta(\tau_1) \approx 7.89898, \quad k_d(\tau_1) = e^{-5\tau_1}, \quad k_p(\tau_1) = e^{-5\tau_1} \left( 5 + \coth\left(\frac{\tau_1}{2}\right) \right).$$

Therefore, one can design (NDE) as follows

$$\dot{y}(t) = -3y(t) + 7.89898 \tanh(y(t)) - k_p(\tau_1)y(t - \tau_1) - k_d(\tau_1)\dot{y}(t - \tau_1). \quad (4.21)$$

We depicted in Figure 11 solutions to equations (4.19), (4.20) and (4.21). As expected, these solutions decay exponentially towards zero as the function  $t \mapsto \exp((-4 + \varepsilon)t)$  for a sufficiently small  $\varepsilon > 0$ .

**Remark 4.4.** We always achieve exponential stability in each case with small values for  $k_p$  and  $k_d$ . This smallness guarantees that the inherent dynamics of the original system are preserved to a great extent. In particular, this is important for neural networks that aim to mimic biological

processes, as it ensures that the control strategy does not overpower the natural behaviours and characteristics of the system.

## 5. DISCUSSION

This paper demonstrates the use of a delayed Proportional-Derivative (PD) controller in a continuous-time modeling of an one-layer Hopfield Neural Network (HNN) to achieve exponential stability in models that are vulnerable to conditions similar to epileptic seizures at the single neuron level. Through a rigorous application of spectral theory analysis, we have developed a methodological framework that improves the stability of neural network models.

We have expanded the spectral theory based on the CRRID property for linear functional differential equations of neutral type to the field of neural dynamics, offering a powerful analytical tool to examine the stability of neural networks based on their spectral properties. This has enabled us to determine the conditions under which the network attains stability, with a primary focus on systems where conventional approaches anticipate instability.

Based on these theoretical insights, integrating a delayed PD controller has shown significant promise in stabilizing the HNN. This approach imitates the natural inhibitory feedback mechanisms within the brain. It offers a biologically inspired method to control and prevent the hyperexcitability that leads to seizures.

It would be beneficial for future research to explore the potential of implementing the delayed PD control strategy and CRRID setting in more complex, multi-layer neural network architectures, which could represent the intricate structures of biological neural systems more accurately. It's important to note that the MID setting developed in [BBNT23, BMN22, MBN21] can address this issue. However, as mentioned in [MBN17], it is widely acknowledged that non-semisimple spectral values are sensitive to minor perturbations due to their splitting mechanism.

The study provides valuable insights into the local stability of the trivial equilibrium in the nonlinear model. However, it is important to recognize that these findings are primarily related to local dynamics. In the future, research should combine the spectral methods used for the linear equation with time-domain approaches based on Lyapunov functionals and linear matrix inequalities. This combined approach would enable a more comprehensive investigation into the global exponential stability of the nonlinear equation with a prescribed decay rate, expanding the applicability and reliability of our findings.

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