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# Prescribed exponential stabilization of scalar neutral differential equations: Application to neural control

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## Abstract

This paper presents a control-oriented delay-based modeling approach for the exponential stabilization of a scalar neutral functional differential equation, which is then applied to the local exponential stabilization of a one-layer neural network of Hopfield type with delayed feedback. The proposed approach utilizes a recently developed partial pole placement method for linear functional differential equations, leveraging the coexistence of real spectral values to explicitly prescribe the exponential decay of the closed-loop solution. While a delayed proportional (P) feedback control may achieve stabilization, it requires higher gains and only allows for a shorter maximum delay compared to the proportional-derivative (PD) feedback control presented in this work. The framework provides a practical illustration of the stabilization strategy, improving upon previous literature results that characterize the solution's exponential decay for simple real spectral values. This approach enhances neural stability in cases where the inherent dynamics are stable and offers a method to achieve local exponential stabilization with a prescribed decay rate when the inherent dynamics are unstable.

**Keywords:** Neural networks, Time-delay controller, Neutral equations, PD-controller, Coexistent-real-roots-induced-dominance, Partial pole placement.

**MSCcodes.** 34K20, 34K60, 37N25, 92B20, 93D15.

## 1. Introduction

In this paper, we focus on the following scalar differential equation with a single constant delay

$$\dot{y}(t) = -(\nu - \mu)y(t) - k_p y(t - \tau) - k_d \dot{y}(t - \tau) \quad (1.1)$$

where  $\nu, \mu, k_p, k_d > 0$  and  $\tau > 0$  represent system parameters. Equation (1.1) is of neutral type, and our primary interest lies in analyzing its stability properties. Specifically, we explore the placement of real characteristic values, with an emphasis on conditions that guarantee exponential stability with a prescribed decay rate.

The stability properties of equation (1.1) have several potential applications, including the prescribed local exponential stabilization of one-layer neural network models of Hopfield type. In particular, these stabilization results can be applied to mitigate the effects of hyperexcitability in such models, which can be associated with seizure-like behavior in individual neurons. Thus, while our focus is on the theoretical stability analysis of equation (1.1), our results can directly affect the design of biologically inspired stable inherent neural network models.

Neural networks exhibit complex dynamics crucial for their biological or artificial functionality. The stability of these systems is essential, as instabilities can lead to dysfunctional behaviors such as seizures in biological systems [8, 32]. The continuous-time one-layer Hopfield-type equation is a classical model that captures the dynamics of a single neuron or a basic

unit of an artificial neural network. The model is described by the following nonlinear differential equation [16]

$$\dot{y}(t) = -\nu y(t) + \mu \tanh(y(t)) + I(t) \quad (1.2)$$

where  $\dot{y}(t)$  denotes the rate of change of the neuron's state at time  $t$ ,  $\nu$  is a positive parameter reflecting the natural decay rate or leakage of the neuron's membrane potential,  $\mu$  is a positive parameter that scales the influence of the activation function, and  $I(t)$  is an input.

When  $I(t) = 0$ , the system dynamics exhibit different stability regimes depending on the relationship between  $\nu$  and  $\mu$ . If  $\nu \geq \mu$ , the zero equilibrium is stable, with exponential stability when  $\nu > \mu$  and asymptotic stability when  $\nu = \mu$ . Conversely, when  $\nu < \mu$ , the system becomes hyperexcitable, resulting in instability of the zero equilibrium and the appearance of additional stable equilibria.

To address the hyperexcitability and prevent destabilizing transitions, we can introduce a delayed proportional-derivative (PD) feedback control

$$I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau) \quad (1.3)$$

where  $k_p > 0$  and  $k_d > 0$  are the proportional and derivative gains, respectively, and  $\tau > 0$  represents a synaptic delay. Incorporating this feedback in (1.2) yields the following delayed differential equation

$$\dot{y}(t) = -\nu y(t) + \mu \tanh(y(t)) - k_p y(t - \tau) - k_d \dot{y}(t - \tau). \quad (1.4)$$

The delayed PD controller introduces time-dependent effects, reflecting physiological synaptic transmission delays [13, 23, 29]. For further insights into using time delays in modeling biological systems, see, for instance, [2, 14, 26]. In particular, it is commonly accepted to include time delays in modeling sensory and motoric neural pathways due to the time lag one observes in communication. See, for instance, [30, 18] and [31] for two delays in neural network modeling. Proportional-integral (PI) controllers have also been proven effective in a firing rate model to control the interspike intervals of a neuron by injecting current through an electrode using a dynamic clamp [22].

It is worth noting that a delayed proportional (P) feedback control, given by  $I(t) = -k_p y(t - \tau)$  with  $k_p > 0$  and  $\tau > 0$ , can potentially stabilize (1.2) in the hyperexcitable regime where  $\mu > \nu$ . However, as we demonstrate in Section 4.2, although the P-controller can achieve the same decay rate as the PD controller in (1.3), it requires higher gains (and, consequently, higher energy costs) and only allows for a shorter maximum allowable delay. This makes it less suitable for stabilizing systems that emulate biological processes, where larger delays are often unavoidable. In contrast, exponential stability is achieved using the delayed PD controller with small values for  $k_p$  and  $k_d$ . The smallness of these gains ensures that the inherent dynamics of the original system are largely preserved, which is crucial for neural networks that aim to mimic biological processes, as it prevents the control strategy from overpowering the system's natural behaviors and characteristics.

Equation (1.1) has been widely discussed in the literature regarding its asymptotic and exponential stability [19, 15, 10, 5, 27]. Sufficient delay-independent conditions for stability have been presented. In [19, Example 1, page 26], the authors proved the global uniform asymptotic stability of equation (1.1) with constant real coefficients (not dependent upon the delay  $\tau$ ) satisfying  $\nu - \mu > 0$ ,  $|k_d| < 1$ , and  $|k_p| < (\nu - \mu) \sqrt{1 - k_d^2}$  using a Lyapunov functional (LF) and a linear matrix inequality (LMI). It's worth noting that asymptotic analysis of (1.1) was already considered in [15, Chapter 9, Section 9.8, page 294] when  $k_p = 0$ ,  $\nu - \mu > 0$ , and  $|k_d| < 1$ , see also [11, Chapter 3, Section 3.3.4, page 69]. Unfortunately, to the best of the authors' knowledge, no known result using a time-domain approach based on an LF and an LMI can be applied to study the asymptotic stability properties of equation (1.1) when  $\nu - \mu \leq 0$ .

In this paper, we employ a frequency-domain approach to analyze the distribution of the roots of the characteristic quasipolynomial associated with equation (1.1). The characteristic quasipolynomial is given by

$$\Delta_0(s) = s + \nu - \mu + e^{-\tau s}(k_d s + k_p), \quad s \in \mathbb{C}. \quad (1.5)$$

Our primary objective is to characterize the spectral properties of the quasipolynomial (1.5) to ensure that the spectral abscissa is negative, thereby guaranteeing the exponential stability of equation (1.1). Recent works have highlighted the importance of *multiplicity-induced dominance (MID)*, where a characteristic root with maximal multiplicity dominates the spectral abscissa [6, 5, 21]. In particular, it appears that a characteristic root of maximal multiplicity (i.e., equal to the degree

of the corresponding quasipolynomial) necessarily defines the spectral abscissa of the system. This property generally occurs in *generic* quasipolynomial and is called GMID. However, in the case of *intermediate* multiplicities, that is, multiplicities which are less than the quasipolynomial's degree, the IMID occurs (the largest assigned root corresponds to the spectral abscissa) under some additional conditions, see for instance [4]. Since these works, the case of the assignment of a characteristic root with maximal multiplicity was recently addressed and thoroughly characterized in [21] (generic retarded case) and in [5] (unifying retarded and neutral cases) for LTI DDEs including a *single delay*. It is essential to point out that the multiplicity of a given spectral value itself is not essential. Still, its connection with the dominance of this root is a meaningful tool for control synthesis. Namely, it is shown that, under appropriate conditions, the coexistence of exactly the maximal number of distinct negative zeros of quasipolynomial of reduced degree guarantees the exponential stability of the zero solution of the corresponding time-delay system, a property called *Coexisting Real Roots Induced Dominance (CRRID)*, see for instance [1, 3, 27, 28]. These properties opened an interesting perspective in control through the so-called *partial pole placement* method, that is, imposing the multiplicity or the coexistence of simple real characteristic root of the closed-loop system by an appropriate choice of the controller gains guarantees the exponential stability of the closed-loop system with a prescribed decay rate.

The stability properties of (1.1) have also been linked to the problem of boundary PI control for the transport equation, as explored in [27]. In that work, the authors demonstrated the effectiveness of the CRRID properties in achieving exponential stabilization of the closed-loop transport equation with a prescribed decay rate. In contrast, the results in [7], which rely on Lyapunov functions, do not offer the same guarantee regarding the decay rate.

The contributions of this paper are as follows: First, it refines recent results on the *Coexisting Real Roots Induced Dominance (CRRID)* property for the first-order neutral functional differential equations, providing simplified proofs and insights into the conditions required for CRRID to hold. Specifically, we provide a more straightforward proof for the Generic CRRID (GCRRID) property, covering arbitrary distributions of real roots, and derive necessary and sufficient conditions for the Intermediate CRRID (ICRRID) property, including cases where unintentional additional real roots coexist. Moreover, we generalize and simplify the proof of the Intermediate MID (IMID) property from [4]. Secondly, we extend the results from [27], providing a detailed characterization of the spectrum distribution when the quasipolynomial has three real spectral values or exactly two real ones. Furthermore, in the single-delay case, when the Frasson-Verduyn Lunel's [12] sufficient condition for the dominance of a simple real spectral value is not met, we prove that the CRRID property remains valid. Finally, by leveraging the CRRID property, we characterize the spectral abscissa's negativity regarding system parameters and establish an explicit and improved exponential estimate for the closed-loop system's solution compared to previous results.

The remainder of this paper is organized as follows: Section 2 provides prerequisites in complex analysis and formulates the problem settings. Section 3 presents the main results and their corresponding proofs. Section 4 translates the main results into the modeling and stabilization of one-layer neural networks.

## 2. Problem settings and prerequisites

Throughout the following, our focus is on studying the asymptotic behavior of solutions of the general scalar neutral functional differential equation (NFDE) in Hale's form

$$\frac{d}{dt}(y(t) + \alpha y(t - \tau)) = -\alpha y(t) - \beta y(t - \tau) \quad (\text{NDE})$$

with initial function  $y_0 \in C^0([-\tau, 0])$ . To this equation corresponds the characteristic quasipolynomial function given by

$$\Delta(s) = s + a + e^{-\tau s}(\alpha s + \beta) \quad (s \in \mathbb{C}) \quad (2.1)$$

where  $(a, \alpha, \beta) \in \mathbb{R}^3$  and  $\tau > 0$ . It is known that the degree of the quasipolynomial  $\Delta$ —the sum of the degrees of the involved polynomials plus the number of delays—is equal to three. Moreover, thanks to the Pólya-Szegő bound [25, Problem 206.2, page 144], the degree of  $\Delta$  is a sharp bound for the number of real roots counting multiplicities of the quasipolynomial  $\Delta$ .

In the next section, we study the spectral properties of the quasipolynomial function  $\Delta$ , focusing on characterizing its rightmost root. Consider a complex value  $s_0 \in \mathbb{C}$  such that  $\Delta(s_0) = 0$ . We say that  $s_0$  is a *dominant* (respectively, *strictly dominant*) root of  $\Delta$  if the following holds:

$$\forall s \in \mathbb{C} \setminus \{s_0\}, \quad \Delta(s) = 0 \implies \begin{cases} \Re(s) \leq \Re(s_0), \\ \text{respectively} \\ \Re(s) < \Re(s_0). \end{cases} \quad (2.2)$$

Despite the challenging question of characterizing the spectral abscissa of (NDE), Frasson-Verduyn Lunel in [12, Lemma B1] provides a test for determining the simplicity and dominance of real spectral values in the multi-delay scalar neutral equations, offering fundamental insights. Notice that such a characterization is closely related to the exponential/asymptotic behavior of the equation (NDE).

According to Frasson-Verduyn Lunel's lemma when restricted to the single-delay case, a real root  $s_0$  of  $\Delta$  is simple and dominant if  $V(s_0) < 1$  where  $V$  is a specific functional construct derived from  $\Delta$ . In other words, the condition is as follows; see, [12, Lemma 5.1].

**Lemma 2.1.** *Suppose that there exists a real zero  $s_0$  of  $\Delta$ . If  $V(s_0) < 1$ , then  $s_0$  is a real simple dominant zero of  $\Delta$ . Here,*

$$V(s_0) = (|\alpha|(1 + |s_0|\tau) + |\beta|\tau)e^{-s_0\tau}$$

where  $(\alpha, \beta) \in \mathbb{R}^2$  and  $\tau > 0$ .

Before starting the study of the localisation of the quasipolynomial  $\Delta$  roots, let us state the following lemma the proof of which can be found in [24, Problem 77, page 46].

**Lemma 2.2** (Descartes' rule of signs). *Let  $a_1, a_2, a_3, \lambda_1, \lambda_2, \lambda_3$  be real constants, such that  $\lambda_1 < \lambda_2 < \lambda_3$ . Denote by  $Z$  the number of real zeros of the entire function*

$$\mathcal{F}(x) = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + a_3 e^{\lambda_3 x} \quad (2.3)$$

and by  $C$ , the number of sign changes in the sequence of numbers  $a_1, a_2, a_3$ . Then,  $C - Z$  is a non-negative even integer.

One also has the following important result, which extends [27, Corollary 2]. It includes the case where we exactly assign two real spectral values to  $\Delta$ .

**Lemma 2.3.** *Let  $\Delta$  be the quasipolynomial defined by (2.1) with real coefficients. Let  $\eta \in \{0, 1\}$ , if  $\Delta$  admits exactly  $3 - \eta$  real roots, then any root  $x + i\omega \in \mathbb{C}$  of  $\Delta$  with  $\omega \neq 0$  satisfies*

$$|\omega| \geq (2 - \eta)\pi/\tau. \quad (2.4)$$

*Proof.* Recall from [25, Problem 206.2, page 144] that if  $M_{\alpha, \beta}$  denotes the number of roots of  $\Delta$  contained in the horizontal strip  $\{s \in \mathbb{C} \mid \alpha \leq \Im(s) \leq \beta\}$  counting multiplicities, then the following bound holds

$$\frac{\tau(\beta - \alpha)}{2\pi} - 3 \leq M_{\alpha, \beta} \leq \frac{\tau(\beta - \alpha)}{2\pi} + 3. \quad (2.5)$$

To complete the proof of the lemma, we argue by contradiction. Assume that  $\Delta$  admits  $3 - \eta$  real roots, and let  $s_0 := x + i\omega \in \mathbb{C}$  be a root of  $\Delta$  with  $\omega \neq 0$  and  $|\omega| < (2 - \eta)\pi/\tau$ . Then there exists  $\varepsilon > 0$  such that  $|\omega| < (2 - \eta)\pi/\tau - \varepsilon$ . Since  $\Delta$  has real coefficients, the complex conjugate of  $s_0$  is also a root of  $\Delta$ , both belonging to the horizontal strip  $\{s \in \mathbb{C} \mid -\frac{(2-\eta)\pi}{\tau} + \varepsilon \leq \Im(s) \leq \frac{(2-\eta)\pi}{\tau} - \varepsilon\}$ . It follows that  $\Delta$  admits at least  $5 - \eta$  roots in this strip, which is inconsistent, since by the Pólya-Szegő bound (2.5), the number of zero in this strip satisfies

$$M_{-\frac{(2-\eta)\pi}{\tau} + \varepsilon, \frac{(2-\eta)\pi}{\tau} - \varepsilon} \leq 5 - \eta - \frac{\varepsilon\tau}{\pi} < 5 - \eta. \quad \square$$

Let us investigate the coexistence of—non-necessarily equidistributed—three real roots for the quasipolynomial  $\Delta$ . Due to the linearity of  $\Delta$  with respect to its coefficients  $a, \alpha$  and  $\beta$ , one reduces the system  $\Delta(s_1) = \Delta(s_2) = \Delta(s_3) = 0$  to the linear system

$$A_{\tau,3}(s_1, s_2, s_3)X = B, \quad (2.6)$$

where  $B = -(s_1 e^{\tau s_1}, s_2 e^{\tau s_2}, s_3 e^{\tau s_3})^t$ ,  $X = (\alpha, \beta, a)^t$  and

$$A_{\tau,3}(s_1, s_2, s_3) = \begin{bmatrix} s_1 & 1 & e^{\tau s_1} \\ s_2 & 1 & e^{\tau s_2} \\ s_3 & 1 & e^{\tau s_3} \end{bmatrix}. \quad (2.7)$$

Using [3, Theorem 2], one immediately obtains that the determinant of the structured Vandermonde-type matrix  $A_{\tau,3}(s_1, s_2, s_3)$  is given by

$$D_{\tau,3}(s_1, s_2, s_3) = \tau^2(s_1 - s_2)(s_1 - s_3)(s_2 - s_3)F_{-\tau,2} \quad (2.8)$$

where  $F_{-\tau,2} := F_{-\tau,2}(s_1, s_2, s_3)$  is defined by

$$F_{-\tau,2} = \int_0^1 \int_0^1 (1-t_1) e^{\tau(t_1 s_1 + (1-t_1)(t_2 s_2 + (1-t_2)s_3))} dt_1 dt_2 > 0. \quad (2.9)$$

By integrating (2.9), one may carefully check that

$$F_{-\tau,2} = \frac{e^{\tau s_3}(s_1 - s_2) + e^{\tau s_2}(s_3 - s_1) + e^{\tau s_1}(s_2 - s_3)}{\tau^2(s_1 - s_3)(s_2 - s_3)(s_1 - s_2)}. \quad (2.10)$$

For distinct real spectral values  $s_3 \neq s_2 \neq s_1$ , one deduces from (2.8) and (2.9) that for every  $\tau > 0$ ,  $D_{\tau,3}(s_1, s_2, s_3) \neq 0$ . It follows that (2.6) is a Cramer system, and one can immediately compute the coefficients  $\alpha, \beta$ , and  $a$  using the Cramer formula. More precisely, one has the following.

**Lemma 2.4.** *For a fixed  $\tau > 0$ , the quasipolynomial  $\Delta$  admits three distinct real spectral values  $s_3, s_2$  and  $s_1$  if, and only if, the coefficients  $\beta, \alpha$  and  $a$  are respectively given by*

$$\beta(\tau) = \frac{1}{D_{\tau,3}(s_1, s_2, s_3)} \det \begin{bmatrix} s_1 & -s_1 e^{\tau s_1} & e^{\tau s_1} \\ s_2 & -s_2 e^{\tau s_2} & e^{\tau s_2} \\ s_3 & -s_3 e^{\tau s_3} & e^{\tau s_3} \end{bmatrix}. \quad (2.11)$$

$$\alpha(\tau) = \frac{1}{D_{\tau,3}(s_1, s_2, s_3)} \det \begin{bmatrix} -s_1 e^{\tau s_1} & 1 & e^{\tau s_1} \\ -s_2 e^{\tau s_2} & 1 & e^{\tau s_2} \\ -s_3 e^{\tau s_3} & 1 & e^{\tau s_3} \end{bmatrix} \quad (2.12)$$

$$= \frac{F_{-\tau,2}(s_1 + s_2, s_1 + s_3, s_2 + s_3)}{F_{-\tau,2}(s_1, s_2, s_3)}.$$

$$a(\tau) = \frac{1}{D_{\tau,3}(s_1, s_2, s_3)} \det \begin{bmatrix} s_1 & 1 & -s_1 e^{\tau s_1} \\ s_2 & 1 & -s_2 e^{\tau s_2} \\ s_3 & 1 & -s_3 e^{\tau s_3} \end{bmatrix} \quad (2.13)$$

$$= -s_1 - \frac{F_{-\tau,1}(s_2, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)}$$

$$= -s_2 - \frac{F_{-\tau,1}(s_1, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)}$$

$$= -s_3 - \frac{F_{-\tau,1}(s_1, s_2)}{\tau F_{-\tau,2}(s_1, s_2, s_3)}.$$

Here, one has for all  $(u, v) \in \mathbb{R}^2$ ,

$$F_{-\tau,1}(u, v) := \int_0^1 e^{\tau(u+(1-t)v)} dt = \frac{e^{\tau u} - e^{\tau v}}{\tau(u - v)} > 0. \quad (2.14)$$

*Proof.* The Relation (2.11) and the first identities in relations (2.12)-(2.13) follow directly by applying the Cramer formulas to the Cramer system (2.6). To obtain the others in (2.12)-(2.13), one may use (2.8), (2.10), (2.14) and the following.

$$\Lambda_1 = e^{\tau(s_1+s_2)}(s_1 - s_2) + e^{\tau(s_1+s_3)}(s_3 - s_1) + e^{\tau(s_2+s_3)}(s_2 - s_3)$$

$$= \tau^2(s_1 - s_2)(s_1 - s_3)(s_2 - s_3)F_{-\tau,2}(s_1 + s_2, s_1 + s_3, s_2 + s_3)$$

$$\Lambda_2 = e^{\tau s_3}(s_2 - s_1)s_3 + e^{\tau s_2}(s_1 - s_3)s_2 + e^{\tau s_1}(s_3 - s_2)s_1$$

$$= -s_1 D_{\tau,3}(s_1, s_2, s_3) - (e^{\tau s_2} - e^{\tau s_3})(s_1 - s_2)(s_1 - s_3)$$

$$= -s_2 D_{\tau,3}(s_1, s_2, s_3) - (e^{\tau s_1} - e^{\tau s_3})(s_1 - s_2)(s_2 - s_3)$$

$$= -s_3 D_{\tau,3}(s_1, s_2, s_3) - (e^{\tau s_1} - e^{\tau s_2})(s_1 - s_3)(s_2 - s_3)$$

where

$$\Lambda_1 := \det \begin{bmatrix} -s_1 e^{\tau s_1} & 1 & e^{\tau s_1} \\ -s_2 e^{\tau s_2} & 1 & e^{\tau s_2} \\ -s_3 e^{\tau s_3} & 1 & e^{\tau s_3} \end{bmatrix}, \quad \Lambda_2 := \det \begin{bmatrix} s_1 & 1 & -s_1 e^{\tau s_1} \\ s_2 & 1 & -s_2 e^{\tau s_2} \\ s_3 & 1 & -s_3 e^{\tau s_3} \end{bmatrix}.$$

This completes the proof of the lemma.  $\square$

The next lemma is a key ingredient in simplifying the proofs of our main results in Section 3.1.

**Lemma 2.5.** *Assume that the quasipolynomial  $\Delta$  has three real spectral values  $s_3 < s_2 < s_1$ . Then, the following holds*

$$0 < \alpha < e^{\tau s_1}. \quad (2.15)$$

$$\alpha < e^{\tau x} \quad (\forall x \geq s_1).$$

*Proof.* Let  $x \geq s_1$ . Then, one has  $e^{\tau x} - \alpha > 0$  since

$$e^{\tau x} - \alpha = \frac{e^{\tau x} F_{-\tau,2}(s_2, s_3, s_1) - F_{-\tau,2}(s_1 + s_2, s_2 + s_3, s_1 + s_3)}{F_{-\tau,2}(s_2, s_3, s_1)}$$

and  $F_{-\tau,2}(s_1, s_2, s_3) = F_{-\tau,2}(s_2, s_3, s_1)$  by [3, Lemma 4] and the fact that for every  $(t_1, t_2) \in [0, 1]^2$ , one has

$$x + t_1 s_2 + (1 - t_1)(t_2 s_3 + (1 - t_2)s_1) - t_1(s_1 + s_2) -$$

$$(1 - t_1)(t_2(s_2 + s_3) - (1 - t_2)(s_1 + s_3)) =$$

$$x - t_1 s_1 - t_2(1 - t_1)s_2 - (1 - t_1)(1 - t_2)s_3 >$$

$$(1 - t_1)s_1 - t_2(1 - t_1)s_1 - (1 - t_1)(1 - t_2)s_1 = 0. \quad \square$$

**Remark 2.6.** From Lemmas 2.4 and 2.5, if  $s_1 < 0$  then for every  $\tau > 0$ , one has  $\beta(\tau) = -\alpha s_1 + \zeta e^{\tau s_1} > 0$  and  $a(\tau)$  may change sign. Here,

$$\zeta := \zeta(\tau) = \frac{F_{-\tau,1}(s_2, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)} > 0. \quad (2.16)$$

### 3. Main results

In this section, we establish our main results. Section 3.1 discusses the GCRRID property for the quasipolynomial  $\Delta$  and derives a new and simpler proof for its validity. In Section 3.2, the ICRRID property is investigated. Next, Section 3.3 provides an example of a simple dominant spectral value violating the sufficient condition established by Frasson-Verduyn Lunel, thereby emphasizing the non-necessary nature of the latter condition. Additionally, we will present a simpler proof of the exponential estimates for solutions of equation (NDE) in Section 3.4.

#### 3.1. Assigning three distinct real spectral values

This section discusses the GCRRID property for the quasipolynomial  $\Delta$ . It involves assigning the maximal number of distinct real spectral values  $s_3 < s_2 < s_1$  to  $\Delta$  and proving that  $s_1$  is a dominant root. We shall then use this result to establish necessary and sufficient conditions to ensure that  $s_1$  is negative, which is essential to the exponential stability of equation (NDE). Finally, we explicitly characterize the spectrum of  $\Delta$  when it admits three real roots  $s_3 < s_2 < s_1$ .

The first main result of this section provides a comprehensive proof of [27, Theorem 5].

**Theorem 3.1** (Dominance of a real root). *Assume that  $\Delta$  admits three real spectral values  $s_3 < s_2 < s_1$ . Then, the spectral value  $s_1$  is a strictly dominant root of  $\Delta$ .*

*Proof.* Fix  $\tau > 0$ . It follows from Lemma 2.4 that  $a + s_1 < 0$  and  $\alpha > 0$ . We argue by contradiction. Assume that there exists  $s_0 := x + i\omega \in \mathbb{C}$  such that  $\Delta(s_0) = 0$  and  $x \geq s_1$ . In particular,  $\omega \neq 0$ , since we have already assigned three real spectral values  $s_3 < s_2 < s_1$  to  $\Delta$ . From  $\Delta(s_1) = 0$ , one gets  $\beta = -\alpha s_1 - e^{\tau s_1}(a + s_1)$ . Therefore,  $\Delta(s_0) = 0$  if, and only if,

$$e^{\tau s_0} s_0 + e^{\tau s_0} a + \alpha s_0 = \alpha s_1 + e^{\tau s_1}(a + s_1). \quad (3.1)$$

By taking the real and imaginary parts in (3.1), one gets

$$\begin{aligned} (a + x) \cos(\tau\omega) - \omega \sin(\tau\omega) &= e^{-\tau(x-s_1)}(a + s_1) - \alpha(x - s_1)e^{-\tau x} \\ (a + x) \sin(\tau\omega) + \omega \cos(\tau\omega) &= -\alpha\omega e^{-\tau x}. \end{aligned} \quad (3.2)$$

By squaring each equality in (3.2) and adding them yields

$$\omega^2 = \frac{(e^{\tau s_1}(a + s_1) - \alpha(x - s_1))^2 - (a + x)^2 e^{2\tau x}}{e^{2\tau x} - \alpha^2} \quad (3.3)$$

which is well defined for every  $x \geq s_1$  by Lemma 2.5. Let us prove that  $\omega^2$  given by (3.3) satisfies  $\omega^2 < 1/\tau^2$ . To do so, define for all  $x \geq s_1$  the function

$$\chi(x) = (e^{\tau s_1}(a + s_1) - \alpha(x - s_1))^2 - (a + x)^2 e^{2\tau x} - \frac{(e^{2\tau x} - \alpha^2)}{\tau^2}. \quad (3.4)$$

We want to prove that  $\chi(x) < 0$  for every  $x \geq s_1$ . On the one hand, one has

$$\chi(s_1) = -(e^{2\tau s_1} - \alpha^2)\tau^{-2} < 0, \quad \lim_{x \rightarrow \infty} \chi(x) = -\infty \quad (3.5)$$

owing to Lemma 2.5. On the other hand, function  $\chi$  is infinitely derivable on  $[s_1, \infty)$ , so that for all  $x \geq s_1$ ,

$$\chi''(x) = -2e^{2\tau x} \left( 2(\tau(a + x) + 1)^2 + e^{-2\tau x}(e^{2\tau x} - \alpha^2) \right) < 0 \quad (3.6)$$

since the two terms between the big brackets of  $\chi''(x)$  are positive by Lemma 2.5. One also has for all  $x \geq s_1$ ,

$$\begin{aligned} \chi'(x) &= -2\alpha((a + s_1)e^{\tau s_1} - \alpha(x - s_1)) - 2\tau(a + x)^2 e^{2\tau x} \\ &\quad - 2(a + x)e^{2\tau x} - 2\tau^{-1}e^{2\tau x} \end{aligned} \quad (3.7)$$

$$\begin{cases} \chi'(s_1) = -2\tau \left( (a + s_1)e^{\tau s_1} + \frac{\alpha + e^{\tau s_1}}{2\tau} \right)^2 - \frac{(e^{\tau s_1} - \alpha)(\alpha + 3e^{\tau s_1})}{2\tau} \\ \lim_{x \rightarrow \infty} \chi'(x) = -\infty. \end{cases}$$

Here,  $\chi'(s_1) < 0$ , as the sum of two negative terms by Lemma 2.5. It follows from (3.6) and (3.7) that

$$\chi'(x) < 0 \quad (\forall x \geq s_1). \quad (3.8)$$

Finally, by combining (3.8) and (3.5), one gets that

$$\chi(x) < 0 \quad (\forall x \geq s_1). \quad (3.9)$$

The latter is equivalent to  $\omega^2 < \tau^{-2}$  with  $\omega^2$  defined in (3.3). So, we have shown that if  $x + i\omega$  is a root of  $\Delta$  with  $x \geq s_1$  and  $\omega \neq 0$ , then  $|\omega| < \tau^{-1}$ . The latter is inconsistent since one has necessarily  $|\omega| \geq 2\pi/\tau$  owing to Lemma 2.3.  $\square$

**Remark 3.2.** It is worth noting that the key property on  $\alpha$  required in the proof of Theorem 3.1 is that  $e^{2\tau x} - \alpha^2 \geq 0$  for all  $x \geq s_1$  which is satisfied owing to Lemma 2.5.

The second main result provides the necessary and sufficient conditions on the delay  $\tau$  and the coefficient  $a$  to guarantee that a dominant root  $s_1$  is negative when three real spectral values  $s_3 < s_2 < s_1$  are formally assigned to  $\Delta$ .

**Theorem 3.3** (Negativity of a dominant root). *Assume that  $\Delta$  admits three real spectral values  $s_3 < s_2 < s_1$ . Then,  $s_1 < 0$  if, and only if, there is a unique  $\tau_* > 0$  such that  $a(\tau_*) = 0$ , i.e.,*

$$a(\tau) \begin{cases} = 0 & \text{if } \tau = \tau_* \\ < 0 & \text{if } \tau < \tau_* \\ > 0 & \text{if } \tau > \tau_*. \end{cases} \quad (3.10)$$

More precisely, it holds

$$s_1 \begin{cases} = -\zeta(\tau) & \text{if } \tau = \tau_* \\ > -\zeta(\tau) & \text{if } \tau < \tau_* \\ < -\zeta(\tau) & \text{if } \tau > \tau_*, \end{cases} \quad \zeta(\tau) := \frac{F_{-\tau,1}(s_2, s_3)}{\tau F_{-\tau,2}(s_1, s_2, s_3)}. \quad (3.11)$$

*Proof.* Let us assume that  $s_1 < 0$ . Since  $\tau \in (0, \infty) \mapsto a(\tau)$  is continuous with respect to  $\tau$ , one may use the intermediate value theorem. Combining (2.13) and (2.10), one gets

$$a(\tau) = \frac{e^{-\tau(s_1-s_3)}(s_1-s_2)s_3 + e^{-\tau(s_1-s_2)}(s_3-s_1)s_2 + (s_2-s_3)s_1}{e^{-\tau(s_1-s_3)}(s_2-s_1) + e^{-\tau(s_1-s_2)}(s_1-s_3) + (s_3-s_2)}.$$

Consequently,  $a(\tau) \rightarrow -s_1 > 0$  when  $\tau \rightarrow \infty$ . Next, (2.9), (2.14), and the continuity of  $F_{-\tau,1}$  and  $F_{-\tau,2}$  with respect to  $\tau$  imply that  $a(\tau) \rightarrow -\infty$  as  $\tau \rightarrow 0$ . It follows that there exists at least one  $\tau_* > 0$  such that  $a(\tau_*) = 0$ . To show that  $\tau_*$  is unique, one applies Lemma 2.2. We observe that  $a(\tau) = 0$  if, and only if, its numerator vanishes at  $\tau$ . Now, let

$$F(\tau) = e^{\tau s_3}(s_1 - s_2)s_3 + e^{\tau s_2}(s_3 - s_1)s_2 + e^{\tau s_1}(s_2 - s_3)s_1 \quad (3.12)$$

be the numerator of  $e^{\tau s_1}a(\tau)$ . Then,  $F$  is analytic in  $\tau$ , and one has from  $s_3 < s_2 < s_1$  that  $(s_1 - s_2)s_3 < 0$ ,  $(s_3 - s_1)s_2 > 0$  and  $(s_2 - s_3)s_1 < 0$ . Let  $C$  denote the number of sign changes in the sequence of real numbers  $(s_1 - s_2)s_3$ ,  $(s_3 - s_1)s_2$ , and  $(s_2 - s_3)s_1$ , then  $C = 2$ . Similarly, if  $Z$  represents the number of real zeros of the entire function  $F$ , then  $C - Z = 2 - Z \geq 0$  and therefore  $Z = 2$  by Lemma 2.2. Since  $F(0) = 0$  and  $F(\tau_*) = 0$ , the uniqueness of  $\tau_* > 0$  follows.

Conversely, assume the existence of a unique  $\tau_* > 0$  such that  $a(\tau_*) = 0$ . One immediately deduces from (2.9), (2.13) and (2.14) that

$$s_1 = -\frac{F_{-\tau_*,1}(s_2, s_3)}{\tau_* F_{-\tau_*,2}(s_1, s_2, s_3)} < 0. \quad \square$$

**Remark 3.4.** Theorem 3.3 states that the exponential stability of equation (NDE) may be achieved with a prescribed decay rate, even though  $a \leq 0$ . Time-domain techniques based on a Lyapunov functional and LMI do not cover this case. See, for instance, [19, 15, 10].

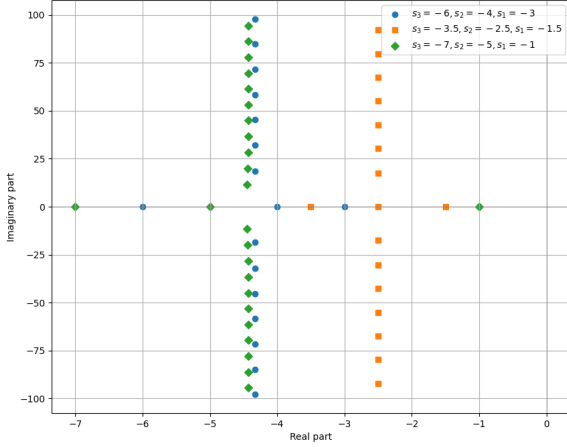


Figure 1: Spectrum of the quasipolynomial  $\Delta$  for various parameters when  $\tau = 1$ . This is obtained by assigning three real spectral values  $s_3 < s_2 < s_1$  to  $\Delta$  and computing each case's system parameters  $a$ ,  $\alpha$ , and  $\beta$ .

In the case of equidistributed real spectral values  $s_3 < s_2 < s_1$ , the delay  $\tau_* > 0$  that enables the design of  $s_1 < 0$  may be explicitly computed, as is stated hereafter.

**Corollary 3.5.** *Assume that the quasipolynomial  $\Delta$  admits three equidistributed real spectral values  $s_k = s_1 - (k - 1)d$ , for  $d > 0$ . Then,  $s_1 < 0$  if, and only if, the delay  $\tau_* > 0$  in Theorem 3.1 reads*

$$\tau_* = \frac{1}{d} \ln \left( \frac{s_3}{s_1} \right) > 0. \quad (3.13)$$

*Proof.* By Theorem 3.3,  $s_1 < 0$ , if and only if, there exists one, and only one,  $\tau_* > 0$  such that  $a(\tau_*) = 0$ , i.e.,

$$s_1 = -\frac{F_{-\tau_*,1}(s_2, s_3)}{\tau_* F_{-\tau_*,2}(s_1, s_2, s_3)}. \quad (3.14)$$

Letting  $s_2 = s_1 - d$  and  $s_3 = s_1 - 2d$  with  $d > 0$ , one finds

$$F_{-\tau_*,2}(s_1, s_2, s_3) = \frac{e^{\tau_* s_1} (1 - e^{-\tau_* d})^2}{2\tau_*^2 d^2} \quad (3.15)$$

$$F_{-\tau_*,1}(s_2, s_3) = \frac{e^{\tau_* s_1} e^{-\tau_* d} (1 - e^{-\tau_* d})}{\tau_* d}. \quad (3.16)$$

By injecting (3.15)-(3.16) into (3.14), one gets

$$s_1 = \frac{-2d}{e^{\tau_* d} - 1} \iff \tau_* = \frac{1}{d} \ln \left( \frac{s_3}{s_1} \right) > 0, \quad (3.17)$$

since  $s_3 < s_1 < 0$ .  $\square$

In the specific case where a single real spectral value  $s_3 = s_2 = s_1$  is assigned to  $\Delta$ , the GMID is demonstrated in [5]. Furthermore, the analysis fully characterizes the remaining spectrum of  $\Delta$  as

$$s = s_1 + i \frac{\omega}{\tau}, \quad \text{where} \quad \tan \left( \frac{\omega}{2} \right) = \frac{\omega}{2}. \quad (3.18)$$

In the following, we fully characterize the spectrum of  $\Delta$  when it admits three real spectral values.

**Theorem 3.6.** *Assume that  $\Delta$  admits three real spectral values  $s_3 < s_2 < s_1$ . Then, the remaining spectrum of  $\Delta$  is given by*

$$s = s_2 + \frac{\ln(\theta)}{\tau} + i \frac{\omega}{\tau} \quad (3.19)$$

if and only if

$$\ln(\theta) = \xi \frac{(\theta - 1)}{2\theta}. \quad (3.20)$$

The imaginary part  $\omega \neq 0$  solves

$$\frac{(\theta + 1)}{2\theta} \tan \left( \frac{\omega}{2} \right) = \frac{\omega}{\xi} \quad (3.21)$$

where  $\theta := \alpha e^{-\tau s_2}$  with  $0 < \alpha < e^{\tau s_1}$  given by (2.12) and

$$\xi := \frac{F_{-\tau,1}(s_1, s_3)}{F_{-\tau,2}(s_1, s_2, s_3)} > 0. \quad (3.22)$$

As an immediate consequence of Theorem 3.6, one has the following.

**Corollary 3.7.** *Assume that  $\Delta$  admits three real spectral values  $s_3 < s_2 < s_1$ . If (3.20) is not satisfied, then the remaining spectrum of  $\Delta$  forms a chain asymptotic to the vertical line*

$$\Re(s) = \ln(\alpha) \quad (3.23)$$

where  $0 < \alpha < e^{\tau s_1}$  is given by (2.12). More precisely, there exists  $s_0 \in \mathbb{C}$  such that the following holds

$$\Delta(s_0) = 0 \quad \text{and} \quad \Re(s_0) \neq \ln(\alpha). \quad (3.24)$$

Before proving Theorem 3.6, let us illustrate it in the specific case where the quasipolynomial  $\Delta$  has three equidistributed real spectral values.

**Corollary 3.8.** *Assume that  $\Delta$  admits three equidistributed real spectral values  $s_k = s_1 - (k - 1)d$ , for  $d > 0$  and  $k = 1, 2, 3$ . Then, the remaining spectrum of  $\Delta$  is characterized by*

$$s = s_2 + i \frac{\omega}{\tau} \quad (3.25)$$

where the imaginary part  $\omega \neq 0$  solves

$$\tan \left( \frac{\omega}{2} \right) = \frac{\omega}{\xi(\tau, d)}, \quad \xi(\tau, d) = \tau d \coth \left( \frac{\tau d}{2} \right). \quad (3.26)$$

Here  $\coth$  is the cotangent hyperbolic function.

*Proof.* Since  $s_3 < s_2 < s_1$  are equidistributed, one has from  $\theta = \alpha e^{-\tau s_2}$ , (2.12) and (3.22) that  $\theta = 1$  and  $\xi := \xi(\tau, d) = \tau d \coth \left( \frac{\tau d}{2} \right)$ . Since  $\theta = 1$ , and  $\xi > 0$ , then (3.20) is satisfied, and the result follows by (3.19) and (3.21).  $\square$

**Remark 3.9.** Since  $\xi(\tau, d) \rightarrow 2$  as  $d \rightarrow 0$ , (3.18) coincides with (3.25), and Corollary 3.8 confirms the intuitive concept that the GMID property is the limiting case of the GCRRID property. We illustrated in Figure 2 values of  $d := s_1 - s_2$  and  $\delta := s_1 - s_3$  where (3.20) is satisfied. We also refer to Figure 1, where we depicted the quasipolynomial  $\Delta$ 's spectrum in three scenarios.

*Proof of Theorem 3.6.* Using (2.11), (2.12) and (2.13), one has

$$\alpha = \theta e^{\tau s_2}, \quad a = -s_2 - \frac{\xi}{\tau}, \quad \beta = -\alpha s_2 + \frac{\xi}{\tau} e^{\tau s_2}. \quad (3.27)$$

Let  $d := s_1 - 2$ ,  $\delta := s_1 - s_3$ , and consider the change of functions

$$\Delta'(z) = \tau \Delta\left(s_2 + \frac{z}{\tau}\right) = z - \xi + e^{-z}(\theta z + \xi) \quad (z \in \mathbb{C}) \quad (3.28)$$

so that  $\Delta'$  has three real roots  $-\tau(\delta - d)$ ,  $0$  and  $\tau d$ . In particular,  $z \in \mathbb{C}$  is a spectral value of  $\Delta'$  if, and only if,  $s_2 + z/\tau$  is a spectral value of  $\Delta$ . Moreover, since  $s_1$  is a dominant root of  $\Delta$ , then  $\tau d$  is a dominant root of  $\Delta'$ . Let now  $z_0 = x_0 + i\omega_0$  be a complex non-real spectral value of  $\Delta'$ . Then,  $x_0 < \tau d$  and  $|\omega_0| \geq 2\pi$  by Lemma 2.3. From  $\Delta'(z_0) = 0$ , one obtains

$$\frac{z - \xi}{\theta z + \xi} = -e^{-z} \implies \sqrt{\frac{(x_0 - \xi)^2 + \omega_0^2}{(\theta x_0 + \xi)^2 + \theta^2 \omega_0^2}} = e^{-x_0}. \quad (3.29)$$

Letting  $|\omega_0| \rightarrow \infty$  in (3.29), one deduces that

$$e^{-x_0} = 1/\theta, \quad \text{since} \quad 0 < \theta < e^{\tau(s_1 - s_2)} = e^{\tau d}. \quad (3.30)$$

It follows that  $x_0 = \ln(\theta)$ , and the rest of the spectrum of  $\Delta'$  is either located on the vertical line  $\Re(z) = \ln(\theta)$  or forms a chain asymptotic to  $\Re(z) = \ln(\theta)$ . But, if (3.20) is satisfied, then (3.28) yields

$$\begin{aligned} \Delta'(2 \ln(\theta) - \bar{z}_0) &= 2 \ln(\theta) - \bar{z}_0 - \xi + \frac{e^{\bar{z}_0}}{\theta^2} (\theta(2 \ln(\theta) - \bar{z}_0) + \xi) \\ &= -\bar{z}_0 - \frac{\xi}{\theta} + \frac{e^{\bar{z}_0}}{\theta} (-\bar{z}_0 + \xi) = -\frac{e^{\bar{z}_0}}{\theta} \Delta'(\bar{z}_0) \end{aligned} \quad (3.31)$$

for every  $z_0 \in \mathbb{C}$ . Therefore, since  $\Delta'$  has real coefficients, a complex number  $\bar{z}_0$  is a spectral value of  $\Delta'$  if, and only if,  $2 \ln(\theta) - \bar{z}_0$ , the reflection of  $z_0$  across the vertical line  $\Re(z) = \ln(\theta)$ , is a spectral value of  $\Delta'$ . As a result, the remaining spectrum of  $\Delta'$  exists along the vertical line  $\Re(z) = \ln(\theta)$ . One deduces that the remaining spectrum of  $\Delta$  exists along the vertical line  $\Re(s) = s_2 + \ln(\theta)/\tau$ . Substituting  $z = \ln(\theta) + i\omega$  for some real number  $\omega \neq 0$  into  $\Delta'(z) = 0$ , yields

$$\ln(\theta) + i\omega = \xi \frac{(\theta - 1)}{2\theta} + i\xi \frac{(\theta + 1)}{2\theta} \tan\left(\frac{\omega}{2}\right) \quad (3.32)$$

after careful computations. Then, (3.20) and (3.21) follow, completing the proof of the theorem.  $\square$

### 3.2. Assigning two distinct real spectral values

This section discusses the ICRRID property of the quasipolynomial  $\Delta$ . The aim is to assign two real spectral values  $s_2 < s_1$  to  $\Delta$  and to find necessary and sufficient conditions on the coefficients of  $\Delta$  guaranteeing that  $s_1$  is a dominant root. Finally, we explicitly characterize the spectrum of  $\Delta$  when it admits exactly two real roots  $s_2 < s_1$ .

Inspired from [27, Theorem 6], we find these conditions in terms of  $(a + s_1)/(s_1 - s_2)$ . The challenge relies on the scenario where a third real spectral value  $x$  coexists in the spectrum of  $\Delta$ , in which case the coefficients  $a$ ,  $\alpha$ , and  $\beta$  are given

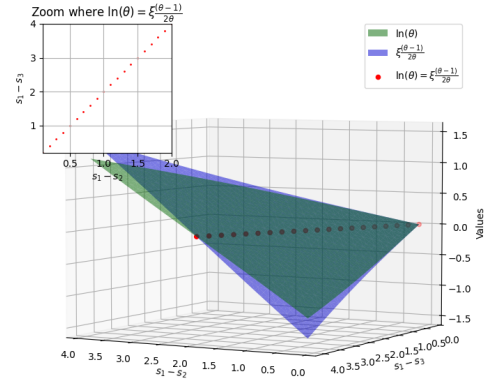


Figure 2: Plot of  $\ln(\theta)$  and  $\xi^{\frac{(\theta-1)}{2\theta}}$  as functions of  $d := s_1 - s_2$  and  $\delta := s_1 - s_3$ . Here  $\theta := \alpha e^{-\tau s_2}$  with  $0 < \alpha < e^{\tau s_1}$  given by (2.12) and  $\xi$  is defined by (3.22). The red dots characterize values of  $d$  and  $\delta$  where (3.20) is satisfied.

in Lemma 2.4. Therefore, the greatest between  $s_2$ ,  $s_1$ , and  $x$  is a dominant root by Theorem 3.1. Since one wants  $s_1$  to be a dominant root of  $\Delta$ , we are looking for the necessary and sufficient conditions on  $(a + s_1)/(s_1 - s_2)$  ensuring that  $x < s_2 < s_1$  or  $s_2 < x < s_1$ . To this end, one deduces from (2.13) that

$$\frac{a + s_1}{s_1 - s_2} = -\frac{(s_1 - x)(e^{\tau s_2} - e^{\tau x})}{e^{\tau x}(s_1 - s_2) + e^{\tau s_2}(x - s_1) + e^{\tau s_1}(s_2 - x)} \quad (3.33)$$

for  $\forall x \in \mathbb{R} \setminus \{s_1, s_2\}$ , which is negative owing to (2.9) and (2.10). On the other hand, introduce for all  $x \in \mathbb{R} \setminus \{s_1, s_2\}$ ,

$$\varphi(x) = \frac{(s_1 - x)(e^{\tau s_2} - e^{\tau x})}{e^{\tau x}(s_1 - s_2) + e^{\tau s_2}(x - s_1) + e^{\tau s_1}(s_2 - x)}. \quad (3.34)$$

Then,  $\varphi(x) > 0$  for every  $x < s_2 < s_1$  or  $s_2 < x < s_1$  or  $x > s_1$ , and satisfies

$$\begin{aligned} \lim_{x \rightarrow -\infty} \varphi(x) &= \frac{1}{e^{\tau(s_1 - s_2)} - 1} \\ \lim_{x \rightarrow s_2} \varphi(x) &= \frac{\tau(s_1 - s_2)}{e^{\tau(s_1 - s_2)} - (1 + \tau(s_1 - s_2))} \\ \lim_{x \rightarrow s_1} \varphi(x) &= \frac{e^{\tau(s_1 - s_2)} - 1}{e^{\tau(s_1 - s_2)}(\tau(s_1 - s_2) - 1) + 1}. \end{aligned}$$

Moreover, letting for all  $x \in \mathbb{R} \setminus \{s_1, s_2\}$ ,

$$\begin{aligned} \psi(x) &= \tau^2(s_2 - x)(s_1 - x)(s_1 - s_2)(F_{-\tau,1}(x, s_1)F_{-\tau,1}(x, s_2) \\ &\quad - e^x F_{-\tau,1}(s_1, s_2)) \end{aligned}$$

one finds

$$\varphi'(x) = \frac{\psi(x)}{[e^{x\tau}(-s_1 + s_2) + e^{s_2\tau}(s_1 - x) + e^{s_1\tau}(-s_2 + x)]^2}$$

thanks to the integral representation (2.14). Using [3, Lemma 7], one finds that

$$F_{-\tau,1}(u, v) - F_{-\tau,1}(u, w) = \tau(v - w)F_{-\tau,2}(u, v, w) > 0 \Leftrightarrow v > w$$

for all  $\tau > 0$  and  $(u, v, w) \in \mathbb{R}^3$ . Therefore, for a fixed  $u \in \mathbb{R}$ ,  $v \mapsto F_{-\tau,1}(u, v)$  is positive and increasing on  $\mathbb{R}$ . One deduces

that  $\varphi'(x) > 0$  for every  $x < s_2 < s_1$  or  $s_2 < x < s_1$  or  $x > s_1$ , and that  $\varphi$  is a positive increasing function for  $x < s_2 < s_1$  or  $s_2 < x < s_1$  or  $x > s_1$ . Therefore, if  $s_2 < s_1$ , then

$$\varphi(x) \geq \frac{1}{e^{\tau(s_1-s_2)} - 1}, \quad (\forall x \in \mathbb{R} \setminus \{s_1, s_2\}).$$

We proved the following.

**Theorem 3.10.** *Assume that the quasipolynomial  $\Delta$  admits two real spectral values  $s_2 < s_1$ . Then, a third real spectral value  $x$  coexists in the spectrum of  $\Delta$  if and only if*

$$\Lambda_3 := \frac{a + s_1}{s_1 - s_2} \leq \frac{-1}{e^{\tau(s_1-s_2)} - 1}. \quad (3.35)$$

Furthermore,

1.  $x < s_2 < s_1$  if and only if

$$\frac{-\tau(s_1 - s_2)}{e^{\tau(s_1-s_2)} - (1 + \tau(s_1 - s_2))} < \Lambda_3 \leq \frac{-1}{e^{\tau(s_1-s_2)} - 1}$$

2.  $s_2 < x < s_1$  if and only if

$$\frac{1 - e^{\tau(s_1-s_2)}}{e^{\tau(s_1-s_2)}(\tau(s_1 - s_2) - 1) + 1} < \Lambda_3 < \frac{-\tau(s_1 - s_2)}{e^{\tau(s_1-s_2)} - (1 + \tau(s_1 - s_2))}$$

3.  $s_2 < s_1 < x$  if and only if

$$\Lambda_3 \leq \frac{1 - e^{\tau(s_1-s_2)}}{e^{\tau(s_1-s_2)}(\tau(s_1 - s_2) - 1) + 1}.$$

Let us now investigate the necessary and sufficient conditions on  $(a + s_1)/(s_1 - s_2)$  guaranteeing the dominance of a simple real spectral value  $s_1$  when exactly two real spectral values  $s_2 < s_1$  are assigned to the quasipolynomial  $\Delta$ .

**Lemma 3.11.** *The quasipolynomial  $\Delta$  admits exactly two distinct real spectral values  $s_2$  and  $s_1$  if, and only if  $a \in \mathbb{R}$ ,*

$$\beta = -\alpha s_1 - e^{\tau s_1}(a + s_1) = -\alpha s_2 - e^{\tau s_2}(a + s_2). \quad (3.36)$$

$$\alpha = \frac{-(a + s_1)e^{\tau s_1} + (a + s_2)e^{\tau s_2}}{s_1 - s_2}. \quad (3.37)$$

**Remark 3.12.** It is an immediate consequence of Theorem 3.10 that, exactly two real spectral values  $s_2 < s_1$  coexist in the spectrum of  $\Delta$ , if and only if, the following inequality holds

$$\frac{-1}{e^{\tau(s_1-s_2)} - 1} < \frac{a + s_1}{s_1 - s_2}. \quad (3.38)$$

The next lemma is a key ingredient in proving our second main result.

**Lemma 3.13.** *Assume that  $\Delta$  admits exactly two real spectral values  $s_2 < s_1$ . Then, the following conditions hold.*

$$\frac{-1}{e^{\tau(s_1-s_2)} - 1} < \frac{a + s_1}{s_1 - s_2} < 1 \iff -e^{\tau s_1} < \alpha < 0. \quad (3.39)$$

In particular, the necessary part of (3.39) implies that

$$-e^{\tau x} < \alpha < 0 \quad \forall x \geq s_1. \quad (3.40)$$

Additionally, if  $s_1 \leq 0$ , then  $\beta \leq 0$  whenever

$$0 \leq \frac{a + s_1}{s_1 - s_2} < 1. \quad (3.41)$$

*Proof.* Firstly, from (3.37), one gets

$$\alpha = -\frac{a + s_1}{s_1 - s_2}(e^{\tau s_1} - e^{\tau s_2}) - e^{\tau s_2}. \quad (3.42)$$

Therefore, the equivalence (3.39) holds owing to

$$-(e^{\tau s_1} - e^{\tau s_2}) - e^{\tau s_2} < \alpha < \frac{e^{\tau s_1} - e^{\tau s_2}}{e^{\tau(s_1-s_2)} - 1} - e^{\tau s_2} \iff -e^{\tau s_1} < \alpha < 0.$$

Inequality (3.40) is a consequence of the necessary part of (3.39) and that  $x \in \mathbb{R} \mapsto e^x$  is increasing. Finally, if  $s_1 \leq 0$ , then  $\beta \leq 0$  follows immediately from (3.36),  $\alpha < 0$  and (3.41).  $\square$

One can now prove the following result, which gives necessary and sufficient conditions for the dominance of a real spectral value when  $\Delta$  admits exactly two real spectral values.

**Theorem 3.14.** *Assume that exactly two real spectral values  $s_2 < s_1$  coexist in the spectrum of  $\Delta$ . Then,  $s_1$  is a strictly dominant root of  $\Delta$  if, and only if, either the sufficient or the necessary part of (3.39) are satisfied.*

*Proof.* Firstly, it follows from Remark 3.12 that  $\Delta$  admits exactly two real spectral values if, and only if, the sufficient part in (3.39) is satisfied. Let  $s_0 := x + i\omega \in \mathbb{C}$  be such that  $\Delta(s_0) = 0$ . In particular,  $\omega \neq 0$ . From  $\Delta(s_1) = 0$ , one gets  $\beta = -\alpha s_1 - e^{\tau s_1}(a + s_1)$ , where  $\alpha$  and  $a$  are given by Lemma 3.11. It follows that  $\Delta(s_0) = 0$  if, and only if,

$$e^{\tau s_0} s_0 + e^{\tau s_0} a + \alpha s_0 = \alpha s_1 + e^{\tau s_1}(a + s_1). \quad (3.43)$$

By taking the real and imaginary parts of both sides in (3.43), one gets

$$\begin{aligned} e^{\tau x}((a + x) \cos(\tau\omega) - \omega \sin(\tau\omega)) &= e^{\tau s_1}(a + s_1) - \alpha(x - s_1) \\ e^{\tau x}((a + x) \sin(\tau\omega) + \omega \cos(\tau\omega)) &= -\alpha\omega. \end{aligned} \quad (3.44)$$

By squaring each equality in (3.44) and adding them, one obtains

$$\omega^2 = \frac{(e^{\tau s_1}(a + s_1) - \alpha(x - s_1))^2 - (a + x)^2 e^{2\tau x}}{e^{2\tau x} - \alpha^2}, \quad (3.45)$$

which is well-defined for every  $x \geq s_1$  by Lemma 3.13. Therefore, by performing the exact same steps of the proof of dominance in Theorem 3.1, one gets that  $\omega^2 < 1/\tau^2$ , i.e.,  $|\omega| < 1/\tau$ . The latter is inconsistent since one has necessarily  $|\omega| \geq \pi/\tau$  owing to Lemma 2.3. Hence,  $s_1$  is a dominant root of  $\Delta$ . Conversely, assume that either the sufficient or the necessary part of (3.39) is not satisfied. Then, another real root  $x$  coexists in the spectrum of  $\Delta$  by Theorem 3.10 or  $s_1$  and  $s_2$  are the only real spectral values of  $\Delta$ , and  $s_1$  is not a strictly dominant root of  $\Delta$  by Lemma 3.13 and the sufficient part of this theorem.  $\square$

**Remark 3.15.** In the particular case of exactly two real spectral values  $s_2 < s_1$  coexisting in the spectrum of  $\Delta$ , and  $\alpha = -e^{\tau s_1}$  then  $s_1$  is a dominant (not strict) root of  $\Delta$  since its remaining spectrum is given by  $s = s_1 + i\frac{2\pi k}{\tau}$ ,  $k \in \mathbb{Z}$ . Indeed, if  $\alpha = -e^{\tau s_1}$ , one has  $\Delta(s) = (s - s_2)(1 - e^{-\tau(s-s_1)})$  for every  $s \in \mathbb{C}$ , from

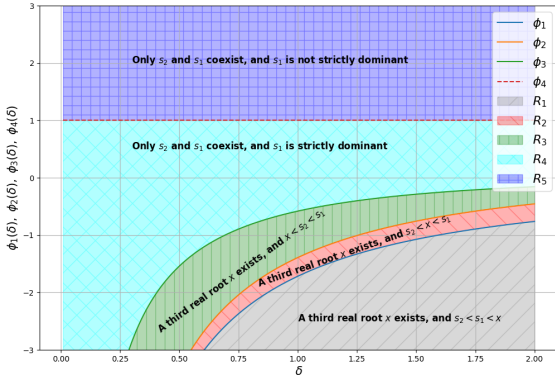


Figure 3: Complete characterization of the regions with respect to the values of  $(a + s_1)/\delta$  that are necessary and sufficient to ensure the strict dominance or not of the real spectral value  $s_1$  when only two real spectral values  $s_2 < s_1$  are assigned to  $\Delta$ . The figure is depicted when the delay  $\tau = 1$  while  $\delta := s_1 - s_2$  ranges between 0.01 and 2. The functions used are  $\phi_1(\delta) = (1 - e^\delta)/(e^\delta(\delta - 1) + 1)$ ,  $\phi_2(\delta) = -\delta/(e^\delta - (1 + \delta))$ ,  $\phi_3(\delta) = -1/(e^\delta - 1)$ , and  $\phi_4(\delta) = 1$ . We define the regions  $R_1$  as the range of  $\delta$  where  $\frac{a+s_1}{\delta} \leq \phi_1(\delta)$ ,  $R_2$  as the range where  $\phi_1(\delta) < \frac{a+s_1}{\delta} \leq \phi_2(\delta)$ ,  $R_3$  as the range where  $\phi_2(\delta) < \frac{a+s_1}{\delta} \leq \phi_3(\delta)$ ,  $R_4$  as the range where  $\phi_3(\delta) < \frac{a+s_1}{\delta} < \phi_4(\delta)$ , and  $R_5$  as the range where  $\frac{a+s_1}{\delta} \geq \phi_4(\delta)$ .

which one obtains the desired result. When  $\alpha = 0$ ,  $s_1$  is still a strictly dominant root of  $\Delta$ . Indeed, it is a dominant root by [3, Theorem 12]. Next, one finds in this case

$$\Delta(s) = \frac{e^{\tau(s_1-s_2)}(s-s_1) + (s_1-s_2)(e^{-\tau(s-s_1)} - 1)}{e^{\tau(s_1-s_2)} - 1}.$$

Letting  $s = s_1 + i\omega$ , one deduces that  $\Delta(s) = 0$  if and only if

$$\cos(\tau\omega) = 1, \quad \omega e^{\tau(s_1-s_2)} = (s_1 - s_2) \sin(\tau\omega).$$

where from one obtains  $\omega = 0$ . It follows that a non-real root  $s$  of  $\Delta$  necessary satisfies  $\Re(s) < s_1$ .

Aside from the necessary and sufficient conditions on the dominance of the assigned rightmost root  $s_1$ , which have already been established in [27, Theorem 6], Theorem 3.10 and Theorem 3.14 provide a complete description of such conditions with respect to the number of assigned roots and the potential coexistence of a third real root. We refer to Figure 3 for a full visualization. The following corollary is an immediate consequence of Theorem 3.14 and Remark 3.15. It generalizes and gives a straightforward proof of the IMID property provided in [4, Theorem 3.1-item 2].

**Corollary 3.16.** *Assume that  $\Delta$  admits two real spectral values  $s_2 < s_1$ . Then,  $s_1$  is a dominant root of  $\Delta$  of multiplicity equal to two if and only if*

$$-1 \leq \tau(a + s_1) \leq 0 \quad (\forall a \in \mathbb{R}). \quad (3.46)$$

*In particular,  $s_1$  is a strictly dominant root of  $\Delta$  if and only if the right inequality in (3.46) is strict.*

*Proof.* Assume that  $\Delta$  admits two real spectral values  $s_2 < s_1$ . By Theorem 3.14 and Remark 3.15, the real root  $s_1$  is a dominant root of  $\Delta$  if and only if  $-e^{\tau s_1} \leq \alpha \leq 0$ . If in addition  $\Delta'(s) = 0$ , then  $\alpha = -(\tau(a + s_1) + 1)e^{\tau s_1}$ . The result follows.  $\square$

As in Theorem 3.6, let us end this section by fully characterizing the remaining spectrum of  $\Delta$  when exactly two real roots coexist in its spectrum. The proof follows *mutatis mutandis* that of Theorem 3.6.

**Theorem 3.17.** *Assume that  $\Delta$  has exactly two real spectral values  $s_2 < s_1$ . Then, its remaining spectrum is given by*

$$s = s_2 + \frac{\ln(-\theta)}{\tau} + i\frac{\omega}{\tau} \quad (3.47)$$

*if and only if*

$$\ln(-\theta) = \xi \frac{(\theta - 1)}{2\theta}. \quad (3.48)$$

*The imaginary part  $\omega \neq 0$  solves*

$$\frac{(\theta + 1)}{2\theta} \cot\left(\frac{\omega}{2}\right) = -\frac{\omega}{\xi} \quad (3.49)$$

*where  $\theta := \alpha e^{-\tau s_2}$  with  $-e^{\tau s_1} < \alpha < 0$  given by (3.37) and*

$$\xi = \frac{(\alpha + e^{\tau s_1})}{F_{-1,\tau}(s_1, s_2)} > 0. \quad (3.50)$$

As an immediate consequence of Theorem 3.17, one has the following.

**Corollary 3.18.** *Assume that  $\Delta$  admits exactly two real spectral values  $s_2 < s_1$ . If (3.48) is not satisfied, then the remaining spectrum of  $\Delta$  forms a chain asymptotic to the vertical line*

$$\Re(s) = \ln(-\alpha) \quad (3.51)$$

*where  $-e^{\tau s_1} < \alpha < 0$  is given by (3.37). More precisely, there exists  $s_0 \in \mathbb{C}$  such that the following holds*

$$\Delta(s_0) = 0 \quad \text{and} \quad \Re(s_0) \neq \ln(-\alpha). \quad (3.52)$$

### 3.3. On Frasson-Verduyn Lunel's sufficient conditions for dominance and beyond

This section builds upon Frasson-Verduyn Lunel's seminal work [12, Lemma 5.1], which established a sufficient condition for the dominance of a simple real spectral value of quasipolynomials with multiple delays. While Frasson-Verduyn Lunel's lemma offers a fundamental method for assessing spectral dominance, its applicability is restricted to specific conditions that may only address certain dynamic scenarios encountered in complex systems. Restricted to the single-delay case, when Frasson-Verduyn Lunel's condition is not met, the CRRID property still provides a guarantee for the dominance of a simple real root.

The first result of this section states as follows.

**Theorem 3.19.** *Let  $\tau > 0$ . There exist  $s_1 \leq 0$  and  $d > 0$ , such that if  $\Delta$  admits three equidistributed real spectral values  $s_3 = s_1 - 2d$ ,  $s_2 = s_1 - d$  and  $s_1$ , then*

$$V(s_1) = (|\alpha|(1 + |s_1|\tau) + |\beta|\tau)e^{-s_1\tau} \quad (3.53)$$

*satisfies  $V(s_1) \geq 1$  and  $s_1$  is a dominant root of  $\Delta$ .*

*Proof.* Assume that the quasipolynomial  $\Delta$  admits three equidistributed real spectral values  $s_k = s_1 - (k-1)d$ , for some  $d > 0$  (that we will choose later on) and  $k = 1, 2, 3$  with  $s_1 \leq 0$ . Necessarily,  $s_k$  is simple since  $\Delta$  cannot admit more than three real roots counting multiplicities [25, Problem 206.2, page 144]. Hence, Theorem 3.1 states that  $s_1$  is a strictly dominant root of  $\Delta$ . Moreover, one deduces from Lemma 2.4 that

$$\alpha = e^{\tau(s_1-d)}, \quad \beta = e^{\tau(s_1-d)}(-s_1 + d + d \coth(\tau d/2)),$$

$$V(s_1) = (1 - 2\tau s_1 + \tau d + \tau d \coth(\tau d/2))e^{-\tau d}. \quad (3.54)$$

Introducing  $g(s_1) := V(s_1) - 1$ , one finds

$$g(s_1) = e^{-\tau d}(\tau d \coth(\tau d/2) + \tau d + 1 - e^{\tau d} - 2\tau s_1).$$

It follows that  $g(s_1) \geq 0$  if, and only if,

$$s_1 \leq (\tau d \coth(\tau d/2) + \tau d + 1 - e^{\tau d})/2\tau. \quad (3.55)$$

In particular, knowing that Frasson-Verduyn Lunel's sufficient condition is not met the first time when  $V(s_1) = 1$ , owing to inequality (3.55) the latter is equivalent to ( $v := \tau s_1$  and  $u := \tau d$ )

$$v(u) = (u \coth(u/2) + u + 1 - e^u)/2 \quad (u > 0). \quad (3.56)$$

A straightforward analysis of the function  $u \mapsto v(u)$  for  $u > 0$  shows that  $v$  is strictly decreasing on  $(0, \infty)$  and satisfies

$$\lim_{u \rightarrow 0} v(u) = 1, \quad \lim_{u \rightarrow \infty} v(u) = -\infty. \quad (3.57)$$

Therefore, there exist  $s_1 \leq 0$  and  $d > 0$  satisfying (3.55) such that  $V(s_1) \geq 1$ .  $\square$

Theorem 3.19 shows that the sufficient condition of Lemma 2.1 is not necessary for the dominance of simple real spectral values when  $\Delta$  admits three equidistributed real roots.

The function  $V$  defined in (3.54) initially seems to depend on three separate parameters: the delay  $\tau$ , the distance  $d$ , and a dominant simple real value  $s_1$ . However, upon closer inspection, it becomes apparent that  $V$  can be expressed solely in terms of the products  $\tau d$  and  $\tau s_1$ , denoted by  $u$  and  $v$ , respectively. As a result, the function reduces to

$$W(u, v) := \left(1 - 2v + u + u \coth\left(\frac{u}{2}\right)\right)e^{-u}. \quad (3.58)$$

For visualization, it is sufficient to consider plots of  $W$  as a function of  $u = \tau d$  and  $v = \tau s_1$ , treating  $\tau$  as a positive constant scaling factor. Refer to Figure 4 (image on the left).

The second main result of this section is the following.

**Theorem 3.20.** *Let  $\tau > 0$ . There exist  $s_1 \leq 0$  and  $\delta > 0$  such that if  $\Delta$  admits exactly two simple real spectral values  $s_1$  and  $s_2 = s_1 - \delta$ , and inequality (3.41) is satisfied, then*

$$Y(s_1) = (|\alpha|(1 + |s_1|\tau) + |\beta|\tau)e^{-s_1\tau} \quad (3.59)$$

satisfies  $Y(s_1) \geq 1$  and  $s_1$  is a dominant root of  $\Delta$ .

*Proof.* If  $\Delta$  admits exactly two simple real spectral values  $s_2 < s_1$  and inequality (3.41) is satisfied, then  $s_1$  is a strictly dominant root of  $\Delta$  by Theorem 3.14. Moreover,  $\alpha < 0$ , and if  $s_1 \leq 0$  then  $\beta \leq 0$  by Lemma 3.13. If  $s_2 = s_1 - \delta$  for some  $\delta > 0$  that will be chosen later on, then one has from Lemma 3.11,

$$\alpha = -e^{\tau s_1} \left( \frac{(a + s_1)}{\delta} (1 - e^{-\tau\delta}) + e^{-\tau\delta} \right)$$

$$\beta = -\alpha s_1 - e^{\tau s_1} (a + s_1)$$

so that

$$Y(s_1) = \frac{e^{-\tau\delta}(a + s_1 - \delta)(-1 + 2\tau s_1) + (a + s_1)(1 - 2\tau s_1 + \tau\delta)}{\delta}.$$

Upon closer inspection, one finds that  $Y$  can be expressed solely in terms of the products  $\tau a$ ,  $\tau\delta$ , and  $\tau s_1$ . Setting  $A := \tau a$ ,  $u := \tau\delta$ ,  $v := \tau s_1$ , then  $A \in \mathbb{R}$ ,  $u > 0$  and  $v < 0$ . Introducing the functions

$$Z(A, u, v) := Y(s_1), \quad h(A, u, v) = Z(A, u, v) - 1 \quad (3.60)$$

one gets

$$uh(A, u, v) = -(2 - 2e^{-u})v^2 - ((1 - e^{-u})(2A - u - 1) + ue^{-u})v + (A + e^{-u} - 1)u + A(1 - e^{-u})$$

which is a second-degree polynomial in  $v$ . Let  $N(A, u, v)$  be the numerator of  $h(A, u, v)$ . From  $N(A, u, v) = 0$ , one finds that its discriminant  $D_1(A, u)$  admits two real roots  $A_0(u) < 0$  for every  $u > 0$  and  $A_1(u)$  that can change sign for  $u > 0$ . One infers that for every  $u > 0$ ,  $D_1(A, u) \geq 0$  for  $A \leq A_0(u)$  and  $A \geq A_1(u)$ . For simplicity, assume from now on that  $A \geq A_1(u)$ . Hence, equation  $N(A, u, v) = 0$  admits two real roots

$$v_1(A, u) = \frac{-C_b(A, u) - \sqrt{D_1(A, u)}}{2C_a(u)}$$

$$v_2(A, u) = \frac{-C_b(A, u) + \sqrt{D_1(A, u)}}{2C_a(u)}. \quad (3.61)$$

Here

$$C_a(u) = -2(1 - e^{-u}), \quad C_c(A, u) = (1 - e^{-u} + u)A - (1 - e^{-u})u,$$

$$C_b(A, u) = -2(1 - e^{-u})A + (1 + u)(1 - e^{-u}) - ue^{-u}$$

are the coefficients of  $N(A, u, v)$  considered as a second-degree polynomial in  $v$ , so that  $D_1(A, u) = C_b^2(A, u) - 4C_a(u)C_c(A, u)$ . Letting

$$A_2(u) = \frac{(1 - e^{-u})u}{1 - e^{-u} + u}, \quad A_3(u) = \frac{(1 + u)(1 - e^{-u}) - ue^{-u}}{2(1 - e^{-u})}$$

one can check that for all  $u > 0$ , it holds  $A_2(u) > 0$ ,  $A_3(u) > 0$  and  $A_1(u) \leq A_2(u) \leq A_3(u)$ . Moreover, one gets the following sign tab.

$A$	$-\infty$	$A_2(u)$	$A_3(u)$	$+\infty$
$C_b(A, u)$	+	+	0	-
$C_c(A, u)$	-	0	+	+

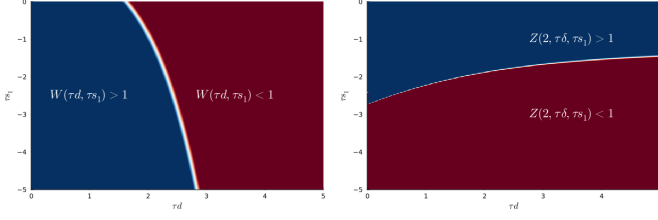


Figure 4: Depiction of the functions  $W(\tau\delta, \tau s_1)$  and  $Z(2, \tau\delta, \tau s_1)$  defined respectively in (3.58) and (3.66). The *red*-colored region in the parameters space  $(\tau\delta, \tau s_1)$  corresponds to the region where Frasson-Verduyn Lunel sufficient condition for the dominance of a simple real spectral value is satisfied. The CRRID property extends the conditions by the *blue* region.

Hence, for all  $A \leq A_2(u)$ , one has  $v_1 \leq 0$  and  $v_2 > 0$ . Since  $A_1(u) \leq A_2(u)$ , and  $C_a(u) < 0$  for every  $u > 0$ , one deduces that for all  $A \in [A_1, A_2]$ ,  $u > 0$  and  $v \in [v_1, v_2]$ , one has  $N(A, u, v) \geq 0$ . Therefore, for every  $s_1 \leq 0$  and every  $\delta > 0$ , one has  $Y(s_1) \geq 1$ , whenever

$$\begin{cases} A_1(\tau\delta) \leq \tau a \leq A_2(\tau\delta) \\ \text{and} \\ v_1(\tau a, \tau\delta) \leq \tau s_1 \leq v_2(\tau a, \tau\delta). \end{cases} \quad (3.62) \quad \square$$

Theorem 3.20 shows that the sufficient condition of Lemma 2.1 is not necessary for the dominance of simple real spectral values when  $\Delta$  admits exactly two simple real roots.

**Remark 3.21.** We stress that there are values for  $s_1$  and  $\delta$  (where  $s_1 \leq 0$  and  $\delta > 0$ ) that satisfy (3.62) and still maintain inequality (3.41). From equation (3.62), take, for instance,

$$\tau a = A_2(\tau\delta), \quad \text{and} \quad \tau s_1 = v_1(\tau a, \tau\delta). \quad (3.63)$$

One finds, after careful computations, that

$$\frac{a + s_1}{\delta} = \frac{A_2(\tau\delta) + v_1(\tau a, \tau\delta)}{\tau\delta} = \frac{1}{2} \left( \frac{2 - e^{\tau\delta}}{1 - e^{\tau\delta}} + \frac{1}{\tau\delta} \right). \quad (3.64)$$

Finally, as a function of  $\tau\delta > 0$ , one immediately checks that the right side in (3.64) is bounded between 1/2 and 3/4.

**Remark 3.22.** When two real roots are assigned, and a third root coexists (see, Theorem 3.10), one can use the exact same arguments as in the proof of Theorem 3.19 to show that in some parameters regions, the Frasson-Verduyn Lunel's criteria does not apply. Furthermore, when exactly two real spectral values  $s_2 < s_1$  coexist in the spectrum of  $\Delta$ , and

$$\frac{-1}{e^{\tau(s_1 - s_2)} - 1} < \frac{a + s_1}{s_1 - s_2} \leq 0, \quad (3.65)$$

then  $s_1$  is a strictly dominant root of  $\Delta$  by Theorem 3.14, and we can prove as in Theorem 3.20 that in some parameters regions the Frasson-Verduyn Lunel's criteria does not apply.

In the proof of Theorem 3.20, we introduced the function

$$\begin{aligned} \delta Z(\tau a, \tau\delta, \tau s_1) &= e^{-\tau\delta} (a + s_1 - \delta)(-1 + 2\tau s_1) \\ &\quad + (a + s_1)(1 - 2\tau s_1 + \tau\delta). \end{aligned} \quad (3.66)$$

When  $a = 2$  (admissible owing to (3.62)), we plotted the heatmap of  $Z(2, \tau\delta, \tau s_1)$  in Figure 4 (image on the *right*) showing the regions  $(\tau\delta, \tau s_1)$  where  $Y(s_1) < 1$  and  $Y(s_1) \geq 1$ .

### 3.4. Exponential Estimates

In [17, Section 6], the author establishes exponential estimates of solutions for time delay systems of neutral-type using quadratic Lyapunov functionals and Lyapunov matrices. Although effective, this method may be computationally involved. Additionally, [15][Chapter 1, Theorem 7.6, page 32] also offers an exponential estimate for solutions of a neutral differential equation based on the characteristic quasipolynomial's rightmost root. The proof employs the Cauchy theorem of residues and involves complex analysis arguments, including the periodicity and analyticity of certain functions. However, the explanation may benefit from greater detail to enhance the reader's clarity and ease of understanding.

In the following, we provide an alternative proof of exponential estimates for solutions of (NDE) that integrates the previously outlined spectral analysis of Sections 3.1 and 3.2.

For all  $t \geq 0$ ,  $y_t \in C^0([-\tau, 0])$  denotes the history function, defined for all  $\theta \in [-\tau, 0]$  as

$$y_t(\theta) = y(t + \theta), \quad \text{with} \quad \|y_t\|_\infty := \sup_{\theta \in [-\tau, 0]} |y_t(\theta)|. \quad (3.67)$$

**Theorem 3.23.** Assume that the quasipolynomial  $\Delta$  admits three real spectral values  $s_3 < s_2 < s_1$ . For every  $\varepsilon > 0$ , there exists a constant  $k := k(\varepsilon, s_1, s_2, s_3) \geq 1$  such that the solution  $y(\cdot)$  of (NDE) with initial condition  $y_0 \in C^0([-\tau, 0])$  satisfies

$$|y(t)| \leq k e^{(s_1 + \varepsilon)t} \|y_0\|_\infty \quad (t \geq 0). \quad (3.68)$$

*Proof.* Applying the Laplace transform to both sides of equation (NDE), one gets

$$\hat{y}(s) = \frac{y_0(0) + \alpha y_0(-\tau)}{\Delta(s)} \quad (s \in \mathbb{C}) \quad (3.69)$$

showing that  $\hat{y}$  is an analytic function of  $s$  for  $\Re(s) > s_1$ . Let  $\varepsilon > 0$  and set  $c_1 := s_1 + \varepsilon$ . Then, the function  $y$  is given by the Bromwich complex contour integral for all  $t \geq 0$

$$y(t) = \frac{(y_0(0) + \alpha y_0(-\tau))}{2i\pi} \lim_{T \rightarrow \infty} \int_{c_1 - iT}^{c_1 + iT} \frac{e^{st}}{\Delta(s)} ds. \quad (3.70)$$

Indeed, let  $T > 0$  and  $c_2 > c_1$ , and consider the integration of the function  $e^{st}/\Delta(s)$  over the closed rectangle  $\Gamma$  in the complex plane with vertical boundaries  $V_1 := \{c_1 + i\omega \mid -T \leq \omega \leq T\}$  and  $V_2 := \{c_2 + i\omega \mid -T \leq \omega \leq T\}$ , and horizontal boundaries  $H_1 := \{x + iT \mid c_1 \leq x \leq c_2\}$  and  $H_2 := \{x - iT \mid c_1 \leq x \leq c_2\}$ . Since  $\Delta$  has no zeroes inside the rectangle  $\Gamma$ , the integral over  $\Gamma$  equals zero. It is then sufficient to show the following.

$$\int_{H_j} \frac{e^{st}}{\Delta(s)} ds \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty \quad (j = 1, 2). \quad (3.71)$$

For  $s = x + iT$  with  $c_1 \leq x \leq c_2$  and  $T > 0$ , one has  $\Delta(s) = (1 + \alpha e^{-\tau s})s + (a + \beta e^{-\tau s})$ . Since  $1 - \alpha e^{-\tau x} > 0$  thanks to  $s_1 < c_1 \leq x \leq c_2$  and Lemma 2.5, one chooses  $T_0 > 0$  large enough to have

$$\frac{T}{2} (1 - \alpha e^{-\tau x}) \geq |a| + |\beta| e^{-\tau c_2} \quad (\forall T \geq T_0). \quad (3.72)$$

It follows that for all  $T \geq T_0$ , it holds

$$|\Delta(s)| \geq T(1 - \alpha e^{-\tau x}) - (|a| + |\beta|e^{-\tau x}) \geq \frac{T}{2}(1 - \alpha e^{-\tau x}). \quad (3.73)$$

Therefore, one has

$$\left| \int_{H_1} \frac{e^{st}}{\Delta(s)} ds \right| \leq \frac{2e^{tc_2}}{T} \int_{c_1}^{c_2} \frac{e^{\tau x}}{(e^{\tau x} - \alpha)} dx = \frac{2}{T} \frac{e^{tc_2}}{\tau} \ln \left( \frac{e^{\tau c_2} - \alpha}{e^{\tau c_1} - \alpha} \right)$$

which tends to 0 as  $T \rightarrow \infty$ . In the same fashion, one proves that the integral over  $H_2$  tends to 0 when  $T \rightarrow \infty$ . It follows that (3.70) defines properly the signal  $y(t)$  for every  $t \geq 0$ . One claims that if  $s = c_1 + iT$ , then

$$|\Delta(s)| \geq \frac{|T|}{2}(1 - \alpha e^{-\tau s_1}), \quad \forall |T| \geq \frac{4\zeta}{1 - \alpha e^{-\tau s_1}} =: T_1 \quad (3.74)$$

where  $\zeta > 0$  is defined by (2.16) and  $1 - \alpha e^{-\tau s_1} > 0$  owing to Lemma 2.5. Indeed, since  $a = -s_1 - \zeta$  and  $\beta = -\alpha s_1 + \zeta e^{\tau s_1}$ , one has  $\Delta(s) = (1 + \alpha e^{-\tau c_1} e^{-i\tau T})(\varepsilon + iT) - \zeta(1 - e^{-\tau \varepsilon} e^{-i\tau T})$ . Therefore,

$$|\Delta(s)| \geq |T|(1 - \alpha e^{-\tau c_1}) - \zeta(1 + e^{-\tau \varepsilon}) \geq |T|(1 - \alpha e^{-\tau s_1}) - 2\zeta$$

thanks to the reverse triangle inequality, which completes the proof of the claim. Setting for all  $t \geq 0$ ,

$$z(t) := \mathcal{L}^{-1} \left\{ \frac{1}{\Delta(s)} \right\} (t) = \frac{1}{2i\pi} \lim_{T \rightarrow \infty} \int_{c_1 - iT}^{c_1 + iT} \frac{e^{st}}{\Delta(s)} ds \quad (3.75)$$

one immediately observes that for every  $s \in \mathbb{C}$  such that  $\Re(s) > s_1$ , one has

$$\frac{1 + \alpha e^{-\tau s}}{\Delta(s)} = \zeta \frac{1 - e^{-\tau(s-s_1)}}{(s-s_1)\Delta(s)} + \frac{1}{s-s_1}$$

so that, taking the inverse Laplace transform and the fact that  $\mathcal{L}^{-1}\{1/(s-s_1)\} = e^{s_1 t}$ , one obtains

$$z(t) + \alpha z(t - \tau) = \zeta \mathcal{L}^{-1} \left\{ \frac{1 - e^{-\tau(s-s_1)}}{(s-s_1)\Delta(s)} \right\} (t) + e^{s_1 t}. \quad (3.76)$$

Thanks to (3.74), one has

$$\begin{aligned} \Lambda_4 &:= \left| \zeta \mathcal{L}^{-1} \left\{ \frac{1 - e^{-\tau(s-s_1)}}{(s-s_1)\Delta(s)} \right\} (t) \right| \\ &\leq \frac{\zeta e^{tc_1}}{\pi} \int_{-T_1}^{T_1} \frac{dT}{|\Delta(c_1 + iT)| \sqrt{\varepsilon^2 + T^2}} \\ &\quad + \frac{4\zeta e^{tc_1}}{\pi(1 - \alpha e^{-\tau s_1})} \int_{T_1}^{\infty} \frac{dT}{T^2} \\ &= \frac{\zeta e^{tc_1}}{\pi} \int_{-T_1}^{T_1} \frac{dT}{|\Delta(c_1 + iT)| \sqrt{\varepsilon^2 + T^2}} + \frac{e^{tc_1}}{\pi} \leq k_0 e^{tc_1} \end{aligned} \quad (3.77)$$

where  $k_0 := k_0(\varepsilon, s_1, s_2, s_3) > 0$  satisfies

$$k_0 \leq \frac{1}{\pi} \left( 1 + 2\zeta T_1 \min_{|T| \leq T_1} (|\Delta(c_1 + iT)| \sqrt{\varepsilon^2 + T^2}) \right). \quad (3.78)$$

Combining (3.76) and (3.77), one obtains

$$|z(t) - \alpha z(t - \tau)| \leq |z(t) + \alpha z(t - \tau)| \leq (1 + k_0)e^{(s_1 + \varepsilon)t} \quad (3.79)$$

which yields

$$|z(t)| \leq (1 + k_0)e^{(s_1 + \varepsilon)t} \sum_{j=0}^{\infty} \alpha^j e^{-j(s_1 + \varepsilon)\tau} = \frac{(1 + k_0)e^{(s_1 + \varepsilon)t}}{1 - \alpha e^{-\tau(s_1 + \varepsilon)}} \quad (3.80)$$

since  $0 < 1 - \alpha e^{-\tau(s_1 + \varepsilon)} < 1$  owing to Lemma 2.5. As a consequence, the result follows since

$$|y(t)| = |y_0(0) + \alpha y_0(-\tau)| |z(t)| \leq \frac{(1 + k_0)(1 + \alpha)}{1 - \alpha e^{-\tau(s_1 + \varepsilon)}} e^{(s_1 + \varepsilon)t} \|y_0\|_{\infty}$$

and  $k(\varepsilon, s_1, s_2, s_3) := (1 + k_0)(1 + \alpha)/(1 - \alpha e^{-\tau(s_1 + \varepsilon)}) \geq 1$ .  $\square$

**Remark 3.24.** It is important to note that (3.68) only make sense when  $k \geq 1$ . The GRRID setting derived in Section 3.1 allows us to explicitly determine  $k \geq 1$ , a property which is not explicitly stated in [15][Chapter 1, Theorem 7.6, page 32].

Even when two real spectral values coexist in the spectrum of  $\Delta$ , one has the same exponential estimates as stated in what follows. The proof is a slight adaptation of that of Theorem 3.23, but we present it for completeness.

**Proposition 3.25.** Assume that  $\Delta$  has two real spectral values  $s_2 < s_1$ . For every  $\varepsilon > 0$ , there exists a constant  $k := k(\varepsilon, s_1, s_2) \geq 1$  such that the solution  $y(\cdot)$  of (NDE) with initial condition  $y_0 \in C^0([-\tau, 0])$  satisfies

$$|y(t)| \leq k e^{(s_1 + \varepsilon)t} \|y_0\|_{\infty} \quad (t \geq 0). \quad (3.81)$$

*Proof.* First, if  $s_1$  and  $s_2$  are simple, then a third simple real root  $s_3$  may coexist in the spectrum of  $\Delta$ , in which case the result follows by Theorem 3.23. If such an  $s_3$  does not exist, then  $s_1$  may be a double real root but remains dominant by Corollary 3.16, and one has  $-e^{\tau s_1} \leq \alpha \leq 0$ .

If  $\alpha = 0$ , then we are in the retarded case, and the exponential estimate (3.81) is proven in [15][Chapter 1, Theorem 5.2, page 20].

If  $-e^{\tau s_1} < \alpha < 0$ , then  $s_2 < s_1$  are the only simple real spectral values in the spectrum of  $\Delta$  and  $s_1$  is dominant by Theorem 3.14. Then, to prove (3.81), one argues as in the proof of Theorem 3.23 as follows. Let  $\varepsilon > 0$  and set  $c_1 := s_1 + \varepsilon$ , and  $c_2 > c_1$ . To prove (3.71), one can replace (3.72) by

$$\frac{T}{2}(1 + \alpha e^{-\tau x}) \geq |a| + |\beta|e^{-\tau c_2}, \quad \forall T \geq T_0 \quad (3.82)$$

with  $T_0 > 0$  large enough. In fact, for  $s = x + iT$  with  $s_1 < c_1 \leq x \leq c_2$ , one gets from  $\Delta(s) = (1 + \alpha e^{-\tau s})s + (a + \beta e^{-\tau s})$  that

$$\begin{aligned} |\Delta(s)| &\geq T \sqrt{1 + 2\alpha e^{-\tau x} \cos(\tau T) + \alpha^2 e^{-2\tau x}} - (|a| + |\beta|e^{-\tau x}) \\ &\geq T(1 + \alpha e^{-\tau x}) - (|a| + |\beta|e^{-\tau x}) \geq \frac{T}{2}(1 + \alpha e^{-\tau x}) \end{aligned}$$

owing to (3.82), since  $\alpha < 0$  and  $1 + \alpha e^{-\tau x} > 0$  by (3.40). Therefore, one deduces

$$\left| \int_{H_1} \frac{e^{st}}{\Delta(s)} ds \right| \leq \frac{2e^{tc_2}}{T} \int_{c_1}^{c_2} \frac{e^{\tau x}}{(e^{\tau x} + \alpha)} dx = \frac{2}{T} \frac{e^{tc_2}}{\tau} \ln \left( \frac{e^{\tau c_2} + \alpha}{e^{\tau c_1} + \alpha} \right)$$

which yields (3.71). By (3.40), one has

$$\kappa := \frac{\alpha + e^{\tau s_1}}{\tau F_{-\tau,1}(s_1, s_2)} > 0 \quad (3.83)$$

where  $F_{-\tau,1}(s_1, s_2)$  is defined by (2.14). Then, if  $s = c_1 + iT$ , one can replace (3.74) by

$$|\Delta(s)| \geq \frac{|T|}{2}(1 + \alpha e^{-\tau s_1}), \quad \forall |T| \geq \frac{4\kappa}{1 + \alpha e^{-\tau s_1}} =: T_2. \quad (3.84)$$

In fact, one obtains from (3.36) and (3.37) that  $a = -s_2 - \kappa$  and  $\beta = -\alpha s_2 + \kappa e^{\tau s_2}$  so that  $\Delta(s) = (1 + \alpha e^{-\tau c_1} e^{-iT}) (\delta + \varepsilon + iT) - \kappa(1 - e^{-\tau(\delta+\varepsilon)} e^{-iT})$  where  $\delta := s_1 - s_2 > 0$ . It follows that

$$|\Delta(s)| \geq |T|(1 + \alpha e^{-\tau c_1}) - \kappa(1 + e^{-\tau(\delta+\varepsilon)}) \geq |T|(1 + \alpha e^{-\tau s_1}) - 2\kappa$$

as in the proof of (3.82). Hence, (3.84) follows immediately. Now, let  $t \geq 0$  and introduce  $z(t)$  as in (3.75). One has from  $a = -s_2 - \kappa$  and  $\beta = -\alpha s_2 + \kappa e^{\tau s_2}$  the following

$$\frac{1 + \alpha e^{-\tau s}}{\Delta(s)} = \kappa \frac{1 - e^{-\tau(s-s_2)}}{(s-s_2)\Delta(s)} + \frac{1}{s-s_2} \quad (3.85)$$

for every  $s \in \mathbb{C}$  such that  $\Re(s) > s_1$ . Therefore,

$$z(t) + \alpha z(t-\tau) = \kappa \mathcal{L}^{-1} \left\{ \frac{1 - e^{-\tau(s-s_2)}}{(s-s_2)\Delta(s)} \right\} (t) + e^{s_2 t}. \quad (3.86)$$

Arguing as in the proof of (3.77), one obtains from (3.84) that

$$\left| \kappa \mathcal{L}^{-1} \left\{ \frac{1 - e^{-\tau(s-s_2)}}{(s-s_2)\Delta(s)} \right\} (t) \right| \leq k_1 e^{t c_1} \quad (3.87)$$

where  $k_1 := k_1(\varepsilon, s_1, s_2) > 0$  satisfies

$$k_1 \leq \frac{1}{\pi} \left( 1 + 2\kappa T_2 \min_{|T| \leq T_2} (|\Delta(c_1 + iT)| \sqrt{(s_1 - s_2 + \varepsilon)^2 + T^2}) \right).$$

Therefore, (3.86) and (3.87), yield

$$|z(t)| + \alpha |z(t-\tau)| \leq |z(t) + \alpha z(t-\tau)| \leq (1 + k_1) e^{(s_1 + \varepsilon)t} \quad (3.88)$$

since  $\alpha < 0$  by (3.40). It follows that

$$|z(t)| \leq (1 + k_1) e^{(s_1 + \varepsilon)t} \sum_{j=0}^{\infty} (-\alpha)^j e^{-j(s_1 + \varepsilon)\tau} = \frac{(1 + k_1) e^{(s_1 + \varepsilon)t}}{1 + \alpha e^{-\tau(s_1 + \varepsilon)}}$$

since  $0 < 1 + \alpha e^{-\tau(s_1 + \varepsilon)} < 1$  by (3.40). This proves (3.81) since

$$\|y(t)\| = \|(y_0(0) + \alpha y_0(-\tau))\| |z(t)| \leq \frac{(1 + k_1)(1 + |\alpha|)}{1 + \alpha e^{-\tau(s_1 + \varepsilon)}} e^{(s_1 + \varepsilon)t} \|y_0\|_{\infty}$$

and  $k(\varepsilon, s_1, s_2) := (1 + k_1)(1 + |\alpha|)/(1 + \alpha e^{-\tau(s_1 + \varepsilon)}) \geq 1$ .

Finally, if  $\alpha = -e^{\tau s_1}$ , the proof follows *mutatis mutandis* the arguments.  $\square$

#### 4. Application to the prescribed local exponential stabilization of a one-layer neural network and beyond

Building on the spectral theory developed in the previous sections, we apply our theoretical insights to the practical design of an exponentially stable one-layer neural network subject to a delayed PD controller in Section 4.1. Section 4.2 demonstrates that while the proportional (P) controller can achieve the same decay rate as the PD controller, it requires higher gain and only allows for a shorter maximum delay.

##### 4.1. Implementing delayed PD controller

In this section, based on the CRRID setting of Section 3.1, we practically implement a delayed PD controller to enhance the stability of the Hopfield neural network when  $\nu \geq \mu$  and to achieve stabilization when  $\nu < \mu$ . This is accomplished by determining the gain parameters  $k_p, k_d$ , and the delay  $\tau > 0$ , with  $\mu$  and  $\nu$  being known.

###### 4.1.1. Improving the decay rate of an exponentially stable one-layer neural network

Consider the one-layer neural network of the form

$$\dot{y}(t) = -2y(t) + \tanh(y(t)) + I(t), \quad (4.1)$$

which is a particular case of (1.2) with  $\nu = 2$  and  $\mu = 1$ . Under no external input ( $I(t) = 0$ ), all solutions of (4.1) are globally exponentially stable, meaning that the trivial equilibrium is globally exponentially stable with a decay rate equal to  $\nu - \mu = 1$ . A classical control problem is to choose the control  $I(t)$  in the feedback form to improve (locally at least) the stability properties of (4.1). By letting  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$  and linearizing around zero, the problem reduces to studying the localization of the spectrum of the quasipolynomial function  $Q_0(s) = s + 1 + e^{-\tau s}(k_d s + k_p)$ .

To simplify the control design, we assign three equidistributed real spectral values  $s_1 = -5, s_2 = -6$ , and  $s_3 = -7$  to  $Q_0$ . Then, Lemma 2.4, yields

$$\tau = \ln(3/2), \quad k_d = e^{-6\tau}, \quad k_p = e^{-6\tau}(6 + \coth(\tau/2)) \quad (4.2)$$

where  $\coth$  is the hyperbolic cotangent function.

As per the CRRID properties provided in Section 3.1, the parameters in (4.2) ensure the local exponential stability of solutions to equation (4.1) with the PD controller  $I(t) = -k_p y(t - \tau) - k_d \dot{y}(t - \tau)$ , with a decay rate equal to  $-5 + \varepsilon$ , for a sufficiently small  $\varepsilon > 0$ , as stated in Theorem 3.23. Refer to Figure 5 (image on the left) for a visualization.

###### 4.1.2. Stabilizing an asymptotically stable one-layer neural network exponentially

Consider the one-layer neural network described by

$$\dot{y}(t) = -y(t) + \tanh(y(t)) + I(t). \quad (4.3)$$

This equation corresponds to a specific case of (1.2) when  $\nu = \mu = 1$ . Using a Lyapunov function [16], it can be verified that all solutions of this equation asymptotically converge to zero when  $I(t) = 0$ . To ensure that the solutions converge to zero with a prescribed exponential decay rate, we control the system using the PD controller (1.3). Linearizing around zero reduces the problem to studying the spectrum of the quasipolynomial  $Q_1(s) = s + e^{-\tau s}(k_d s + k_p)$ .

Assuming three non-equidistributed real spectral values  $s_3 < s_2 < s_1$ , we let  $s_1 = -3, s_2 = -4$ , and  $s_3 = -6$ . Using Lemma 2.4, we find:

$$\tau = \ln\left(\frac{1 + \sqrt{5}}{2}\right), \quad k_d = -20 + 9\sqrt{5}, \quad k_p = -66 + 30\sqrt{5}. \quad (4.4)$$

Figure 5 (image on the right) illustrates the solutions of (4.3) with and without the PD controller.

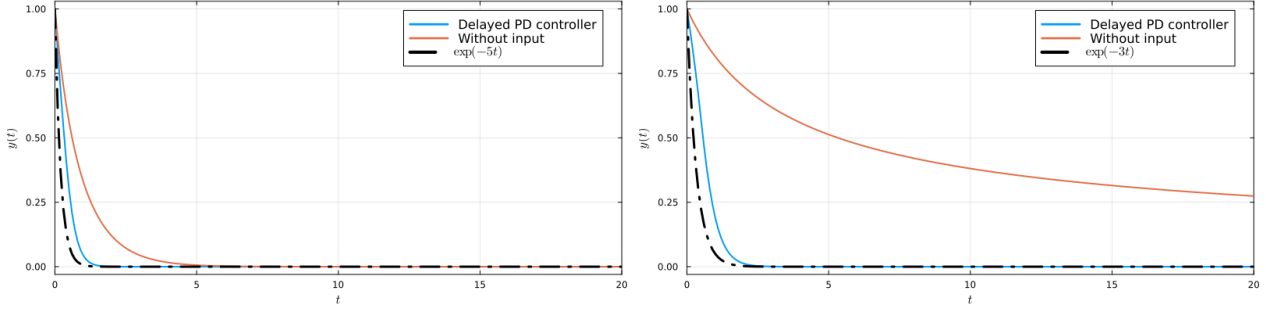


Figure 5: Solutions to (4.1) (left) and (4.3) (right) when  $I = 0$  and when  $I$  is the delayed PD controller given by (1.3) where  $\tau$ ,  $k_d$  and  $k_p$  are given by (4.2) (left) and (4.4) (right) respectively. The initial condition is taken as  $y_0 = 1$  for the solution with  $I = 0$ , and  $y_0(t) = 1 + \sin(t)$  for the solution with the delayed PD controller.

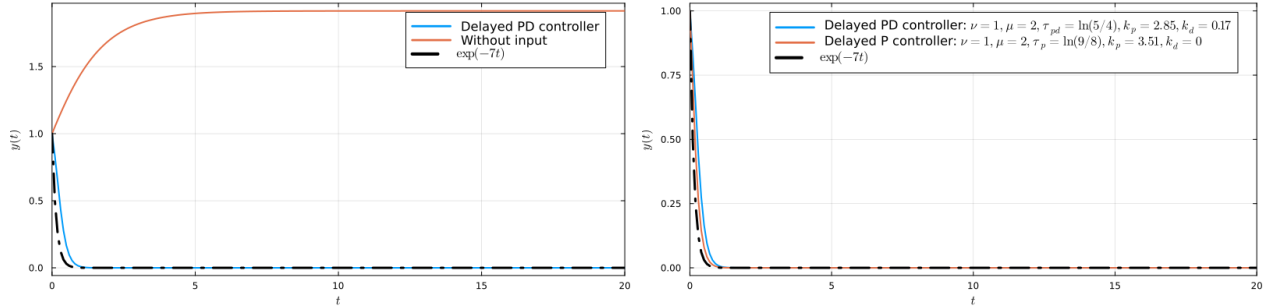


Figure 6: On the left, solutions to (4.5) when  $I = 0$  and when  $I$  is the delayed PD controller (1.3) where  $\tau$ ,  $k_d$ , and  $k_p$  are given by (4.6). On the right, a comparison between the solution to (4.5) with the delayed P and PD controllers. This shows whether they achieve exponential stabilization with the same decay, the total gains of the delayed PD controller are less than that of the delayed P controller. Moreover, the delayed PD controller allows for larger delays than the delayed P controller. See Section 4.2. The initial condition is taken as  $y_0 = 1$  for the solution with  $I = 0$ , and  $y_0(t) = 1 + \sin(t)$  for the solution with the delayed P and PD controllers.

#### 4.1.3. Stabilizing an unstable one-layer neural network exponentially

This section focuses on the scenario where the interaction strength surpasses the neural system's natural decay rate, leading to inherent instability. Specifically, consider the one-layer neural network

$$\dot{y}(t) = -y(t) + 2 \tanh(y(t)) + I(t), \quad (4.5)$$

which is a particular case of (1.2) with  $\nu = 1$  and  $\mu = 2$ . Linear stability analysis suggests that the trivial equilibrium of (4.5) is unstable when  $I = 0$ . To stabilize the system, we apply a PD controller (1.3).

By linearizing the equation, the local asymptotic behavior of solutions is equivalent to studying the spectrum of the quasipolynomial function  $Q_2(s) = s - 1 + e^{-\tau s}(k_d s + k_p)$ .

Assuming three equidistributed real spectral values  $s_1 = -7$ ,  $s_2 = -8$ , and  $s_3 = -9$ , Lemma 2.4 yields

$$\tau = \ln(5/4), \quad k_d = e^{-8\tau}, \quad k_p = e^{-8\tau} (8 + \coth(\tau/2)) \quad (4.6)$$

where  $\coth$  is the hyperbolic cotangent function. Figure 6 depicts the solutions of (4.5) with and without the PD controller.

#### 4.2. Performance comparison between proportional and proportional-derivative delayed controllers

In this section, we illustrate why the delayed PD controller,  $I(t) = k_p y(t - \tau_{pd}) - k_d y'(t - \tau_{pd})$ , is more advantageous for the prescribed local exponential stabilization of the one-layer neural network (1.2) than the delayed proportional (P) controller,

$I(t) = -k'_p y(t - \tau_p)$ . Recall that using  $I(t) = -k'_p y(t - \tau_p)$  to locally stabilize  $\dot{y}(t) = -\nu y(t) + \mu \tanh(y(t)) + I(t)$  in the hyperexcitable regime where  $\mu > \nu$ , reduces the problem to localizing the spectral abscissa of the quasipolynomial

$$\Delta_2(s) = a + s + k'_p e^{-\tau_p s}, \quad s \in \mathbb{C}, \quad (a := \nu - \mu) \quad (4.7)$$

in the left half of the complex plane. By the Pólya-Szegő bound [25, Problem 206.2, page 144],  $\Delta_2$  is of degree 2. Assigning two real roots  $s_2 < s_1$  to  $\Delta_2$  yields

$$a(\tau_p) = -s_1 - \frac{e^{\tau_p s_2}}{\tau_p F_{-\tau_p, 1}(s_1, s_2)}, \quad k'_p(\tau_p) = \frac{e^{\tau_p (s_1 + s_2)}}{\tau_p F_{-\tau_p, 1}(s_1, s_2)} \quad (4.8)$$

where  $F_{-\tau_p, 1}(s_1, s_2)$  is defined in (2.14). From [3, Theorem 12],  $s_1$  is the dominant root of  $\Delta_2$ , and there exists a unique  $\tau_p^* > 0$  such that  $s_1 < 0$  if, and only if,  $a(\tau_p^*) = 0$ , i.e.,

$$\tau_p^* = \frac{1}{(s_1 - s_2)} \ln\left(\frac{s_2}{s_1}\right) > 0. \quad (4.9)$$

For the delayed PD controller, Corollary 3.5 shows that assigning three equidistributed real roots  $s_3 < s_2 < s_1$  to the corresponding quasipolynomial  $\Delta_3(s) = s + a + e^{-\tau_{pd} s}(k_d s + k_p)$  yields

$$\tau_{pd}^* = \frac{1}{(s_1 - s_2)} \ln\left(\frac{s_3}{s_1}\right) > 0. \quad (4.10)$$

Therefore, when  $s_1 < 0$ , the two controllers ensure the prescribed local exponential stabilization of (1.2) with the same decay rate as shown in Theorem 3.23 and Proposition 3.25, see,

for instance, image on the *right* in Figure 6. Comparing (4.9) and (4.10), it follows that the delayed PD controller allows for larger delays than the delayed P controller, as  $\tau_{pd}^* > \tau_p^*$ . This inequality holds even in non-equidistributed scenarios, though determining  $\tau_{pd}^*$  requires numerical computations, as it cannot be easily derived analytically.

Finally, when the inherent dynamics are unstable ( $\mu > \nu$ ), the delayed P controller requires a significantly larger gain  $k'_p$  than the combined gains  $k_p$  and  $k_d$  of the delayed PD controller. For simplicity, assume  $\nu = 1$  and  $\mu = 2$ . Assigning three equidistributed real roots  $s_3 = -9$ ,  $s_2 = -8$ , and  $s_1 = -7$  for the PD controller gives

$$\tau_{pd} = \ln(5/4) \approx 0.22, \quad k_d = e^{-8\tau_{pd}} \approx 0.17, \quad (4.11)$$

$$k_p = e^{-8\tau_{pd}} (8 + \coth(\tau_{pd}/2)) \approx 2.85. \quad (4.12)$$

In contrast, assigning two real roots  $s_2 = -8$  and  $s_1 = -7$  for the P controller yields

$$\tau_p = \ln(9/8) \approx 0.12, \quad k'_p \approx 3.51. \quad (4.13)$$

Clearly,  $\tau_{pd} > \tau_p$ ,  $k'_p > \max(k_p, k_d)$ ,  $k'_p > k_p + k_d$ , and  $k'_p > \sqrt{k_p^2 + k_d^2}$ . Moreover, decreasing  $s_2$  in the P controller design decreases  $\tau_p$  and increases the gain  $k'_p$ , making the design even less favorable compared to the PD controller. For instance, with  $s_2 = -9$  and  $s_1 = -7$ , we find

$$\tau_p = \ln(\sqrt{5}/2) \approx 0.11, \quad k'_p \approx 3.66. \quad (4.14)$$

Conversely, increasing  $s_2$  increases the delay  $\tau_p$  and decreases the gain  $k'_p$ . Furthermore, even in the limiting case where  $s_2 = s_1 = -7$ , [21, Theorem 3.1] ensures that  $-7$  remains the dominant root of  $\Delta_2(s) = s - 1 + k'_p e^{-\tau_p s}$ , yielding

$$\tau_p = \frac{1}{8} = 0.125, \quad k'_p = 3.33. \quad (4.15)$$

This scenario still satisfies  $\tau_{pd} > \tau_p$ ,  $k'_p > \max(k_p, k_d)$ ,  $k'_p > k_p + k_d$ , and  $k'_p > \sqrt{k_p^2 + k_d^2}$ , where  $\tau_{pd}$ ,  $k_p$ , and  $k_d$  are given by (4.12) and (4.11). However, even this “better” scenario ( $s_2 = s_1 = -7$ ) is not appropriate in practice, as noted in [20], because non-semi-simple spectral values are highly sensitive to minor perturbations due to their splitting mechanism.

Let  $k$  be the gain vector ( $k \in \mathbb{R}$  for the delayed P controller and  $k \in \mathbb{R}^2$  for the delayed PD controller). If one wants to stabilize with a constraint  $|k| \leq \gamma$  for some  $\gamma > 0$  where  $|k|$  is the Euclidean norm of  $k$ , then the delayed PD controller is more likely to satisfy this constraint. Note that constrained stabilization of scalar linear delay systems with two delays has been considered in [9, Section 5].

## 5. Discussion

This paper demonstrates the use of a delayed Proportional-Derivative (PD) controller in the continuous-time modeling of a one-layer Hopfield-type neural network to achieve local exponential stabilization. Through rigorous spectral analysis, we

have developed a methodological framework that enhances the stability of the model, focusing on achieving a prescribed exponential decay rate. We have extended the spectral theory based on the CRRID property for linear functional differential equations of neutral type to neural dynamics, providing a powerful analytical tool for examining the stability of neural networks through their spectral properties. This has allowed us to determine conditions under which the network attains stability, even in systems where conventional methods predict instability.

Our analysis reveals that while a delayed proportional controller can achieve the same decay rate as the delayed PD controller, it requires significantly higher gains and permits a shorter maximum allowable delay.

Future research should explore the implementation of the delayed PD control strategy and the CRRID setting in more complex, multi-layer neural network architectures, which could better represent the intricate structures of biological neural systems. The MID setting developed in [4, 5, 21] may also address this issue. As noted in [20], non-semi-simple spectral values are sensitive to minor perturbations due to their splitting mechanism, which presents a challenge in practical implementations.

This study provides valuable insights into the local stability of the zero equilibrium in the nonlinear model. It is important to note that these findings primarily concern local dynamics. Future research should combine the spectral methods used for the linear equation with time-domain approaches based on Lyapunov functionals and linear matrix inequalities. Such a combined approach would facilitate a more comprehensive investigation into the global exponential stability of the nonlinear equation with a prescribed decay rate, broadening the applicability and reliability of the results presented here.

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