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THE GENERALIZED EULERIAN POWER SUMS $\sum_{k=1}^n k^m z^k$

RAYMOND MORTINI

ABSTRACT. We give a representation of the generalized Eulerian power sums $\sum_{k=1}^n k^m z^k$.

1. INTRODUCTION

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$. It is a very ancient result, may be already known by Euler, that for $m \in \mathbb{N}^*$

$$(1.1) \quad E_m(z) := \sum_{k=1}^{\infty} k^m z^k = \frac{\sum_{n=1}^m \begin{bmatrix} m \\ n \end{bmatrix} z^n}{(1-z)^{m+1}},$$

where $\begin{bmatrix} m \\ n \end{bmatrix}$ are the Eulerian numbers given by

$$\begin{bmatrix} m \\ n \end{bmatrix} := \sum_{i=0}^{n-1} (-1)^i \binom{m+1}{i} (n-i)^m \quad (n \leq m).$$

For a proof see [5, 6] as well as [1] and [7, p.143]. Note that $\begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{N}$, as the binomial coefficients have this property. **Let us point out that the Eulerian numbers should not to be mixed up with the Euler numbers, both are quite different classes of numbers (see [7]). Eulerian numbers also appear in combinatorics counting certain permutations (see [3] or [7, p. 144f]).** Properties of the Eulerian numbers are given in [5, 3, 7] and are of course listed on wikipedia. One important one is the symmetry:

$$(1.2) \quad \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m \\ m-n+1 \end{bmatrix}.$$

The intention of our small note is to give for $m \in \mathbb{N}$ a similar representation of the finite Taylor sums

$$E_{n,m}(z) := \sum_{k=1}^n k^m z^k$$

of the functions E_m . We call these functions *generalized Eulerian power sums*, as they generalize of course the numbers $\sum_{k=1}^n k^m$.

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On Mathstackexchange [9] it was asked for a general formula for $E_{n,m}$. For instance, as every undergraduate student of mathematics should know,

$$E_{n,0}(z) = \frac{z - z^{n+1}}{1 - z}.$$

As well illustrated in [9], the explicit formulas get very difficult with increasing power m . For instance

$$E_{n,1}(z) = \frac{nz^{n+2} - (n+1)z^{n+1} + z}{(1-z)^2},$$

and

$$E_{n,2}(z) = \frac{-n^2z^{n+3} + (2n^2 + 2n - 1)z^{n+2} - (n-1)^2z^{n+1} + z^2 + z}{(1-z)^3}.$$

When dealing with this question, I was surprised that one could readily give such a general formula, and I posted this on [9] under my abbreviated prename ‘‘Ray’’. Here I present the details, hoping that the readers of these Mathematics Newsletters of the Ramanujan Mathematical society will enjoy seeing how to develop such a formula.

2. THE EXPLICIT VALUE OF THE GENERALIZED EULERIAN SUMS

Our proof will be based on the method given in [5], which amounts in using the difference operator D^p . That is, let $(c_n)_{n \in \mathbb{Z}}$ be a double sided sequence of real numbers and put

$$\begin{aligned} D^0 c_n &= c_n, & D^1 c_n &= c_n - c_{n-1} \\ D^{p+1} c_n &= D^1(D^p c_n), & (p \in \mathbb{N}^*). \end{aligned}$$

It is easily seen by induction (and of course well known) that

$$(2.1) \quad D^p c_n = \sum_{j=0}^p (-1)^j \binom{p}{j} c_{n-j}.$$

The crux is now the following fact, which is easily proven by induction, too: if $c_{-n} = 0$ for $n \in \mathbb{N}^*$ and $\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} \leq 1$ then

$$(2.2) \quad (1-z)^p \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} (D^p c_n) z^n, \quad |z| < 1.$$

We are now ready to give the value for the generalized Eulerian sums.

Theorem 2.1. *Let $m \in \mathbb{N}$. Then, for $z \in \mathbb{C} \setminus \{1\}$,*

$$(2.3) \quad \sum_{k=1}^n k^m z^k = \frac{1}{(1-z)^{m+1}} \left(\sum_{j=0}^{\max\{1,m\}} (a_j - b_{j,n}) z^j + (1 - z^{n+1}) \sum_{j=0}^{\max\{1,m\}} b_{j,n} z^j \right),$$

where $a_0 := a_0(m) := 0$,

$$a_j := a_j(m) := \sum_{i=0}^{j-1} (-1)^i \binom{m+1}{i} (j-i)^m = \begin{bmatrix} m \\ j \end{bmatrix}, \quad (j \in \mathbb{N}^*),$$

and

$$b_{j,n} := b_{j,n}(m) := \sum_{i=0}^j (-1)^i \binom{m+1}{i} (j-i+1+n)^m, \quad (j \in \mathbb{N}).$$

For technical reasons, and in order to better compare the a_j with the $b_{j,n}$, we replaced here $\begin{bmatrix} m \\ j \end{bmatrix}$ by a_j (these are the Eulerian numbers). Note, though, that all the coefficients a_j and $b_{j,n}$ depend on the parameter m .

Proof. We first take $|z| < 1$. Then we may write $E_{n,m}(z)$ as

$$\begin{aligned} \sum_{k=1}^n k^m z^k &= \sum_{k=1}^{\infty} k^m z^k - \sum_{k=n+1}^{\infty} k^m z^k \\ &= \sum_{k=1}^{\infty} k^m z^k - z^{n+1} \sum_{k=0}^{\infty} (k+n+1)^m z^k. \end{aligned}$$

Hence, by (2.2),

$$\begin{aligned} (1-z)^{m+1} \sum_{k=1}^n k^m z^k &= (1-z)^{m+1} \sum_{k=1}^{\infty} k^m z^k - z^{n+1} (1-z)^{m+1} \sum_{k=0}^{\infty} (k+n+1)^m z^k \\ &= (1-z)^{m+1} E_m(z) - z^{n+1} \sum_{k=0}^{\infty} (D^{m+1} c_k) z^k, \end{aligned}$$

$$\text{where } c_k = \begin{cases} (k+n+1)^m & \text{if } k \in \mathbb{N} \\ 0 & \text{if } k < 0. \end{cases}$$

By equation (2.1)

$$b_{k,n} := D^{m+1} c_k = \sum_{i=0}^k (-1)^i \binom{m+1}{i} (k-i+1+n)^m.$$

Now let $m \geq 1$. Since for positive indices c_k is a polynomial of degree m , $D^{m+1}(c_k) = 0$ for $k > m$, and by using (1.1), we conclude that

$$(2.4) \quad (1-z)^{m+1} \sum_{k=1}^n k^m z^k = \sum_{k=0}^m a_k z^k - z^{n+1} \sum_{k=0}^m b_{k,n} z^k,$$

from which we deduce the assertion of the theorem whenever $m \geq 1$. If $m = 0$, then

$$a_k(0) = \sum_{i=0}^{k-1} (-1)^i \binom{1}{i} (k-i)^0 = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2. \end{cases}$$

and

$$b_{k,n}(0) = \sum_{i=0}^k (-1)^i \binom{1}{i} (k-i+1+n)^0 = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1. \end{cases}$$

Hence the right hand side in (2.3) equals $\frac{z - z^{n+1}}{1-z}$, which coincides with $E_{n,0}(z)$.

Thus formula (2.3) holds for $|z| < 1$. The unicity theorem for holomorphic functions now shows the validity of the formula for all $z \in \mathbb{C}$, $z \neq 1$. \square

Remark 2.2. Here we make the following observations (note that $a_k = a_k(m) = \begin{bmatrix} m \\ k \end{bmatrix}$ and $b_{k,n} = b_{k,n}(m)$):

(1) By taking $z = 1$ in (2.4), we see that for every $n \in \mathbb{N}^*$ and $m \in \mathbb{N}^*$,

$$\sum_{k=0}^m b_{k,n}(m) = \sum_{k=0}^m a_k(m) = m!,$$

where the last equality comes from [5, formula (3)]. An amazing fact!

(2) Recall that for $n, m \in \mathbb{N}^*$ and $\begin{bmatrix} m \\ 0 \end{bmatrix} := 0$,

$$E_{n,m}(1) = \sum_{k=1}^n k^m = \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix} \binom{n+j}{m+1}.$$

In fact, by Wopitzky's formula (see [8], [3, p. 255]¹, and [7, p. 139]) for $m \in \mathbb{N}^*$ and $x \in \mathbb{R}$,

$$x^m = \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix} \binom{x+j-1}{m}.$$

Hence

$$\begin{aligned} \sum_{k=1}^n k^m &= \sum_{k=1}^n \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix} \binom{k+j-1}{m} = \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix} \sum_{k=1}^n \binom{k+j-1}{m} \\ &= \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix} \binom{n+j}{m+1}. \end{aligned}$$

¹ Attention: the symbol $\langle \begin{smallmatrix} n \\ p \end{smallmatrix} \rangle$ in [3] corresponds to our $\begin{bmatrix} n \\ p+1 \end{bmatrix}$.

A similar formula for $E_{n,m}(1)$ in terms of the Stirling numbers of the second kind is given e.g. in [2, p. 456] respectively [7, p. 212], and the standard Bernoulli formula is nicely presented in [7, p. 211].

By the way, a nice natural proof (similar to that in [4]) of Worpitzky's formula can be given as follows, by posing $\begin{bmatrix} m \\ j \end{bmatrix} = 0$ if $m \in \mathbb{N}^*$ and $j > m$:

$$\begin{aligned}
\sum_{k=1}^{\infty} k^m z^k &\stackrel{(1.1)}{=} \frac{\sum_{n=1}^m \begin{bmatrix} m \\ n \end{bmatrix} z^n}{(1-z)^{m+1}} \\
&= \left(\sum_{n=0}^m \begin{bmatrix} m \\ n \end{bmatrix} z^n \right) \left(\sum_{n=0}^{\infty} (-1)^n \binom{-m-1}{n} z^n \right) \\
&= \left(\sum_{n=0}^{\infty} \begin{bmatrix} m \\ n \end{bmatrix} z^n \right) \left(\sum_{n=0}^{\infty} \binom{m+n}{n} z^n \right) \\
&\stackrel{\text{Cauchy prod.}}{=} \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \begin{bmatrix} m \\ j \end{bmatrix} \binom{m+n-j}{n-j} \right) z^n.
\end{aligned}$$

A comparison of the coefficients and the facts that $\begin{bmatrix} m \\ j \end{bmatrix} = 0$ for $j > m$ as well as $\begin{bmatrix} m \\ 0 \end{bmatrix} = 0$ for $m \in \mathbb{N}^*$ yields

$$n^m = \sum_{j=1}^{\min\{n,m\}} \begin{bmatrix} m \\ j \end{bmatrix} \binom{m+n-j}{m}.$$

Now if $n < m$, then $\binom{m+n-j}{m} = 0$ for $n < j \leq m$. Hence, using (1.2),

$$\begin{aligned}
n^m &= \sum_{j=1}^m \begin{bmatrix} m \\ j \end{bmatrix} \binom{m+n-j}{m} = \sum_{j=1}^m \begin{bmatrix} m \\ m-j+1 \end{bmatrix} \binom{m+n-j}{m} \\
&\stackrel{k:=m-j+1}{=} \sum_{k=1}^m \begin{bmatrix} m \\ k \end{bmatrix} \binom{n+k-1}{m}.
\end{aligned}$$

Now consider the polynomials $p(x) = x^m$ and

$$q(x) = \sum_{k=1}^m \begin{bmatrix} m \\ k \end{bmatrix} \binom{x+k-1}{m}.$$

Then p and q have degree at most m and coincide for $x = n \in \mathbb{N}^*$. Hence they are equal.

(3) By taking $z = -1$ in (2.4), we obtain for $n, m \in \mathbb{N}^*$

$$\begin{aligned} E_{n,m}(-1) &= \sum_{k=1}^n k^m (-1)^k = 2^{-m-1} \left(\sum_{k=0}^m (-1)^k (a_k(m) + (-1)^n b_{k,n}(m)) \right) \\ &= 2^{-m-1} \left(\sum_{k=0}^m (-1)^k \left(\begin{bmatrix} m \\ k \end{bmatrix} + (-1)^n b_{k,n}(m) \right) \right). \end{aligned}$$

A further representation of $E_{n,m}(-1)$ is given in [7, p. 219] in terms of the Euler polynomials. It would be interesting to give the exact relations between the Euler polynomials and our coefficients $b_{j,m}(n)$.

Recall from Remark 2.2, that

$$\sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix} = m!.$$

The associated alternating sum can also be computed, (see [7, p. 222]):

$$C_m := \sum_{k=0}^m (-1)^{k-1} \begin{bmatrix} m \\ k \end{bmatrix} = 2^{m+1} (2^{m+1} - 1) \frac{B_{m+1}}{m+1},$$

where B_n is the n -th Bernoulli number, defined to be $f^{(n)}(0)$ for the holomorphic function

$$f(z) = \begin{cases} \frac{z}{e^z - 1} & \text{if } 0 < |z| < 2\pi \\ 1 & \text{if } z = 0. \end{cases}$$

In particular, $C_m = 0$ if m is even. This last assertion also follows from the symmetry of the Eulerian numbers $\begin{bmatrix} m \\ k \end{bmatrix}$. Usually, the numbers $T_n := |C_{2n-1}|$ are called the *tangent numbers*, because

$$\sum_{n=1}^{\infty} T_n \frac{x^{2n-1}}{(2n-1)!} = \tan x, \quad (|x| < \pi).$$

We therefore call C_m the *signed tangent number*. It is easy to see that $C_m \in \mathbb{N}$ (since the Eulerian coefficients belong to \mathbb{N}).

Here are the first Eulerian, Bernoulli and tangent numbers:

\backslash	$\begin{bmatrix} m \\ 1 \end{bmatrix}$	$\begin{bmatrix} m \\ 2 \end{bmatrix}$	$\begin{bmatrix} m \\ 3 \end{bmatrix}$	$\begin{bmatrix} m \\ 4 \end{bmatrix}$	$\begin{bmatrix} m \\ 5 \end{bmatrix}$	$\begin{bmatrix} m \\ 6 \end{bmatrix}$	$\begin{bmatrix} m \\ 7 \end{bmatrix}$
$m=1$	1						
$m=2$	1	1					
$m=3$	1	4	1				
$m=4$	1	11	11	1			
$m=5$	1	26	66	26	1		
$m=6$	1	57	302	302	57	1	
$m=7$	1	120	1191	2416	1191	120	1
\vdots							\ddots

$$\begin{aligned}
 B_0 &= 1 & B_1 &= -\frac{1}{2} & B_2 &= \frac{1}{6} & B_4 &= -\frac{1}{30} & B_6 &= \frac{1}{42} \\
 B_8 &= -\frac{1}{30} & B_{10} &= \frac{5}{66} & B_{12} &= -\frac{691}{2730} & B_{14} &= \frac{7}{6} & B_{16} &= -\frac{3617}{510} \\
 B_{18} &= \frac{43867}{798} & B_{20} &= -\frac{174611}{330} & B_{22} &= \frac{854513}{138} & B_{24} &= -\frac{236364091}{2730} & & \dots
 \end{aligned}$$

Note that $B_{2n+1} = 0$ for all $n \in \mathbb{N}^*$. And finally, we conclude this remark by giving the first signed tangent numbers:

$$\begin{aligned}
 C_1 &= 1 & C_3 &= -2 & C_5 &= 16 & C_7 &= -272 & C_9 &= 7\,936 & C_{11} &= -353\,792 \\
 & & C_{13} &= 22\,368\,256 & C_{15} &= -1\,903\,757\,312 & & \dots & & & &
 \end{aligned}$$

Remark 2.3. Instead of the finite differences calculus we applied here to obtain our formula 2.3, one may also use the differential operator $(z \frac{d}{dz})^m$ to obtain a formula for $\sum_{k=1}^n k^m z^k$. This is based though on an inductive argument, necessitating an a priori knowledge of the formula. Its proof is lengthier. So I find our approach here more natural. This differential calculus approach to $E_m(z) = \sum_{k=1}^{\infty} k^m z^k$ was done for instance in [6].

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