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# LINEAR DYNAMICS OF MULTIPLICATION AND COMPOSITION OPERATORS ON $\text{Hol}(\mathbb{D})$

L. OGER

ABSTRACT. We give a complete description of the linear dynamics of multiplication  $M_m$  and composition operators  $C_\varphi$  on the space  $\text{Hol}(\mathbb{D})$  of all holomorphic maps on the unit disc. We show that  $M_m$  is never supercyclic, and cyclic if and only if the map  $m$  is injective. For composition operators, we prove that if  $\varphi$  has a fixed point in  $\mathbb{D}$ , then  $C_\varphi$  is either not cyclic, or cyclic but not supercyclic on  $\text{Hol}(\mathbb{D})$ . On the other hand, if  $\varphi$  does not have any fixed point in the unit disc, then  $C_\varphi$  is hypercyclic on  $\text{Hol}(\mathbb{D})$ . We provide explicit expressions of cyclic and hypercyclic vectors. Finally, we make some observations on weighted composition operators on  $\text{Hol}(\mathbb{D})$ .

## CONTENTS

|  |    |
|--|----|
| 1. Introduction                              | 1  |
| 2. Multiplication operators                  | 3  |
| 3. General results on composition operators  | 5  |
| 4. Elliptic case                             | 12 |
| 5. Hyperbolic case                           | 14 |
| 6. Parabolic case                            | 15 |
| 7. Remarks on weighted composition operators | 17 |
| References                                   | 19 |

## 1. INTRODUCTION

Let  $\varphi$  be a holomorphic self-map of the unit disc  $\mathbb{D}$  of  $\mathbb{C}$ , and  $m$  a holomorphic map on  $\mathbb{D}$ . The composition operator associated with  $\varphi$  is defined as  $C_\varphi(f) = f \circ \varphi$ . The multiplication operator associated with  $m$  is defined as  $M_m(f) = mf$ .

Motivated by the fairly recent papers dealing with spectral properties of (weighted) composition operators on the Fréchet space  $\text{Hol}(\mathbb{D})$  of all holomorphic functions on the unit disc ([3, 2]), we study their linear dynamic properties. Linear dynamics on various topological spaces is a recent branch of functional analysis. It is a very active research area, closely linked to the invariant subspace problem and universality. See for example the seminal contributions of Godefroy and Shapiro ([13]), Grosse-Erdmann ([14]) and Kitai ([17]). Let us begin by recalling the first definitions ([5]).

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*Key words and phrases.* composition operator; multiplication operator; cyclicity; hypercyclicity; Fréchet space.

**Definition 1.1.** Let  $X$  be a topological vector space, and  $T : X \rightarrow X$  a linear operator.

- We say that  $T$  is *cyclic* on  $X$  if there exists  $x \in X$  (a *cyclic vector*) such that  $\text{Span}\{T^n(x) : n \geq 0\}$  is a dense subset of  $X$ .
- We say that  $T$  is *supercyclic* on  $X$  if there exists  $x \in X$  (a *supercyclic vector*) such that  $\{\lambda T^n(x) : \lambda \in \mathbb{C}, n \geq 0\}$  is a dense subset of  $X$ .
- We say that  $T$  is *hypercyclic* on  $X$  if there exists  $x \in X$  (a *hypercyclic vector*) such that  $\{T^n(x) : n \geq 0\}$  is a dense subset of  $X$ .

In Banach spaces of holomorphic functions (weighted Hardy spaces, Bergman spaces, Fock spaces, ...), many results about the cyclicity of composition operators were found. See for instance [6, 7, 12, 23].

The paper is organised as follows. In Section 2, we focus on multiplication operators, which are the first natural linear continuous operators on  $\text{Hol}(\mathbb{D})$ . We first show (Proposition 2.2) that  $M_m$  is cyclic if and only if the map  $m$  is injective. By contrast, no multiplication operator is supercyclic (Proposition 2.3).

In Sections 3 to 6, we study composition operators. In order to analyse these operators directly on  $\text{Hol}(\mathbb{D})$ , we have to categorise the symbols  $\varphi$ . Indeed, depending on the nature of  $\varphi$ , the properties of  $C_\varphi$  vary a lot. To do this classification, we use Denjoy-Wolff and Julia-Wolff-Carathéodory theorems ([1, 8, 10]).

- If  $\varphi$  has a fixed point in  $\mathbb{D}$ , then we say that  $\varphi$  is *elliptic*.
- Otherwise, there exist  $\xi \in \mathbb{T}$  (known as *Denjoy-Wolff point* of  $\varphi$ ) such that for an appropriate notion of limit (angular or non-tangential limit, see [8, Definition 1.5.1] for more details), we may write  $\varphi(\xi) = \xi$ . Moreover, if we consider

$$\delta = \liminf_{z \rightarrow \xi} \frac{1 - |\varphi(z)|}{1 - |z|} \leq 1,$$

we say that  $\varphi$  is *hyperbolic* if  $\delta < 1$ , and *parabolic* if  $\delta = 1$ . The number  $\delta$  is the *dilation coefficient* of  $\varphi$  at the point  $\xi$ .

In Section 3, we collect general results that will be used specifically for each type of symbol  $\varphi$ . We begin with necessary conditions that are well known in the Hardy spaces ([7]): we adapt them in the context of  $\text{Hol}(\mathbb{D})$ . Then, we emphasize links between the study of linear dynamics and spectral properties (Theorem 3.4), isometries (Proposition 3.6) and functional equations (Theorem 3.7).

Section 4 is devoted to elliptic composition operators. The bijective composition operators are the easiest ones to handle: they are never supercyclic, and cyclic if and only if the symbol is an *aperiodic automorphism*, that is a self-map of the disc conjugated to an aperiodic rotation. The most interesting case is the non-automorphic one. We prove that it is never supercyclic, and cyclic if and only if  $\varphi'(\alpha) \neq 0$ , where  $\alpha \in \mathbb{D}$  is the fixed point of  $\varphi$ . This extends the case of diagonalisable operators in finite-dimensional spaces.

Sections 5 and 6 deal with the hyperbolic and parabolic composition operators. By means of the solutions of Schröder and Abel functional equations, we recover the results of Grosse-Erdmann and Mortini ([15]): these operators are always hypercyclic on

$\text{Hol}(\mathbb{D})$ . The added value of our approach is that we obtain explicit hypercyclic vectors, thanks to Valiron and Pommerenke's maps. We obtain the following table:

| Nature of $\varphi$                             | Cyclicity | Supercyclicity | Hypercyclicity |
|---|-----------|----------------|----------------|
| $\varphi$ periodic automorphism                 | ✗         | ✗              | ✗              |
| $\varphi$ elliptic, not a periodic automorphism | ✓         | ✗              | ✗              |
| $\varphi$ non elliptic                          | ✓         | ✓              | ✓              |

TABLE 1. Linear dynamics of  $C_\varphi$  on  $\text{Hol}(\mathbb{D})$ ,  $\varphi$  injective

Finally, the last section is an overview of what happens when we combine multiplication and composition operators. The *weighted composition operator* with weight  $m$  and symbol  $\varphi$  is defined as  $W_{m,\varphi}(f) = m(f \circ \varphi)$ . The linear dynamics of these operators is much more difficult. Nevertheless, we obtain one necessary condition on  $m$  and  $\varphi$  to make  $W_{m,\varphi}$  cyclic (Theorem 7.1), and we give five examples to show that the behaviour of  $W_{m,\varphi}$  is erratic in general.

## 2. MULTIPLICATION OPERATORS

Let us begin by a useful lemma, stated in a particular case in [3, Proposition 3.5].

**Lemma 2.1.** *Let  $\sigma \in \text{Hol}(\mathbb{D})$  be an injective map.*

*The set  $\text{Span}\{\sigma^n : n \geq 0\}$  is a dense subset of  $\text{Hol}(\mathbb{D})$ .*

*Proof.* Consider  $\Omega = \sigma(\mathbb{D})$ . Since  $\sigma$  is injective, the set  $\Omega$  is a simply connected domain. By Runge's theorem ([20, 21]), the set of all polynomial functions on  $\Omega$  is a dense subset of  $\text{Hol}(\Omega)$ . Composition by  $\sigma$  gives the density of  $\text{Span}\{\sigma^n : n \geq 0\}$  on  $\text{Hol}(\mathbb{D})$ .  $\square$

We start by considering the cyclicity of multiplication operators.

**Proposition 2.2.** *Let  $m \in \text{Hol}(\mathbb{D})$ . The operator  $M_m$  is cyclic on  $\text{Hol}(\mathbb{D})$  if and only if the map  $m$  is injective. Moreover, in this case, a map  $\phi \in \text{Hol}(\mathbb{D})$  is cyclic for  $M_m$  if and only if  $\phi(z) \neq 0$  for all  $z \in \mathbb{D}$ .*

*Proof.* First, note that for all  $f \in \text{Hol}(\mathbb{D})$ ,

$$\text{Span}\{M_m^n f : n \geq 0\} = \text{Span}\{m^n f : n \geq 0\}.$$

Assume that  $m$  is injective. Then, by Lemma 2.1,  $\text{Span}\{m^n : n \geq 0\}$  is a dense subset of  $\text{Hol}(\mathbb{D})$ . Hence, the constant map  $\mathbb{1}$  is a cyclic vector of  $M_m$ .

Conversely, if  $M_m$  is cyclic, then there exists  $\sigma \in \text{Hol}(\mathbb{D})$  such that  $\text{Span}\{m^n \sigma : n \geq 0\}$  is a dense subset of  $\text{Hol}(\mathbb{D})$ . Hence, for all  $h \in \text{Hol}(\mathbb{D})$ , there exists  $(a_n) \subset \mathbb{C}$  such that

$$h = \sum_{n \geq 0} a_n M_m^n(\sigma) = \sigma \sum_{n \geq 0} a_n m^n.$$

Thus,  $\sigma(z) \neq 0$  for all  $z \in \mathbb{D}$  - otherwise, if  $\sigma(z_0) = 0$  for some  $z_0 \in \mathbb{D}$ , then  $h(z_0) = 0$  for all  $h \in \text{Hol}(\mathbb{D})$  - and dividing by  $\sigma$ , the constant map  $\mathbb{1}$  is a cyclic vector of  $M_m$ , so  $\text{Span}\{m^n : n \geq 0\}$  is a dense subset of  $\text{Hol}(\mathbb{D})$ . However, if  $m$  is not injective,

let  $z_1$  and  $z_2$  be two distinct points of  $\mathbb{D}$  such that  $m(z_1) = m(z_2)$ . Then, for all  $g \in \text{Hol}(\mathbb{D}) = \text{Span}\{m^n : n \geq 0\}$ ,  $g(z_1) = g(z_2)$ , a contradiction.

Finally, if  $m$  is injective and  $\phi(z) \neq 0$  for all  $z \in \mathbb{D}$ , let  $f \in \text{Hol}(\mathbb{D})$ . There exists  $g \in \text{Hol}(\mathbb{D})$  such that  $f = \phi g$ , and

$$g = \sum_{n \geq 0} a_n m^n \implies f = \sum_{n \geq 0} a_n m^n \phi,$$

for some sequence  $(a_n) \subset \mathbb{C}$ . Thus,  $\phi$  is a cyclic vector of  $M_m$ .

Conversely, if  $\phi(z_0) = 0$  for some  $z_0 \in \mathbb{D}$ , then

$$\text{Span}\{m^n \phi : n \geq 0\} \subset \{f \in \text{Hol}(\mathbb{D}) : f(z_0) = 0\},$$

which is not a dense subset of  $\text{Hol}(\mathbb{D})$ .  $\square$

Now, we show that no multiplication operator is supercyclic on  $\text{Hol}(\mathbb{D})$ .

**Proposition 2.3.** *Let  $m \in \text{Hol}(\mathbb{D})$ . The operator  $M_m$  is not supercyclic on  $\text{Hol}(\mathbb{D})$ .*

*Proof.* From Proposition 2.2, we only consider injective weights  $m$ .

Assume that  $M_m$  is a supercyclic operator on  $\text{Hol}(\mathbb{D})$ , and denote by  $\sigma$  one of its supercyclic vectors. Then,  $\{\lambda m^n \sigma : \lambda \in \mathbb{C}, n \in \mathbb{N} \cup \{0\}\}$  is a dense subset of  $\text{Hol}(\mathbb{D})$ . Hence, for all map  $h \in \text{Hol}(\mathbb{D})$  and  $K \subset \mathbb{D}$  compact, there exist two sequences  $(\lambda_k) \subset \mathbb{C}$  and  $(n_k) \subset \mathbb{N} \cup \{0\}$  such that

$$\|h - \lambda_k m^{n_k} \sigma\|_{\infty, K} \xrightarrow{k \rightarrow +\infty} 0.$$

Thus,  $\sigma(z) \neq 0$  for all  $z \in \mathbb{D}$ . Therefore, dividing by  $\sigma$ , the constant map  $\mathbb{1}$  is a supercyclic vector of  $M_m$ , so  $\{\lambda m^n : \lambda \in \mathbb{C}, n \in \mathbb{N} \cup \{0\}\}$  is a dense subset of  $\text{Hol}(\mathbb{D})$ .

Now, we want to approach all maps with functions of the form  $\lambda m^n$ . Note that since  $m$  is not constant (because  $m$  is injective), then  $m(\mathbb{D})$  is an open set. Let  $w_0 \in m(\mathbb{D})$ . There exists  $\delta > 0$  such that  $D(w_0, \delta) \subset m(\mathbb{D})$ . Hence,  $\{w_0 + it : -\delta < t < \delta\} \subset m(\mathbb{D})$ . Let  $z_1 = m^{-1}(w_0)$  and  $z_2 = m^{-1}(w_0 + it)$ , for some  $0 < t < \delta$ . Then,  $|m(z_1)| \neq |m(z_2)|$  and  $\Re(m(z_1)) = \Re(m(z_2))$ .

Considering a compact subset  $K$  of  $\mathbb{D}$  containing  $z_1$  and  $z_2$ , there exist  $(\lambda_k) \subset \mathbb{C}$  and  $(n_k) \subset \mathbb{N} \cup \{0\}$  such that

$$\|e^m - \lambda_k m^{n_k}\|_{\infty, K} \xrightarrow{k \rightarrow +\infty} 0.$$

In particular,  $\lambda_k m(z_1)^{n_k} \rightarrow e^{m(z_1)}$  and  $\lambda_k m(z_2)^{n_k} \rightarrow e^{m(z_2)}$ , as  $k$  goes to  $+\infty$ . Denoting by  $r_i = |m(z_i)|$  and  $\rho = |e^{m(z_1)}| = e^{\Re(m(z_1))} = e^{\Re(m(z_2))} = |e^{m(z_2)}|$ , we get

$$|\lambda_k| \underset{k \rightarrow +\infty}{\sim} \frac{\rho}{r_1^{n_k}} \quad \text{and} \quad |\lambda_k| \underset{k \rightarrow +\infty}{\sim} \frac{\rho}{r_2^{n_k}}.$$

We may choose the values  $r_1$  and  $r_2$  very close to each other (since  $t$  can be very close to 0), and such that  $1/r_1$  is not an integer power of  $1/r_2$ . That gives a contradiction.  $\square$

## 3. GENERAL RESULTS ON COMPOSITION OPERATORS

To begin this section, we remark that the results found in the papers cited in the introduction will help us a lot. Indeed, we have the following fundamental lemma, whose proof is repeated here.

**Lemma 3.1** ([12]). *Let  $X, Y$  be two metric spaces, such that  $X$  embeds continuously in  $Y$  and is a dense subset of  $Y$ . Let  $T$  be a linear operator, continuous on  $X$  and on  $Y$  (for their respective topologies), such that  $T(X) \subset X$  and  $T(Y) \subset Y$ .*

*If  $T$  is cyclic (resp. supercyclic, hypercyclic) on  $X$ , then  $T$  is cyclic (resp. supercyclic, hypercyclic) on  $Y$ .*

*Proof.* Let  $\gamma \in X$  be a cyclic vector of the operator  $T$ . Then, for all  $f \in X$ , there exists a sequence  $(a_j) \subset \mathbb{C}$  such that

$$d_X \left( f, \sum_{j=0}^n a_j T^j(\gamma) \right) \xrightarrow{n \rightarrow \infty} 0.$$

Since  $X$  embeds continuously in  $Y$ , we obtain

$$d_Y \left( f, \sum_{j=0}^n a_j T^j(\gamma) \right) \xrightarrow{n \rightarrow \infty} 0.$$

Moreover, since  $X$  is a dense subset of  $Y$ , for all  $g \in Y$  and  $\varepsilon > 0$ , there exists  $f_0 \in X$  such that  $d_Y(f_0, g) < \varepsilon$ . Hence, denoting by  $(a_{j,0})$  the sequence  $(a_j)$  associated with  $f_0$ ,

$$\limsup_{n \rightarrow \infty} d_Y \left( g, \sum_{j=0}^n a_{j,0} T^j(\gamma) \right) < \varepsilon.$$

This inequality being valid for  $\varepsilon > 0$ , the vector  $\gamma \in X \subset Y$  is a cyclic vector of  $T$ , so  $T$  is cyclic on  $Y$ . Supercyclicity and hypercyclicity follow from similar arguments (considering  $a_n T^n(\gamma)$  for the first one,  $T^n(\gamma)$  for the last one).  $\square$

For instance, when  $X = H^2(\mathbb{D})$ ,  $Y = \text{Hol}(\mathbb{D})$  and  $T = C_\varphi$  for a certain  $\varphi \in H^2(\mathbb{D})$ , Lemma 3.1 is valid. Hence, the results of [7] and [12] will give us results on  $\text{Hol}(\mathbb{D})$ .

### 3.1. Necessary conditions for cyclicity.

We now consider « basic » conditions, that were given in the Hardy space  $H^2(\mathbb{D})$  in [7]. We adapt here in the context of  $\text{Hol}(\mathbb{D})$ . For a set  $X \subset \text{Hol}(\mathbb{D})$ , we denote by  $\overline{X}$  its closure.

**Lemma 3.2.** *Assume that  $C_\varphi$  is cyclic on  $\text{Hol}(\mathbb{D})$ . Then,  $\text{codim}(\overline{C_\varphi(\text{Hol}(\mathbb{D}))}) \leq 1$ .*

*Proof.* Let  $\gamma$  be a cyclic vector of  $C_\varphi$ . Then, for all  $f \in \text{Hol}(\mathbb{D})$ , there exists a sequence  $(a_k) \subset \mathbb{C}$  such that

$$f = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k (\gamma \circ \varphi^{[k]}) = a_0 \gamma + \underbrace{\lim_{n \rightarrow \infty} \left[ \sum_{k=1}^n a_k (\gamma \circ \varphi^{[k-1]}) \right]}_{\in \overline{C_\varphi(\text{Hol}(\mathbb{D}))}} \circ \varphi.$$

This decomposition finishes the proof.  $\square$

**Proposition 3.3.** *Assume that  $C_\varphi$  is cyclic on  $\text{Hol}(\mathbb{D})$ . Then  $\varphi$  is injective.*

*Proof.* Let  $\varphi$  be a non injective map. Then, there exist two distinct points  $a, b \in \mathbb{D}$  such that  $\varphi(a) = \varphi(b) = w$ . Thus, for all  $g \in \overline{C_\varphi(\text{Hol}(\mathbb{D}))}$ , there exists a sequence  $(g_k) \subset C_\varphi(\text{Hol}(\mathbb{D}))$  such that  $g = \lim g_k$ . Hence, denoting  $g_k = h_k \circ \varphi$ ,

$$g(a) - g(b) = \lim_{k \rightarrow \infty} (h_k \circ \varphi)(a) - \lim_{k \rightarrow \infty} (h_k \circ \varphi)(b) = \lim_{k \rightarrow \infty} [h_k(w) - h_k(w)] = 0.$$

We have shown that  $\overline{C_\varphi(\text{Hol}(\mathbb{D}))} \subset \{f \in \text{Hol}(\mathbb{D}) : f(a) = f(b)\}$ . Consider

$$E = \text{Hol}(\mathbb{D}) / \{f \in \text{Hol}(\mathbb{D}) : f(a) = f(b)\}.$$

We may find two linearly independent functions on  $E$  (for instance,  $z \mapsto z$  and  $z \mapsto \frac{z-6}{z+1}$ ). Hence,  $\dim(E) \geq 2$ , so  $\text{codim}(\overline{C_\varphi(\text{Hol}(\mathbb{D}))}) \geq 2$ , so  $C_\varphi$  is not cyclic by Lemma 3.2.  $\square$

### 3.2. A spectral condition for cyclicity.

Since the authors of [3] described the spectra of composition operators on  $\text{Hol}(\mathbb{D})$ , it is very natural to take advantage of the knowledge of the eigenvalues of  $C_\varphi$ . The following theorem will be very important in Section 4.

**Theorem 3.4.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic map. Assume that there exist a constant  $\lambda \in \mathbb{D} \setminus \{0\}$  and an injective function  $\sigma \in \text{Hol}(\mathbb{D})$  such that  $\sigma \circ \varphi = \lambda\sigma$ . Then, the operator  $C_\varphi$  is cyclic on  $\text{Hol}(\mathbb{D})$ .*

*Proof.* Step 1: Let  $\psi = \exp(\sigma) = \sum_{n \geq 0} \sigma^n / n!$ . Then

$$\begin{pmatrix} \psi \\ C_\varphi(\psi) \\ \vdots \\ C_\varphi^k(\psi) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots \\ 1 & \lambda & \cdots & \lambda^k & \cdots \\ \vdots & \vdots & & \vdots & \\ 1 & \lambda^k & \cdots & \lambda^{2k} & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix} \begin{pmatrix} 1 \\ \sigma \\ \vdots \\ \sigma^k / k! \\ \vdots \end{pmatrix}.$$

Hence, we obtained an infinite Vandermonde matrix. Let us study the properties of its coefficients. For  $k, n \in \mathbb{N} \cup \{0\}$  such that  $n \geq k$ , denote

$$\alpha_{k,n} = \prod_{j=n-k+1}^n (1 - \lambda^j).$$

Then, for  $n \geq k$ , the following properties are valid:

$$(\mathbf{P}_1) : a_{0,n} = 1, \quad (\mathbf{P}_2) : \alpha_{k,k} = (1 - \lambda^k)\alpha_{k-1,k-1}, \quad (\mathbf{P}_3) : \alpha_{k+1,n} = (1 - \lambda^{n-k})\alpha_{k,n}.$$

It immediately follows from the definition of  $\alpha_{n,k}$ . Moreover, for all  $\ell \in \mathbb{N} \cup \{0\}$ , we have

$$(\mathbf{P}_4) : \alpha_{k+\ell,n+\ell} - (1 - \lambda^k)\alpha_{k-1,n-1}\alpha_{\ell,n+\ell} = \lambda^k\alpha_{k,n-1}\alpha_{\ell,n+\ell}$$

Indeed, since  $\llbracket n - k + 1, n + \ell \rrbracket = \llbracket n - k + 1, n - 1 \rrbracket \cup \{n\} \cup \llbracket n + 1, n + \ell \rrbracket$  (with the

convention  $\llbracket a, b \rrbracket = \emptyset$  for  $a > b$ ), we get

$$\begin{aligned}
 & \prod_{j=n-k+1}^{n+\ell} (1 - \lambda^j) - (1 - \lambda^k) \prod_{j=n-k+1}^{n-1} (1 - \lambda^j) \prod_{j=n+1}^{n+\ell} (1 - \lambda^j) \\
 &= [(1 - \lambda^n) - (1 - \lambda^k)] \prod_{j=n-k+1}^{n-1} (1 - \lambda^j) \prod_{j=n+1}^{n+\ell} (1 - \lambda^j) \\
 &= \lambda^k (1 - \lambda^{n-k}) \prod_{j=n-k+1}^{n-1} (1 - \lambda^j) \prod_{j=n+1}^{n+\ell} (1 - \lambda^j) = \lambda^k \prod_{j=n-k}^{n-1} (1 - \lambda^j) \prod_{j=n+1}^{n+\ell} (1 - \lambda^j).
 \end{aligned}$$

Step 2: Set  $\psi_0 = \psi$ , and for all  $k \geq 0$ , let us define  $\psi_{k+1} = \psi_k - \lambda^{-k} C_\varphi(\psi_k)$ . By induction, we show that for all  $k \geq 0$ ,

$$(*) \quad \psi_k = \sum_{n \geq k} \frac{1}{n!} \alpha_{k,n} \sigma^n.$$

- The case  $k = 0$  immediately follows from the definition of  $\exp$  and from  $(\mathbf{P}_1)$ .
- Assume that the formula  $(*)$  is valid for a certain  $k$ . Note that  $C_\varphi(\sigma^n) = \lambda^n \sigma^n$ . Using  $(\mathbf{P}_3)$ ,

$$\begin{aligned}
 \psi_{k+1} &= \psi_k - \lambda^{-k} C_\varphi(\psi_k) = \sum_{n \geq k} \frac{1}{n!} \alpha_{k,n} (\sigma^n - \lambda^{-k} C_\varphi(\sigma^n)) \\
 &= \sum_{n \geq k} \frac{1}{n!} [\alpha_{k,n} (1 - \lambda^{n-k})] \sigma^n = \sum_{n \geq k+1} \frac{1}{n!} \alpha_{k+1,n} \sigma^n.
 \end{aligned}$$

Indeed, if  $n = k$ , then  $1 - \lambda^{n-k} = 0$ .

Step 3: For all  $n \in \mathbb{N}$ ,  $\text{Span}(\psi_k : k \leq n) = \text{Span}(C_\varphi^k \psi : k \leq n)$ . We prove it by induction.

- The case  $n = 0$  is easy since  $\psi_0 = \psi = C_\varphi^0 \psi$ .
- Assume that the equality is valid for  $n$ . Since  $\llbracket 1, n \rrbracket \subset \llbracket 1, n+1 \rrbracket$ , it is sufficient to show that  $\psi_{n+1} \in \text{Span}\{C_\varphi^k(\psi) : k \leq n+1\}$  and  $C_\varphi^{n+1}(\psi) \in \text{Span}\{\psi_k : k \leq n+1\}$ . By inductive hypothesis, there exist  $a_0, \dots, a_n \in \mathbb{C}$  such that

$$\psi_n = \sum_{k=0}^n a_k C_\varphi^k(\psi).$$

Hence,

$$\begin{aligned}
 \psi_{n+1} &= \sum_{k=0}^n a_k C_\varphi^k(\psi) - \sum_{k=0}^n \frac{a_k}{\lambda^n} C_\varphi^{k+1}(\psi) \\
 &= a_0 \psi + \sum_{k=1}^n \left( a_k - \frac{a_{k-1}}{\lambda^n} \right) C_\varphi^k(\psi) - \frac{a_n}{\lambda^n} C_\varphi^{n+1}(\psi) \in \text{Span}\{C_\varphi^k(\psi) : k \leq n+1\}.
 \end{aligned}$$



Conversely, we write

$$\begin{aligned} C_\varphi^{n+1}(\psi) &= \frac{\lambda^n}{a_n} \left( a_0 \psi + \sum_{k=1}^n \left( a_k - \frac{a_{k-1}}{\lambda^n} \right) C_\varphi^k(\psi) - \psi_{n+1} \right) \\ &\in \text{Span}(\{C_\varphi^k(\psi) : k \leq n\} \cup \{\psi_{n+1}\}) = \text{Span}\{\psi_k : k \leq n+1\}. \end{aligned}$$

Step 4: For all  $\ell \in \mathbb{N}$ , set  $g_{\ell,0} = \alpha_{\ell,\ell}^{-1} \psi_\ell$ , and for all  $k \geq 1$ , let us define

$$g_{\ell,k} = g_{\ell,k-1} + \frac{(-1)^k \lambda^{k(k-1)/2}}{\alpha_{k,k} \alpha_{\ell,\ell}} \psi_{k+\ell}.$$

By induction on  $k$ , we show that for all  $k, \ell \geq 0$ ,

$$g_{\ell,k} = \sigma^\ell + \frac{(-1)^k \lambda^{k(k+1)/2}}{\alpha_{k,k} \alpha_{\ell,\ell}} \sum_{n \geq k+1} \frac{\alpha_{k,n-1} \alpha_{\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell}.$$

- $k = 0$  : Using (\*) and (**P**<sub>1</sub>),

$$g_{\ell,0} = \frac{1}{\alpha_{\ell,\ell}} \psi_\ell = \frac{1}{\alpha_{\ell,\ell}} \sum_{n \geq \ell} \frac{\alpha_{\ell,n}}{n!} \sigma^n = \frac{1}{\alpha_{\ell,\ell}} \sum_{n \geq 0} \frac{\alpha_{\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell} = \sigma^\ell + \frac{(-1)^0 \lambda^0}{\alpha_{0,0} \alpha_{\ell,\ell}} \sum_{n \geq 1} \frac{\alpha_{0,n-1} \alpha_{\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell}.$$

- Assume that the formula is valid for  $k-1$ . Using (\*),

$$\begin{aligned} g_{\ell,k} &= g_{\ell,k-1} + \frac{(-1)^k \lambda^{\frac{k(k-1)}{2}}}{\alpha_{k,k} \alpha_{\ell,\ell}} \psi_{k+\ell} \\ &= \sigma^\ell + \frac{(-1)^{k-1} \lambda^{\frac{k(k-1)}{2}}}{\alpha_{k-1,k-1} \alpha_{\ell,\ell}} \sum_{n \geq k} \frac{\alpha_{k-1,n-1} \alpha_{\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell} + \frac{(-1)^k \lambda^{\frac{k(k-1)}{2}}}{\alpha_{k,k} \alpha_{\ell,\ell}} \sum_{n \geq k} \frac{\alpha_{k+\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell} \\ &= \sigma^\ell + \frac{(-1)^k \lambda^{\frac{k(k-1)}{2}}}{\alpha_{\ell,\ell}} \left( \frac{1}{\alpha_{k,k}} \sum_{n \geq k} \frac{\alpha_{k+\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell} - \frac{1}{\alpha_{k-1,k-1}} \sum_{n \geq k} \frac{\alpha_{k-1,n-1} \alpha_{\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell} \right) \\ &= \sigma^\ell + \frac{(-1)^k \lambda^{\frac{k(k-1)}{2}}}{\alpha_{k,k} \alpha_{\ell,\ell}} \left( \sum_{n \geq k} \frac{\alpha_{k+\ell,n+\ell} - (1-\lambda^k) \alpha_{k-1,n-1} \alpha_{\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell} \right) \quad \text{by } (\mathbf{P}_2) \\ &= \sigma^\ell + \frac{(-1)^k \lambda^{\frac{k(k-1)}{2}}}{\alpha_{k,k} \alpha_{\ell,\ell}} \left( \sum_{n \geq k} \frac{\lambda^k \alpha_{k,n-1} \alpha_{\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell} \right) \quad \text{by } (\mathbf{P}_4) \\ &= \sigma^\ell + \frac{(-1)^k \lambda^{\frac{k(k+1)}{2}}}{\alpha_{k,k} \alpha_{\ell,\ell}} \left( \sum_{n \geq k} \frac{\alpha_{k,n-1} \alpha_{\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell} \right). \end{aligned}$$

We deduce that

$$|g_{\ell,k} - \sigma^\ell| = \frac{|\lambda|^{\frac{k(k+1)}{2}}}{|\alpha_{k,k}| |\alpha_{\ell,\ell}|} \left| \sum_{n \geq k} \frac{\alpha_{k,n-1} \alpha_{\ell,n+\ell}}{(n+\ell)!} \sigma^{n+\ell} \right|.$$

Moreover, since  $0 < |\lambda| < 1$ , for all  $j \geq 1$ ,  $1 - |\lambda| \leq |1 - \lambda^j| \leq 2$ . Hence, for all  $n \geq k$ , since  $\alpha_{k,n}$  is a product with  $k$  terms, we have  $(1 - |\lambda|)^k \leq |\alpha_{k,n}| \leq 2^k$ . Finally,

$$|g_{\ell,k} - \sigma^\ell| \leq |\lambda|^{\frac{k(k+1)}{2}} \left( \frac{2}{1 - |\lambda|} \right)^{k+\ell} \left| \sum_{n \geq k} \frac{1}{(n + \ell)!} \sigma^{n+\ell} \right|.$$

**Conclusion:** Let  $\varepsilon > 0$ ,  $r \in (0, 1)$  and  $f \in \text{Hol}(\mathbb{D})$ . By Lemma 2.1, there exist an integer  $N \in \mathbb{N}$  and a map  $h \in \text{Span}(\sigma^\ell : \ell \leq N)$  such that

$$\|f - h\|_{\infty,r} := \sup_{|z| \leq r} |F(z) - h(z)| \leq \varepsilon.$$

In the following, denote  $h = \sum_{\ell=0}^N c_\ell \sigma^\ell$ , with  $c_\ell \in \mathbb{C}$ . Then, for all  $k \in \mathbb{N}$ , using Step 4,

$$\begin{aligned} \left\| h - \sum_{\ell=0}^N c_\ell g_{\ell,k} \right\|_{\infty,r} &= \left\| \sum_{\ell=0}^N c_\ell (\sigma^\ell - g_{\ell,k}) \right\|_{\infty,r} \\ &\leq \sum_{\ell=0}^N \left[ |c_\ell| |\lambda|^{\frac{k(k+1)}{2}} \left( \frac{2}{1 - |\lambda|} \right)^{k+\ell} \left\| \sum_{n \geq k} \frac{1}{(n + \ell)!} \sigma^{n+\ell} \right\|_{\infty,r} \right] \xrightarrow{k \rightarrow +\infty} 0, \end{aligned}$$

since we last term of the sum is the rest of a uniformly converging series on  $r\overline{\mathbb{D}}$ .

Thus, there exists  $k_0 \in \mathbb{N}$  such that if  $g = \sum_{\ell=0}^N c_\ell g_{\ell,k_0}$ , then

$$\|h - g\|_{\infty,r} \leq \varepsilon \implies \|f - g\|_{\infty,r} \leq 2\varepsilon.$$

To conclude, since for all  $k, \ell \in \mathbb{N}$ ,  $g_{\ell,k} \in \text{Span}\{\psi_p : p \leq k + \ell\}$ , then

$$g \in \text{Span}\{\psi_p : p \leq N + k_0\} = \text{Span}\{C_\varphi^p(\psi) : p \leq N + k_0\},$$

using Step 2. Therefore,  $\text{Span}\{C_\varphi^p(\psi) : p \in \mathbb{N}\}$  is a dense subset of  $\text{Hol}(\mathbb{D})$ , which proves that  $\psi$  is a cyclic vector of  $C_\varphi$  on  $\text{Hol}(\mathbb{D})$ .  $\square$

This theorem is a generalisation of the following easy observation, dealing with diagonalisable operators in finite dimension with all distinct eigenvalues. Its proof is included for the sake of completeness.

**Proposition 3.5.** *Let  $T \in \mathcal{M}_n(\mathbb{C})$  be a diagonalisable matrix. Are equivalent:*

- (i)  $T$  is cyclic.
- (ii) The eigenvalues of  $T$  are non-zero and all distinct.

*In such case, the cyclic vectors of  $T$  are the  $x = (x_j) \in \mathbb{C}^n$  such that  $x_j \neq 0$  for all  $j$ .*

*Proof.* Since the cyclic properties are invariant by conjugation, we assume that the matrix  $T$  is diagonal, that is  $T = \text{diag}(\lambda_1, \dots, \lambda_n)$ .

If all the  $\lambda_i$  are distinct and non-zero, then there exists a basis  $(v_1, \dots, v_n)$  of  $\mathbb{C}^n$  such that for all  $1 \leq i \leq n$ ,  $T(v_i) = \lambda_i v_i$ . Let  $v \in \mathbb{C}^n$ . We may write  $v = \sum_{i=1}^n a_i v_i$ .

Assume that  $b = (b_1, \dots, b_n) \in \mathbb{C}^n$  satisfies

$$\sum_{j=1}^n b_j T^{j-1}(v) = 0.$$

Then we obtain

$$0 = \sum_{j=1}^n b_j \sum_{i=1}^n a_i T^{j-1}(v_i) = \sum_{j=1}^n \sum_{i=1}^n b_j a_i \lambda_i^{j-1} v_i = \sum_{i=1}^n \left( a_i \sum_{j=1}^n b_j \lambda_i^{j-1} \right) v_i.$$

Since  $(v_1, \dots, v_n)$  is a basis of  $\mathbb{C}^n$ , for all  $i \in \{1, \dots, n\}$ ,

$$a_i \sum_{j=1}^n b_j \lambda_i^{j-1} = 0.$$

Assume that  $a_i \neq 0$  for all  $i$ . Then  $b$  satisfies

$$\underbrace{\begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{pmatrix}}_{=:M} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix  $M$  being invertible (as a Vandermonde matrix with distinct coefficients),  $b = 0$ . Hence,  $\{v, T(v), \dots, T^{n-1}(v)\}$  is a basis of  $\mathbb{C}^n$ , and the matrix  $T$  is cyclic, with  $v$  as a cyclic vector. Conversely, if there exists  $i$  such that  $a_i = 0$ , then there exists  $b \neq \vec{0}$  such that  $a_i \sum_{j=1}^n b_j \lambda_i^{j-1} = 0$ . Thus,

$$\text{Span}(\{v, T(v), \dots, T^{n-1}(v)\}) \subset \text{Span}(v_j : j \neq i),$$

which is  $(n-1)$ -dimensional, so the vector  $v$  is not cyclic for  $T$ . Finally, note that since  $\{v_1, \dots, v_n\}$  is a basis of  $\mathbb{C}^n$ , we have  $x_i \neq 0$  for all  $i$  if and only if  $a_j \neq 0$  for all  $j$ .

Assume now that there exists  $i \in \{1, \dots, n\}$  such that  $\lambda_i = 0$ . Then, for all  $x \in \mathbb{C}^n$ ,  $(Tx)_i = 0$ , so  $\text{Span}(\{x, T(x), \dots, T^{n-1}(x)\}) \subset \{y \in \mathbb{C}^n : y_i = 0\}$ , which is  $(n-1)$ -dimensional. In the same way, if  $\lambda_i = \lambda_j$  for some  $i, j \in \{1, \dots, n\}$ , then for all  $x \in \mathbb{C}^n$ ,  $\text{Span}(\{x, T(x), \dots, T^{n-1}(x)\}) \subset \{y \in \mathbb{C}^n : y_i = y_j\}$ , once again a  $(n-1)$ -dimensional subspace of  $\mathbb{C}^n$ . Hence, in these two cases  $T$  has no cyclic vector.  $\square$

### 3.3. Supercyclicity and isometries.

We now consider the following result, about the cyclicity of isometries.

**Proposition 3.6.** *Let  $X$  be a Fréchet space, and  $T : X \rightarrow X$  a continuous linear operator. Denote by  $(\|\cdot\|_p)$  a family of seminorms associated with  $X$ . Assume that the operator  $T$  has a non-trivial invariant subspace, and that for all  $p \in \mathbb{N}$ ,  $T$  is isometric for  $\|\cdot\|_p$ . Then,  $T$  is not supercyclic on  $X$ .*

*Proof.* Assume that  $T$  is supercyclic, denote by  $\gamma$  a supercyclic vector. Let  $x, y \in X$ . There exist two sequences  $(c_j) \subset \mathbb{C}$  and  $(n_j) \subset \mathbb{N}$  such that for all  $p \in \mathbb{N}$ ,

$$\|c_j T^{n_j}(\gamma) - x\|_p \xrightarrow{j \rightarrow \infty} 0.$$

Let  $p \in \mathbb{N}$  and  $\varepsilon > 0$ . Upon renormalising, we may assume that  $\|\gamma\|_p = 1$ . First, note that for  $j$  sufficiently large,

$$|c_j| = |c_j| \|T^{n_j}(\gamma)\|_p = \|c_j T^{n_j}(\gamma)\|_p \leq 1 + \|x\|_p.$$

Hence, the sequence  $(c_j)$  is bounded, let us say by  $A$ . Choose  $\lambda \in \mathbb{C}$  and  $k \in \mathbb{N}$  so that

$$\|\lambda T^k(\gamma) - y\|_p < \frac{\varepsilon}{A + \varepsilon} \frac{\|y\|_p}{2}.$$

Using once again the isometric property,  $|\lambda| > \|y\|_p/2$ . Hence, for  $j$  large enough,  $n_j > k$ , we obtain

$$\|\lambda T^{n_j}(\gamma) - T^{n_j-k}(y)\|_p < \frac{\varepsilon}{A + \varepsilon} \frac{\|y\|_p}{2}.$$

Multiplying on each side by  $|c_j|/|\lambda|$ , we get

$$\left\| c_j T^{n_j}(\gamma) - \frac{c_j}{\lambda} T^{n_j-k}(y) \right\|_p < \varepsilon \frac{|c_j|}{A + \varepsilon} \frac{\|y\|_p}{2|\lambda|} < \varepsilon.$$

Finally,

$$\limsup_{j \rightarrow \infty} \left\| x - \frac{c_j}{\lambda} T^{n_j-k}(y) \right\|_p \leq \varepsilon.$$

This inequality being valid for all  $\varepsilon > 0$ ,

$$\left\| x - \frac{c_j}{\lambda} T^{n_j-k}(y) \right\|_p \xrightarrow{j \rightarrow \infty} 0.$$

Since it is valid for all  $p \in \mathbb{N}$ , the vector  $y$  is supercyclic for  $C_\varphi$ .

Thus, every vector of  $X$  is supercyclic for the operator  $T$ , so  $T$  cannot have any non-trivial invariant subspace, a contradiction.  $\square$

### 3.4. Cyclicity and functional equations.

To conclude this section, we give a sufficient condition for  $C_\varphi$  to be cyclic, using functional equations. This theorem will be used several times in Sections 5 and 6. It follows from the ideas of Bourdon and Shapiro [7, Theorem 0.6] in the Hardy space  $H^2$ .

**Theorem 3.7.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic map. Assume that there exist two maps  $\sigma : \mathbb{D} \rightarrow \Omega$  injective, and  $\psi : \Omega \rightarrow \Omega$  bijective such that*

$$\sigma \circ \varphi = \psi \circ \sigma.$$

*If the operator  $C_\psi$  is hypercyclic on  $\text{Hol}(\Omega)$ , then  $C_\varphi$  is hypercyclic on  $\text{Hol}(\mathbb{D})$ .*

*Proof.* Let  $F \in \text{Hol}(\Omega)$  be a hypercyclic vector of  $C_\psi$ . Then,

$$\{C_\varphi^n(F \circ \sigma) : n \geq 0\} = \{F \circ \sigma \circ \varphi^{[n]} : n \geq 0\} = \{F \circ \psi^{[n]} \circ \sigma : n \geq 0\}.$$

Let  $f \in \text{Hol}(\mathbb{D})$ ,  $\varepsilon > 0$  and  $K$  a compact subset of  $\mathbb{D}$ . Denote  $L = \sigma(K)$ .

- Since  $\sigma$  is injective, by Lemma 2.1, the set  $\text{Span}\{\sigma^n : n \geq 0\}$  is a dense subset of  $\text{Hol}(\mathbb{D})$ . Hence, since  $\text{Span}\{\sigma^n : n \geq 0\} = \text{Span}\{z^n \circ \sigma : n \geq 0\} \subset C_\sigma(\text{Hol}(\mathbb{D}))$ , then  $C_\sigma(\text{Hol}(\mathbb{D}))$  is also a dense subset of  $\text{Hol}(\mathbb{D})$ . Therefore, there exists a map  $g = h \circ \sigma \in C_\sigma(\text{Hol}(\mathbb{D}))$  such that

$$\|g - f\|_{\infty, K} := \sup_{z \in K} |g(z) - f(z)| \leq \varepsilon.$$

- Moreover, since the map  $F$  is a hypercyclic vector of  $C_\psi$ , there exists a function  $H \in \{F \circ \psi^{[n]} : n \geq 0\}$  such that  $\|H - h\|_{\infty, L} \leq \varepsilon$ . If we denote  $G = H \circ \sigma$ , we get

$$\begin{aligned} \|G - g\|_{\infty, K} &= \sup_{z \in K} |G(z) - g(z)| = \sup_{z \in K} |H(\sigma(z)) - h(\tau^{-1}(\sigma(z)))| \\ &\leq \sup_{w \in L} |H(w) - h(w)| \\ &= \|H - h\|_{\infty, L} \leq \varepsilon. \end{aligned}$$

To conclude,  $G = H \circ \sigma \in \{F \circ \psi^{[n]} \circ \sigma : n \geq 0\}$ , and

$$\|G - f\|_{\infty, K} \leq \|G - g\|_{\infty, K} + \|g - f\|_{\infty, K} \leq 2\varepsilon.$$

Hence,  $\{F \circ \psi^{[n]} \circ \sigma : n \geq 0\} = \{C_\varphi^n(F \circ \sigma) : n \geq 0\}$  is a dense subset of  $\text{Hol}(\mathbb{D})$ . The operator  $C_\varphi$  is hypercyclic on  $\text{Hol}(\mathbb{D})$ , with  $F \circ \sigma$  as a hypercyclic vector.  $\square$

#### 4. ELLIPTIC CASE

In this section, we consider composition operators  $C_\varphi$ , with an elliptic symbol  $\varphi$ . Hence,  $\varphi$  has a fixed point in the unit disc  $\mathbb{D}$ . Upon conjugating, in the following, we will assume that  $\varphi(0) = 0$ .

##### 4.1. Bijective symbols.

Let us start by studying invertible symbols. To do this, note that using Schwarz's lemma ([1]), the function  $\varphi$  is defined by  $\varphi(z) = \beta z$ , for a certain  $\beta \in \mathbb{T}$ . First, we recall that the composition operators associated with those  $\varphi$  are isometric on  $\text{Hol}(\mathbb{D})$ .

**Proposition 4.1** ([9]). *Let  $T : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D})$  be a continuous linear operator. Then, the operator  $T$  is isometric for all the seminorms associated with  $\text{Hol}(\mathbb{D})$  if and only if there exist  $\alpha, \beta \in \mathbb{T}$  such that*

$$T = T_{\alpha, \beta} : f(z) \mapsto \alpha f(\beta z).$$

Hence, we obtain the following theorem.

**Theorem 4.2.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be defined by  $\varphi(z) = \beta z$ , for some  $|\beta| = 1$ .*

*The operator  $C_\varphi$  is not supercyclic on  $\text{Hol}(\mathbb{D})$ , and is cyclic on  $\text{Hol}(\mathbb{D})$  if and only if  $\beta$  is aperiodic, i.e. for all  $n \geq 1$ ,  $\beta^n \neq 1$ .*

*Proof.* We start by studying supercyclicity.

By Proposition 4.1, the operator  $C_\varphi$  is an isometry for all the seminorms associated with the space  $\text{Hol}(\mathbb{D})$ . Moreover, the set of all constant functions is a non-trivial invariant subspace of  $C_\varphi$ . Hence, by Proposition 3.6,  $C_\varphi$  is not supercyclic.

We now focus on cyclicity.

- If  $\beta$  is periodic, that is if there exists  $n_0 \geq 2$  such that  $\beta^{n_0} = 1$ , then

$$\{C_\varphi^n(f) : n \geq 0\} = \{f \circ \varphi^{[n]} : n \geq 0\} = \{f \circ \varphi^{[n]} : 0 \leq n \leq N - 1\}.$$

Hence, the space  $\text{Span}\{C_\varphi^n(f) : n \geq 0\}$  is finite-dimensional. Thus, it cannot be a dense subset of  $\text{Hol}(\mathbb{D})$ .

- If  $\beta$  is aperiodic, using [7, Proposition 2.1], the operator  $C_\varphi$  is cyclic on  $H^2(\mathbb{D})$ . Hence, it is cyclic on  $\text{Hol}(\mathbb{D})$  by Lemma 3.1.  $\square$

#### 4.2. Non automorphic symbols.

If the symbol  $\varphi$  is not bijective, then by Schwarz's lemma,  $0 \leq |\varphi'(0)| < 1$ . First, we focus on the easy case: what happens if  $\varphi'(0) = 0$  ?

**Lemma 4.3.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic map such that  $\varphi(0) = \varphi'(0) = 0$ . Then,  $C_\varphi$  is not cyclic on  $\text{Hol}(\mathbb{D})$ .*

*Proof.* If  $\varphi'(0) = 0$ , then  $\varphi$  is not injective. Proposition 3.3 concludes the proof.  $\square$

All that remains is to study symbols satisfying  $0 < |\varphi'(0)| < 1$ . Note that if  $\varphi$  is not injective, then by Proposition 3.3,  $C_\varphi$  is not cyclic.

Deters and Seubert ([11]) proved that diagonal operators with respect to the basis  $\{z^n : n \geq 0\}$  are cyclic on  $\text{Hol}(\mathbb{D})$  if and only if all their eigenvalues are distinct. In particular, we recover the result about composition operators associated with rotations, which are then cyclic if and only if they are aperiodic.

In our context, the operator  $C_\varphi$  is now diagonal with respect to the basis  $\{\kappa^n : n \geq 0\}$ , where  $\kappa$  is the Koenigs' map of  $\varphi$  (cf. [3, 18]). The proof of cyclicity obtained in [11] cannot be adapted in this situation, since it uses technical estimates of the Taylor coefficients of analytic maps on the disc, which are unknown when we replace  $z^n$  with  $\kappa^n$ . Indeed, the explicit expression of  $\kappa$  is out of scope. The key here is Theorem 3.4.

**Theorem 4.4.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic injective map such that  $\varphi(0) = 0$  and  $0 < |\varphi'(0)| < 1$ . Then, the operator  $C_\varphi$  is cyclic but not supercyclic on  $\text{Hol}(\mathbb{D})$ .*

*Proof.* Assume that  $C_\varphi$  is supercyclic on  $\text{Hol}(\mathbb{D})$ , and denote by  $f$  a supercyclic vector of  $C_\varphi$ . First, if  $f(0) = 0$ , then for all  $\lambda \in \mathbb{C}$  and  $n \in \mathbb{N}$ , we would have  $\lambda(f \circ \varphi^{[n]})(0) = 0$ . The set  $\{\lambda C_\varphi^n(f) : \lambda \in \mathbb{C}, n \geq 0\}$  would not be a dense subset of  $\text{Hol}(\mathbb{D})$ .

In the following, upon renormalizing, we assume  $f(0) = 1$ . Let  $g \in \text{Hol}(\mathbb{D})$ . Then there exist two sequences  $(n_k) \subset \mathbb{N}$  and  $(\lambda_{n_k}) \subset \mathbb{C}$  such that

$$g = \lim_{k \rightarrow \infty} \lambda_{n_k} (f \circ \varphi^{[n_k]}),$$

for the topology of uniform convergence on all compact subset of  $\mathbb{D}$ . In particular, we also have pointwise convergence, so

$$g(0) = \lim_{k \rightarrow \infty} \lambda_{n_k} (f \circ \varphi^{[n_k]})(0) = \lim_{k \rightarrow \infty} \lambda_{n_k} f(0) = \lim_{k \rightarrow \infty} \lambda_{n_k}.$$

Since  $\varphi$  is not an automorphism, using Denjoy-Wolff's theorem ([1, 10]), the sequence  $(\varphi^{[n]})$  of iterates of  $\varphi$  converges uniformly on all compact subsets of  $\mathbb{D}$  (hence, pointwise) to 0. Hence, for all  $z \in \mathbb{D}$ ,

$$g(z) = \lim_{k \rightarrow \infty} \lambda_{n_k} (f \circ \varphi^{[n_k]})(z) = \lim_{k \rightarrow \infty} \lambda_{n_k} \times \lim_{k \rightarrow \infty} (f \circ \varphi^{[n_k]})(z) = g(0) \times f(0) = g(0).$$

We have shown that every map of  $\text{Hol}(\mathbb{D})$  is a constant map, a contradiction.

Hence,  $C_\varphi$  is not a supercyclic operator on  $\text{Hol}(\mathbb{D})$ . Now, we show that it is cyclic.

By Koenigs' theorem ([3, 18]), there exists a map  $\kappa \in \text{Hol}(\mathbb{D})$  such that  $\kappa \circ \varphi = \varphi'(0)\kappa$ . Moreover,  $\kappa$  is defined as the (uniform on all compact subsets of  $\mathbb{D}$ ) limit of the sequence  $(\kappa_n)$  defined by

$$\kappa_n = \frac{\varphi^{[n]}}{\varphi'(0)^n}.$$

Since  $\varphi$  is injective, the iterates of  $\varphi$  have the same property. Hence, for all  $n \geq 0$ ,  $\kappa_n$  is injective. Using Hurwitz's theorem ([21, Chapter 10]), the function  $\kappa$  is either injective or constant. But  $\kappa'(0) = 1$ , so  $\kappa$  is not constant.

Hence,  $\kappa$  is an injective map, and  $\varphi'(0) \in \mathbb{D} \setminus \{0\}$  satisfies  $\kappa \circ \varphi = \varphi'(0)\kappa$ . By Theorem 3.4, the operator  $C_\varphi$  is cyclic on  $\text{Hol}(\mathbb{D})$ .  $\square$

## 5. HYPERBOLIC CASE

Now, we focus on composition operators with a hyperbolic symbol  $\varphi$ . Then,  $\varphi$  does not have any fixed point in the unit disc, and there exists  $\xi \in \mathbb{T}$  such that  $\varphi(\xi) = \xi$ . Upon conjugating, we will assume that  $\varphi(1) = 1$ .

In order to study the map  $\varphi$ , it is necessary to change the point of view.

**Definition 5.1.** The *right half-plane* is defined by  $\mathbb{H} = \{z \in \mathbb{C} : \Re(z) > 0\}$ , and the *Cayley transform* is defined by

$$\tau(z) = \frac{1+z}{1-z}.$$

It is a biholomorphism from  $\mathbb{D}$  to  $\mathbb{H}$ . For any holomorphic map  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ , we define the *Cayley conjugation* of  $\varphi$  by  $\Phi = \tau \circ \varphi \circ \tau^{-1}$ .

When  $\varphi$  is hyperbolic, the main result is Valiron's theorem ([2, 22]).

**Theorem 5.2.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic hyperbolic map, with Denjoy-Wolff point 1, and dilation coefficient  $\delta < 1$  at the point 1. Denote by  $\Phi$  the Cayley conjugation of  $\varphi$ , and for  $n \geq 0$ ,  $z_n = x_n + iy_n = \Phi^{[n]}(1)$ .*

*There exists a constant  $\gamma$  such that the sequence  $(\sigma_n)$  defined by*

$$\sigma_n(z) = \frac{\Phi^{[n]}(\tau^{-1}(z)) - iy_n}{x_n} + \gamma$$

*has a subsequence converging uniformly on all compact subsets of  $\mathbb{D}$  to a holomorphic function  $\sigma : \mathbb{D} \rightarrow \mathbb{H}$  satisfying  $\sigma \circ \varphi = \delta^{-1}\sigma$ .*

Let us begin with the easy part: when  $\varphi$  is bijective.

**Proposition 5.3.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a hyperbolic automorphism.*

*Then the operator  $C_\varphi$  is hypercyclic on  $\text{Hol}(\mathbb{D})$ .*

*Proof.* Using [7, Theorem 2.3] or [12, Table I], if  $\varphi$  is a hyperbolic automorphism, then  $C_\varphi$  is hypercyclic on  $H^2(\mathbb{D})$ . By Lemma 3.1,  $C_\varphi$  is also hypercyclic on  $\text{Hol}(\mathbb{D})$ .  $\square$

Now, assume that  $\varphi$  is injective (otherwise,  $C_\varphi$  would not be cyclic by Proposition 3.3), but not bijective. We show that the map  $\sigma$  defined in Theorem 5.2, called *Valiron's map* of  $\varphi$ , is injective.

**Lemma 5.4.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic hyperbolic and injective map, with Denjoy-Wolff point 1. The Valiron's map of  $\varphi$  is injective.*

*Proof.* If  $\varphi$  is injective, by conjugating, so is the case of the map  $\Phi$  and all its iterates. Moreover,  $\tau^{-1}$  is bijective, so by composing, for all  $n \in \mathbb{N}$ ,  $\Phi^{[n]} \circ \tau^{-1}$  is injective. After adding the constant functions, we deduce that for all  $n$ , the map  $\sigma_n$  defined in Theorem 5.2 is injective. We conclude using Hurwitz's theorem, since  $\sigma$  is not a constant map.  $\square$

Finally, we obtain the following theorem.

**Theorem 5.5.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic hyperbolic and injective map.*

*Then, the operator  $C_\varphi$  is hypercyclic on  $\text{Hol}(\mathbb{D})$ , and the Valiron's map  $\sigma$  of  $\varphi$  is a hypercyclic vector of  $C_\varphi$ .*

*Proof.* Using Valiron's theorem (Theorem 5.2), denoting  $\psi(z) = \delta^{-1}z$ , there exists an injective map  $\sigma : \mathbb{D} \rightarrow \mathbb{H}$  (Lemma 5.4) such that

$$\sigma \circ \varphi = \psi \circ \sigma.$$

Moreover,  $\psi$  is a hyperbolic automorphism of  $\mathbb{H}$  since  $\psi(0) = 0$ ,  $\psi(\infty) = \infty$ , and  $\psi^{-1}(z) = \delta z$ . Thus,  $C_\psi$  is hypercyclic on  $\text{Hol}(\mathbb{H}^+)$  using Proposition 5.3 and conjugating by the Cayley transform. Finally,  $C_\varphi$  is hypercyclic on  $\text{Hol}(\mathbb{D})$  by Theorem 3.7.  $\square$

## 6. PARABOLIC CASE

This section intends to study the cyclic properties of the composition operator  $C_\varphi$  when  $\varphi$  is parabolic. As in the previous sections, we will assume that the Denjoy-Wolff point of  $\varphi$  is 1. However, we need to split the study. Indeed, we have the following theorem, from Baker and Pommerenke ([2, 4, 19]).

**Theorem 6.1.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic parabolic map, with Denjoy-Wolff point 1, and dilation coefficient  $\delta < 1$  at the point 1. Denote by  $\Phi$  the Cayley conjugation of  $\varphi$ , and for  $n \geq 0$ ,  $z_n = x_n + iy_n = \Phi^{[n]}(1)$ . Set*

$$\ell = \lim_{n \rightarrow +\infty} \frac{z_{n+1} - z_n}{z_{n+1} + \bar{z}_n} \geq 0.$$

(i) *If  $\ell > 0$ , then the sequence  $(\sigma_n)$  defined by*

$$\sigma_n(z) = \frac{\Phi^{[n]}(\tau^{-1}(z)) - iy_n}{x_n}$$

*converges uniformly on all compact subsets of  $\mathbb{D}$  to a holomorphic map  $\sigma : \mathbb{D} \rightarrow \mathbb{H}^+$  satisfying  $\sigma \circ \varphi = \sigma + ib$  for some  $b \in \mathbb{R}^*$ .*

(ii) *If  $\ell = 0$ , then the sequence  $(\theta_n)$  defined by*

$$\theta_n(z) = \frac{\Phi^{[n]}(\tau^{-1}(z)) - z_n}{z_{n+1} - z_n}$$

*converges uniformly on all compact subsets of  $\mathbb{D}$  to a holomorphic map  $\theta : \mathbb{D} \rightarrow \mathbb{C}$  satisfying  $\theta \circ \varphi = \theta + 1$ .*



However, this does not make the study harder than the hyperbolic case. Actually, the proofs will be very similar to those in Section 5. Once again, let us start with the easy case.

**Proposition 6.2.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a parabolic automorphism.*

*Then the operator  $C_\varphi$  is hypercyclic on  $\text{Hol}(\mathbb{D})$ .*

*Proof.* Using [7, Theorem 2.3] or [12, Table I], if  $\varphi$  is a parabolic automorphism, then  $C_\varphi$  is hypercyclic on  $H^2(\mathbb{D})$ . By Lemma 3.1,  $C_\varphi$  is also hypercyclic on  $\text{Hol}(\mathbb{D})$ .  $\square$

Now, assume that  $\varphi$  is injective, but not bijective. Then, we show that the map  $\sigma$  (if  $\ell > 0$ ) or  $\theta$  (if  $\ell = 0$ ) defined in Theorem 6.1, called *Pommerenke's map* of  $\varphi$ , is injective.

**Lemma 6.3.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic parabolic and injective map, with Denjoy-Wolff point 1. The Pommerenke's map of  $\varphi$  is injective.*

*Proof.* The maps  $\sigma$  and  $\theta$  defined in Theorem 6.1 are constructed as in the hyperbolic case with Valiron's map (Lemma 5.4), that is, the limit of a sequence whose terms are of the form  $\Phi^{[n]} \circ \tau^{-1}$  (which are injective) with constants. Hence, by Hurwitz's theorem, we obtain the same conclusion as in the hyperbolic case.  $\square$

To conclude, we obtain the following result.

**Theorem 6.4.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic parabolic and injective map.*

*Then, the operator  $C_\varphi$  is hypercyclic on  $\text{Hol}(\mathbb{D})$ , and the Pommerenke's map of  $\varphi$  is a hypercyclic vector of  $C_\varphi$ .*

*Proof.* We need to consider separately the cases  $\ell > 0$  and  $\ell = 0$ .

If  $\ell > 0$ , by Pommerenke's theorem (Theorem 6.1), denoting  $\psi(z) = z + ib$  (with  $b$  depending on  $\varphi$ ), there exists an injective map  $\sigma : \mathbb{D} \rightarrow \mathbb{H}$  (Lemma 6.3) such that

$$\sigma \circ \varphi = \psi \circ \sigma.$$

Moreover,  $\psi$  is a parabolic automorphism of  $\mathbb{H}$  since  $\infty$  is the only fixed point of  $\psi$ , and  $\psi^{-1}(z) = z - ib$ . Thus,  $C_\psi$  is hypercyclic on  $\text{Hol}(\mathbb{H}^+)$  using Proposition 6.2 and conjugating by the Cayley transform. Hence,  $C_\varphi$  is hypercyclic on  $\text{Hol}(\mathbb{D})$ , using Theorem 3.7.

If  $\ell = 0$ , then Theorem 6.1 gives an injective map  $\theta : \mathbb{D} \rightarrow \mathbb{C}$  (Lemma 6.3) such that

$$\theta \circ \varphi = \psi \circ \theta,$$

with  $\psi(z) = z + 1$ . Then,  $\psi$  is an automorphism of  $\mathbb{C}$ , and the operator  $C_\psi$  is well-known as Birkhoff's operator, which is hypercyclic on  $\text{Hol}(\mathbb{C})$  (see [5, Example 1.4] or [16, Exercice 2.20]). Therefore,  $C_\varphi$  is hypercyclic on  $\text{Hol}(\mathbb{D})$  by Theorem 3.7.  $\square$

## 7. REMARKS ON WEIGHTED COMPOSITION OPERATORS

We finish with some notes and examples about the mix of multiplication and composition operators. Let  $\varphi$  be a holomorphic self-map of the disc, and  $m \in \text{Hol}(\mathbb{D})$ . The *weighted composition operator* with weight  $m$  and symbol  $\varphi$  is defined as

$$W_{m,\varphi}(f) = m(f \circ \varphi),$$

for  $f \in \text{Hol}(\mathbb{D})$ . It is a linear and continuous operator.

Lemma 3.2 is also valid when we consider  $W_{m,\varphi}$  instead of  $C_\varphi$ . Here we present one case where it is useful.

**Proposition 7.1.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic map, and  $m \in \text{Hol}(\mathbb{D})$ .*

*If  $m$  vanishes at least twice on the unit disc  $\mathbb{D}$  (after counting the multiplicities), then  $W_{m,\varphi}$  is not cyclic on  $\text{Hol}(\mathbb{D})$ .*

*Proof.* The goal is to show that  $\text{codim}(\overline{W_{m,\varphi}(\text{Hol}(\mathbb{D}))}) \geq 2$ . The process is the following:

- Find a closed subset  $E$  of  $\text{Hol}(\mathbb{D})$  such that  $W_{m,\varphi}(\text{Hol}(\mathbb{D})) \subset E$ .
- Consider  $F = \text{Hol}(\mathbb{D})/E$ , and find two maps  $g, h \in F$  that are linearly independent. To do this, we need to show that  $g \notin E$ ,  $h \notin E$ , and for all  $\lambda \in \mathbb{C}$ ,  $h - \lambda g \notin E$ .

The set  $E$  and the maps  $g$  and  $h$  depend on the behaviour of  $m$ .

Assume that there exist two distinct points  $a, b \in \mathbb{D}$  such that  $m(a) = m(b) = 0$ . Then,  $E = \{f \in \text{Hol}(\mathbb{D}) : f(a) = f(b) = 0\}$ ,  $g : z \mapsto 1$  and  $h : z \mapsto z - b$  satisfy the conditions. Indeed,  $g(b) = 1 \neq 0$ ,  $h(a) = 3 \neq 0$ , and for all  $\lambda \in \mathbb{C}$ ,  $(h - \lambda g)(z) = z - b - \lambda$ . Hence,

$$(h - \lambda g)(a) = (h - \lambda g)(b) = 0 \implies \lambda = a - b = 0,$$

a contradiction.

In the same way, assume that there exists  $a \in \mathbb{D}$  such that  $m(a) = m'(a) = 0$ . In this case,  $E = \{f \in \text{Hol}(\mathbb{D}) : f(a) = f'(a) = 0\}$ ,  $g : z \mapsto 1$  and  $h : z \mapsto z - a$  are good choices, since  $g(a) = 1 \neq 0$ ,  $h'(a) = 1 \neq 0$ , and  $(h - \lambda g)'(a) = 1 \neq 0$  for all  $\lambda \in \mathbb{C}$ .

Finally, since  $\text{codim}(\overline{W_{m,\varphi}(\text{Hol}(\mathbb{D}))}) \geq 2$ , the operator  $W_{m,\varphi}$  is not cyclic on  $\text{Hol}(\mathbb{D})$  by Lemma 3.2 applied to  $W_{m,\varphi}$ .  $\square$

The complete study of the linear dynamics of weighted composition operators seems to be out of scope. When  $m$  vanishes once, both behaviours are possible.

**Example 7.2.** Let  $m(z) = z$  and  $\varphi(z) = e^{i\theta}z$ , with  $\theta \in \mathbb{R}$ .

We show that  $W_{m,\varphi}$  is cyclic on  $\text{Hol}(\mathbb{D})$ . Note that for all  $n \in \mathbb{N}$ ,

$$W_{m,\varphi}^n(f) = m_n(f \circ \varphi^{[n]}), \quad m_n = \prod_{k=0}^{n-1} (m \circ \varphi^{[k]}).$$

In this case,  $m_n(z) = e^{in(n-1)\theta/2}z^n$ . Let  $\sigma = \mathbb{1}$  be the constant map equal to 1. Let  $f \in \text{Hol}(\mathbb{D})$ . There exists a sequence  $(\lambda_n)_{n \geq 0}$  such that

$$f(z) = \sum_{n \geq 0} \lambda_n z^n = \sum_{n \geq 0} \frac{\lambda_n}{e^{in(n-1)\theta/2}} m_n(z) = \sum_{n \geq 0} \frac{\lambda_n}{e^{in(n-1)\theta/2}} W_{m,\varphi}^n(\mathbb{1}).$$

For  $\mu_n = \lambda_n/e^{in(n-1)\theta/2}$ , we have  $|\mu_n| = |\lambda_n|$ , so  $f \in \text{Span}(W_{m,\varphi}^n(\mathbf{1}) : n \geq 0)$ . Thus,  $W_{m,\varphi}$  is cyclic on  $\text{Hol}(\mathbb{D})$ , with  $\mathbf{1}$  as a cyclic vector.

**Example 7.3.** Let  $\varphi$  be an elliptic and injective self-map of the disc, such that  $\varphi(0) = 0$ . Let  $\kappa$  be the Koenigs' map ([2, 18]) of  $\varphi$ , which satisfies  $\kappa \circ \varphi = \varphi'(0)\kappa$ . Then,  $\kappa(0) = 0$  since  $\varphi'(0) \neq 0$ , and  $\kappa$  is injective (see the end of the proof of Theorem 4.4).

Consider the weighted composition operator  $W_{\kappa,\varphi}$ . Then, note that for  $n \geq 0$ ,

$$W_{\kappa,\varphi}^n(\mathbf{1}) = \prod_{k=0}^{n-1} (\kappa \circ \varphi^{[k]}) = \prod_{k=0}^{n-1} \varphi'(0)^k \kappa.$$

Therefore,  $\text{Span}\{W_{\kappa,\varphi}^n(\mathbf{1}) : n \geq 0\} = \text{Span}\{\kappa^n : n \geq 0\} = \text{Hol}(\mathbb{D})$ , using Lemma 2.1. Hence,  $W_{\kappa,\varphi}$  is cyclic on  $\text{Hol}(\mathbb{D})$ , with  $\mathbf{1}$  as a cyclic vector.

**Example 7.4.** Let  $m(z) = z$  and  $\varphi(z) = z^2$ . Then  $m_n(z) = z^{2^n-1}$  and  $\varphi^{[n]}(z) = z^{2^n}$  for all  $n \in \mathbb{N}$ . Let  $\sigma \in \text{Hol}(\mathbb{D})$ . We write  $\sigma$  as

$$\sigma(z) = \sum_{k \geq 0} b_k z^k.$$

The computations of  $W_{m,\varphi}^n(\sigma)$  give

$$W_{m,\varphi}^n(\sigma)(z) = \sum_{k \geq 0} b_k z^{(k+1)2^n-1}.$$

Let  $f \in \text{Span}(W_{m,\varphi}^n(\sigma) : n \geq 0)$ . There exist two sequences  $(\lambda_n)$  and  $(a_j)$  such that

$$f(z) = \sum_{n \geq 0} \sum_{k \geq 0} \lambda_n b_k z^{(k+1)2^n-1} = \sum_{j \geq 0} a_j z^j.$$

Note that  $(k+1)2^n-1 = 0$  if and only if  $(n, k) = (0, 0)$ , and  $(k+1)2^n-1 = 2$  if and only if  $(n, k) = (0, 2)$ . Hence,  $a_0 = \lambda_0 b_0$  and  $a_2 = \lambda_0 b_2$ . Thus,

$$\text{Span}(W_{m,\varphi}^n(\sigma) : n \geq 0) \subset \{f \in \text{Hol}(\mathbb{D}) : 2f(0) = f''(0)\}.$$

This proves that  $W_{m,\varphi}$  is not cyclic on  $\text{Hol}(\mathbb{D})$ .

Both behaviours exist even if  $m$  does not vanish on  $\mathbb{D}$ .

**Example 7.5.** Let  $m(z) = z+1$  and  $\varphi(z) = e^{i\theta}z$ , with  $\theta \in \mathbb{R}$ . Then, for all  $n \in \mathbb{N} \cup \{0\}$ ,  $m_n$  is a polynomial with degree  $n$ . Hence,

$$\text{Span}(W_{m,\varphi}^k(\mathbf{1}) : 0 \leq k \leq n) = \text{Span}(m_k : 0 \leq k \leq n) = \mathbb{C}_n[X],$$

where  $\mathbb{C}_n[X]$  denotes the set of all polynomials with degree at most  $n$ .

Let  $K$  be a compact subset of  $\mathbb{D}$ ,  $f \in \text{Hol}(\mathbb{D})$  and  $\varepsilon > 0$ . Without loss of generality, we will assume that  $K$  is a disc, so that  $\mathbb{C} \setminus K$  is connected. By Runge's theorem ([21, Theorem 13.7]), there exists a polynomial  $P$  such that

$$\|f - P\|_{\infty, K} := \sup_{z \in K} |f(z) - P(z)| \leq \varepsilon.$$

There exists  $n \in \mathbb{N} \cup \{0\}$  such that  $P \in \mathbb{C}_n[X]$ , so we obtain  $P \in \text{Span}(W_{m,\varphi}^k(\mathbf{1}) : 0 \leq k \leq n)$  for this particular  $n$ . Hence,  $W_{m,\varphi}$  is cyclic on  $\text{Hol}(\mathbb{D})$ .

**Example 7.6.** Let  $m(z) = z + 1$  and  $\varphi(z) = z^2$ . Then  $m_n(z) = \frac{1-z^{2^n}}{1-z}$  and  $\varphi^{[n]}(z) = z^{2^n}$  for all  $n \in \mathbb{N}$ . Let  $\sigma \in \text{Hol}(\mathbb{D})$ . Once again, we write

$$\sigma(z) = \sum_{k \geq 0} b_k z^k.$$

Computing  $W_{m,\varphi}^n(\sigma)$  gives this time

$$W_{m,\varphi}^n(\sigma)(z) = \sum_{k \geq 0} \sum_{j \geq 0} b_k z^{k2^n+j} (1 - z^{2^n}).$$

Let  $f \in \text{Span}(W_{m,\varphi}^n(\sigma) : n \geq 0)$ . There exist two sequences  $(\lambda_n)$  and  $(a_\ell)$  such that

$$f(z) = \sum_{n \geq 0} \sum_{k \geq 0} \sum_{j \geq 0} \lambda_n b_k (z^{k2^n+j} - z^{(k+1)2^n+j}) = \sum_{\ell \geq 0} a_\ell z^\ell.$$

Note that

- $k2^n + j = 0 \iff j = k = 0$ ,
- $k2^n + j = 1 \iff (j = 1, k = 0) \text{ or } (j, k, n) = (0, 1, 0)$ ,
- $k2^n + j = 2 \iff (j = 2, k = 0) \text{ or } (j, k, n) \in \{(1, 1, 0), (0, 2, 0), (0, 1, 1)\}$ ,
- $k2^n + j = 3 \iff (j = 3, k = 0) \text{ or } (j, k, n) \in \{(2, 1, 0), (1, 1, 1), (1, 2, 0), (0, 3, 0)\}$ .

Hence, we can show that

$$a_0 = \sum_{n \geq 0} \lambda_n b_0, \quad \begin{cases} a_1 = \lambda_0(b_1 - b_0) + a_0, \\ a_2 = \lambda_0(b_2 - b_0) + \lambda_1(b_1 - b_0) + a_0, \\ a_3 = \lambda_0(b_3 - b_0) + \lambda_1(b_1 - b_0) + a_0. \end{cases}$$

Thus,  $a_1 - a_0 = \lambda_0(b_1 - b_0)$  and  $a_3 - a_2 = \lambda_0(b_3 - b_2)$ . If  $b_0 = b_1$ , then  $W_{m,\varphi}$  is not cyclic on  $\text{Hol}(\mathbb{D})$  since

$$\text{Span}(W_{m,\varphi}^n(\sigma) : n \geq 0) \subset \{f \in \text{Hol}(\mathbb{D}) : f(0) = f'(0)\}.$$

In the same way, if  $b_2 = b_3$ , then  $W_{m,\varphi}$  is not cyclic on  $\text{Hol}(\mathbb{D})$ . Finally, if  $b_0 \neq b_1$  and  $b_2 \neq b_3$ , then

$$\text{Span}(W_{m,\varphi}^n(\sigma) : n \geq 0) \subset \left\{ f = \sum_{n \geq 0} a_n z^n \in \text{Hol}(\mathbb{D}) : a_3 = a_2 + (a_1 - a_0) \frac{b_3 - b_2}{b_1 - b_0} \right\}.$$

Therefore,  $W_{m,\varphi}$  is not cyclic on  $\text{Hol}(\mathbb{D})$ .

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## REFERENCES

- [1] M. Abate. *Iteration theory of holomorphic maps on taut manifolds*. Res. Lect. Notes Math. Com-menda di Rende (Italy): Mediterranean Press, 1989.
- [2] W. Arendt, E. Bernard, B. Célariès, and I. Chalendar. Denjoy-Wolff theory and spectral properties of weighted composition operators on  $\text{Hol}(\mathbb{D})$ . *Ill. J. Math.*, 66(4):463–489, 2022.
- [3] W. Arendt, B. Célariès, and I. Chalendar. In Koenigs' footsteps: diagonalization of composition operators. *J. Funct. Anal.*, 278(2):24, 2020. Id/No 108313.

- [4] I. N. Baker and C. Pommerenke. On the iteration of analytic functions in a half-plane II. *J. Lond. Math. Soc., II. Ser.*, 20:255–258, 1979.
- [5] F. Bayart and É. Matheron. *Dynamics of linear operators*, volume 179 of *Camb. Tracts Math.* Cambridge: Cambridge University Press, 2009.
- [6] F. Bayart and S. Tapiá-García. Cyclicity of composition operators on the fock space, 2022.
- [7] P. S. Bourdon and J. H. Shapiro. *Cyclic phenomena for composition operators*, volume 596 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 1997.
- [8] F. Bracci, M. D. Contreras, and S. Díaz-Madrigal. *Continuous semigroups of holomorphic self-maps of the unit disc.* Springer Monogr. Math. Cham: Springer, 2020.
- [9] I. Chalendar, L. Oger, and J. R. Partington. Linear isometries of  $\text{Hol}(\mathbb{D})$ . *J. Math. Anal. Appl.*, page 128619, 2024.
- [10] C. C. Cowen and B. D. MacCluer. *Composition operators on spaces of analytic functions.* Boca Raton, FL: CRC Press, 1995.
- [11] I. N. Deters and S. M. Seubert. Cyclic vectors of diagonal operators on the space of functions analytic on a disk. *J. Math. Anal. Appl.*, 334(2):1209–1219, 2007.
- [12] E. A. Gallardo-Gutiérrez and A. Montes-Rodríguez. *The role of the spectrum in the cyclic behavior of composition operators*, volume 791 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 2004.
- [13] G. Godefroy and J. H. Shapiro. Operators with dense, invariant, cyclic vector manifolds. *J. Funct. Anal.*, 98(2):229–269, 1991.
- [14] K.-G. Grosse-Erdmann. Universal families and hypercyclic operators. *Bull. Am. Math. Soc., New Ser.*, 36(3):345–381, 1999.
- [15] K.-G. Grosse-Erdmann and R. Mortini. Universal functions for composition operators with non-automorphic symbol. *J. Anal. Math.*, 107:355–376, 2009.
- [16] K.-G. Grosse-Erdmann and A. Peris Manguillot. *Linear chaos.* Univer-sitext. Berlin: Springer, 2011.
- [17] C. Kitai. *Invariant Closed Sets for Linear Operators.* PhD thesis, University of Toronto, 1982.
- [18] G. Koenigs. Recherches sur les intégrales de certaines équations fonctionnelles. *Annales Scientifiques de l'École Normale Supérieure*, 3.1:3–41, 1884.
- [19] C. Pommerenke. On the iteration of analytic functions in a halfplane. I. *J. Lond. Math. Soc., II. Ser.*, 19:439–447, 1979.
- [20] R. Remmert. *Funktionentheorie. 2*, volume 6 of *Grundwiss. Math.* Berlin: Springer-Verlag, 2., korr. Aufl. edition, 1995.
- [21] W. Rudin. *Real and complex analysis.* New York, NY: McGraw-Hill, 3rd ed. edition, 1987.
- [22] G. Valiron. Sur l'itération des fonctions holomorphes dans un demi-plan. *Bull. Sci. Math., II. Sér.*, 55:105–128, 1931.
- [23] N. Zorboska. Cyclic composition operators on smooth weighted Hardy spaces. *Rocky Mt. J. Math.*, 29(2):725–740, 1999.

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