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Polynomial regression on Lie groups

Johan Aubray^{1,*}  and Florence Nicol¹ 

¹ ENAC, Université de Toulouse, 7, Avenue Edouard Belin, Toulouse, France; firstname.lastname@enac.fr

* Correspondence: johan.aubray@enac.fr

Abstract: In this paper, we address the problem of estimating the position of a mobile such as a drone from noisy position measurements using the framework of Lie groups. To model the motion of a rigid body, the relevant Lie group happens to be the Special Euclidean group $SE(n)$, with $n = 2$ or 3 . Our work takes place in the parametric framework developed by Hinkle et al. (2014), who derived equations for geodesic regression and polynomial regression on Riemannian manifolds. Based on this approach, our goal was to implement this technique to the Lie group $SE(3)$ context. Given a set of noisy points in $SE(3)$ representing measurements on the trajectory of a mobile, one wants to find the geodesic that best fits those points in a Riemannian least squares sense. Finally, applications to simulated data are proposed to illustrate this work. The limitations of such a method and future perspectives are discussed.

Keywords: Lie groups, $SE(3)$, Geodesic regression, Riemannian manifolds, drones, Air Traffic Management

1. Introduction

Estimating the position of an aircraft in the context of Air Traffic Management (ATM) is necessary in two situations. In the first one, the goal is to present to air traffic controllers a clean image of the aircraft positions in their sectors. In this case, only the past trajectory is known and one wants to give the best estimate of the current position. This is the radar tracking problem, that has been dealt with for decades using Kalman filtering and its extensions [1], which has recently been studied in the framework of Lie groups [2]. In [3], [4], the symmetries of the state space seen as a manifold are used to improve estimation. In the second situation, a database of full or partial trajectories is available, so that the estimate of a position may be computed using past and future measurements. The corresponding noise removal procedure is known as regression and generally relies on a simple, often local model of the trajectory whose parameters are estimated so as to minimize a least squares criterion.

When the data belongs to an Euclidean space, statistical regression analyses are usually of two kinds: parametric methods such as linear and polynomial regression and non-parametric methods such as kernel-based technique, spline smoothing and local polynomial regression. As an alternative, due to the functional nature of trajectories mapping time to position, the problem arising from the targeted application may also be described in the general framework of functional data statistics in which mobile trajectories are functional objects belonging to a given Hilbert space. Usually, in the absence of the notion of density probability in the original infinite dimensional Hilbert space, the classical approach is to project on a finite dimensional subspace. However, this works only if it possible to find a suitable basis in low dimension representing well all then possible trajectories, which is not the case in the context of highly maneuvering aircraft.

Rather than considering trajectories in the state space, we may use the framework of Lie groups by studying the motion of the mobiles in another representation space. Lie groups are continuous groups of transformations such as scalings, rotations, translations. For poses of a rigid object, the appropriate Lie group is the Special Euclidean group $SE(n)$,

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which in the last few years has been used in the fields of robotics (ref), computational anatomy [5], automation ([6], [7]), among others.

Recently, non-parametric estimation methods on Lie groups have also been explored, such as [8] who developed non-parametric kernel-based regression where the response variable is real-valued and the Lie group-valued predictors are contaminated by measurement errors.

Our work is guided by the applications and is situated in a parametric framework of the geodesic regression method developed in [9]. This method was extended in [10] to higher order polynomials on Riemannian manifolds, which include Lie groups with a left or right invariant Riemannian metric. The algorithm for geodesic and polynomial regression is explicitly derived in $SO(3)$. Our goal is to follow the same idea by modelling the motion of rigid bodies and rigid transformations with the Special Euclidean group $SE(n)$, $n = 2$ or 3 .

In [10], geodesic and polynomial calculations were carried out using the Levi-Civita connection, the main advantage of which is that it is compatible with the metric. In the present contribution, we extend this principle to any connection, since many applications use connections other than the Levi-Civita connection. For example, [11] uses Cartan-Schouten connections to express geodesics in Lie groups as one-parameter subgroups. Similarly, α -connections ([12]) are often used in statistical models on manifolds. The advantage of using any given connection can be seen in two ways: either it simplifies the writing of the problem, or it allows us to express a non-intrinsic quantity, i.e. one whose writing depends on a specific coordinate system (e.g. this is the case for polynomial curves, which are expressed differently depending on the coordinate system chosen).

The paper is organized as follows. In the second section, elements of differential geometry necessary to lay the theoretical background. The third section presents the notion of a polynomial function in a Lie group using jet bundles. The fourth section expresses a mathematical formulation for the regression problem by using differential forms. Finally, several applications to simulated data are proposed. The limitations of the method is discussed and future perspectives to improve the estimation are proposed.

2. Elements of differential geometry

2.1. Manifolds, Lie groups and vector bundles

We assume the reader is familiar with the definition of a manifold as a submanifold of \mathbb{R}^n , and recall the definition of an abstract manifold here:

Definition 1. An n -dimensional smooth manifold is a second countable, Hausdorff set \mathcal{M} endowed with an equivalence class of n -dimensional atlases of class C^∞ on \mathcal{M} . A n -dimensional atlas of class C^∞ on \mathcal{M} is a set of pairs $\{(U_i, \phi_i)\}_{i \in I}$ satisfying the following axioms:

- Each U_i is an open subset of \mathcal{M} and, $\mathcal{M} \subset \bigcup_i U_i$,
- Each ϕ_i is a bijection from U_i into an open subset $\phi_i(U_i)$ of \mathbb{R}^n , and $\phi_i(U_i \cup U_j) \subset \mathbb{R}^n$ is open for every i and j ,
- For every pair (i, j) , the map $\phi_j \circ \phi_i^{-1}: \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ is a C^∞ diffeomorphism.

A Lie group inherits the algebraic properties of a group and the topological properties of a manifold.

Definition 2. A Lie group G is a smooth manifold and a group such that the multiplication $m: G \times G \rightarrow G, (x, y) \mapsto x \circ y$ and the inversion $i: G \rightarrow G, x \mapsto x^{-1}$ are smooth operations.

We shall use the following notations:

- $L_g: G \rightarrow G, x \mapsto gx$ for the left translation,
- $R_g: G \rightarrow G, x \mapsto xg$ for the right translation,
- $e \in G$ for the unit element

Definition 3. Let E, \mathcal{M} be smooth manifolds. A real vector bundle on \mathcal{M} is a triple (E, π, \mathcal{M}) with $\pi : E \rightarrow \mathcal{M}$ a smooth onto mapping such that:

- (i) For all $p \in \mathcal{M}$, $\pi^{-1}(p)$ is a real vector space isomorphic to \mathbb{R}^k , k a fixed integer.
- (ii) For all $p \in \mathcal{M}$, there exists an open set $U \subset \mathcal{M}$, $p \in U$ and a diffeomorphism $\phi_U : U \times \mathbb{R}^k \rightarrow \pi^{-1}(U)$ such that:
 - $\forall q \in U, v \in \mathbb{R}^k, \pi(\phi_U(q, v)) = q$,
 - $\forall q \in U$, the mapping $v \in \mathbb{R}^k \mapsto \phi_U(q, v)$ is an isomorphism on $\phi^{-1}(q)$.

Remark 4. The definition 3 can be utilized verbatim to define vector bundles with other fields of scalars, particularly complex or quaternionic vector bundles. However, this additional generality will not be required for the purposes of this paper.

Notation 1. A bundle (E, π_E, \mathcal{M}) is often represented by its total space E along, π_E being implicit.

Definition 5. Let $U \subset \mathcal{M}$ be an open set and (E, π, \mathcal{M}) a vector bundle on \mathcal{M} . A smooth mapping $s : U \rightarrow E$ such that $\forall p \in \mathcal{M}$, $\pi(s(p)) = p$ is called a local section. When $U = \mathcal{M}$, s is designated as a global section or simply a section.

Proposition 6. Let $U \subset \mathcal{M}$ be an open set and (E, π, \mathcal{M}) a vector bundle on \mathcal{M} . The set of sections on U has the structure of a real vector space and a $C^\infty(U, \mathbb{R})$ -module.

Notation 2. The $C^\infty(U, \mathbb{R})$ -module of local sections on U is designated by $\Gamma(U; E)$. When $U = \mathcal{M}$, it is simply denoted by $\Gamma(E)$.

Remark 7. One can define equivalently a vector bundle as the sheaf generated by the local sections.

Remark 8. A vector bundle of dimension 1 is called a line bundle. Its sections are the $C^\infty(\mathcal{M}, \mathbb{R})$ -mappings.

Definition 9. A triple (\mathcal{M}, E, ∇) with \mathcal{M} a smooth manifold, E a vector bundle on \mathcal{M} and ∇ a Koszul connection on E is called a gauge structure.

Proposition 10. Let E, F be two vector bundles over \mathcal{M} . The pointwise operations on sections $\oplus, \otimes, \text{hom}(\cdot, \cdot)$ define the respective vector bundles $E \oplus F, E \otimes F, \text{hom}(E, F)$.

Proof. See [13], Theorem 6.2, p. 67. \square

Definition 11. The hom bundle with $F = \mathbb{R}$ is called the dual bundle of E and is denoted by E^* .

2.2. Koszul connections

Definition 12. Let (E, π, \mathcal{M}) be a vector bundle over \mathcal{M} . A Koszul connection is an \mathbb{R} -linear mapping:

$$\nabla : \Gamma(E) \rightarrow \Gamma(TM^* \otimes E)$$

such that:

$$\forall s \in \Gamma(E), \forall f \in C^\infty(\mathcal{M}, \mathbb{R}), \nabla(fs) = df \otimes s + f\nabla s \quad (1)$$

A Koszul connection is often called a covariant derivative.

Remark 13. Applying definition 12 on a local frame $\{e_i\}$ yields the so-called Christoffel symbols:

$$\forall i, j, k \in 1, \dots, \dim(\mathcal{M}), \nabla_{e_j} e_i = \Gamma_{ij}^k e_k$$

The Christoffel symbols obviously characterize uniquely the connection ∇ .

Remark 14. There is no reason for a locally constant section to vanish under the action of a Koszul connection.

Definition 15. Given a local frame of sections $\{e_i\}$, the connection forms are the one forms ω_j^i such that: 129
130

$$\nabla e_j = e_i \otimes \omega_j^i \quad (2)$$

Proposition 16. If E, F are two vector bundles and ∇_E, ∇_F are respectively connections on E, F , there exists two canonical connections: 131

$$\nabla_E \oplus \nabla_F: s_E \oplus s_F \mapsto (\nabla_E s_E) \oplus (\nabla_F s_F), \quad (3)$$

$$\nabla_E \otimes \nabla_F: s_E \otimes s_F \mapsto (\nabla_E s_E) \otimes s_F + s_E \otimes (\nabla_F s_F). \quad (4)$$

Proposition 17. Let E be a vector bundle over \mathcal{M} and let E^* be its dual bundle. If ∇ is a Koszul connection on E , then the dual connection ∇^* on E^* is defined as: 132

$$\nabla_X^*(\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y), \quad X, Y \in \Gamma(TM), \omega \in \Gamma(E^*). \quad (5)$$

Definition 18 ([14]). Let $f \in C^\infty(\mathcal{M}, \mathbb{R})$, that is f is a section of the trivial bundle $\mathcal{M} \times \mathbb{R}$. The covariant derivative of f , still denoted by ∇f , is defined as: 133
134

$$\nabla f = df. \quad (6)$$

Propositions 17, 16 and definition 18 can be combined in order to obtain the covariant derivative of arbitrary tensors. 135
136

Definition 19. Let $(E, \pi_E, \mathcal{M}), (F, \pi_F, \mathcal{N})$ be two vector bundles. A bundle morphism is a couple of mappings $\tilde{\phi}: E \rightarrow F, \phi: \mathcal{M} \rightarrow \mathcal{N}$ such that that, for all $x \in \mathcal{M}$, the mapping $u \in \pi_E^{-1}(x) \rightarrow \phi_F^{-1}(\phi(x))$ is linear and makes the next diagram commute: 137
138
139

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\phi}} & F \\ \downarrow \pi_E & & \downarrow \pi_F \\ \mathcal{M} & \xrightarrow{\phi} & \mathcal{N} \end{array} \quad (7)$$

Definition 20. Let $\phi: \mathcal{M} \rightarrow \mathcal{N}$ be a smooth mapping and (F, π_F, \mathcal{N}) a vector bundle on \mathcal{N} . The pullback bundle ϕ^*F is the vector bundle on \mathcal{M} generated by the local sections of the form $x \mapsto s(\phi(x))$ where s is a local section of F . Furthermore, there exists a bundle morphism $\tilde{\phi}: E \rightarrow F$ sending a generating section $s \circ \phi$ to s . 140
141
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Definition 21. Let \mathcal{M}, \mathcal{N} be smooth manifolds and $\phi: \mathcal{M} \rightarrow \mathcal{N}$ a smooth mapping. Let E be a vector bundle on \mathcal{N} equipped with a Koszul connection ∇ . The pullback connection of ∇ by ϕ , designated by ∇^ϕ , is defined on generating sections $s \circ \phi$ by: 144
145
146

$$\nabla_X^\phi s \circ \phi = \nabla_{D\phi X} s \quad (8)$$

where $X \in TM$. 147

2.3. Covariant derivatives and parallel sections 148

Definition 22. Let \mathcal{M} be a smooth manifold, E a vector bundle on \mathcal{M} and ∇ a Koszul connection on E . Let $\gamma:]0, 1[\rightarrow \mathcal{M}$ a smooth mapping, that is a curve on \mathcal{M} . The covariant derivative of $s \in \Gamma(E)$ along γ is the mapping: 149
150
151

$$t \in]0, 1[\rightarrow \nabla_1^\gamma s \circ \gamma(t) \quad (9)$$

where 1 is the unit constant vector in $]0, 1[$. 152

Notation 3. The covariant derivative of a section s along a curve γ is often denoted by: 153

$$\nabla_\gamma s$$

Proposition 23. With the same writing conventions as in definition 22, the covariant derivative of a section $s = s^i e_i$ expressed in a local frame (e_1, \dots, e_n) is given by: 154

$$ds^i(\dot{\gamma}(t))e_i + \Gamma_{ij}^k s^i \dot{\gamma}^j(t)e_n \quad (10) \quad 155$$

Proof. This result is a direct application of definition 21 with $\phi = \gamma$. \square 156

Remark 24. The first term in equation 10 can be written, by a common abuse of notation: 157

$$\frac{d}{dt} s^i(t) e_i \quad (11)$$

where $s^i e_i$ stands for the section in the pullback bundle associated to s . 158

Definition 25. Let (\mathcal{M}, E, ∇) be a gauge structure and let $\gamma:]0, 1[\rightarrow \mathcal{M}$ be a smooth curve. A section $s \in \Gamma(E)$ is said to be ∇ -parallel along γ if: 159

$$\nabla_{\dot{\gamma}} s = 0. \quad 160$$

Remark 26. A parallel section is, in a local frame, solution of an ordinary differential equation (ODE): 161

$$\frac{d}{dt} s^k(t) + \Gamma_{ij}^k s^i \dot{\gamma}^j(t) = 0, k = 1 \dots n \quad (12) \quad 162$$

This gives a practical way to determine a parallel section from an initial condition. 163

Notation 4. The parallel transport of a section s from a to b will be denoted $\Pi_a^b s$. 164

Given a gauge structure $(\mathcal{M}, T\mathcal{M}, \nabla)$, $x \in \mathcal{M}$ and a tangent vector $X \in T_x \mathcal{M}$, the next ODE defines locally a smooth curve γ : 165

$$\begin{cases} \dot{\gamma}(t) = s \circ \gamma(t) \\ \nabla_{\dot{\gamma}} s = 0 \\ \gamma(0) = x \end{cases} \quad (13) \quad 166$$

Definition 27. Let $(\mathcal{M}, T\mathcal{M}, \nabla)$ be a gauge structure. For any $x \in \mathcal{M}$, there exists an open neighborhood U of 0 in $T_x \mathcal{M}$ such that the curve γ defined by equation 13 exists to time 1 for a tangent vector $X \in U$. The mapping: 167

$$X \mapsto \exp_x X = \gamma(1) \quad (14) \quad 168$$

where γ is the solution curve to equation 13 is called the exponential map. 169

Definition 28. Let (e_1, \dots, e_n) be a basis of $T_x \mathcal{M}$ such that the exponential map exists for all the e_i , $i = 1 \dots n$. The mapping 170

$$(t_1, \dots, t_n) \in [0, 1]^n \mapsto \exp_x \left(\sum_{i=1}^n t_i e_i \right) \quad (15) \quad 171$$

defines a local coordinate system in a neighborhood of x called the normal coordinates at x . 172

Remark 29. By the very definition of a flow, it comes: 173

$$\frac{d}{dt} \exp_x tX = s(\gamma(t)) \quad (16) \quad 174$$

with γ, s the solution to ODE 13. Taking the second derivative is a little bit awkward since the corresponding tangent vector would lie in $TT\mathcal{M}$. However, it makes sense to consider the covariant derivative along γ : 175

$$\nabla_{\dot{\gamma}} s = 0. \quad (17) \quad 176$$

The normal coordinate system thus has a vanishing second (covariant) derivative. The generalization of this property to higher covariant derivatives will be used later as an extension of the notion of polynomial curves.

Notation 5. A normal coordinate system at x is often denoted by $x^i, i = 1 \dots n$ and the associated vector fields by ∂_i . A vector field X is said to be a coordinate vector field if it can be written as $X = a^i \partial_i$ with $a_i, i = 1 \dots n$ constant mappings [14].

Proposition 30. Any two coordinate vector fields X, Y have a vanishing commutator $[X, Y] = 0$.

Proof. Writing $X = X^i \partial_i, Y = Y^i \partial_i$, it comes, using the fact that $X^i, Y^i, i = 1 \dots n$ are constant mappings:

$$[X, Y] = \left((\partial_j Y^i) X^j - (\partial_j X^i) Y^j \right) \partial_i = 0$$

□

2.4. Musical isomorphisms

In the sequel, we will often have to toggle between matching differential forms and vectors. We define the musical isomorphisms using the metric g .

Definition 31. Let (\mathcal{M}, g) be a Riemannian manifold. Let $\alpha \in T^* \mathcal{M}$. α^\sharp is the only element of $T\mathcal{M}$ such that, for all $X \in T\mathcal{M}$:

$$\alpha(X) = g(\alpha^\sharp, X) \quad (18)$$

and we note it as $\alpha^\sharp := \sharp(\alpha)$.

In a similar way, for $Y \in T\mathcal{M}$, Y^\flat is the only element of $T^* \mathcal{M}$ such that, for all $X \in T\mathcal{M}$:

$$Y^\flat(X) = g(X, Y) \quad (19)$$

and we note it as $Y^\flat := \flat(Y)$.

Both isomorphisms can be expressed in local coordinates:

$$\begin{aligned} \alpha^\sharp &= \sharp(\alpha) = \sharp(\alpha_i e^i) = \underbrace{g^{ij} \alpha_j}_{X^i} e_i, \\ X^\flat &= \flat(X) = \flat(X^i e_i) = \underbrace{g_{ij} X^j}_{\alpha_i} e^i. \end{aligned}$$

where g_{ij} and g^{ij} are respectively the i, j coefficient of the metric matrix and the i, j coefficient of the inverse of the metric matrix.

3. Polynomial functions

As a consequence of the Taylor formula, the notion of a polynomial function of degree k on \mathbb{R}^n is well defined as a smooth mapping $P: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial^{|I|} P}{\partial^{I_1} x_1 \dots \partial^{I_n} x_n} = 0$$

for any multi-index $I = (I_1, \dots, I_n)$ such that $|I| = \sum_{i=1}^n I_i > k$. On a manifold, this kind of property may hold locally, that is, for example, a function has a finite length Taylor expansion in terms of a local coordinate system, but generally fails globally, due to the transition functions between local charts. In this work, a curve γ will be considered as polynomial of order k if:

$$\nabla_{\dot{\gamma}}^k \dot{\gamma} = 0 \quad (20)$$

Remark 32. All covariant derivatives of order $p \geq k$ vanish.

4. Optimality conditions 208

In this section, the first and second order local optimality conditions for a real valued mapping with domain a Riemannian manifold are briefly introduced. 209
210

Lemma 33. *Let M be a smooth manifold and $f: M \rightarrow \mathbb{R}$ a smooth mapping. For f to be locally optimal at $p \in M$, it is necessary that the differential of f at p vanishes: $df_p = 0$.* 211
212

Proof. We recall that f is said to admit a local minimum at $p \in M$ if there exists an open neighborhood U of p such that, for any $q \in U$, $f(q) \geq f(p)$. Without loss of generality, U may be assumed to be path-connected, so that for any $q \in U$, one can find a smooth path $c: [0, 1] \rightarrow M$ such that $c(0) = p, c(1) = q$. Let X_p be a tangent vector in T_pM such that $c'(0) = X_p$. Then $f \circ c: [0, 1] \rightarrow \mathbb{R}$ admits a local minimum at 0, which implies $(f \circ c)'(0) = df(c(0))c'(0) = df_p(X_p) = 0$, therefore $df_p = 0$. \square 213
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In fact, if $\gamma:] - \epsilon, +\epsilon[\rightarrow M$ is a smooth curve on M such that $\gamma(0) = p$, then $df_p(\dot{\gamma}(0))$ is the first order variation of f in the direction of the tangent vector $\dot{\gamma}(0) \in T_pM$. This approach can be extended by iterating the derivative. 219
220
221

Proposition 34. *Let M be a smooth manifold, ∇ a Koszul connection on TM and $f: M \rightarrow \mathbb{R}$ a smooth mapping. For $x \in M$ to be a local minimum of f , it is necessary that the Hessian ∇df be non-negative and sufficient that it is positive definite.* 222
223
224

Proof. The proof is essentially the same as in the \mathbb{R}^n case, recalling that for any Koszul connection, $\nabla_X f = X(f)$. \square 225
226

5. The geodesic regression problem 227

5.1. Polynomial curves 228

We recall that in a Lie group G with its Lie algebra \mathfrak{g} , there is a canonical left invariant metric derived from the Euclidean inner product: 229
230

$$\langle\langle X_g, Y_g \rangle\rangle_g = \langle T_g L_g^{-1} X_g, T_g L_g^{-1} Y_g \rangle \quad (21)$$

Please note that any symmetric positive-definite matrix may be used to produce different left invariant metrics: 231
232

$$\langle\langle X_g, Y_g \rangle\rangle_{A,g} = \langle T_g L_g^{-1} A X_g, T_g L_g^{-1} A Y_g \rangle \quad (22)$$

Likewise, a right-invariant metric can also be defined. A minimizing geodesic between two points g_0, g_1 with respect to $\langle\langle \cdot, \cdot \rangle\rangle_g$ is a curve $\gamma: [0, 1] \rightarrow G$, such that $\gamma(0) = g_0, \gamma(1) = g_1$ minimizing the functional: 233
234
235

$$L(\gamma) = \frac{1}{2} \int_0^1 \langle\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle\rangle_g dt \quad (23)$$

Due to the left invariance of the metric, it is enough to seek a solution of the reduced problem: 236
237

$$L(X) = \frac{1}{2} \int_0^1 \langle\langle \dot{X}(t), \dot{X}(t) \rangle\rangle_e dt \quad (24)$$

where $X:] - \epsilon, +\epsilon[\rightarrow \mathfrak{g}$ is \mathfrak{g} -valued curve. It is well known that X satisfies the so-called Euler-Arnold equation [15]: 238
239

$$\dot{X}(t) = \text{ad}_{X(t)}^t X(t) \quad (25)$$

From the solution of 25, the original geodesic is reconstructed with the help of the left translation: 240
241

$$\dot{\gamma}(t) = T_e L_{\gamma(t)} X(t) \quad (26)$$

which simplifies in a matrix Lie group as:

$$\gamma(t) = \exp\left(\int_0^t X(u)du\right) \quad (27)$$

Minimizing geodesics are Riemannian equivalents to straight lines, and are thus the natural object to consider for extending the linear regression to a Lie group G . Furthermore, if ∇^{LC} is a connection on G , then a minimizing geodesic γ has to satisfy:

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0 \quad (28)$$

and so the Riemannian equivalent to a degree k polynomial is a curve γ such that:

$$\left(\nabla_{\dot{\gamma}(t)}\right)^k \dot{\gamma}(t) = 0 \quad (29)$$

Here again, by the left invariance of the metric, the Levi-Civita connection in $T_g G$ has to be considered only in a basis of left-invariant vector fields $T_e L_g e_1, \dots, T_e L_g e_n$. Applying the Koszul formula to the basis vectors e_1, \dots, e_n , it comes:

$$\Gamma_{ij}^k = \frac{1}{2} \left(c_{ij}^k + c_{ki}^j - c_{jk}^i \right) \quad (30)$$

where the $c_{ij}^k = \langle [e_i, e_j], e_k \rangle$ are the structure constants.

Using this form, one can compute a degree p polynomial curve X in \mathfrak{g} by solving an augmented differential equation:

$$v_1(t) = \nabla_{X(t)} v_0(t), v_2(t) = \nabla_{X(t)} v_1(t), \dots, \nabla_{X(t)} v_{p-1}(t) = 0 \quad (31)$$

with the convention $v_0 = X$. Please note that the previous system can be written in the form:

$$\frac{d}{dt} \begin{pmatrix} v_0 \\ \vdots \\ v_{p-1} \end{pmatrix} = F(v_0, \dots, v_{p-1}) \quad (32)$$

where the vector v_0, \dots, v_{p-1} is in \mathbb{R}^{pn} , with n the dimension of \mathfrak{g} as a real vector space.

5.2. Regression in a Lie group

An extended notion of parametric linear regression on Riemannian manifolds, called intrinsic geodesic regression, has been independently developed in [9] and [16]. Let (y_1, \dots, y_N) be a set of points representing the trajectory of the mobile on a Riemannian manifold (\mathcal{M}, g) . The idea is to find a geodesic curve $\gamma(t)$ on the manifold that best fits those points in a "linear" manner at known times t_1, \dots, t_N . The estimate is found by minimizing a least squares criterion based on a Riemannian distance between the model $\gamma(t)$ and the data.

More formally, the geodesic regression model is expressed as follows in [9]

$$Y = \text{Exp}_{\gamma(t)}(\varepsilon) = \text{Exp}_{\text{Exp}_p(tv)}(\varepsilon)$$

where $\gamma(t) = \text{Exp}_p(tv)$ is the geodesic curve given by the initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, and ε denotes a Gaussian random variable taking values in the tangent space at $\gamma(t)$.

In [10], this geodesic regression model is generalized to a polynomial one

$$Y = \text{Exp}_{\gamma(t)}(\varepsilon)$$

where the curve $\gamma(t)$ is a Riemannian polynomial of order k , as defined by equation 29. The Riemannian least mean squares estimate is obtained by minimizing over the decision vectors $(\gamma(0), \dot{\gamma}(0), \dots, \dot{\gamma}^{(i)}(0), \dots, \dot{\gamma}^{(k)}(0))$ the next criterion:

$$E(\gamma) = \frac{1}{N} \sum_{j=1}^N d(\gamma(t_j), y_j)^2 \quad (33)$$

where d is a geodesic distance between the model $\gamma(t_j)$ and the data $y_j, j = 1, \dots, N$.

This minimisation problem is then performed under the constraints that the model curve γ has, in the geodesic regression problem, a parallel first derivative: $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$ or, more generally, a k -th order polynomial constraint $(\nabla_{\dot{\gamma}(t)})^k \dot{\gamma}(t) = 0$ that is uniquely solved by adding initial conditions $\gamma(0), \dot{\gamma}(0), \dots, \dot{\gamma}^{(i)}(0), \dots, \dot{\gamma}^{(k)}(0)$.

Using properties of translation given by equations 24 and 26 in a Lie group, by denoting $X_1, \dots, X_i, \dots, X_k$ the i -th order velocities of in \mathfrak{g} , we get the "forward" polynomial equations

$$\begin{cases} \frac{d}{dt} \gamma(t) &= X_1(t) \gamma(t) \\ &\vdots \\ \frac{d}{dt} X_i(t) &= \overline{\nabla}_{X_1} X_i(t) + X_{i+1}(t) \\ &\vdots \\ \frac{d}{dt} X_k(t) &= \overline{\nabla}_{X_1} X_k(t) \end{cases} \quad (34)$$

Remark 35.

Let us now move to minimising the criterion given by equation 33. The next lemma is a well-known result that can be proven using normal coordinates.

Lemma 36. Let $p, q \in M$ such that $\log_p q = V \in T_p M$ is well defined. The differential at p of the mapping $m \mapsto d^2(m, q)$ is the 1-form $\alpha_q: X \in T_p M \mapsto 2g(X, V)$.

Proposition 37 (Variational formula). Let Φ_X be the flow of a vector field X , i.e., $\Phi(t, x)$ is the value of the integral curve of X at time t , with $\Phi(0, x) = x$. Then, the derivative $\partial_x \Phi$ of Φ with respect to the initial condition x obeys the differential equation:

$$\frac{d}{dt} \partial_x \Phi(t, x) = DX(\Phi(t, x)) \cdot \partial_x \Phi(t, x) \quad (35)$$

Where $DX: TM \rightarrow TTM$ is the derivative of the section $X: M \rightarrow TM$ and the initial condition $\partial_x \Phi(0, x) = \text{Id}$.

Going back to the original problem, that is to find a local minimum of the criterion 33, and assuming that γ is the integral curve of a vector field X with initial condition $\gamma(0) = p$, the application of the lemma 36 yields the next expression for the differential at $T_p M$ of $E(\gamma)$ with respect to the initial conditions:

$$Y \in T_p M \mapsto \frac{2}{N} \sum_{i=1}^N g(\partial_p \gamma(t_i, p) \cdot Y, \log_{\gamma(t_i, p)} y_i) = \frac{2}{N} \sum_{i=1}^N g(Y, \partial_p \gamma(t_i, p)^* \log_{\gamma(t_i, p)} y_i) \quad (36)$$

where $\partial_p \gamma(t_i, p)^*$ is the adjoint of $\partial_p \gamma(t_i, p)$ with respect to the metric. The (Riemannian) gradient at $T_p M$ is thus given by:

$$\frac{2}{N} \sum_{i=1}^n \partial_p \gamma(t_i, p)^* \log_{\gamma(t_i, p)} y_i \quad (37)$$

and the corresponding differential form is :

$$dE = \frac{2}{N} \left(\sum_{i=1}^n \partial_p \gamma(t_i, p)^* \log_{\gamma(t_i, p)} y_i \right)^b \tag{38}$$

5.3. Geodesic regression in $SE(3)$

Coming back to the original problem, which is to estimate a trajectory (i.e. a time stamped curve) from raw positioning data belonging to $SE(3)$. We remind here some specific elements about $SE(n)$ and its Lie algebra $\mathfrak{se}(n)$:

Definition 38. The set of affine maps $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f(x) = Rx + T$ where R is a rotation matrix ($R \in SO(n)$) and T is a vector of \mathbb{R}^n is called the Special Euclidean group $SE(n)$, and is a Lie group.

$SE(n)$ is often called the group of rigid motions, or rigid-body motions. An element $A \in SE(n)$ can be represented by a $(n+1)$ -by- $(n+1)$ matrix of the form $\begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}$ where $R \in SO(n)$ and $T \in \mathbb{R}^n$. It is a group, where the product of two elements of $SE(3)$ is given by:

$$\begin{pmatrix} R_1 & T_1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} R_2 & T_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_1 R_2 & R_1 T_2 + T_1 \\ 0 & 1 \end{pmatrix}$$

and the inverse of an element is:

$$\begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^T & -R^T T \\ 0 & 1 \end{pmatrix}$$

We have $\dim(SE(n)) = \frac{n(n+1)}{2}$, therefore $\dim(SE(3)) = 6$

Definition 39. The vector space of real $(n+1)$ -by- $(n+1)$ matrices of the form

$$X = \begin{pmatrix} K & V \\ 0 & 0 \end{pmatrix} \tag{39}$$

where K is a skew-symmetric n -by- n matrix and V a vector in \mathbb{R}^n is the Lie algebra of $SE(n)$

Remark 40. It can be easily shown that the matrix exponential of any matrix defined by 39 belongs to $SE(n)$. It can also be proven by the exp map $\mathfrak{se}(n) \rightarrow SE(n)$ is onto, but not one-to-one due to the rotation part of the matrix being the same with angles differing by an integer multiple of 2π .

Since $\dim(\mathfrak{se}(3)) = \dim(SE(3)) = 6$, a basis for $\mathfrak{se}(3)$ corresponding to the infinitesimal rotations and translations along all three axes can be written as follows :

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There is a canonical isomorphism between $\mathfrak{se}(3)$ and \mathbb{R}^6 often denoted \wedge , its inverse being denoted \vee :

$$\wedge : \mathbb{R}^6 \rightarrow \mathfrak{se}(3), \vee : \mathfrak{se}(3) \rightarrow \mathbb{R}^6$$

With these elements in mind, we can solve the regression. [10] gave a closed form expression for geodesics in groups like $SO(3)$ that have a bi-invariant metric, but no such closed-form can be derived for a higher order polynomial. Unfortunately, $SE(3)$ has no such bi-invariant metric [5] and numerical integration is the only solution available, even for order 1 polynomials, i.e. geodesics.

6. Implementation

We now show here how the solution is numerically implemented. The general idea of the algorithm is an iterative two-step algorithm with:

- a forward equation corresponding to the numerical integration of 34,
- a backward equation corresponding to the cancellation of the differential form written in 38.

The forward equation is, as previously indicated, solved in \mathfrak{g} then solved in G by applying the matrix exp map.

As far as the backward equation goes, as a reminder, the differential of E is given by 38 :

$$dE = \frac{2}{N} \left(\sum_{j=1}^N \partial_p \gamma(t_j, p)^* \log_{\gamma(t_j, p)} y_j \right)^b$$

We have:

$$dE = \frac{2}{N} \left(\sum_{j=1}^N \partial_p \gamma(t_j, p)^{\dagger} \text{Log}_{\gamma(t_j, p)} y_j \right)^b = \frac{2}{N} \sum_{j=1}^N dE_j, \quad (40)$$

where for $j = 1, \dots, n$,

$$dE_j(Y) = g(\partial_p \gamma(t_j, p).Y, \text{Log}_{\gamma(t_j, p)} y_j). \quad (41)$$

However, each term dE_j of the sum of equation 40 is expressed at time t_j , whereas the equation 36 is expressed at 0. It is thus necessary to pull all these terms back to 0 by parallel transport. Thus, at each instant t_j , we can write

$$\nabla_{\dot{\gamma}}(dE_j(Y)) = dE_j(\nabla_{\dot{\gamma}} Y) + (\nabla_{\dot{\gamma}} dE_j)(Y) = 0, \quad (42)$$

where the transport of dE_j by ∇ can be written using the metric and gives rise to the dual connection ∇^* defined by proposition 17.

The first term cancels out, the second term representing the parallel transport of the differential form dE_j towards the point of origin has to be cancelled.

Let us write each term dE_j in local coordinates (x^1, \dots, x^n) around p , and let :

$$\begin{cases} A_{(j)} := A_{(j)i}^l dx^i \otimes \partial_l = \partial_p \gamma(t_j, p) \\ L_{(j)} := L_{(j)}^k \partial_k = \text{Log}_{\gamma(t_j, p)} y_j \end{cases} \quad (43)$$

We then have, using the expression for dE as written in (36):

$$\begin{aligned} \partial_p \gamma(t_j, p).Y &= A_{(j)i}^l dx^i \otimes \partial_l (Y^k \partial_k) \\ &= Y^k A_{(j)i}^l dx^i \otimes \partial_l (\partial_k) \\ &= Y^k A_{(j)k}^l \partial_l. \end{aligned}$$

To simplify the expressions, we choose a normal coordinate system, which implies that $g(\partial_l, \partial_i) = \delta_l^i$. Then

$$dE_j(Y) = g(Y^k A_{(j)k}^l \partial_l, L_{(j)}^i \partial_i) = \sum_k Y^k (A^T)_{(j)i}^k L_{(j)}^i,$$

and therefore

$$dE_j = \sum_k (A^T)_i^k L_{(j)}^i dx^k \quad (44)$$

Let us now apply the parallel transport $\prod_{t_j}^0 dE_j$: we want to solve

$$\nabla_{\dot{\gamma}} dE_j = 0 \quad (45)$$

$$\begin{aligned} 0 &= \nabla_{\dot{\gamma}} dE_j \\ &= \nabla_{\dot{\gamma}^m \partial_m} \left(\sum_k (A^T)_i^k L_{(j)}^i dx^k \right) \\ &= \dot{\gamma}^m \nabla_{\partial_m} \left(\sum_k (A^T)_i^k L_{(j)}^i dx^k \right) \\ &= \partial_m \left(\sum_k (A^T)_i^k L_{(j)}^i dx^k \right) + \sum_k (A^T)_i^k L_{(j)}^i \Gamma_{mk}^n dx^n \end{aligned} \quad (46)$$

This gives a system of linear differential equations with non-constant coefficients of the first order. The complete system is written :

$$\begin{cases} \nabla_{\dot{\gamma}} dE_1 = 0 \\ \nabla_{\dot{\gamma}} dE_2 = 0 \\ \dots \\ \nabla_{\dot{\gamma}} dE_N = 0 \end{cases} \quad (47)$$

and allows us to determine new decision vectors at the point of origin.

Here we explain the principles of implementation using the pseudocode described in the [1](#) algorithm. It describes the procedure for determining the optimal geodesic. This algorithm can easily be generalised to the case of a polynomial curve of order p .

Algorithm 1 Determining the optimum curve

```

1:  $p \leftarrow p_0$  (initialisation)
2:  $v \leftarrow v_0$  (initialisation)
3: while  $\|\delta\| > \delta_{max}$  do
4:   Solve the equation  $(\nabla_{\dot{\gamma}}^{(k)}) \dot{\gamma}(t) = 0$  for  $t \in [0, 1]$ 
5:   for  $j=1:\text{NumberOfDataPoints}$  do
6:     Compute  $dE_j$  using the gradient  $\partial_p \gamma(t_i, p)^\dagger \text{Log}_{\gamma(t_i, p)} y_i$ 
7:   end for
8:    $\delta = \sum_j \left( \prod_{t_j}^0 dE_j \right)^\sharp$  following equation (46)
9:    $v \leftarrow v - \epsilon \delta$ 
10: end while

```

Note that :

- step 1 may be done several ways: a naive initialisation consists in setting $p_0 = e$, but the Fréchet mean is a likely better alternative,
- in step 8, transporting the differential form dE via the ∇ connection is done numerically by transporting the gradient dE^\sharp associated with dE (see definition 18) via the dual ∇^* connection. We have :

$$\delta = \sum_j \left(\prod_{t_j}^0 dE_j \right)^\sharp = \sum_j \prod_{t_j}^0 * dE_j^\sharp \quad (48)$$

where Π^* denotes the parallel transport via the dual connection. 355

The algorithm was tested on several test trajectories generated with the MatLab software: 356

- a straight line, 357
- a constant radius turn, 358
- an helix, 359

with different noise variances. 360

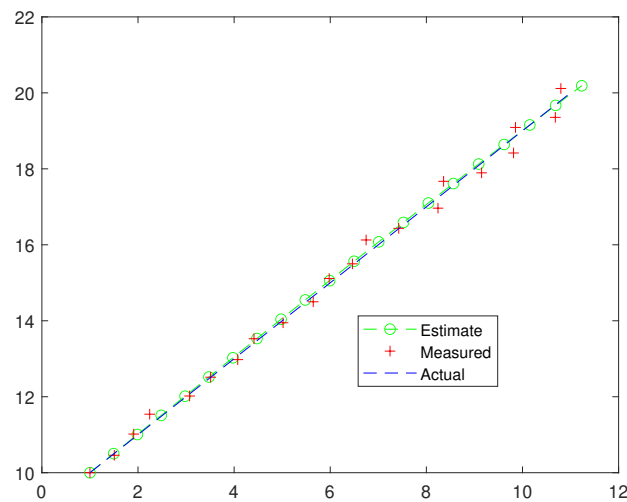


Figure 1. A straight line, with a "small" additive noise 361

The first examples are shown in figures 1 and 2. We can see the estimate is close to the actual trajectory, even though a slightly curving move is present due to the estimated rotation part of the trajectory in $SE(3)$ being close to zero, but not exactly zero. This could be corrected by post-processing by a flagging of the trajectory. 362

Figure 4 displays a trajectory that is highly unlikely for a civilian aircraft, but could be plausible for a UAV. This kind of trajectory would be difficult to track by a Kalman filter, unless the state variables also lie on a Lie group. 363

We can measure how well the model fits the data using a coefficient of determination R^2 . Following [9], since the data does not belong to an Euclidean space, we will have to use the Fréchet variance as the denominator: 364

$$\text{Var}_F = \frac{1}{N} \min_{y \in \mathcal{M}} \sum_j d(y, y_j)^2, \quad \text{365}$$

leading to 366

$$R^2 = 1 - \frac{SSE}{\text{sample Fréchet variance}} = 1 - \frac{\sum_j d(\gamma(t_j), y_j)^2}{\min_{y \in \mathcal{M}} \sum_j d(y, y_j)^2} \quad (49) \quad \text{369}$$

But since the trajectories can also be considered in the state space, which is Euclidean (generally, \mathbb{R}^2 or \mathbb{R}^3), we can also use the standard Euclidean R^2 coefficient. In both cases, an R^2 close to 0 will indicate the estimator does not provide a better estimate than the sample mean (Fréchet mean for manifold-valued data), whereas an R^2 equal to 1 will indicate a perfect fit between the model and the data. 370

7. Conclusion and future work 371

In this work, we have written polynomial regression in Lie groups as a solution of a k times iterated covariant derivative. For that, the iterated connection is not necessarily the 372

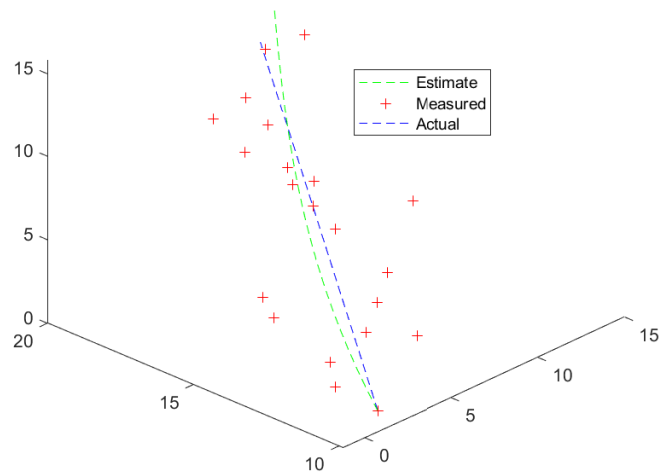


Figure 2. A straight line, with a "large" additive noise

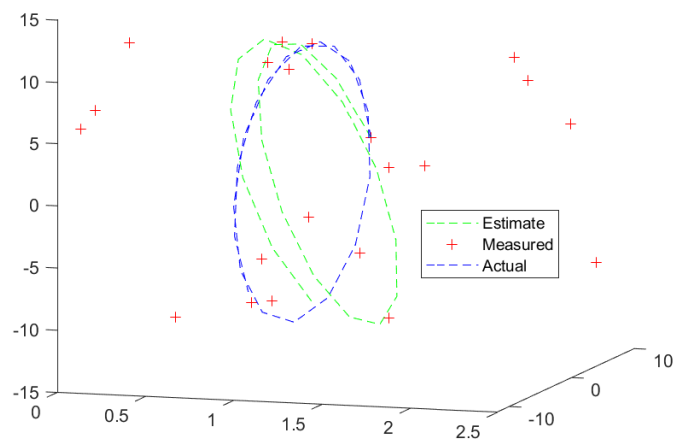
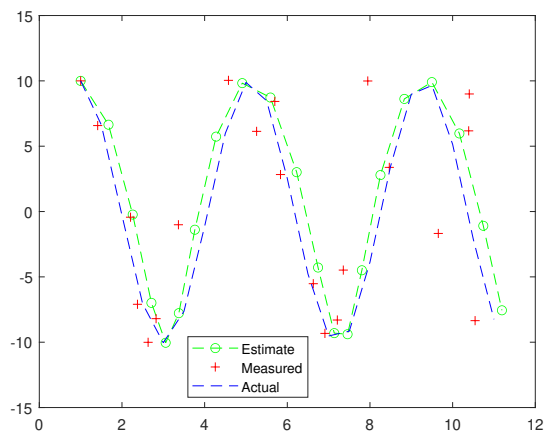
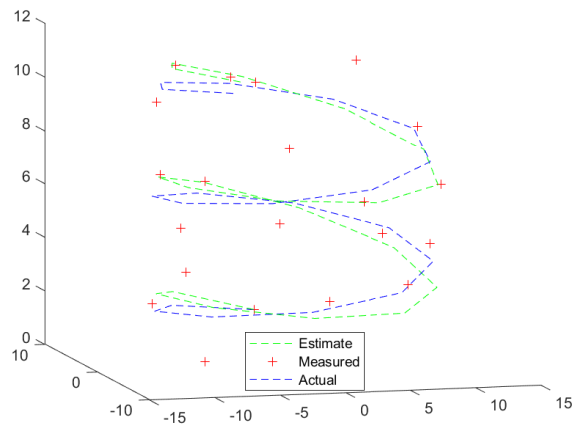


Figure 3. A constant turn



(a) An helix in 2-D view



(b) An helix in 3D-view

Figure 4. An helix

Levi-Civita connection. The solution was tested in the specific case of the Special Euclidean group $SE(3)$ that models rigid-body motions.

From that, since the connection exponential defines a local coordinate system, we can go further and find coordinate systems that flatten the manifold, thus greatly simplifying the regression problem and have it behave as in the Euclidean case. Also, further work would consist in characterising the statistical quantities associated with the estimation process using polynomial regression. Determining the limit law to be considered in this case is a problem in itself. Another axis would involve using local G -valued polynomials to track the changes in maneuvers and adapt the parameters of the optimum fitting curve.

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