Decomposition of the displacements of a plate with very high thickness contrast

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Abstract

The aim of this paper is to give a decomposition of the displacements of plates with a very high thickness contrast. Estimates of all terms of the decomposition with respect to the norm of the strain tensor are obtained. Weighted Poincaré-Wirtinger and Korn inequalities are also given.

Key Words: linear elasticity, displacements decomposition, elementary displacement, Kirchhoff-Love displacement, weighted Poincaré-Wirtinger inequality, weighted Korn inequality, plate with very high thickness contrast.
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1 Introduction

The aim of this paper is to extend the tools developed in our previous articles (see [12, 13, 14]), to plates with very high thickness contrasts. An aircraft wing or a lens (in optics) can be considered as plates with a very high thickness contrast. The plates we are studying not only vary widely in thickness, but also have uneven top and bottom surfaces.

Let ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary. Here, the open set ω plays the same role as the mid-surface of a plate of constant thickness. The plate with very high thickness contrast is Ω_ε, its thickness varying continuously from one part of its boundary of thickness of the order ε to another part of thickness of the order ε^2.

In this paper we show that every displacement u belonging to $W^{1,p}(\Omega_\varepsilon)^3$, $p \in (1,\infty)$, is written as the sum of an elementary displacement $U_{e\ell}$ and a residual displacement $\overline{u}$ (the warping),

$$\begin{aligned} u(x) &= U_{e\ell}(x) + \overline{u}(x) \\ U_{e\ell}(x) &= \mathcal{U}(x') + x_3\mathcal{R}(x') \end{aligned} \qquad \text{for a.e. } x = (x', x_3) \in \Omega_\varepsilon$$

so that ($\alpha \in \{1,2\}$)

$$\int_{F^0_\varepsilon(x')} \overline{u}(x', x_3)dx_3 = 0, \quad \int_{F^0_\varepsilon(x')} \overline{u}_\alpha(x', x_3)x_3dx_3 = 0 \quad \text{for a.e. } x' \text{ in } \omega \tag{1.1}$$

where $F^0_\varepsilon(x')$ is a part of the fiber of Ω_ε passing through $x' \in \omega$. The segment $F^0_\varepsilon(x')$ is symmetric with respect to ω and, like all the fibers of Ω_ε, is perpendicular to ω.
The above conditions (1.1) determine $\mathcal{U}(x')$ and $\mathcal{R}(x')$ in terms of u and integrals on $F^0_\varepsilon(x')$ (see (4.3) and also [12]). The field $\mathcal{U}$ belongs to $W^{1,p}(\omega)^3$, it represents the displacement of the 2D domain $\omega\times\{0\}$, while $\mathcal{R} = \mathcal{R}_1\mathbf{e}_1 + \mathcal{R}_2\mathbf{e}_2$ belongs to $W^{1,p}(\omega)^2$. The term $x_3\mathcal{R}(x')$ stands for a "small rotation"

of the fiber passing trough x'. The axis of this rotation is directed by $-\mathcal{R}_2(x')\mathbf{e}_1 + \mathcal{R}_1(x')\mathbf{e}_2$ and its angle is approximately the euclidian norm of this vector.
The last term of our decomposition, $\overline{u}$, gives information about the deformations of the fibers of the plate. Compare to the other terms of the decomposition ($\mathcal{U}$ and $x_3\mathcal{R}$) it is very small (see Theorems 4.2 and 5.1). In early work on elasticity in the 19th century, this term was neglected, as were certain terms in the stress tensor. Below, in Section 6, we show that it is as important as the other two terms $\mathcal{U}$ and $\mathcal{R}$ in the strain tensor and therefore in the stress tensor. It is interesting to note that if we study the asymptotic behavior of a sequence of displacements, the strain and stress tensor terms are all determined using the limits of the three terms in our decomposition.

As a general reference on elasticity, we refer to [21, 7, 22]. A general introduction to the mathematical modeling of elastic plates can be found in [8, 10]. For the study of thin plates with rapidly varying thickness we refer to [18, 19]. For the decomposition of the displacements or deformations we refer to [12, 3, 4, 14], some applications of these decompositions for the study of junctions can be found in [11, 16, 15]. The study of a beam with very high thickness contrast was carried out in [17]. The Poincaré-Wirtinger and Korn inequalities have been extensively studied, here are a few articles devoted to them [6, 23, 20, 5, 2, 1]

The paper is organized as follow:

- Section 2 is devoted to the main notations,
- in Section 3 we recall some important results about the geometry of the structure and the decompositions of the displacements of a thin plate via Kirchhoff-Love displacements,
- the main decomposition result of this paper is given in Section 4, Theorem 4.2,
- in Section 5, Theorem 5.1, for a clamped plate, we give the full estimates of a displacement and a weighted Korn inequality with respect to the norm of the stain tensor,
- in Section 6, we introduce the rescaling operator Π_ε and we study the asymptotic behavior of a sequence of displacements,
- Section 7 is devoted to applying the results of the previous sections to solve a classical elasticity problem,
- in Appendix we prove some technical lemmas and weighted Poincaré-Wirtinger and Korn inequalities.

In this work, the constants appearing in the estimates will always be independent from ε. If this is not stated, p is a real number belonging to $(1, \infty)$ and p' denotes the dual exponent of p, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. As a rule the Latin indices i, j, k and l take values in $\{1, 2, 3\}$ while the Greek indices α and β in $\{1, 2\}$. We also use the Einstein convention of summation over repeated indices.

2 Notations

The euclidian space $\mathbb{R}^3$ is referred to the orthonormal frame $\big(O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\big)$ and $\mathbb{R}^2$ to the frame $\big(O; \mathbf{e}_1, \mathbf{e}_2\big)$. We denote $|\cdot|_2$ the euclidian norm in $\mathbb{R}^2$ and $\cdot$ its scalar product.

In this paper ε is a small parameter belonging to $(0, \varepsilon_0]$, $0 < \varepsilon_0 < 1$), it will tend to 0.

Denote

- ω a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary, below it is improperly referred to as the "mid-surface" of the plate,

- $\rho \in W^{1,\infty}(\mathbb{R}^2)$ the function defined by

$$\rho(x') = \inf\big\{1, \mathrm{dist}(x', \partial\omega)\big\} \qquad \text{for all } x' \in \mathbb{R}^2,$$

- γ a non-empty subset of $\partial\omega$, possibly containing only one point,
- $\rho_\gamma \in W^{1,\infty}(\mathbb{R}^2)$ the function defined by

$$\rho_\gamma(x') = \inf\big\{1, \mathrm{dist}(x', \gamma)\big\} \qquad \text{for all } x' \in \mathbb{R}^2,$$

- $\rho_\varepsilon \in W^{1,\infty}(\mathbb{R}^2)$ the function defined by

$$\rho_\varepsilon(x') = \varepsilon + \big(1-\varepsilon\big)\rho_\gamma(x') \qquad \text{for all } x' \in \mathbb{R}^2,$$

- ϕ, ψ two functions belonging to $W^{1,\infty}(\omega)$ and satisfying

$$\forall \big(x', z'\big) \in \overline{\omega}^2 \qquad \begin{cases} |\boldsymbol{\phi}(x') - \phi(z')| \le K_0 |x' - z'|_2, \quad \phi(x') \ge 2C_0, \quad \forall x' \in \omega, \\ |\boldsymbol{\psi}(x') - \psi(z')| \le K_0 |x' - z'|_2, \quad \psi(x') \ge 2C_0, \quad \forall x' \in \omega, \end{cases}$$

  where K_0 and C_0 are strictly positive constants
- $C_1 = \max\{\|\phi\|_{L^\infty(\omega)}, \|\psi\|_{L^\infty(\omega)}\}$,
- Ω_ε the plate with a very high thickness contrast

$$\Omega_\varepsilon \doteq \Big\{x = (x', x_3) \in \omega \times \mathbb{R} \mid \ -\varepsilon\rho_\varepsilon(x')\psi(x') < x_3 < \varepsilon\rho_\varepsilon(x')\phi(x')\Big\},$$

  the plate Ω_ε is much thinner in the neighborhood of γ,
- Ω'_ε the plate with a non-planar top and bottom surfaces and thickness of order ε

$$\Omega'_\varepsilon \doteq \Big\{x = (x', x_3) \in \omega \times \mathbb{R} \mid \ -\varepsilon\psi(x') < x_3 < \varepsilon\phi(x')\Big\},$$

- γ_0 the clamped part of the "mid-surface", $\gamma_0 \subset \partial\omega \setminus \overline{\gamma}$, γ_0 has a non-null measure,
- $\Gamma_{0,\varepsilon} \doteq \big(\gamma_0 \times \mathbb{R}\big) \cap \partial\Omega_\varepsilon$ the clamped part of Ω_ε,
- $\Omega \doteq \Big\{(x_1, x_2, X_3) \in \omega \times \mathbb{R} \ \mid \ -\rho_\gamma(x')\psi(x') < X_3 < \rho_\gamma(x')\phi(x')\Big\}$ the re-scaled plate and $\Gamma_0 \doteq \big(\gamma_0 \times \mathbb{R}\big) \cap \partial\Omega$ the clamped part of Ω,
- $e(v)$ the strain tensor of $v \in W^{1,p}(\Omega_\varepsilon)^3$, $1 \le p \le \infty$,

$$e(v) = \frac{1}{2}\Big((\nabla v)^T + \nabla v\Big), \qquad e_{ij}(v) = \frac{1}{2}\big(\partial_j v_i + \partial_i v_j\big).$$

  $e(v)$ is the 3×3 symmetric matrix whose entries are the $e_{ij}(v)$'s,
- $\partial_i = \dfrac{\partial}{\partial x_i}$, $\partial^2_{ij} = \dfrac{\partial^2}{\partial x_i \partial x_j}$, $(i,j) \in \{1,2,3\}^2$.

Note that the function ρ_ε satisfies

$$\varepsilon \le \rho_\varepsilon(x') \le 1 \quad \text{and} \quad |\rho_\varepsilon(x') - \rho_\varepsilon(z')| \le |x' - z'|_2 \qquad \forall (x', z') \in \omega^2. \tag{2.1}$$

3 Preliminary reminders

The following lemma is proved in [13].

Lemma 3.1. *Let $B_\varepsilon \subset \omega$ be a domain of diameter less than εR, star-shaped with respect to the disc $D(O, \varepsilon R_1)$. If $R \leq \dfrac{C_0}{K_0}$ then the domain*

$$\mathcal{D}_\varepsilon = \Big\{ x = (x', x_3) \in B_\varepsilon \times \mathbb{R} \mid -\varepsilon\psi(x') < x_3 < \varepsilon\phi(x') \Big\}$$

has a diameter less than $\varepsilon(R + 2C_1)$ and is star-shaped with respect to the ball $B(O, \varepsilon R_1')$ where

$$R_1' = \min\big\{C_0, R_1\big\}. \tag{3.1}$$

Denote

$$\omega_\delta \doteq \big\{ x' \in \omega \mid \rho(x') > \delta \big\}, \qquad \delta \geq 0.$$

The boundary of ω being Lipshitz, there exist constants A, B and $\mathfrak{c}_0$ strictly positive and a finite number N of local coordinate systems (x_{1n}, x_{2n}) in orthonormal frames $\big(O_n; \mathbf{e}_{1n}, \mathbf{e}_{2n}\big)$ and maps $f_n \in W^{1,\infty}(0, A)$, $1 \leq n \leq N$, such that

$$\begin{aligned}
&\partial\omega = \bigcup_{n=1}^{N} \Big\{ (x_{1n}, x_{2n}) \in \mathbb{R}^2 \mid x_{2n} = f_n(x_{1n}), \quad x_{1n} \in (0, A) \Big\}, \\
&\mathcal{B}_n \doteq \Big\{ (x_{1n}, x_{2n}) \in \mathbb{R}^2 \mid f_n(x_{1n}) < x_{2n} < f_n(x_{1n}) + B, \quad x_{1n} \in (0, A) \Big\} \subset \omega, \\
&\omega \setminus \overline{\omega_{\mathfrak{c}_0}} \subset \bigcup_{n=1}^{N} \mathcal{B}_n.
\end{aligned}$$

Note that

$$\omega = \omega_{\mathfrak{c}_0} \cup \Big(\bigcup_{n=1}^{N} \mathcal{B}_n \Big).$$

Since $f_n \in W^{1,\infty}(0, A)$, $1 \leq n \leq N$, there exists $K_1 > 0$ such that

$$|f_n(t) - f_n(t')| \leq K_1 |t - t'|, \qquad \forall (t, t') \in [0, A]^2, \qquad \forall n \in \{1, \dots, N\}.$$

Without loss of generality, we assume that

$$B \leq 1 \qquad \text{and} \qquad K_1 \geq \frac{1}{\sqrt{3}}. \tag{3.2}$$

Now, set

$$x_k = k\kappa\varepsilon, \quad k \in \Big\{0, \dots, N_\varepsilon - 1\Big\} \quad \text{and} \quad x_{N_\varepsilon} = A - \kappa\varepsilon$$

where $N_\varepsilon = \Big[\dfrac{A}{\kappa\varepsilon}\Big]$, $[t] \in \mathbb{N}$ is the integer part of $t \in \mathbb{R}^+$, κ is a strictly positive constant which will be given below (see (3.5)).
Denote (see Figure 1)

$$\mathcal{B}_{n,k,\varepsilon} \doteq \Big\{ (x_{1n}, x_{2n}) \in \mathbb{R}^2 \mid \begin{array}{c} x_k < x_{1n} < x_k + \kappa\varepsilon, \\ f_n(x_{1n}) < x_{2n} < f_n(x_{1n}) + B \end{array} \Big\} \quad k \in \big\{0, \dots, N_\varepsilon\big\}.$$

We cover $\mathcal{B}_{n,k,\varepsilon}$, $(n, k) \in \{1, \dots, N\} \times \{0, \dots, N_\varepsilon\}$ as follows:

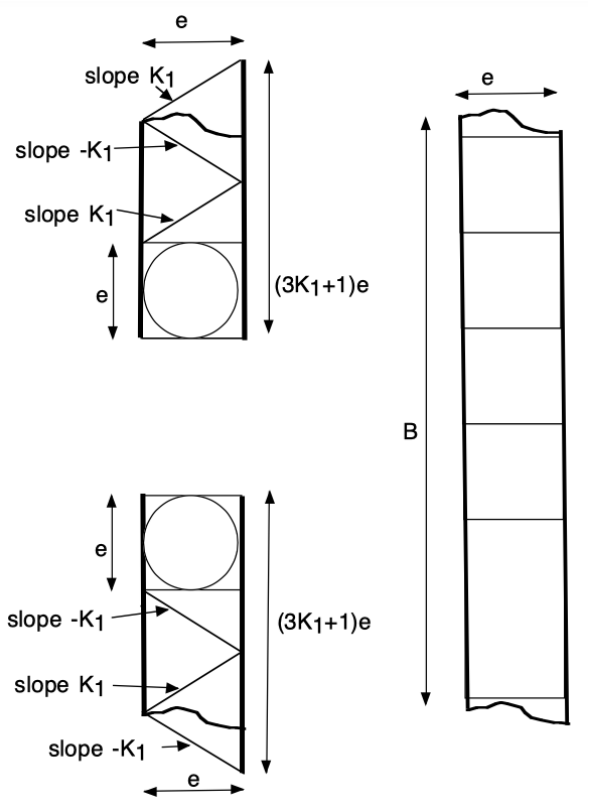

Figure 1: The domains $\mathcal{B}^{\rm Top}_{n,k,\varepsilon}$, $\mathcal{B}^{\rm Bottom}_{n,k,\varepsilon}$ and $\mathcal{B}_{n,k,\varepsilon}$. In these drawings $e = \kappa\varepsilon$.

- first, its top and bottom with the small domains $\mathcal{B}^{\rm Top}_{n,k,\varepsilon}$ and $\mathcal{B}^{\rm Bottom}_{n,k,\varepsilon}$ (see Figure 1 left) which are star-shaped with respect to discs of radius $\kappa\varepsilon/2$ and whose diameters are less than $(3K_1+2)\kappa\varepsilon$,
- then, between $\max\limits_{x_{1n}\in[x_k,x_k+\kappa\varepsilon]} f_n(x_{1n})$ and $\min\limits_{x_{1n}\in[x_k,x_k+\kappa\varepsilon]} f_n(x_{1n}) + B$, we cover by squares whose edges have length $\kappa\varepsilon$ and possibly we add a rectangle whose width is $\kappa\varepsilon$ and length lies between $\kappa\varepsilon$ and $2\kappa\varepsilon$ (see Figure 1 right). These quadrilaterals are star-shaped with respect to discs of radius $\kappa\varepsilon/2$, and their diameters are less than $3\kappa\varepsilon$.

Denote

$$Y \doteq (0,1)^2, \quad \Xi_\varepsilon \doteq \big\{\xi \in \mathbb{Z}^2 \mid Y_{\xi,\varepsilon} \subset \omega\big\}, \quad Y_{\xi,\varepsilon} \doteq \kappa\varepsilon(\xi + Y), \quad \widehat{\omega}_\varepsilon \doteq \text{Interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon} \overline{Y}_{\xi,\varepsilon}\Big).$$

The squares $Y_{\xi,\varepsilon}$, $\xi \in \Xi_\varepsilon$ which are included in $\widehat{\omega}_\varepsilon$ cover most of ω.

Below are the conditions that will enable us to determine κ.

To take account of the conditions imposed by the covering of the domains $\mathcal{B}_n$, $n \in \{1,\dots,N\}$, and Lemma 3.1, we assume that

$$\kappa\varepsilon \le \kappa\varepsilon_0 \le A, \quad (6K_1+2)\kappa\varepsilon \le (6K_1+2)\kappa\varepsilon_0 \le B \quad \text{and} \quad \kappa \max\big\{(3K_1+2), 3\big\} \le \frac{C_0}{K_0}. \tag{3.3}$$

For the squares $Y_{\xi,\varepsilon}$, we need

$$\kappa\sqrt{2} \le \frac{C_0}{K_0} \tag{3.4}$$

to take account of the condition imposed by Lemma 3.1.
We also assume

$$\kappa\varepsilon\sqrt{2} \leq \kappa\varepsilon_0\sqrt{2} < \mathfrak{c}_0$$

to get $\widehat{\omega}_\varepsilon \subset \omega_{\mathfrak{c}_0}$.
The above conditions (3.3)-(3.4) allow to give κ

$$\kappa = \frac{1}{\varepsilon_0}\min\Big\{A, \frac{B}{6K_1+2}, \frac{\mathfrak{c}_0}{\sqrt{2}}, \frac{C_0\varepsilon_0}{K_0\max\big\{(3K_1+2),3\big\}}\Big\}. \tag{3.5}$$

Finally, any point of the open set ω belongs to at most $C_N = 3N+1$ domains of the ω covering

$$\mathcal{B}^{\rm Top}_{n,k,\varepsilon}, \quad \mathcal{B}^{\rm Bottom}_{n,k,\varepsilon}, \qquad (n,k) \in \{1,\dots,N\}\times\{0,\dots,N_\varepsilon\} \tag{3.6}$$

and quadrilaterals included in $\widehat{\omega}_\varepsilon$ or $\mathcal{B}_{n,k,\varepsilon}$. The diameters of these small domains are less than $3(K_1+1)\kappa\varepsilon$ and all these domains are star-shaped with respect to discs of radius $\kappa\varepsilon/2$.
Denote $\mathbf{Q}_\varepsilon$ a generic quadrilateral included in $\widehat{\omega}_\varepsilon$ or $\mathcal{B}_{n,k,\varepsilon}$ and

$$\widetilde{\mathcal{B}^{\rm Top}_{n,k,\varepsilon}} \doteq \big(\mathcal{B}^{\rm Top}_{n,k,\varepsilon}\times\mathbb{R}\big)\cap\Omega'_\varepsilon, \qquad \widetilde{\mathcal{B}^{\rm Bottom}_{n,k,\varepsilon}} \doteq \big(\mathcal{B}^{\rm Bottom}_{n,k,\varepsilon}\times\mathbb{R}\big)\cap\Omega'_\varepsilon, \qquad \widetilde{\mathbf{Q}}_\varepsilon \doteq \big(\mathbf{Q}_\varepsilon\times\mathbb{R}\big)\cap\Omega'_\varepsilon. \tag{3.7}$$

These domains, included in Ω'_ε, have a diameter less than $\big(3(K_1+1)\kappa+2C_1\big)\varepsilon$ and by construction they are star-shaped with respect to balls of radius $\min\Big\{C_0, \dfrac{\kappa}{2}\Big\}\varepsilon$ (κ is given by (3.5)).
As a consequence of [13, Theorem 4.1] we have

Theorem 3.2. *Every displacement u in $W^{1,p}(\Omega'_\varepsilon)^3$ is decomposed as the sum of a Kirchhoff-Love displacement and a residual displacement*

$$u(x) = U^\diamond_{KL}(x) + \mathfrak{u}^\diamond(x) = \underbrace{\begin{pmatrix} \mathcal{U}^\diamond_1(x') - x_3\partial_1\mathcal{U}^\diamond_3(x') \\ \mathcal{U}^\diamond_2(x') - x_3\partial_2\mathcal{U}^\diamond_3(x') \\ \mathcal{U}^\diamond_3(x') \end{pmatrix}}_{\text{Kirchhoff-Love displacement}} + \underbrace{\mathfrak{u}^\diamond(x)}_{\text{residual displacement}} \qquad \text{for a.e. } x \text{ in } \Omega'_\varepsilon \tag{3.8}$$

where $\mathcal{U}^\diamond_m = \mathcal{U}^\diamond_1\mathbf{e}_1 + \mathcal{U}^\diamond_2\mathbf{e}_2 \in W^{1,p}(\omega)^2$, $\mathcal{U}^\diamond_3 \in W^{2,p}(\omega)$ and $\mathfrak{u}^\diamond \in W^{1,p}(\Omega'_\varepsilon)^3$.
The terms $\mathcal{U}^\diamond$ and $\mathfrak{u}^\diamond$ satisfy

$$\begin{aligned} &\|e_{\alpha\beta}(\mathcal{U}^\diamond_m)\|_{L^p(\omega)} \leq \frac{C}{\varepsilon^{1/p}}\|e(u)\|_{L^p(\Omega'_\varepsilon)}, \\ &\|D^2\mathcal{U}^\diamond_3\|_{L^p(\omega)} \leq \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega'_\varepsilon)}, \\ &\|e(\mathfrak{u}^\diamond)\|_{L^p(\Omega'_\varepsilon)} \leq C\|e(u)\|_{L^p(\Omega'_\varepsilon)} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\|\mathfrak{u}^\diamond\|_{L^p(\Omega'_\varepsilon)} \leq C\,\varepsilon\,\|e(u)\|_{L^p(\Omega'_\varepsilon)}, \\ &\|\nabla\mathfrak{u}^\diamond\|_{L^p(\Omega'_\varepsilon)} \leq C\|e(u)\|_{L^p(\Omega'_\varepsilon)} \end{aligned} \tag{3.10}$$

The constants depend only on K_0, K_1, C_0, C_1, A, B, $\mathfrak{c}_0$, N and p.

In the proof of [13, Theorem 4.1], the coverings of ω and Ω'_ε given by (3.6)-(3.7) play an important role via the ratio of the diameter of an element of these coverings to the radius of the ball for which it is star-shaped, that is $\dfrac{3(K_1+1)\kappa+2C_1}{\min\{C_0,\kappa/2\}}$ for both coverings (κ is given by (3.5)).

4 Decomposition of the displacements of plates with a very high thickness contrast via elementary displacements

Denote

$$\begin{aligned} &F_\varepsilon(x') \doteq \{x'\}\times(-\varepsilon\rho_\varepsilon(x')\psi(x')\,,\,\varepsilon\rho_\varepsilon(x')\phi(x')\big),\\ &F^0_\varepsilon(x') \doteq \{x'\}\times(-\varepsilon\rho_\varepsilon(x')C_0\,,\,\varepsilon\rho_\varepsilon(x')C_0\big), \quad F^0_\varepsilon(x')\subset F_\varepsilon(x'), \end{aligned} \qquad x'\in\omega.$$

$F_\varepsilon(x')$ is the fiber passing through the point x' of ω.

Definition 4.1. *An elementary displacement of the plate Ω_ε is a displacement $v\in L^1(\Omega_\varepsilon)^3$ written in the form*

$$v(x',x_3)=\mathcal{V}(x')+x_3\mathcal{A}(x') \quad \textit{for a.e. } x=(x',x_3)\in\Omega_\varepsilon.$$

The component $\mathcal{V}$ belongs to $L^1(\omega)^3$ while $\mathcal{A}=\mathcal{A}_1\mathbf{e}_1+\mathcal{A}_2\mathbf{e}_2$ is in $L^1(\omega)^2$.

Here, $\mathcal{V}$ gives the "mid-surface"[1] displacement and $x_3\mathcal{A}(x')$ represents a "small rotation" of the fiber $F_\varepsilon(x')$, whose axis is directed by $-\mathcal{A}_2(x')\mathbf{e}_1+\mathcal{A}_1(x')\mathbf{e}_2$ and whose angle is approximately $|\mathcal{A}(x')|_2$ (see [9, Chapter 11] or [11, 12] for more details).

To any displacement $u\in L^1(\Omega_\varepsilon)^3$ we associate an elementary displacement $U_{e\ell}\in L^1(\Omega_\varepsilon)^3$ and a warping $\overline{u}\in L^1(\Omega_\varepsilon)^3$

$$\begin{aligned} u(x)&=U_{e\ell}(x)+\overline{u}(x)\\ U_{e\ell}(x)&=\mathcal{U}(x')+x_3\mathcal{R}(x') \end{aligned} \qquad \text{for a.e. } x=(x',x_3)\in\Omega_\varepsilon \tag{4.1}$$

so that ($\alpha\in\{1,2\}$)

$$\int_{F^0_\varepsilon(x')}\overline{u}(x',x_3)dx_3=0, \quad \int_{F^0_\varepsilon(x')}\overline{u}_\alpha(x',x_3)x_3dx_3=0 \quad \text{for a.e. } x' \text{ in } \omega. \tag{4.2}$$

The above equalities determine $\mathcal{U}(x')$ and $\mathcal{R}(x')$ in terms of u and integrals on the fiber $F^0_\varepsilon(x')$ (see [12]). We have

$$\begin{aligned} \mathcal{U}(x')&=\frac{1}{|F^0_\varepsilon(x')|}\int_{F^0_\varepsilon(x')}u(x',x_3)dx_3,\\ \mathcal{R}(x')&=\frac{12}{|F^0_\varepsilon(x')|^3}\int_{F^0_\varepsilon(x')}x_3\big(u_1(x',x_3)\mathbf{e}_1+u_2(x',x_3)\mathbf{e}_2\big)dx_3, \end{aligned} \qquad \text{for a.e. } x'\in\omega \tag{4.3}$$

where $|F^0_\varepsilon(x')|=2\varepsilon\rho_\varepsilon(x')C_0$ is the length of the fiber $F^0_\varepsilon(x')$.

Theorem 4.2. *Let u be a displacement in $W^{1,p}(\Omega_\varepsilon)^3$ decomposed as (4.1). The terms $\mathcal{U}$, $\mathcal{R}$ and $\overline{u}$ of this decomposition satisfy*

$$\begin{aligned} &\mathcal{U}\in W^{1,p}(\omega)^3, \quad \mathcal{R}\in W^{1,p}(\omega)^2, \quad \overline{u}\in W^{1,p}(\Omega_\varepsilon)^3,\\ &\|\rho_\varepsilon^{1/p}e_{\alpha\beta}(\mathcal{U}_m)\|_{L^p(\omega)}\le\frac{C}{\varepsilon^{1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)},\\ &\big\|\rho_\varepsilon^{1/p}\big(\nabla\mathcal{U}_3+\mathcal{R}\big)\big\|_{L^p(\omega)}\le\frac{C}{\varepsilon^{1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}\\ &\|\rho_\varepsilon^{1+1/p}\nabla\mathcal{R}\|_{L^p(\omega)}\le\frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)},\\ &\Big\|\frac{\overline{u}}{\rho_\varepsilon}\Big\|_{L^p(\Omega_\varepsilon)}\le C\varepsilon\|e(u)\|_{L^p(\Omega_\varepsilon)}, \qquad \|\nabla\overline{u}\|_{L^p(\Omega_\varepsilon)}\le C\|e(u)\|_{L^p(\Omega_\varepsilon)} \end{aligned} \tag{4.4}$$

where $\mathcal{U}_m=\mathcal{U}_1\mathbf{e}_1+\mathcal{U}_2\mathbf{e}_2\in W^{1,p}(\omega)^2$.
The constants do not depend on ε.[2]

[1] Here the set $\omega\times\{0\}$.
[2] They depend on K_0, K_1, C_0, C_1, A, B, $\mathfrak{c}_0$, N and p.

Proof. Since $\rho_\varepsilon \in W^{1,\infty}(\omega)$ and $u \in W^{1,p}(\Omega_\varepsilon)^3$, the function $\mathcal{U}$ belongs to $W^{1,p}(\omega)^3$ and $\mathcal{R}$ to $W^{1,p}(\omega)^2$. So, $\overline{u}$ belongs to $W^{1,p}(\Omega_\varepsilon)^3$.
Let $\mathbf{Y}_\varepsilon$ be an element of the ω covering, of type $\mathcal{B}^{\rm Top}_{n,k,\varepsilon}$, $\mathcal{B}^{\rm Bottom}_{n,k,\varepsilon}$ or $\mathbf{Q}_\varepsilon$, and $\mathbf{a}_\varepsilon$ a point in $\mathbf{Y}_\varepsilon$.

Step 1. Preliminary considerations about $\mathbf{Y}_\varepsilon$.

Denote

- εR the diameter of $\mathbf{Y}_\varepsilon$ ($R \leq 3(K_1+1)\kappa + 2C_1$),
- $\overline{\rho_\varepsilon} = \min\limits_{x\in\mathbf{Y}_\varepsilon} \rho_\varepsilon(x)$, $\overline{\overline{\rho_\varepsilon}} = \max\limits_{x\in\mathbf{Y}_\varepsilon} \rho_\varepsilon(x)$.

We have

$$1 \leq \frac{\overline{\overline{\rho_\varepsilon}}}{\overline{\rho_\varepsilon}} \leq R \qquad \text{and} \qquad \varepsilon \leq \overline{\rho_\varepsilon} \leq 1. \tag{4.5}$$

We define $\mathbf{Y} \subset \mathbb{R}^2$ by

$$\mathbf{Y}_\varepsilon = \mathbf{a}_\varepsilon + \varepsilon\mathbf{Y}.$$

Now, consider the portion of Ω_ε

$$\widetilde{\mathbf{Y}}_\varepsilon = \big(\mathbf{Y}_\varepsilon \times \mathbb{R}\big) \cap \Omega_\varepsilon.$$

The domain $\widetilde{\mathbf{Y}}_\varepsilon$ is a small plate with non-planar top and bottom surfaces. We transform $\widetilde{\mathbf{Y}}_\varepsilon$ by a dilation of center $\mathbf{a}_\varepsilon$ and ratio $1/\varepsilon$, we obtain the domain $\widehat{\mathbf{Y}}_\varepsilon$

$$\widehat{\mathbf{Y}}_\varepsilon = \Big\{ y \in \mathbf{Y}\times\mathbb{R} \;\Big|\; -\overline{\rho_\varepsilon}\psi_\varepsilon(y') < y_3 < \overline{\rho_\varepsilon}\phi_\varepsilon(y') \Big\}$$

where

$$\psi_\varepsilon(y') = \frac{\big(\rho_\varepsilon\psi\big)(\mathbf{a}_\varepsilon + \varepsilon y')}{\overline{\rho_\varepsilon}}, \qquad \phi_\varepsilon(y') = \frac{\big(\rho_\varepsilon\phi\big)(\mathbf{a}_\varepsilon + \varepsilon y')}{\overline{\rho_\varepsilon}}, \qquad \forall y' = (y_1, y_2) \in \mathbf{Y}.$$

The functions ψ_ε and ϕ_ε belong to $W^{1,\infty}(\mathbf{Y})$ and they satisfy

$$\begin{aligned}
&\max\{\|\phi_\varepsilon\|_{L^\infty(\mathbf{Y})}, \|\psi_\varepsilon\|_{L^\infty(\mathbf{Y})}\} \leq RC_1, \qquad \phi_\varepsilon(y') \geq 2C_0, \quad \psi_\varepsilon(y') \geq 2C_0,\\
&|\phi_\varepsilon(y') - \phi_\varepsilon(z')| \leq \big(RK_0\varepsilon_0 + C_1\big)|y' - z'| \qquad \forall (y', z') \in \mathbf{Y}^2,\\
&|\psi_\varepsilon(y') - \psi_\varepsilon(z')| \leq \big(RK_0\varepsilon_0 + C_1\big)|y' - z'| \qquad \forall (y', z') \in \mathbf{Y}^2.
\end{aligned}$$

Case 1. $\mathbf{Y}_\varepsilon$ is a domain of type $\mathcal{B}^{\rm Top}_{n,k,\varepsilon}$ (resp. $\mathcal{B}^{\rm Bottom}_{n,k,\varepsilon}$).
In this case $\mathbf{Y}$ is included in a rectangle whose width is κ and length less than $(3K_1+1)\kappa$.
Firstly, we cover most of $\mathbf{Y}$ with squares of side length $\kappa_1\overline{\rho_\varepsilon}$ (κ_1 is given below by (4.6)). These squares are star-shaped with respect to discs of radius $\kappa_1\overline{\rho_\varepsilon}/2$, the diameter of these small squares are less than $2\kappa_1\overline{\rho_\varepsilon}$.
Then, since the top of this domain (resp. the bottom) is given by a map defined on the interval $[0,\kappa]$ which has a Lipschitz constant $\varepsilon K_1 \leq \varepsilon_0 K_1 \leq K_1$, we cover the neighborhood of the top (resp. the bottom) of $\mathbf{Y}$ with domains similar to $\mathcal{B}^{\rm Top}_{n,k,\varepsilon}$ (resp. $\mathcal{B}^{\rm Bottom}_{n,k,\varepsilon}$). These domains have width equal to $\kappa_1\overline{\rho_\varepsilon}$ and length between $(2K_1+1)\kappa_1\overline{\rho_\varepsilon}$ and $(3K_1+1)\kappa_1\overline{\rho_\varepsilon}$. Their diameters are less than $(3K_1+2)\kappa_1\overline{\rho_\varepsilon}$ and they are star-shaped with respect to discs of radius $\kappa_1\overline{\rho_\varepsilon}/2$.

So, as we did for $\mathcal{B}^{\rm Top}_{n,k,\varepsilon}$ (resp. $\mathcal{B}^{\rm Bottom}_{n,k,\varepsilon}$) in Section 3, we determine κ_1 taking into account the condition given by Lemma 3.1 and the simple conditions to get these domains included in $\mathbf{Y}$. That is

$$(3K_1+2)\kappa_1 \leq \frac{C_0}{RK_0\varepsilon_0 + C_1}, \qquad \kappa_1\overline{\rho_\varepsilon} \leq \kappa_1 \leq \kappa, \qquad (3K_1+1)\kappa_1\overline{\rho_\varepsilon} \leq (3K_1+1)\kappa_1 \leq (2K_1+1)\kappa.$$

We set

$$\kappa_1 = \min\Big\{ \frac{C_0}{(RK_0\varepsilon_0 + C_1)(3K_1+2)}, \frac{(2K_1+1)\kappa}{3K_1+1} \Big\}. \tag{4.6}$$

So, every point of $\mathbf{Y}$ belongs to at most 4 domains of this covering.

Case 2. $\mathbf{Y}_\varepsilon$ is a quadrilateral.

In this case $\mathbf{Y}$ is also a quadrilateral whose width is κ and whose length is between κ and 2κ. We cover it with squares of side length $\kappa_1\overline{\rho_\varepsilon}$ (κ_1 is given above by (4.6)). These squares are star-shaped with respect to discs of radius $\kappa_1\overline{\rho_\varepsilon}/2$, the diameter of these small squares are less than $2\kappa_1\overline{\rho_\varepsilon}$. Every point of $\mathbf{Y}$ belongs to at most 4 squares of this covering.

Step 2. From a Kirchhoff-Love displacement to an elementary displacement.

Let u be a displacement belonging to $W^{1,p}(\Omega_\varepsilon)^3$. We can now apply Theorem 3.2 to the displacement $u(\mathbf{a}_\varepsilon + \varepsilon\cdot)$ of the plate $\widehat{\mathbf{Y}}_\varepsilon$.
First, we replace

- ω by $\mathbf{Y}$,
- ε by $\overline{\rho_\varepsilon}$,
- ϕ by ϕ_ε, ψ by ψ_ε,
- K_0 by $RK_0\varepsilon_0 + C_1$,
- C_0 remains C_0,
- C_1 by RC_1,
- κ by κ_1.

Then, we decompose the displacement $\mathbf{u} = u(\mathbf{a}_\varepsilon + \varepsilon\cdot)$ as the sum of a Kirchhoff-Love displacement and a residual displacement

$$\mathbf{u}(y) = U^*_{KL}(y) + \mathfrak{u}^*(y) = \begin{pmatrix} \mathcal{U}^*_1(y') - y_3\dfrac{\partial \mathcal{U}^*_3}{\partial y_1}(y') \\ \mathcal{U}^*_2(y') - y_3\dfrac{\partial \mathcal{U}^*_3}{\partial y_2}(y') \\ \mathcal{U}^*_3(y') \end{pmatrix} + \mathfrak{u}^*(y) \quad \text{for a.e. } y \text{ in } \widehat{\mathbf{Y}}_\varepsilon \tag{4.7}$$

where $\mathcal{U}^*_m = \mathcal{U}^*_1\mathbf{e}_1 + \mathcal{U}^*_2\mathbf{e}_2 \in W^{1,p}(\mathbf{Y})^2$, $\mathcal{U}^*_3 \in W^{2,p}(\mathbf{Y})$ and $\mathfrak{u}^* \in W^{1,p}(\widehat{\mathbf{Y}}_\varepsilon)^3$.
We have (see Theorem 3.2)

$$\begin{aligned} \|e_{\alpha\beta,y'}(\mathcal{U}^*_m)\|_{L^p(\mathbf{Y})} &\le \frac{C^\diamond}{\overline{\rho_\varepsilon}^{1/p}}\|e_y(\mathbf{u})\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)}, \\ \|D^2_{y'}\mathcal{U}^*_3\|_{L^p(\mathbf{Y})} &\le \frac{C^\diamond}{\overline{\rho_\varepsilon}^{1+1/p}}\|e_y(\mathbf{u})\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)}, \\ \|e_y(\mathfrak{u}^*)\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)} &\le C^\diamond\|e_y(\mathbf{u})\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \|\mathfrak{u}^*\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)} &\le C^\diamond\,\overline{\rho_\varepsilon}\,\|e_y(\mathbf{u})\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)}, \\ \|e_y(\mathfrak{u}^*)\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)} &\le C^\diamond\|e_y(\mathbf{u})\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)}, \\ \|\nabla_y\mathfrak{u}^*\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)} &\le C^\diamond\|e_y(\mathbf{u})\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)}. \end{aligned} \tag{4.9}$$

The constant $C^\diamond$ depends only on $K_0,\ K_1,\ C_0,\ C_1,\ A,\ B, \mathfrak{c}_0$ and p via the constants introduced in the previous step and summarised above.
We have

$$\|e_y(\mathbf{u})\|_{L^p(\widehat{\mathbf{Y}}_\varepsilon)} = \varepsilon^{1-3/p}\|e(u)\|_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}. \tag{4.10}$$

Denote

$$\widetilde{F}^0_\varepsilon(y') = \{y'\}\times\big(-C_0\rho_\varepsilon(\mathbf{a}_\varepsilon+\varepsilon y'), C_0\rho_\varepsilon(\mathbf{a}_\varepsilon+\varepsilon y')\big)$$

the portion of fiber of $\widehat{\mathbf{Y}}_\varepsilon$ obtained by scaling from the fiber $F^0_\varepsilon(x')$.
From (4.3) and (4.7) we obtain for a.e. $y'\in\mathbf{Y}$

$$\begin{aligned}
\mathcal{U}(\mathbf{a}_\varepsilon+\varepsilon y') &= \mathcal{U}^*(y') + \frac{1}{|\widetilde{F}^0_\varepsilon(y')|}\int_{\widetilde{F}^0_\varepsilon(y')}\mathfrak{u}^*(y)dy_3,\\
\varepsilon\mathcal{R}(\mathbf{a}_\varepsilon+\varepsilon y') &= -\Big(\frac{\partial\mathcal{U}^*_3}{\partial y_1}(y')\mathbf{e}_1+\frac{\partial\mathcal{U}^*_3}{\partial y_2}(y')\mathbf{e}_2\Big)+\frac{12}{|\widetilde{F}^0_\varepsilon(y')|^3}\int_{\widetilde{F}^0_\varepsilon(y')}(\mathfrak{u}^*_1\mathbf{e}_1+\mathfrak{u}^*_2\mathbf{e}_2)(y)y_3dy_3,\\
\varepsilon\partial_\alpha\mathcal{U}(\mathbf{a}_\varepsilon+\varepsilon y') &= \frac{\partial\mathcal{U}^*}{\partial y_\alpha}(y')+\frac{1}{|\widetilde{F}^0_\varepsilon(y')|}\int_{\widetilde{F}^0_\varepsilon(y')}\frac{\partial\mathfrak{u}^*}{\partial y_\alpha}(y)dy_3,\\
\varepsilon^2\partial_\alpha\mathcal{R}(\mathbf{a}_\varepsilon+\varepsilon y') &= -\Big(\frac{\partial^2\mathcal{U}^*_3}{\partial y_1\partial y_\alpha}(y')\mathbf{e}_1+\frac{\partial^2\mathcal{U}^*_3}{\partial y_2\partial y_\alpha}(y')\mathbf{e}_2\Big)+\frac{12}{|\widetilde{F}^0_\varepsilon(y')|^3}\int_{\widetilde{F}^0_\varepsilon(y')}\frac{\partial(\mathfrak{u}^*_1\mathbf{e}_1+\mathfrak{u}^*_2\mathbf{e}_2)}{\partial y_\alpha}(y)y_3dy_3.
\end{aligned}\tag{4.11}$$

Step 3. We prove inequality $(4.4)_1$.
From equality $(4.11)_3$ we deduce

$$\varepsilon e_{\alpha\beta}(\mathcal{U}_m)(\mathbf{a}_\varepsilon+\varepsilon y') = e_{\alpha\beta,y'}(\mathcal{U}^*_m)(y')+\frac{1}{|\widetilde{F}^0_\varepsilon(y')|}\int_{\widetilde{F}^0_\varepsilon(y')}e_{\alpha\beta,y}(\mathfrak{u}^*)(y)dy_3.$$

We elevate the above equality to the power p, multiply it by $\rho_\varepsilon(\mathbf{a}_\varepsilon+\varepsilon y')$ and then use Hölder inequality. We obtain

$$\varepsilon^p\rho_\varepsilon(\mathbf{a}_\varepsilon+\varepsilon y')|e_{\alpha\beta}(\mathcal{U}_m)(\mathbf{a}_\varepsilon+\varepsilon y')|^p \le 2^{p-1}\overline{\overline{\rho_\varepsilon}}|e_{\alpha\beta,y'}(\mathcal{U}^*_m)(y')|^p + 2^{p-1}\overline{\overline{\rho_\varepsilon}}\frac{1}{2C_0\overline{\rho_\varepsilon}}\int_{\widetilde{F}^0_\varepsilon(y')}|e_{\alpha\beta,y}(\mathfrak{u}^*)(y)|^pdy_3.$$

Now, we integrate with respect to y' and use (4.5)-(4.8)-(4.10). This yields

$$\begin{aligned}
\varepsilon^{p-2}\|\rho_\varepsilon^{1/p}e_{\alpha\beta}(\mathcal{U}_m)\|^p_{L^p(\mathbf{Y}_\varepsilon)} &\le 2^{p-1}\overline{\overline{\rho_\varepsilon}}\|e_{\alpha\beta,y'}(\mathcal{U}^*_m)\|^p_{L^p(\mathbf{Y})}+\frac{2^{p-1}}{2C_0}R\|e_{\alpha\beta,y}(\mathfrak{u}^*)\|^p_{L^p(\widehat{\mathbf{Y}}_\varepsilon)}\\
&\le 2^{p-1}(C^\diamond)^pR\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}+\frac{2^{p-1}(C^\diamond)^p}{2C_0}R\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}.
\end{aligned}$$

Thus, we obtain

$$\|\rho_\varepsilon^{1/p}e_{\alpha\beta}(\mathcal{U}_m)\|^p_{L^p(\mathbf{Y}_\varepsilon)} \le \Big(1+\frac{1}{2C_0}\Big)(2C^\diamond)^pR\varepsilon^{-1}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)} \le (C^{2\diamond})^p\varepsilon^{-1}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}.$$

The constant $C^{2\diamond}$ depends on $C^\diamond$, C_0, R and p. The sum over all the elements of the ω covering leads to

$$\|\rho_\varepsilon^{1/p}e_{\alpha\beta}(\mathcal{U}_m)\|^p_{L^p(\omega)} \le C_N(C^{2\diamond})^p\varepsilon^{-1}\|e(u)\|^p_{L^p(\Omega_\varepsilon)}.$$

Step 4. We prove inequality $(4.4)_2$.
From equality $(4.11)_{2,3}$ we get

$$\varepsilon\partial_\alpha\mathcal{U}_3(\mathbf{a}_\varepsilon+\varepsilon y')+\varepsilon\mathcal{R}_\alpha(\mathbf{a}_\varepsilon+\varepsilon y') = \frac{1}{|\widetilde{F}^0_\varepsilon(y')|}\int_{\widetilde{F}^0_\varepsilon(y')}\frac{\partial\mathfrak{u}^*_3}{\partial y_\alpha}(y)dy_3+\frac{12}{|\widetilde{F}^0_\varepsilon(y')|^3}\int_{\widetilde{F}^0_\varepsilon(y')}\mathfrak{u}^*_\alpha(y)y_3dy_3.$$

We elevate the above equality to the power p, multiply it by $\rho_\varepsilon(\mathbf{a}_\varepsilon+\varepsilon y')$ and then use Hölder inequality and (4.5). This leads to

$$\rho_\varepsilon(\mathbf{a}_\varepsilon+\varepsilon y')\varepsilon^p\big|\partial_\alpha\mathcal{U}_3(\mathbf{a}_\varepsilon+\varepsilon y')+\mathcal{R}_\alpha(\mathbf{a}_\varepsilon+\varepsilon y')\big|^p \le \frac{2^{p-1}}{2C_0}R\Big(\int_{\widetilde{F}^0_\varepsilon(y')}\Big|\frac{\partial\mathfrak{u}^*}{\partial y_\alpha}(y)\Big|^pdy_3+\frac{6^p}{C_0^p\overline{\rho_\varepsilon}^p}\int_{\widetilde{F}^0_\varepsilon(y')}|\mathfrak{u}^*(y)|^pdy_3\Big).$$

We integrate with respect to y'. This yields

$$\varepsilon^{p-2}\Big\|\rho_\varepsilon^{1/p}\big(\partial_\alpha\mathcal{U}+\mathcal{R}_\alpha\big)\Big\|^p_{L^p(\mathbf{Y}_\varepsilon)}\leq\frac{2^{p-1}}{2C_0}R\Big(\|\nabla_y\mathfrak{u}^*\|^p_{L^p(\widehat{\mathbf{Y}}_\varepsilon)}+\Big(\frac{6}{C_0}\Big)^p\frac{1}{\overline{\rho_\varepsilon}^p}\|\mathfrak{u}^*\|^p_{L^p(\widehat{\mathbf{Y}}_\varepsilon)}\Big).$$

Then, with (4.9)-(4.10) we obtain

$$\big\|\rho_\varepsilon^{1/p}\big(\partial_\alpha\mathcal{U}+\mathcal{R}_\alpha\big)\big\|^p_{L^p(\mathbf{Y}_\varepsilon)}\leq(C^{3\diamond})^p\varepsilon^{-1}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}.$$

The constant $C^{3\diamond}$ depends on $C^\diamond$, C_0, R and p. The sum over all the elements of the ω covering leads to

$$\big\|\rho_\varepsilon^{1/p}\big(\partial_\alpha\mathcal{U}+\mathcal{R}_\alpha\big)\big\|^p_{L^p(\omega)}\leq C_N(C^{3\diamond})^p\varepsilon^{-1}\|e(u)\|^p_{L^p(\Omega_\varepsilon)}.$$

Step 5. We prove inequality $(4.4)_3$.
From equality $(4.11)_4$, we have

$$\varepsilon^2\partial_\alpha\mathcal{R}_\beta(\mathbf{a}_\varepsilon+\varepsilon y')=-\frac{\partial^2\mathcal{U}_3^*}{\partial y_\beta\partial y_\alpha}(y')+\frac{12}{|\widetilde{F}^0_\varepsilon(y')|^3}\int_{\widetilde{F}^0_\varepsilon(y')}\frac{\partial\mathfrak{u}^*_\beta}{\partial y_\alpha}(y)y_3dy_3.$$

We elevate the above equality to the power p, multiply it by $\rho_\varepsilon(\mathbf{a}_\varepsilon+\varepsilon y')^{p+1}$ and then use Hölder inequality. This gives

$$\varepsilon^{2p}\rho_\varepsilon(\mathbf{a}_\varepsilon+\varepsilon y')^{p+1}\big|\partial_\alpha\mathcal{R}_\beta(\mathbf{a}_\varepsilon+\varepsilon y')\big|^p\leq2^{p-1}\overline{\overline{\rho_\varepsilon}}^{p+1}\Big(\Big|\frac{\partial^2\mathcal{U}_3^*}{\partial y_\beta\partial y_\alpha}(y')\Big|^p+\frac{6^p}{2C_0^{p+1}\overline{\rho_\varepsilon}^{p+1}}\int_{\widetilde{F}^0_\varepsilon(y')}\Big|\frac{\partial\mathfrak{u}^*_\beta}{\partial y_\alpha}(y)\Big|^pdy_3\Big).$$

We integrate with respect to y'. This yields

$$\varepsilon^{2p-2}\big\|\rho_\varepsilon^{1+1/p}\partial_\alpha\mathcal{R}_\beta\big\|^p_{L^p(\mathbf{Y}_\varepsilon)}\leq2^{p-1}\overline{\overline{\rho_\varepsilon}}^{p+1}\Big\|\frac{\partial^2\mathcal{U}_3^*}{\partial y_\beta\partial y_\alpha}\Big\|^p_{L^p(\mathbf{Y})}+\frac{12^p}{C_0^{p+1}}\frac{\overline{\overline{\rho_\varepsilon}}^{p+1}}{\overline{\rho_\varepsilon}^{p+1}}\Big\|\frac{\partial\mathfrak{u}^*_\beta}{\partial y_\alpha}\Big\|^p_{L^p(\widehat{\mathbf{Y}}_\varepsilon)}.$$

Due to (4.5)-$(4.8)_2$-$(4.9)_2$ and (4.10), this implies

$$\varepsilon^{2p-2}\big\|\rho_\varepsilon^{1+1/p}\partial_\alpha\mathcal{R}_\beta\big\|^p_{L^p(\mathbf{Y}_\varepsilon)}\leq2^{p-1}(C^\diamond)^pR^{p+1}\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}+\frac{12^p}{C_0^{p+1}}(C^\diamond)^pR^{p+1}\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}.$$

Hence

$$\big\|\rho_\varepsilon^{1+1/p}\partial_\alpha\mathcal{R}_\beta\big\|^p_{L^p(\mathbf{Y}_\varepsilon)}\leq(C^{4\diamond})^p\varepsilon^{-p-1}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}.$$

The constant $C^{4\diamond}$ depends on $C^\diamond$, C_0, R and p. The sum over all the elements of the ω covering leads to

$$\big\|\rho_\varepsilon^{1+1/p}\partial_\alpha\mathcal{R}_\beta\big\|^p_{L^p(\omega)}\leq C_N(C^{4\diamond})^p\varepsilon^{-p-1}\|e(u)\|^p_{L^p(\Omega_\varepsilon)}.$$

Step 6. We prove inequality $(4.4)_{4,5}$.
By proceeding as the above steps and using equalities (4.11) together with estimates (4.9) we obtain

$$\begin{aligned}
\|\mathcal{U}(\mathbf{a}_\varepsilon+\varepsilon\cdot)-\mathcal{U}^*\|^p_{L^p(\mathbf{Y})}&\leq\frac{(C^\diamond)^p}{2C_0}\overline{\rho_\varepsilon}^{p-1}\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)},\\
\Big\|\varepsilon\partial_\alpha\mathcal{U}(\mathbf{a}_\varepsilon+\varepsilon\cdot)-\frac{\partial\mathcal{U}^*}{\partial y_\alpha}\Big\|^p_{L^p(\mathbf{Y})}&\leq\frac{(C^\diamond)^p}{2C_0\overline{\rho_\varepsilon}}\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)},\\
\Big\|\varepsilon\mathcal{R}_\alpha(\mathbf{a}_\varepsilon+\varepsilon\cdot)+\frac{\partial\mathcal{U}_3^*}{\partial y_\alpha}\Big\|^p_{L^p(\mathbf{Y})}&\leq\frac{6^p(C^\diamond)^p}{C_0^{p+1}}\frac{\varepsilon^{p-3}}{\overline{\rho_\varepsilon}}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)},\\
\Big\|\varepsilon^2\partial_\alpha\mathcal{R}_\beta(\mathbf{a}_\varepsilon+\varepsilon\cdot)+\frac{\partial^2\mathcal{U}_3^*}{\partial y_\beta\partial y_\alpha}\Big\|^p_{L^p(\mathbf{Y})}&\leq\frac{6^p(C^\diamond)^p}{C_0^{p+1}}\frac{\varepsilon^{p-3}}{\overline{\rho_\varepsilon}^{p+1}}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}.
\end{aligned}\tag{4.12}$$

We have

$$\overline{u}(\mathbf{a}_\varepsilon+\varepsilon y)=-\big(\mathcal{U}(\mathbf{a}_\varepsilon+\varepsilon y')-\mathcal{U}^*(y')\big)-y_3\big(\varepsilon\mathcal{R}_\alpha(\mathbf{a}_\varepsilon+\varepsilon y')+\frac{\partial\mathcal{U}_3^*}{\partial y_\alpha}(y')\big)\mathbf{e}_\alpha+\mathfrak{u}^*(y)\qquad\text{for a.e. } y\in\widetilde{\mathbf{Y}}_\varepsilon.$$

Remind that the length of the fibers of $\widehat{\mathbf{Y}}_\varepsilon$ are less than $2C_1\overline{\overline{\rho_\varepsilon}}$.
Now, elevate the above equality to the power p, then divide it by $\rho_\varepsilon(\mathbf{a}_\varepsilon+\varepsilon y')^p$ and use Hölder inequality. Thanks to estimates $(4.12)_{1,3}$ and (4.9)-(4.10), this gives

$$\begin{aligned}\Big\|\frac{\overline{u}(\mathbf{a}_\varepsilon+\varepsilon\cdot)}{\rho_\varepsilon(\mathbf{a}_\varepsilon+\varepsilon\cdot)}\Big\|^p_{L^p(\widehat{\mathbf{Y}}_\varepsilon)}&\le 3^{p-1}\Big((C^\diamond)^p\frac{C_1}{C_0}\frac{\overline{\overline{\rho_\varepsilon}}}{\overline{\rho_\varepsilon}}\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}\\&\quad+2\Big(\frac{C_1\overline{\overline{\rho_\varepsilon}}}{\overline{\rho_\varepsilon}}\Big)^{p+1}\frac{6^p(C^\diamond)^p}{C_0^{p+1}}\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}+(C^\diamond)^p\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}\Big).\end{aligned}$$

Hence, with the help of (4.5) we obtain

$$\Big\|\frac{\overline{u}}{\rho_\varepsilon}\Big\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}\le(C^{5\diamond})^p\varepsilon^p\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}.$$

The constant $C^{5\diamond}$ depends on $C^\diamond$, C_0, C_1, R and p. The sum over all the elements of the ω covering leads to

$$\Big\|\frac{\overline{u}}{\rho_\varepsilon}\Big\|^p_{L^p(\Omega_\varepsilon)}\le C_N(C^{5\diamond})^p\varepsilon^p\|e(u)\|^p_{L^p(\Omega_\varepsilon)}.$$

Now, we estimate the partial derivatives of $\overline{u}$. We have

$$\begin{aligned}\varepsilon\partial_\beta\overline{u}(\mathbf{a}_\varepsilon+\varepsilon y)&=-\Big(\varepsilon\partial_\beta\mathcal{U}(\mathbf{a}_\varepsilon+\varepsilon y')-\frac{\partial\mathcal{U}}{\partial y_\beta}(y')\Big)-y_3\Big(\varepsilon^2\partial_\beta\mathcal{R}_\alpha(\mathbf{a}_\varepsilon+\varepsilon y)+\frac{\partial^2\mathcal{U}_3}{\partial y_\beta\partial y_\alpha}(y)\Big)\mathbf{e}_\alpha+\frac{\partial\mathfrak{u}}{\partial y_\beta}(y)\\\varepsilon\partial_3\overline{u}(\mathbf{a}_\varepsilon+\varepsilon y)&=-\Big(\varepsilon\mathcal{R}_\alpha(\mathbf{a}_\varepsilon+\varepsilon y')+\frac{\partial\mathcal{U}_3}{\partial y_\alpha}(y')\Big)\mathbf{e}_\alpha+\frac{\partial\mathfrak{u}}{\partial y_3}(y).\end{aligned}$$

Thanks to the estimates $(4.12)_{2,3,4}$, we obtain

$$\begin{aligned}\varepsilon^{p-3}\big\|\partial_\alpha\overline{u}\big\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}&\le 3^{p-1}\Big((C^\diamond)^p\frac{C_1}{C_0}\frac{\overline{\overline{\rho_\varepsilon}}}{\overline{\rho_\varepsilon}}\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}\\&\quad+2\Big(\frac{C_1\overline{\overline{\rho_\varepsilon}}}{\overline{\rho_\varepsilon}}\Big)^{p+1}\frac{6^p(C^\diamond)^p}{C_0^{p+1}}\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}+(C^\diamond)^p\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}\Big)\end{aligned}$$

and

$$\varepsilon^{p-3}\big\|\partial_3\overline{u}\big\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}\le 2^{p-1}\frac{6^p(C^\diamond)^p}{C_0^{p+1}}\frac{2C_1\overline{\overline{\rho_\varepsilon}}}{\overline{\rho_\varepsilon}}\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}+2^{p-1}(C^\diamond)^p\varepsilon^{p-3}\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}.$$

Thus

$$\|\nabla\overline{u}\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}\le(C^{6\diamond})^p\|e(u)\|^p_{L^p(\widetilde{\mathbf{Y}}_\varepsilon)}.$$

The constant $C^{6\diamond}$ depends on $C^\diamond$, C_0, C_1, R and p. The sum over all the elements of the ω covering leads to

$$\|\nabla\overline{u}\|^p_{L^p(\Omega_\varepsilon)}\le C_N(C^{6\diamond})^p\|e(u)\|^p_{L^p(\Omega_\varepsilon)}.$$

This completes the proof of Theorem 4.2. □

5 Korn's type inequality

Denote

$$W^{1,p}_{\Gamma_{0,\varepsilon}}(\Omega_\varepsilon) \doteq \Big\{ v \in W^{1,p}(\Omega_\varepsilon) \mid v = 0 \quad \text{a.e. on } \Gamma_{0,\varepsilon} \Big\},$$
$$W^{1,p}_{\gamma_0}(\omega) \doteq \Big\{ V \in W^{1,p}(\omega) \mid V = 0 \quad \text{a.e. on } \gamma_0 \Big\}.$$

Let u be a displacement belonging to $W^{1,p}_{\Gamma_{0,\varepsilon}}(\Omega_\varepsilon)^3$ decomposed as (4.1). The boundary conditions imply

$$\mathcal{U} = 0, \quad \mathcal{R} = 0 \quad \text{a.e. on } \gamma_0 \quad \text{and} \quad \overline{u} = 0 \quad \text{a.e. on } \Gamma_{0,\varepsilon}.$$

Theorem 5.1. *For every displacement $u \in W^{1,p}_{\Gamma_{0,\varepsilon}}(\Omega_\varepsilon)^3$ decomposed as (4.1) we have*

$$\begin{aligned}
&\|\mathcal{U}_m\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1/p}} \|e(u)\|_{L^p(\Omega_\varepsilon)}, \\
&\big\|\rho_\varepsilon^{1/p}\mathcal{R}\big\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}} \|e(u)\|_{L^p(\Omega_\varepsilon)}, \quad \big\|\rho_\varepsilon^{1/p}\nabla\mathcal{U}_3\big\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}} \|e(u)\|_{L^p(\Omega_\varepsilon)}, \\
&\big\|\mathcal{U}_3\big\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}} \|e(u)\|_{L^p(\Omega_\varepsilon)}.
\end{aligned} \tag{5.1}$$

Moreover,

$$\begin{aligned}
&\|\rho_\varepsilon^{1/p}\,\nabla\mathcal{U}_m\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1/p}} \|e(u)\|_{L^p(\Omega_\varepsilon)} && \text{if } p \in (1,2), \\
&\|\rho_\varepsilon^{1/q}\,\nabla\mathcal{U}_m\|_{L^2(\omega)} \le \frac{C}{\varepsilon^{1/2}} \|e(u)\|_{L^2(\Omega_\varepsilon)} && \text{if } p = 2, \quad q \in (1,2), \\
&\|\nabla\mathcal{U}_m\|_{L^q(\omega)} \le \frac{C}{\varepsilon^{1/p}} \|e(u)\|_{L^p(\Omega_\varepsilon)} && \text{if } p \in (2,\infty), \quad q \in (1,p/2),
\end{aligned} \tag{5.2}$$

In addition, we have

$$\begin{aligned}
&\|u_1\|_{L^p(\Omega_\varepsilon)} + \|u_2\|_{L^p(\Omega_\varepsilon)} + \varepsilon\|u_3\|_{L^p(\Omega_\varepsilon)} \le C\|e(u)\|_{L^p(\Omega_\varepsilon)}, \\
&\sum_{i=1}^{3} \big\|\partial_i u_i\big\|_{L^p(\Omega_\varepsilon)} + \sum_{i=1}^{2} \varepsilon\big(\big\|\partial_3 u_i\big\|_{L^p(\Omega_\varepsilon)} + \big\|\partial_i u_3\big\|_{L^p(\Omega_\varepsilon)}\big) \le C\|e(u)\|_{L^p(\Omega_\varepsilon)}
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\big\|\partial_2 u_1\big\|_{L^p(\Omega_\varepsilon)} + \big\|\partial_1 u_2\big\|_{L^p(\Omega_\varepsilon)} \le C\|e(u)\|_{L^p(\Omega_\varepsilon)} && \text{if } p \in (1,2), \\
&\big\|\partial_2 u_1\big\|_{L^q(\Omega_\varepsilon)} + \big\|\partial_1 u_2\big\|_{L^q(\Omega_\varepsilon)} \le C\|e(u)\|_{L^p(\Omega_\varepsilon)} && \text{if } p \in [2,\infty), \ \ q \in (1,\max\{2,p/2\}).
\end{aligned} \tag{5.4}$$

The constants do not depend on ε[3].

Proof. Step 1. Preliminary estimates.
First of all, since $\rho_\varepsilon \ge \varepsilon$, from (4.4) we obtain

$$\begin{aligned}
&\|e_{\alpha\beta}(\mathcal{U}_m)\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{2/p}} \|e(u)\|_{L^p(\Omega_\varepsilon)}, \\
&\big\|\nabla\mathcal{U}_3 + \mathcal{R}\big\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{2/p}} \|e(u)\|_{L^p(\Omega_\varepsilon)} \\
&\|\rho_\varepsilon \nabla\mathcal{R}\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+2/p}} \|e(u)\|_{L^p(\Omega_\varepsilon)}.
\end{aligned} \tag{5.5}$$

Then, the above estimate $(5.5)_1$, the $2D$-Korn inequality and the fact that $\mathcal{U}_m$ vanishes on γ_0 yield

$$\|\mathcal{U}_m\|_{W^{1,p}(\omega)} \le C\|e(\mathcal{U}_m)\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{2/p}} \|e(u)\|_{L^p(\Omega_\varepsilon)}. \tag{5.6}$$

[3] The constants of the estimates (5.2) and (5.4) depend on p and q.

Observe that $(1-\varepsilon)\rho_\gamma = \rho_\varepsilon - \varepsilon$ and $0 \le \rho_\varepsilon - \varepsilon \le \rho_\varepsilon$, then from estimate $(4.4)_3$ and the fact that $0 < 1-\varepsilon_0 \le 1-\varepsilon \le 1$ we obtain

$$(1-\varepsilon_0)\|\rho_\gamma \nabla \mathcal{R}\|_{L^p(\omega)} \le (1-\varepsilon)\|\rho_\gamma \nabla \mathcal{R}\|_{L^p(\omega)} \le \|\rho_\varepsilon \nabla \mathcal{R}\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+2/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

This above estimate together with (8.16) lead to

$$\|\mathcal{R}\|_{L^p(\omega)} \le C\|\rho_\gamma \nabla \mathcal{R}\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+2/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}. \tag{5.7}$$

Now, from this inequality, $(5.5)_2$ and then the Poincaré inequality we obtain

$$\|\mathcal{U}_3\|_{W^{1,p}(\omega)} \le \frac{C}{\varepsilon^{1+2/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}. \tag{5.8}$$

Step 2. We prove $(5.1)_1$.
Proceeding as in Step 1 and using the estimate $(4.4)_1$, we obtain

$$(1-\varepsilon_0)^{1/p}\|\rho_\gamma^{1/p}\, e(\mathcal{U}_m)\|_{L^p(\omega)} \le (1-\varepsilon)^{1/p}\|\rho_\gamma^{1/p}\, e(\mathcal{U}_m)\|_{L^p(\omega)} \le \|\rho_\varepsilon^{1/p}\, e(\mathcal{U}_m)\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

Then, Corollary 8.9 in Appendix gives

$$\|\mathcal{U}_m\|_{L^p(\omega)} \le C\|\rho_\gamma^{1/p}\, e(\mathcal{U}_m)\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}. \tag{5.9}$$

So, $(5.1)_1$ is proved.
Step 3. We prove $(5.1)_{2,3,4}$.
We have

$$(1-\varepsilon)^{1+1/p}\big\|\rho_\gamma^{1+1/p}\nabla\mathcal{R}\big\|_{L^p(\omega)} = \big\|(\rho_\varepsilon-\varepsilon)^{1+1/p}\nabla\mathcal{R}\big\|_{L^p(\omega)}.$$

Then, $(4.4)_3$ yields

$$(1-\varepsilon_0)^{1+1/p}\big\|\rho_\gamma^{1+1/p}\nabla\mathcal{R}\big\|_{L^p(\omega)} \le \big\|\rho_\varepsilon^{1+1/p}\nabla\mathcal{R}\big\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

The above estimate and the one below from (8.14) in Appendix

$$\big\|\rho_\gamma^{1/p}\mathcal{R}\big\|_{L^p(\omega)} \le C\big\|\rho_\gamma^{1+1/p}\nabla\mathcal{R}\big\|_{L^p(\omega)} \tag{5.10}$$

lead to

$$\big\|\rho_\gamma^{1/p}\mathcal{R}\big\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

Thus

$$\big\|(\rho_\varepsilon-\varepsilon)^{1/p}\mathcal{R}\big\|_{L^p(\omega)} = \big\|(1-\varepsilon)^{1/p}\rho_\gamma^{1/p}\mathcal{R}\big\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}$$

which, in turn, with (5.7) and the fact that $\rho_\varepsilon^{1/p} \le (\rho_\varepsilon-\varepsilon)^{1/p} + \varepsilon^{1/p}$ give $(5.1)_2$.
Now, from $(4.4)_2$ and $(5.1)_2$ we get

$$\|\rho_\varepsilon^{1/p}\nabla\mathcal{U}_3\|_{L^p(\omega)} \le \big\|\rho_\varepsilon^{1/p}\mathcal{R}\big\|_{L^p(\omega)} + \big\|\rho_\varepsilon^{1/p}\big(\nabla\mathcal{U}_3+\mathcal{R}\big)\big\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}$$

and then

$$\|\rho_\gamma^{1/p}\nabla\mathcal{U}_3\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

This estimate, those of (5.8) and (8.16) in Appendix yield

$$\|\rho_\varepsilon^{1/p}\nabla\mathcal{U}_3\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}, \qquad \|\mathcal{U}_3\|_{L^p(\omega)} \le C\|\rho_\gamma^{1/p}\nabla\mathcal{U}_3\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

As a consequence, $(5.1)_{3,4}$ are proved.

Step 4. We prove (5.2).

• if $p \in (1,2)$ we choose $s = -\dfrac{1}{p'}$ then, Lemma 8.8 gives the estimate $(5.2)_1$,

• if $p = 2$, we choose $s = \dfrac{1}{q} - 1$ with $q \in (1,2)$. Then we have $2s+1 = \dfrac{2}{q} - 1 > 0$ and $s+1 = \dfrac{1}{q}$. Due to $(8.18)_2$ and $(5.5)_1$ we obtain

$$\|\rho_\gamma^{1/q}\nabla\mathcal{U}_m\|_{L^2(\omega)} \le C\|\rho_\gamma^{1/q}e(\mathcal{U}_m)\|_{L^2(\omega)},$$
$$\|\rho_\gamma^{1/q}e(\mathcal{U}_m)\|_{L^2(\omega)} \le \|\rho_\gamma^{1/2}e(\mathcal{U}_m)\|_{L^2(\omega)} \le C\|\rho_\varepsilon^{1/2}e(\mathcal{U}_m)\|_{L^2(\omega)} \le \frac{C}{\varepsilon^{1/2}}\|e(u)\|_{L^2(\Omega_\varepsilon)}$$

This proves $(5.2)_2$ with a constant which depends on q.

• if $p \in (2,\infty)$, Lemma 8.7 with $s = \dfrac{1}{p}$ implies that for any $q \in (1, p/2)$ one has

$$\|e_{\alpha\beta}(\mathcal{U}_m)\|_{L^q(\omega)} \le C\|\rho_\gamma^{1/p}e(\mathcal{U}_m)\|_{L^p(\omega)} \le C\|\rho_\varepsilon^{1/p}e(\mathcal{U}_m)\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

The constant depends on p and q and is independent of ε. The $2D$ Korn inequality, the above inequality and $(5.1)_1$ lead to

$$\|\nabla\mathcal{U}_m\|_{L^q(\omega)} \le C\big(\|e(\mathcal{U}_m)\|_{L^q(\omega)} + \|\mathcal{U}_m\|_{L^q(\omega)}\big) \le \frac{C}{\varepsilon^{1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

This proves $(5.2)_3$ with a constant which depends on p and q.

Step 5. We prove (5.3) and (5.4).

Estimates (5.3) (resp. (5.4)) are the consequences of those in (4.4) and (5.1) (resp. (5.2)). □

Remark 5.2. *See also Remark 8.11. If ω is a polygonal domain and if γ has a finite number of connected open components then we have*

$$\|\rho_\gamma^{1/p}\nabla\mathcal{U}_m\|_{L^p(\omega)} \le C\|\rho_\gamma^{1/p}e(\mathcal{U}_m)\|_{L^p(\omega)} \quad\Longrightarrow\quad \|\rho_\varepsilon^{1/p}\nabla\mathcal{U}_m\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

Moreover, we have the following Korn type inequalities:

$$\|u_1\|_{L^p(\Omega_\varepsilon)} + \|u_2\|_{L^p(\Omega_\varepsilon)} + \varepsilon\|u_3\|_{L^p(\Omega_\varepsilon)} \le C\|e(u)\|_{L^p(\Omega_\varepsilon)},$$
$$\sum_{i=1}^{3}\big\|\partial_i u_i\big\|_{L^p(\Omega_\varepsilon)} + \sum_{i=1}^{2}\varepsilon\big(\big\|\partial_3 u_i\big\|_{L^p(\Omega_\varepsilon)} + \big\|\partial_i u_3\big\|_{L^p(\Omega_\varepsilon)}\big) \le C\|e(u)\|_{L^p(\Omega_\varepsilon)},$$
$$\big\|\partial_2 u_1\big\|_{L^p(\Omega_\varepsilon)} + \big\|\partial_1 u_2\big\|_{L^p(\Omega_\varepsilon)} \le C\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

The constants do not depend on ε.

Theorem 5.3. *If $\gamma = \partial\omega$ then, for every displacement $u \in W^{1,p}(\Omega_\varepsilon)^3$ there exits a rigid displacement $\mathbf{r}$ such that*

$$\|u_1 - \mathbf{r}_1\|_{L^p(\Omega_\varepsilon)} + \|u_2 - \mathbf{r}_2\|_{L^p(\Omega_\varepsilon)} + \varepsilon\|u_3 - \mathbf{r}_3\|_{L^p(\Omega_\varepsilon)} \le C\|e(u)\|_{L^p(\Omega_\varepsilon)},$$
$$\sum_{i=1}^{3}\big\|\partial_i u_i\big\|_{L^p(\Omega_\varepsilon)} + \big\|\partial_2(u_1 - \mathbf{r}_1)\big\|_{L^p(\Omega_\varepsilon)} + \big\|\partial_1(u_2 - \mathbf{r}_1)\big\|_{L^p(\Omega_\varepsilon)} \le C\|e(u)\|_{L^p(\Omega_\varepsilon)},$$
$$\sum_{i=1}^{2}\big(\big\|\partial_3(u_i - \mathbf{r}_i)\big\|_{L^p(\Omega_\varepsilon)} + \big\|\partial_i(u_3 - \mathbf{r}_3)\big\|_{L^p(\Omega_\varepsilon)}\big) \le \frac{C}{\varepsilon}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

The constants do not depend on ε.

Proof. First, we decompose u as (4.1). The terms $\mathcal{U}$, $\mathcal{R}$ satisfy (4.4). Then, Proposition 8.13 with $s = \dfrac{1}{p}$ gives a rigid $2D$ displacement $\mathbf{R}$ ($\mathbf{R}(x') = (a_1 - b_3 x_2)\mathbf{e}_1 + (a_2 + b_3 x_1)\mathbf{e}_2$) such that

$$\|\mathcal{U}_m - \mathbf{R}\|_{L^p(\omega)} \le C\|\rho^{1/p}\, e(\mathcal{U}_m)\|_{L^p(\omega)},$$
$$\|\rho^{1/p}\, \nabla(\mathcal{U}_m - \mathbf{R})\|_{L^p(\omega)} \le C\|\rho^{1/p}\, e(\mathcal{U}_m\|_{L^p(\omega)}.$$

As in the proof of Theorem 5.1 (see Steps 1-2), we show that

$$\|\mathcal{U}_m - \mathbf{R}\|_{L^p(\omega)} + \|\rho_\varepsilon^{1/p}\, \nabla(\mathcal{U}_m - \mathbf{R})\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1/p}}\|e(u)\|_{L^p(\omega)}.$$

The Poincaré-Wirtinger inequality, given in Lemma 8.3, provides two constant b_2 and b_3 such that

$$\big\|\rho^{1/p}(\mathcal{R}_\alpha - b_\alpha)\big\|_{L^p(\omega)} \le C\|\rho^{1+1/p}\nabla\mathcal{R}_\alpha\|_{L^p(\omega)}.$$

As in the proof of Theorem 5.1 (see Step 3), we obtain

$$\big\|\rho_\varepsilon^{1/p}(\mathcal{R}_\alpha - b_\alpha)\big\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}, \quad \big\|\rho_\varepsilon^{1/p}(\partial_\alpha\mathcal{U}_3 + b_\alpha)\big\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

The Poincaré-Wirtinger inequality provides a constant a_3 such that

$$\|\mathcal{U}_3 - (a_3 - b_1 x_1 - b_2 x_2)\|_{L^p(\omega)} \le C\|\rho_\gamma^{1/p}\nabla\mathcal{U}_3\|_{L^p(\omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u)\|_{L^p(\Omega_\varepsilon)}.$$

Set

$$\mathbf{r}(x) = (a_1 - b_3 x_2 + b_1 x_3)\mathbf{e}_1 + (a_2 + b_3 x_1 + b_2 x_3)\mathbf{e}_1 + (a_3 - b_1 x_1 - b_2 x_2)\mathbf{e}_3.$$

The estimates above and the last two in (4.4) lead to those of the theorem with the rigid displacement above. □

6 Asymptotic behavior of a sequence of displacements

First, we recall the definition of the dimension reduction operator.

Definition 6.1. *For ϕ measurable function on Ω_ε, the dimension reduction operator $\Pi_\varepsilon(\phi)$ is defined as follows:*

$$\Pi_\varepsilon(\phi)(x', X_3) = \phi(x', \varepsilon X_3) \qquad \text{for a.e. } (x', X_3) \in \Omega.$$

$\Pi_\varepsilon(\phi)$ is a measurable function on Ω.

Observe that for all $(x', X_3) \in \Omega$ we have $-\varepsilon\rho_\gamma(x')\psi(x') < \varepsilon X_3 < \varepsilon\rho_\gamma(x')\phi(x')$. So, since $0 \le \rho_\gamma \le \rho_\varepsilon$ we get $(x', \varepsilon X_3) \in \Omega_\varepsilon$. Therefore, the above definition makes sense.
We easily check that

1. for any $\Phi \in L^1(\Omega_\varepsilon)$ such that $\Phi \ge 0$ a.e. in Ω_ε, we have

$$\int_\Omega \Pi_\varepsilon(\Phi)dx'dX_3 \le \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\Phi\, dx. \tag{6.1}$$

2. for any $\Phi \in L^p(\Omega_\varepsilon)$ and $\phi \in L^p(\omega)$, $1 \le p \le \infty$

$$\begin{gathered}\|\Pi_\varepsilon(\Phi)\|_{L^p(\Omega)} \le \frac{1}{(1-\varepsilon_0)^{1/p}}\frac{1}{\varepsilon^{1/p}}\|\Phi\|_{L^p(\Omega_\varepsilon)},\\ \Pi_\varepsilon(\phi) = \phi \quad \text{and} \quad (4C_0)^{1/p}\|\rho_\gamma^{1/p}\phi\|_{L^p(\omega)} \le \|\Pi_\varepsilon(\phi)\|_{L^p(\Omega)} \le (2C_1)^{1/p}\|\rho_\gamma^{1/p}\phi\|_{L^p(\omega)},\end{gathered} \tag{6.2}$$

3. for any $\Phi \in W^{1,p}(\Omega_\varepsilon)$, $1 \le p \le \infty$

$$\partial_\alpha \Pi_\varepsilon(\Phi) = \Pi_\varepsilon\big(\partial_\alpha \Phi\big), \qquad \frac{\partial \Pi_\varepsilon(\Phi)}{\partial X_3} = \varepsilon \Pi_\varepsilon\big(\partial_3 \Phi\big). \tag{6.3}$$

Let u be a displacement belonging to $W^{1,p}(\Omega_\varepsilon)$, decomposed as (4.1).
The strain tensor of u is given by the following 3×3 symmetric matrix defined a.e. in Ω_ε by

$$e(u) = \begin{pmatrix} e_{11}(\mathcal{U}_m) + x_3 \partial_1 \mathcal{R}_1 + e_{11}(\overline{u}) & * & * \\ e_{12}(\mathcal{U}_m) + \dfrac{x_3}{2}\big(\partial_2 \mathcal{R}_1 + \partial_1 \mathcal{R}_2\big) + e_{12}(\overline{u}) & e_{22}(\mathcal{U}_m) + x_3 \partial_2 \mathcal{R}_2 + e_{22}(\overline{u}) & * \\ \dfrac{1}{2}\big(\partial_1 \mathcal{U}_3 + \mathcal{R}_1\big) + e_{13}(\overline{u}) & \dfrac{1}{2}\big(\partial_2 \mathcal{U}_3 + \mathcal{R}_2\big) + e_{23}(\overline{u}) & e_{33}(\overline{u}) \end{pmatrix} \tag{6.4}$$

where $\mathcal{U}_m = \mathcal{U}_1 \mathbf{e}_1 + \mathcal{U}_2 \mathbf{e}_2$.
First, we introduce the spaces

$$\begin{aligned} \mathbf{Warp}_p(\Omega) &\doteq \Big\{ \widehat{\Phi} \in L^p(\Omega)^3 \mid \frac{\partial \widehat{\Phi}}{\partial X_3} \in L^p(\Omega)^3 \Big\}, \\ \mathfrak{W}_p(\Omega) &\doteq \Big\{ \widehat{\Phi} \in \mathbf{Warp}_p(\Omega) \ \mid \ \int_{-\rho_\gamma \psi}^{\rho_\gamma \phi} \widehat{\Phi}(\cdot, X_3) dX_3 = 0 \quad \text{a.e. in } \omega \Big\}. \end{aligned}$$

We equip $\mathfrak{W}_p(\Omega)$ with the semi-norm

$$\|\widehat{\Phi}\|_p \doteq \Big\| \frac{\partial \widehat{\Phi}}{\partial X_3} \Big\|_{L^p(\Omega)}, \qquad \forall \widehat{\Phi} \in \mathfrak{W}_p(\Omega).$$

This semi-norm is equivalent to the usual norm of $\mathbf{Warp}_p(\Omega)$. Indeed the Poincaré-Wirtinger inequality implies

$$\int_{-\rho_\gamma \psi}^{\rho_\gamma \phi} |\widehat{\Phi}(\cdot X_3)|^p dX_3 \le C |\rho_\gamma|^p \int_{-\rho_\gamma \psi}^{\rho_\gamma \phi} \Big| \frac{\partial \widehat{\Phi}}{\partial X_3}(\cdot, X_3) \Big|^p dX_3 \qquad \text{a.e. in } \ \omega$$

from which we deduce that

$$\|\widehat{\Phi}\|_{L^p(\Omega)} \le \Big\| \frac{\widehat{\Phi}}{\rho_\gamma} \Big\|_{L^p(\Omega)} \le C \Big\| \frac{\partial \widehat{\Phi}}{\partial X_3} \Big\|_{L^p(\Omega)}, \qquad \forall \widehat{\Phi} \in \mathfrak{W}_p(\Omega).$$

The above estimate also shows that $\dfrac{\widehat{\Phi}}{\rho_\gamma}$ belongs to $L^p(\Omega)^3$.
Now, for $p \in (1, \infty)$ and $s \in \mathbb{R}$, we denote

$$\begin{aligned} L^p_s(\omega) &\doteq \Big\{ \phi \in L^p_{loc}(\omega) \mid \rho_\gamma^s \phi \in L^p(\omega) \Big\}, \\ W^{1,p}_s(\omega) &\doteq \Big\{ \phi \in W^{1,p}_{loc}(\omega) \ \mid \ \rho_\gamma^s \nabla \phi \in L^p(\omega)^2 \Big\}, \\ M^{1,p}_{1/p}(\omega) &\doteq \Big\{ \Phi \in W^{1,p}_{loc}(\omega)^2 \mid \rho_\gamma^{1/p} e(\Phi) \in L^p(\omega)^4 \Big\}, \\ W^{1,p}_{1+1/p}(\omega) &\doteq \Big\{ \phi \in L^p_{1/p}(\omega) \ \mid \ \rho_\gamma^{1+1/p} \nabla \phi \in L^p(\omega)^2 \Big\}, \\ W^{2,p}_{1+1/p}(\omega) &\doteq \Big\{ \phi \in L^p(\omega) \ \mid \ \rho_\gamma^{1/p} \nabla \phi \in L^p(\omega)^2, \ \ \rho_\gamma^{1+1/p}\, \mathrm{H}(\phi) \in L^p(\omega)^4 \Big\} \end{aligned} \tag{6.5}$$

where $H(\phi)$ is the Hessian matrix of ϕ and

$$\begin{aligned}
\mathbb{W}^{1,p}_{s}(\omega) &\doteq \Big\{\phi \in L^p(\omega) \mid \rho_\gamma^s \nabla\phi \in L^p(\omega)^2, \quad \phi = 0 \quad \text{a.e. on } \gamma_0\Big\},\\
\mathbb{M}^{1,p}_{1/p}(\omega) &\doteq \Big\{\Phi \in M^{1,p}_{1/p}(\omega) \ \mid \ \Phi = 0 \ \text{ a.e. on } \gamma_0\Big\},\\
\mathbb{W}^{1,p}_{1+1/p}(\omega) &\doteq \Big\{\phi \in W^{1,p}_{1+1/p}(\omega) \ \mid \ \phi = 0 \ \text{ a.e. on } \gamma_0\Big\},\\
\mathbb{W}^{2,p}_{1+1/p}(\omega) &\doteq \Big\{\phi \in W^{2,p}_{1+1/p}(\omega) \ \mid \ \phi = 0, \quad \nabla\phi = 0 \ \text{ a.e. on } \gamma_0\Big\}.
\end{aligned} \tag{6.6}$$

Note that in a neighborhood of γ_0 in ω the functions belonging to $W^{1,p}_s(\omega)$ (resp. $M^{1,p}_{1/p}(\omega)$, $W^{1,p}_{1+1/p}(\omega)$, $W^{2,p}_{1+1/p}(\omega)$) belong to the space $W^{1,p}$ (resp. $(W^{1,p})^2$, $W^{1,p}$, $W^{2,p}$) of this neighborhood. Concequently, the spaces introduced in (6.6) make sense.
Note also that if Φ belongs to $\mathbb{M}^{1,p}_{1/p}(\omega)$ then, proceeding as in the proof of Theorem 5.1 (Step 4) we obtain

$$\mathbb{M}^{1,p}_{1/p}(\omega) \subset \begin{cases} \mathbb{W}^{1,p}_{1/p}(\omega)^2 & \text{if } \ p \in (1,2),\\ \mathbb{W}^{1,p}_{1/q}(\omega)^2 & \text{if } \ p = 2, \quad q \in (1,2),\\ W^{1,q}(\omega)^2 & \text{if } \ p \in (2,\infty), \quad q \in (1, p/2). \end{cases}$$

Lemma 6.2. *We have*

$$\begin{aligned}
&L^p_{1/p}(\omega) \subset L^p(\Omega), \qquad W^{1,p}_{1/p}(\omega) \subset W^{1,p}(\Omega),\\
&W^{1,p}_{1+1/p}(\omega) \subset \big\{\phi \in L^p(\Omega) \mid \rho_\gamma \nabla\phi \in L^p(\Omega)^2\big\},\\
&W^{2,p}_{1+1/p}(\omega) \subset \big\{\phi \in W^{1,p}(\Omega) \mid \rho_\gamma H(\phi) \in L^p(\Omega)^4\big\}.
\end{aligned} \tag{6.7}$$

We also have

$$M^{1,p}_{1/p}(\omega) \subset W^{1,q}(\Omega)^2 \quad \textit{where} \quad \begin{cases} q = p & \textit{if } \ p \in (1,2),\\ q \in (1, \max\{2, p/2\}) & \textit{if } \ p \in [2,\infty). \end{cases} \tag{6.8}$$

Proof. The inclusions given by (6.7) are the consequences of the definition of these spaces and inequalities (6.2).
If $p \in (1,2)$ then, the $(6.8)_1$ comes from the equality $M^{1,p}_{1/p}(\omega) = W^{1,p}_{1/p}(\omega)^2$ (see Lemma 8.8) and $(6.7)_2$.
If $p \in [2,\infty)$ then, for any ϕ in $W^{1,p}_{1/q}(\omega)$, the Hölder inequality gives

$$\|\rho_\gamma^{1/q} \nabla\phi\|_{L^q(\omega)} \le |\omega|^{\frac{1}{q}-\frac{1}{p}} \|\rho_\gamma^{1/q} \nabla\phi\|_{L^p(\omega)}.$$

So ϕ belongs to $W^{1,q}_{1/q}(\omega)$ and therefore to $W^{1,q}(\Omega)$. This ends the proof of (6.8). □

For every $\big(\Phi, \widehat{\Phi}\big) \in \mathbb{M}^{1,p}_{1/p}(\omega) \times \mathbb{W}^{2,p}_{1+1/p}(\omega) \times L^p(\omega; W^{1,p}(-1,1))^3$ we denote

$$E(\Phi, \widehat{\Phi}) = \begin{pmatrix} e_{11}(\Phi_m) - X_3\partial^2_{11}\Phi_3 & * & *\\ e_{12}(\Phi_m) - X_3\partial_{12}\Phi_3 & e_{22}(\Phi_m) - X_3\partial^2_{22}\Phi_3 & *\\ \dfrac{1}{2}\dfrac{\partial\widehat{\Phi}_1}{\partial X_3} & \dfrac{1}{2}\dfrac{\partial\widehat{\Phi}_2}{\partial X_3} & \dfrac{\partial\widehat{\Phi}_3}{\partial X_3} \end{pmatrix} \tag{6.9}$$

where $\Phi_m = \Phi_1\mathbf{e}_1 + \Phi_2\mathbf{e}_2$, $\Phi = \Phi_m + \Phi_3\mathbf{e}_3 = \big(\Phi_m, \Phi_3\big)$.

Note that we have

$$\forall\big(\Phi,\widehat{\Phi}\big)\in\mathbb{M}^{1,p}_{1/p}(\omega)\times\mathbb{W}^{2,p}_{1+1/p}(\omega)\times L^p(\omega;W^{1,p}(-1,1))^3,$$

$$C_0'\Big(\|\rho_\gamma^{1/p}e(\Phi_m)\|_{L^p(\omega)}+\big\|\rho_\gamma^{1+1/p}\nabla\Phi_3\big\|_{L^p(\omega)}+\Big\|\frac{\partial\widehat{\Phi}}{\partial X_3}\Big\|_{L^p(\Omega)}\Big)\le\|E(\Phi,\widehat{\Phi})\|_{L^p(\Omega)}\tag{6.10}$$

$$\le C_1'\Big(\|\rho_\gamma^{1/p}e(\Phi_m)\|_{L^p(\omega)}+\big\|\rho_\gamma^{1+1/p}\nabla\Phi_3\big\|_{L^p(\omega)}+\Big\|\frac{\partial\widehat{\Phi}}{\partial X_3}\Big\|_{L^p(\Omega)}\Big).$$

Theorem 6.3. *Let $\{u_\varepsilon\}_\varepsilon$ be a sequence of displacements belonging to $W^{1,p}_{\Gamma_{0,\varepsilon}}(\Omega_\varepsilon)$, decomposed as (4.1) and satisfying*

$$\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)}\le C\varepsilon^{2+1/p}\tag{6.11}$$

where the constant does not depend on ε.
Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$ and $\mathcal{U}_m=\mathcal{U}_1\mathbf{e}_1+\mathcal{U}_2\mathbf{e}_2\in\mathbb{M}^{1,p}_{1/p}(\omega)$, $\mathcal{R}\in\mathbb{W}^{1,p}_{1+1/p}(\omega)^2$, $\mathcal{U}_3\in\mathbb{W}^{2,p}_{1+1/p}(\omega)$ and $\widehat{U}\in\mathfrak{W}_p(\Omega)$ such that

$$\begin{aligned}\frac{1}{\varepsilon^2}\mathcal{U}_{m,\varepsilon}&\rightharpoonup\mathcal{U}_m\quad\textit{weakly in}\ \ \mathbb{M}^{1,p}_{1/p}(\omega),\\ \frac{1}{\varepsilon}\mathcal{U}_{3,\varepsilon}&\rightharpoonup\mathcal{U}_3\quad\textit{weakly in}\ \ \mathbb{W}^{1,p}_{1/p}(\omega),\\ \frac{1}{\varepsilon}\mathcal{R}_{\varepsilon}&\rightharpoonup\mathcal{R}\quad\textit{weakly in}\ \ \mathbb{W}^{1,p}_{1+1/p}(\omega)^2.\end{aligned}\tag{6.12}$$

$\mathcal{U}_3$ and $\mathcal{R}$ are linked by the following condition:

$$\nabla\mathcal{U}_3=-\mathcal{R}\quad\textit{a.e. in }\omega.\tag{6.13}$$

We also have

$$\begin{aligned}&\frac{1}{\varepsilon^2}\Pi_\varepsilon(u_{\alpha,\varepsilon})\rightharpoonup\mathcal{U}_1-X_3\partial_\alpha\mathcal{U}_3\quad\textit{weakly in}\ \ W^{1,q}(\Omega),\ \ \textit{where}\begin{cases}q=p & \textit{if }\ p\in(1,2),\\ q\in(1,\max\{2,p/2\}) & \textit{if }\ p\in[2,\infty).\end{cases}\\ &\frac{1}{\varepsilon}\Pi_\varepsilon(u_{3,\varepsilon})\rightharpoonup\mathcal{U}_3\quad\textit{weakly in}\ \ W^{1,p}(\Omega)\end{aligned}\tag{6.14}$$

and

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(e(u_\varepsilon)\big)\rightharpoonup E\big(\mathcal{U},\widehat{U}\big)\quad\textit{weakly in}\ \ L^p(\Omega)^{3\times3}\tag{6.15}$$

where $\mathcal{U}=(\mathcal{U}_m,\mathcal{U}_3)$.

Proof. Step 1. We prove (6.12) and (6.13).
First, as mention above we have $(1-\varepsilon_0)\rho_\gamma\le(1-\varepsilon)\rho_\gamma\le\rho_\varepsilon$ in ω. Hence, estimates (4.4)-(5.1) and (6.11) give

$$\begin{aligned}\|\mathcal{U}_{m,\varepsilon}\|_{L^p(\omega)}+\|\rho_\gamma^{1/p}e(\mathcal{U}_{m,\varepsilon})\|_{L^p(\omega)}&\le\frac{C}{\varepsilon^{1/p}}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)}\le C\varepsilon^2,\\ \|\mathcal{U}_{3,\varepsilon}\|_{L^p(\omega)}+\|\rho_\gamma^{1/p}\nabla\mathcal{U}_{3,\varepsilon}\|_{L^p(\omega)}&\le\frac{C}{\varepsilon^{1+1/p}}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)}\le C\varepsilon,\\ \|\rho_\gamma^{1/p}\mathcal{R}_\varepsilon\|_{L^p(\omega)}+\|\rho_\gamma^{1+1/p}\nabla\mathcal{R}_\varepsilon\|_{L^p(\omega)}&\le\frac{C}{\varepsilon^{1+1/p}}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)}\le C\varepsilon\\ \|\rho_\gamma^{1/p}(\nabla\mathcal{U}_{3,\varepsilon}+\mathcal{R}_\varepsilon)\|_{L^p(\omega)}&\le\frac{C}{\varepsilon^{1/p}}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)}\le C\varepsilon^2.\end{aligned}\tag{6.16}$$

We also have (see (5.2))

$$\begin{aligned}\|\rho_\varepsilon^{1/p}\,\nabla\mathcal{U}_{m,\varepsilon}\|_{L^p(\omega)}&\le C\varepsilon^2\qquad&&\text{if }\ p\in(1,2),\\ \|\rho_\varepsilon^{1/q}\,\nabla\mathcal{U}_{m,\varepsilon}\|_{L^2(\omega)}&\le C\varepsilon^2\qquad&&\text{if }\ p=2,\quad q\in(1,2),\\ \|\nabla\mathcal{U}_{m,\varepsilon}\|_{L^q(\omega)}&\le C\varepsilon^2\qquad&&\text{if }\ p\in(2,\infty),\quad q\in(1,p/2).\end{aligned}\tag{6.17}$$

Then, up to a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, we have the convergences (6.12) and $\nabla\mathcal{U}_3 = -\mathcal{R}$ a.e. in ω. So, $\mathcal{U}_3$ belongs to $\mathbb{W}^{2,p}_{1+1/p}(\omega)$.

From the above estimate $(6.16)_4$, we also deduce that there exists $\mathcal{Z} \in L^p_{1/p}(\omega)^2$ (it also belongs to $L^p(\Omega)^2$) satisfying the following equivalent convergences:

$$\begin{aligned} &\frac{1}{\varepsilon^2}\rho_\gamma^{1/p}(\nabla\mathcal{U}_{3,\varepsilon} + \mathcal{R}_\varepsilon) \rightharpoonup \rho_\gamma^{1/p}\mathcal{Z} \quad \text{weakly in } \ L^p(\omega)^2, \\ &\frac{1}{\varepsilon^2}\Pi_\varepsilon(\nabla\mathcal{U}_{3,\varepsilon} + \mathcal{R}_\varepsilon) \rightharpoonup \mathcal{Z} \quad \text{weakly in } \ L^p(\Omega)^2. \end{aligned} \tag{6.18}$$

Step 2. We prove (6.14).

From the estimates $(4.4)_{4,5}$ and the properties (6.2)-(6.3) of the operator Π_ε, we obtain

$$\begin{aligned} &\Big\|\frac{1}{\rho_\varepsilon}\Pi_\varepsilon(\overline{u}_\varepsilon)\Big\|_{L^p(\Omega)} \le C\varepsilon^{1-1/p}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \le C\varepsilon^3, \\ &\|\Pi_\varepsilon(\overline{u}_\varepsilon)\|_{L^p(\Omega)} \le C\varepsilon^{1-1/p}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \le C\varepsilon^3, \\ &\big\|\partial_\alpha\Pi_\varepsilon\big(\overline{u}_\varepsilon\big)\big\|_{L^p(\Omega)} \le \frac{C}{\varepsilon^{1/p}}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \le C\varepsilon^2, \\ &\Big\|\frac{\partial\Pi_\varepsilon(\overline{u}_\varepsilon)}{\partial X_3}\Big\|_{L^p(\Omega)} \le \frac{C}{\varepsilon^{1/p}}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \le C\varepsilon^3. \end{aligned}$$

Then, up to a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, there exists $\overline{U} \in L^p(\omega; W^{1,p}(-1,1))^3$ such that

$$\begin{aligned} &\frac{1}{\varepsilon^3}\Pi_\varepsilon(\overline{u}_\varepsilon) \rightharpoonup \overline{U} \quad \text{weakly in } \ \mathbf{Warp}_p(\Omega), \\ &\frac{1}{\varepsilon^2}\partial_\alpha\Pi_\varepsilon\big(\overline{u}_\varepsilon\big) \rightharpoonup 0 \quad \text{weakly in } \ L^p(\Omega)^3. \end{aligned} \tag{6.19}$$

Since the sequence $\Big\{\frac{1}{\rho_\varepsilon}\frac{1}{\varepsilon^3}\Pi_\varepsilon(\overline{u}_\varepsilon)\Big\}_\varepsilon$ is uniformly bounded in $L^p(\Omega)^3$ we also have

$$\frac{1}{\rho_\varepsilon}\frac{1}{\varepsilon^3}\Pi_\varepsilon(\overline{u}_\varepsilon) \rightharpoonup \frac{1}{\rho_\gamma}\overline{U} \quad \text{weakly in } \ L^p(\Omega)^3.$$

Now, the estimates (5.3) yield ($(\alpha,\beta) \in \{1,2\}^2$)

$$\begin{aligned} &\|\Pi_\varepsilon(u_{3,\varepsilon})\|_{L^p(\Omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \le C\varepsilon, \\ &\big\|\partial_\alpha\Pi_\varepsilon\big(u_{3,\varepsilon}\big)\big\|_{L^p(\Omega)} + \Big\|\frac{\partial\Pi_\varepsilon(\overline{u}_\varepsilon)}{\partial X_3}\Big\|_{L^p(\Omega)} \le \frac{C}{\varepsilon^{1+1/p}}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \le C\varepsilon. \end{aligned}$$

Then, the above convergences lead to those of $(6.14)_2$.

Besides, from (5.3) and (5.4) we have

$$\begin{aligned} &\|\Pi_\varepsilon(u_{\alpha,\varepsilon})\|_{L^p(\Omega)} \le \frac{C}{\varepsilon^{1/p}}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \le C\varepsilon^2, \\ &\big\|\partial_\alpha\Pi_\varepsilon\big(u_{\beta,\varepsilon}\big)\big\|_{L^q(\Omega)} + \Big\|\frac{\partial\Pi_\varepsilon(u_{\beta,\varepsilon})}{\partial X_3}\Big\|_{L^p(\Omega)} \le \frac{C}{\varepsilon^{1/p}}\|e(u_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \le C\varepsilon^2. \end{aligned}$$

So the above convergences (6.12), (6.18) and (6.19) lead to that of $(6.14)_1$.

Step 3. We prove (6.15).

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(e(u_\varepsilon)\big) \rightharpoonup \begin{pmatrix} e_{11}(\mathcal{U}_m) - X_3\partial^2_{11}\mathcal{U}_3 & * & * \\ e_{12}(\mathcal{U}_m) - X_3\partial^2_{12}\mathcal{U}_3 & e_{22}(\mathcal{U}_m) - X_3\partial^2_{22}\mathcal{U}_3 & * \\ \frac{1}{2}\Big(\mathcal{Z}_1 + \frac{\partial\overline{U}_1}{\partial X_3}\Big) & \frac{1}{2}\Big(\mathcal{Z}_2 + \frac{\partial\overline{U}_2}{\partial X_3}\Big) & \frac{\partial\overline{U}_3}{\partial X_3} \end{pmatrix} \quad \text{weakly in } \ L^p(\Omega)^{3\times 3}.$$

Set ($\alpha \in \{1,2\}$)

$$\begin{aligned}
&\widehat{U}_\alpha(\cdot,X_3) = X_3\mathcal{Z}_\alpha + \overline{U}_\alpha(\cdot,X_3) - \frac{1}{\rho_\gamma(\phi+\psi)}\int_{-\rho_\gamma\psi}^{\rho_\gamma\phi}\big(X_3\mathcal{Z}_\alpha + \overline{U}_\alpha(\cdot,X_3)\big)dX_3,\\
&\widehat{U}_3(\cdot,X_3) = \overline{U}_3(\cdot,X_3) - \frac{1}{\rho_\gamma(\phi+\psi)}\int_{-\rho_\gamma\psi}^{\rho_\gamma\phi}\overline{U}_3(\cdot,X_3)dX_3,
\end{aligned} \qquad \text{a.e in } \Omega.$$

This gives convergences (6.15) with $\widehat{U} \in \mathfrak{W}_p(\Omega)$. □

7 The linear elasticity problem in Ω_δ

We denote

$$\mathbb{D}_\varepsilon \doteq H^1_{\Gamma_{0,\varepsilon}}(\Omega_\varepsilon), \qquad \mathbb{D}_M \doteq \mathbb{M}^{1,2}_{1/2}(\omega)\times\mathbb{W}^{2,2}_{3/2}(\omega).$$

In this section, $\mathbb{D}_\varepsilon$ is the set of admissible displacements of the plate Ω_ε while $\mathbb{D}_M$ is the space of macroscopic limit displacements.

For $1 \le i,j,k,l \le 3$, let a_{ijkl} be in $L^\infty(\Omega)$ and satisfy the symmetry conditions

$$a_{ijkl}(x',X_3) = a_{jikl}(x',X_3) = a_{klij}(x',X_3) \qquad \text{for a.e. } (x',X_3)\in\Omega,$$

as well as the coercivity condition

$$a_{ijkl}\tau_{ij}\tau_{kl} \ge c_0\,\tau_{ij}\tau_{ij} \qquad \text{a.e. in } \Omega \tag{7.1}$$

for every symmetric matrix $\tau = (\tau_{ij})$ of order 3, where c_0 is a strictly positive constant.

The constitutive law of the materials is the relation between the strain tensor and the stress tensor

$$\sigma_{ij}(v) = a_{ijkl}\,e_{kl}(v), \qquad \forall v \in \mathbb{D}_\varepsilon. \tag{7.2}$$

For simplicity, we consider only applied body forces F_ε.

The displacement $u_\varepsilon \in \mathbb{D}_\varepsilon$ is the solution of the following elasticity problem:

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon}\sigma_{ij}(u_\varepsilon)\,e_{ij}(v)\,dx = \int_{\Omega_\varepsilon}F_\varepsilon\cdot v\,dx \\ \forall v\in\mathbb{D}_\varepsilon. \end{cases} \tag{7.3}$$

We make the assumption that the applied body forces F_ε are of the form

$$F_\varepsilon(x) = \big(\varepsilon^2 f_{\alpha,\varepsilon}(x') + \varepsilon x_3 g_{\alpha,\varepsilon}(x')\big)\mathbf{e}_\alpha + \varepsilon^3 f_{3,\varepsilon}(x')\mathbf{e}_3 \quad \text{ for a.e. } x\in\Omega_\varepsilon, \tag{7.4}$$

where

$$f_\varepsilon = \frac{1}{\rho_\varepsilon}f, \qquad g_\varepsilon = \frac{1}{\rho_\varepsilon^2}g$$

$f = f_i\mathbf{e}_i \in L^2(\omega)^3$ and $g = g_\alpha\mathbf{e}_\alpha \in L^2(\omega)^2$.

Now, for every $u\in\mathbb{D}_\varepsilon$ decomposed as (4.1) one has

$$\begin{aligned}
\int_{\Omega_\varepsilon}F_\varepsilon\cdot U_{e\ell}\,dx = {}& \varepsilon^3\int_\omega f_\alpha\,\mathcal{U}_\alpha\,(\phi+\psi)dx' + \frac{\varepsilon^4}{3}\int_\omega g_\alpha\rho_\varepsilon\mathcal{R}_\alpha\,(\phi^3+\psi^3)dx'\\
&+\frac{\varepsilon^3}{2}\int_\omega g_\alpha\,\mathcal{U}_\alpha\,(\phi^2-\psi^2)dx' + \frac{\varepsilon^4}{2}\int_\omega f_\alpha\rho_\varepsilon\mathcal{R}_\alpha\,(\phi^2-\psi^2)dx'\\
&+\varepsilon^4\int_\omega f_3\,\mathcal{U}_3\,(\phi+\psi)dx'
\end{aligned}$$

and due to $(4.4)_4$

$$\Big|\int_{\Omega_\varepsilon} F_\varepsilon\cdot\overline{u}\,dx\Big|\le C\varepsilon^3\big(\|f\|_{L^2(\omega)}+\|g\|_{L^2(\omega)}\big)\|e(u)\|_{L^2(\Omega_\varepsilon)}. \tag{7.5}$$

From the estimates (5.1) and the one above, we obtain an upper bound on the right-hand side of (7.3)

$$\forall u\in\mathbb{D}_\varepsilon,\qquad \Big|\int_{\Omega_\varepsilon} F_\varepsilon\cdot u\,dx\Big|\le C\varepsilon^{5/2}\big(\|f\|_{L^2(\omega)}+\|g\|_{L^2(\omega)}\big)\|e(u)\|_{L^2(\Omega_\varepsilon)}.$$

Applying the above estimate with $u=u_\varepsilon$ taken as the test function in (7.3), this gives

$$\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)}\le C\varepsilon^{5/2}\big(\|f\|_{L^2(\omega)}+\|g\|_{L^2(\omega)}\big). \tag{7.6}$$

7.1 The limit problem in Ω

In this subsection, we investigate the asymptotic behavior of the sequence $\{u_\varepsilon\}_\varepsilon$, the solution to problem (7.3).

Theorem 7.1. *Let u_ε be the solution to problem (7.3). There exist $\mathcal{U}=(\mathcal{U}_m,\mathcal{U}_3)\in\mathbb{D}_M$ and $\widehat{U}\in\mathfrak{W}_2(\Omega)$ such that*

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{U}_{m,\varepsilon}\rightharpoonup\mathcal{U}_m\quad \textit{weakly in}\ \ \mathbb{M}^{1,2}_{1/2}(\omega),\\
&\frac{1}{\varepsilon}\mathcal{U}_{3,\varepsilon}\rightharpoonup\mathcal{U}_3\quad \textit{weakly in}\ \ \mathbb{W}^{1,2}_{1/2}(\omega),\\
&\frac{1}{\varepsilon}\mathcal{R}_\varepsilon\rightharpoonup\mathcal{R}=-\nabla\mathcal{U}_3\quad \textit{weakly in}\ \ \mathbb{W}^{1,2}_{3/2}(\omega)
\end{aligned} \tag{7.7}$$

and

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(e(u_\varepsilon)\big)\longrightarrow E\big(\mathcal{U},\widehat{U}\big)\quad \textit{strongly in } L^2(\Omega)^{3\times3} \tag{7.8}$$

The pair $(\mathcal{U},\widehat{U})\in\mathbb{D}_M\times\mathfrak{W}_2(\Omega)$ is the unique solution to

$$\begin{cases}
\displaystyle\int_\Omega a_{ijkl}E_{ij}(\mathcal{U},\widehat{U})E_{kl}(\mathcal{V},\widehat{V})\,dx'dX_3\\
\displaystyle=\int_\omega f\cdot\mathcal{V}\,(\phi+\psi)dx'-\frac{1}{3}\int_\omega g\cdot\nabla\mathcal{V}_3\,\rho_\gamma\,(\phi^3+\psi^3)dx'\\
\displaystyle+\frac{1}{2}\int_\omega\big(g\cdot\mathcal{V}_m+f_m\cdot\nabla\mathcal{V}_3\,\rho_\gamma\big)\,(\phi^2-\psi^2)dx'\\
\forall(\mathcal{V},\widehat{V})\in\mathbb{D}_M\times\mathfrak{W}_2(\Omega)
\end{cases} \tag{7.9}$$

where

$$f_m=f_1\mathbf{e}_1+f_2\mathbf{e}_2.$$

Proof. Step 1. The convergences (7.7) and (7.8).

Due to the estimate (7.6), there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\mathcal{U}_m=\mathcal{U}_1\mathbf{e}_1+\mathcal{U}_2\mathbf{e}_2\in\mathbb{M}^{1,2}_{1/2}(\omega)$, $\mathcal{U}_3\in\mathbb{W}^{2,2}_{3/2}(\omega)$ and $\widehat{U}\in\mathfrak{W}_2(\Omega)$ such that the convergences (7.7) and (7.8) hold in the weak sense.

Step 2. The limit problem in Ω.

Let $\mathcal{V}=(\mathcal{V}_m,\mathcal{V}_3)$ be in $\mathbb{D}_M$ and $\widehat{V}\in H^1\big(\omega\times(-C_1,C_1)\big)^3$ such that $\widehat{V}=0$ a.e. on $\gamma_0\times(-C_1,C_1)$. We choose as test displacement

$$v_\varepsilon(x)=\frac{1}{\varepsilon^2}\Big[\big(\mathcal{V}_1(x')-x_3\partial_1\mathcal{V}_3(x')\big)\mathbf{e}_1+\big(\mathcal{V}_2(x')-x_3\partial_2\mathcal{V}_3(x')\big)\mathbf{e}_1+\mathcal{V}_3(x')\mathbf{e}_3+\varepsilon^3\widehat{V}\Big(x',\frac{x_3}{\varepsilon}\Big)\Big]\qquad\text{for a.e. } x \text{ in } \Omega_\varepsilon.$$

A straightforward calculation gives

$$\varepsilon^2\Pi_\varepsilon\big(e(v_\varepsilon)\big) \longrightarrow E(\mathcal{V},\widehat{V}) \quad \text{strongly in} \quad L^2(\Omega)^{3\times 3}$$

and

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} F_\varepsilon\cdot v_\varepsilon\,dx \longrightarrow \int_\omega f\cdot\mathcal{V}\,(\phi+\psi)dx' - \frac{1}{3}\int_\omega g\cdot\nabla\mathcal{V}_3\,\rho_\gamma\,(\phi^3+\psi^3)dx' + \frac{1}{2}\int_\omega \big(g\cdot\mathcal{V}_m + f_m\cdot\nabla\mathcal{V}_3\,\rho_\gamma\big)\,(\phi^2-\psi^2)dx'.$$

We take v_ε as the test displacement in (7.3), then we transform with Π_ε, divide by ε and pass to the limit using the convergences (7.7) and (7.8) (in the weak sense) and those above concerning the test displacement. This leads to (7.9) with the pair $(\mathcal{V},\widehat{V})$.
Then, since the space of restrictions to Ω of fields $\widehat{V}$ belonging to $H^1(\omega\times(-C_1,C_1))^3$ and satisfying $\widehat{\mathfrak{V}}=0$ on $\gamma_0\times(-C_1,C_1)$ is dense in $\mathfrak{W}_2(\Omega)$, problem (7.9) is satisfied for every $\widehat{V}$ in $\mathbf{Warp}_2(\Omega)$ and thus for every field in $\mathfrak{W}_2(\Omega)$.

Step 3. Uniqueness of the solution to the problem (7.9).
The bilinear form over $\mathbb{D}_M\times\mathfrak{W}_2(\Omega)$

$$\big((\mathcal{V},\widehat{V}),\,(\mathcal{W},\widehat{W})\big)\in\big(\mathbb{D}_M\times\mathfrak{W}_2(\Omega)\big)^2 \longmapsto \int_\Omega a_{ijkl}E_{ij}(\mathcal{V},\widehat{V})\,E_{kl}(\mathcal{W},\widehat{W})\,dx'dX$$

is coercive and bounded (see (6.10) and (7.1)). So, problem (7.9) admits a unique solution. Hence, the whole sequences in (7.7) and (7.8) converge to their limits.

Step 4. Strong convergence of the rescaled strain tensor.

Take u_ε as the test displacement in (7.3), then transform using Π_ε and divide by ε^4.
We first have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^5}\int_\Omega \Pi_\varepsilon(F_\varepsilon)\cdot\Pi_\varepsilon(u_\varepsilon)\,dx'dX = \int_\omega f\cdot\mathcal{U}\,(\phi+\psi)dx' - \frac{1}{3}\int_\omega g\cdot\nabla\mathcal{U}_3\,\rho_\gamma\,(\phi^3+\psi^3)dx' + \frac{1}{2}\int_\omega \big(g\cdot\mathcal{U}_m + f_m\cdot\nabla\mathcal{U}_3\,\rho_\gamma\big)\,(\phi^2-\psi^2)dx'.$$

Now, the weak lower semi-continuity of the left-hand side of (7.9), the convergences (7.7)-(7.8) (in the weak sense) together with (6.1) yield

$$\begin{aligned}
&\int_\Omega a_{ijkl}E_{ij}(\mathcal{U},\widehat{U})\,E_{kl}(\mathcal{U},\widehat{U})\,dx'dX_3 \le \liminf_{\varepsilon\to 0}\Big(\int_\Omega a_{ijkl}\,\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(e_{ij}(u_\varepsilon)\big)\,\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(e_{kl}(u_\varepsilon)\big)dx'dX_3\Big)\\
&\le \limsup_{\varepsilon\to 0}\Big(\int_\Omega a_{ijkl}\,\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(e_{ij}(u_\varepsilon)\big)\,\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(e_{kl}(u_\varepsilon)\big)dx'dX_3\Big)\\
&\le \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^5}\int_{\Omega_\varepsilon}\sigma_{ij}(u_\varepsilon)\,e_{ij}(u_\varepsilon)\,dx = \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^5}\int_{\Omega_\varepsilon}F_\varepsilon\cdot u_\varepsilon\,dx\\
&= \int_\omega f\cdot\mathcal{U}\,(\phi+\psi)dx' - \frac{1}{3}\int_\omega g\cdot\nabla\mathcal{U}_3\,\rho_\gamma\,(\phi^3+\psi^3)dx' + \frac{1}{2}\int_\omega \big(g\cdot\mathcal{U}_m + f_m\cdot\nabla\mathcal{U}_3\,\rho_\gamma\big)\,(\phi^2-\psi^2)dx'.
\end{aligned}$$

Hence, the above inequalities are equalities. This proves the strong convergence (7.8) of the strain tensor. □

7.2 The correctors

Set

$$\mathbf{M}^{11}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad \mathbf{M}^{12}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\qquad \mathbf{M}^{22}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}.$$

Below we introduce the six correctors.

$$\chi_m^{\alpha\beta},\ \chi_b^{\alpha\beta} \in \mathfrak{W}_2(\Omega), \qquad (\alpha,\beta) \in \big\{(1,1),(1,2),(2,2)\big\},$$

defined by

$$\left.\begin{aligned} &\int_{-\rho_\gamma\psi}^{\rho_\gamma\phi} a_{ijkl}\big(\mathbf{M}_{ij}^{\alpha\beta} + E_{ij}(0,\chi_m^{\alpha\beta})\big)E_{kl}(0,\widehat{V})dX_3 = 0 \qquad \text{a.e. in } \omega,\\ &\int_{-\rho_\gamma\psi}^{\rho_\gamma\phi} a_{ijkl}\big(X_3\mathbf{M}_{ij}^{\alpha\beta} + E_{ij}(0,\chi_b^{\alpha\beta})\big)E_{kl}(0,\widehat{V})dX_3 = 0 \qquad \text{a.e. in } \omega, \end{aligned}\right\} \quad \forall \widehat{V} \in \mathfrak{W}_2(\Omega). \tag{7.10}$$

Using these correctors, $\widehat{U}$ can be written as follows:

$$\widehat{U} = e_{\alpha\beta}(\mathcal{U}_m)\chi_m^{\alpha\beta} + \partial^2_{\alpha\beta}\mathcal{U}_3\chi_b^{\alpha\beta}. \tag{7.11}$$

7.3 The limit problem posed in the "mid-surface" ω

Theorem 7.2. *The limit displacement $\mathcal{U} = (\mathcal{U}_m, \mathcal{U}_3) \in \mathbb{D}_M$ is the unique solution to*

$$\left\{\begin{aligned} &\int_\omega \Big[A_{\alpha\beta\alpha'\beta'}e_{\alpha\beta}(\mathcal{U}_m)e_{\alpha'\beta'}(\mathcal{V}_m) + B_{\alpha\beta\alpha'\beta'}\Big(e_{\alpha\beta}(\mathcal{U}_m)\partial^2_{\alpha'\beta'}\mathcal{V}_3 + e_{\alpha'\beta'}(\mathcal{V}_m)\partial^2_{\alpha\beta}\mathcal{U}_3\Big)\\ &\qquad + C_{\alpha\beta\alpha'\beta'}\partial^2_{\alpha\beta}\mathcal{U}_3\,\partial^2_{\alpha'\beta'}\mathcal{V}_3\Big]dx'\\ &= \int_\omega f\cdot\mathcal{V}\,(\phi+\psi)dx' - \frac{1}{3}\int_\omega g\cdot\nabla\mathcal{V}_3\,\rho_\gamma\,(\phi^3+\psi^3)dx'\\ &+\frac{1}{2}\int_\omega \big(g\cdot\mathcal{V}_m + f_m\cdot\nabla\mathcal{V}_3\,\rho_\gamma\big)\,(\phi^2-\psi^2)dx', \qquad \forall\mathcal{V}\in\mathbb{D}_M \end{aligned}\right. \tag{7.12}$$

where

$$\begin{aligned} A_{\alpha\beta\alpha'\beta'} &= \int_{-\rho_\gamma\psi}^{\rho_\gamma\phi} a_{ijkl}\big(\mathbf{M}_{ij}^{\alpha\beta} + E_{ij}(0,\chi_m^{\alpha\beta})\big)\mathbf{M}_{kl}^{\alpha'\beta'}dX_3,\\ B_{\alpha\beta\alpha'\beta'} &= \int_{-\rho_\gamma\psi}^{\rho_\gamma\phi} a_{ijkl}\big(X_3\mathbf{M}_{ij}^{\alpha\beta} + E_{ij}(0,\chi_b^{\alpha\beta})\big)\mathbf{M}_{kl}^{\alpha'\beta'}dX_3,\\ C_{\alpha\beta\alpha'\beta'} &= \int_{-\rho_\gamma\psi}^{\rho_\gamma\phi} a_{ijkl}\big(X_3\mathbf{M}_{ij}^{\alpha\beta} + E_{ij}(0,\chi_b^{\alpha\beta})\big)X_3\mathbf{M}_{kl}^{\alpha'\beta'}dX_3. \end{aligned}$$

Proof. In problem (7.9), we choose $\widehat{V} = 0$ and we replace $\widehat{U}$ by its expression given above by (7.11). Then, as in [9, Theorem 11.21], we prove that the bilinear form over $\mathbb{D}_M\times\mathbb{D}_M$

$$\begin{aligned} (\mathcal{U},\mathcal{V}) \longmapsto \int_\Omega \Big[&A_{\alpha\beta\alpha'\beta'}e_{\alpha\beta}(\mathcal{U}_m)e_{\alpha'\beta'}(\mathcal{V}_m) + B_{\alpha\beta\alpha'\beta'}\Big(e_{\alpha\beta}(\mathcal{U}_m)\partial^2_{\alpha'\beta'}\mathcal{V}_3 + e_{\alpha'\beta'}(\mathcal{V}_m)\partial^2_{\alpha\beta}\mathcal{U}_3\Big)\\ &+ C_{\alpha\beta\alpha'\beta'}\partial^2_{\alpha\beta}\mathcal{U}_3\,\partial^2_{\alpha'\beta'}\mathcal{V}_3\Big]dx' \end{aligned}$$

is coercive and bounded. Therefore, problem (7.12) admits a unique solution. □

7.4 Case of a homogeneous and isotropic material

In this subsection, we consider a plate made of a homogeneous and isotropic material. So, we have

$$a_{ijkl} \doteq \lambda\boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{kl} + \mu\big(\boldsymbol{\delta}_{ik}\boldsymbol{\delta}_{jl} + \boldsymbol{\delta}_{il}\boldsymbol{\delta}_{jk}\big), \qquad \{i,j,k,l\}\in\{1,2,3\}^4$$

where $\boldsymbol{\delta}_{ij}$ is the Kronecker symbol and λ, μ the Lamé's constants.
In this case, we can easily obtain $\widehat{U}$ taking $\mathcal{V}=0$ in problem (7.9). We find (see also [14])

$$\begin{aligned}
&\frac{\partial \widehat{U}_1}{\partial X_3} = \frac{\partial \widehat{U}_2}{\partial X_3} = 0,\\
&\frac{\partial \widehat{U}_3}{\partial X_3} = \frac{\lambda}{\lambda+2\mu}\big(-e_{11}(\mathcal{U})-e_{22}(\mathcal{U})+X_3\Delta\mathcal{U}_3\big),
\end{aligned} \qquad \text{a.e. in} \quad \Omega. \tag{7.13}$$

Now, inserting the above value of $\widehat{U}$ in (7.9) leads to the variational problem satisfied by $\mathcal{U}$.

Theorem 7.3. *The field $\mathcal{U}\in\mathbb{D}_M$ is the unique solution to*

$$\begin{cases}
\dfrac{E}{1-\nu^2}\displaystyle\int_\omega \big[(1-\nu)\,e_{\alpha\beta}(\mathcal{U})\,e_{\alpha\beta}(\mathcal{V})+\nu e_{\alpha\alpha}(\mathcal{U})e_{\beta\beta}(\mathcal{V})\big]\,(\phi+\psi)\rho_\gamma\,dx'\\
+\dfrac{E}{2(1-\nu^2)}\displaystyle\int_\omega (1-\nu)\big(e_{\alpha\beta}(\mathcal{U})\,\partial^2_{\alpha\beta}\mathcal{V}_3+e_{\alpha\beta}(\mathcal{V})\,\partial^2_{\alpha\beta}\mathcal{U}_3\big)(\phi^2-\psi^2)\rho_\gamma^2\,dx'\\
+\dfrac{E}{2(1-\nu^2)}\displaystyle\int_\omega \nu\big(e_{\alpha\alpha}(\mathcal{U})\,\Delta\mathcal{V}_3+e_{\alpha\alpha}(\mathcal{V})\,\Delta\mathcal{U}_3\big)(\phi^2-\psi^2)\rho_\gamma^2\,dx'\\
+\dfrac{E}{3(1-\nu^2)}\displaystyle\int_\omega \big[(1-\nu)\partial^2_{\alpha\beta}\mathcal{U}_3\,\partial^2_{\alpha\beta}\mathcal{V}_3+\nu\Delta\mathcal{U}_3\Delta\mathcal{V}_3\big](\phi^3+\psi^3)\rho_\gamma^3\,dx'\\
=\displaystyle\int_\omega f\cdot\mathcal{V}\,(\phi+\psi)dx'-\frac{1}{3}\int_\omega g\cdot\nabla\mathcal{V}_3\,\rho_\gamma\,(\phi^3+\psi^3)dx'\\
+\dfrac{1}{2}\displaystyle\int_\omega\big(g\cdot\mathcal{V}_m+f_m\cdot\nabla\mathcal{V}_3\,\rho_\gamma\big)\,(\phi^2-\psi^2)dx', \qquad \forall\mathcal{V}\in\mathbb{D}_M
\end{cases} \tag{7.14}$$

where $E=\dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is the Young modulus and $\nu=\dfrac{\lambda}{2(\lambda+\mu)}$ the Poisson constant.
Moreover, we have

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(\sigma_{11}(u_\delta)\big)\to\frac{E}{1-\nu^2}\Big[e_{11}(\mathcal{U})-X_3\partial^2_{11}\mathcal{U}_3+\nu\Big(e_{22}(\mathcal{U})-X_3\partial^2_{22}\mathcal{U}_3\Big)\Big] \quad \textit{strongly in } L^2(\Omega),\\
&\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(\sigma_{22}(u_\delta)\big)\to\frac{E}{1-\nu^2}\Big[e_{22}(\mathcal{U})-X_3\partial^2_{22}\mathcal{U}_3+\nu\Big(e_{11}(\mathcal{U})-X_3\partial^2_{11}\mathcal{U}_3\Big)\Big] \quad \textit{strongly in } L^2(\Omega),\\
&\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(\sigma_{12}(u_\delta)\big)\to 2\mu\Big[e_{12}(\mathcal{U})-X_3\partial^2_{12}\mathcal{U}_3\Big] \quad \textit{strongly in } L^2(\Omega),\\
&\frac{1}{\varepsilon^2}\Pi_\varepsilon\big(\sigma_{i3}(u_\delta)\big)\to 0 \quad \textit{strongly in } L^2(\Omega).
\end{aligned} \tag{7.15}$$

Proof. The strong convergences (7.15) are the consequences of (7.8). □

Now, we can reconstruct the solution to problem (7.3). We obtain for a.e. $x\in\Omega_\varepsilon$

$$u_\varepsilon(x)\approx\begin{pmatrix}\varepsilon^2\mathcal{U}_1(x')-\varepsilon x_3\partial_1\mathcal{U}_3(x')\\ \varepsilon^2\mathcal{U}_2(x')-\varepsilon x_3\partial_2\mathcal{U}_3(x')\\ \varepsilon\mathcal{U}_3(x')\end{pmatrix}+\frac{\lambda\varepsilon^2 x_3}{\lambda+2\mu}\Big(-e_{11}(\mathcal{U})(x')-e_{22}(\mathcal{U})(x')+\frac{x_3^2}{2\varepsilon^2}\Delta\mathcal{U}_3(x')\Big).$$

and for the stress tensor, we have

$$\sigma(u_\varepsilon)(x) \approx \varepsilon^2 \begin{pmatrix} \frac{E}{1-\nu^2}\big(e_{11}(\mathcal{U})(x') + \nu e_{22}(\mathcal{U})(x')\big) & 2\mu e_{12}(\mathcal{U})(x') & 0 \\ 2\mu e_{12}(\mathcal{U})(x') & \frac{E}{1-\nu^2}\big(e_{22}(\mathcal{U})(x') + \nu e_{11}(\mathcal{U})(x')\big) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$- \varepsilon x_3 \begin{pmatrix} \frac{E}{1-\nu^2}\Big(\partial^2_{11}\mathcal{U}_3(x') + \nu\partial^2_{22}\mathcal{U}_3(x')\Big) & 2\mu\partial^2_{12}\mathcal{U}_3(x') & 0 \\ 2\mu\partial^2_{12}\mathcal{U}_3(x') & \frac{E}{1-\nu^2}\Big(\partial^2_{22}\mathcal{U}_3(x') + \nu\partial^2_{11}\mathcal{U}_3(x')\Big) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

8 Appendix

8.1 Preliminary lemmas

Lemma 8.1. *For any $\chi \in W^{1,p}(0,d)$, $d>0$ satisfying $\chi(d)=0$ and any $\mathfrak{b}\geq 0$, if $sp+1>0$ then we have*

$$\int_0^d (t+\mathfrak{b})^{sp}|\chi(t)|^p\,dt \leq \Big(\frac{p}{sp+1}\Big)^p \int_0^d (t+\mathfrak{b})^{(s+1)p}|\chi'(t)|^p\,dt. \tag{8.1}$$

Proof. We start with $\chi \in \mathcal{C}^1([0,d])$ satisfying $\chi(d)=0$. We have

$$p\int_0^d (t+\mathfrak{b})^{sp+1} sign\big(\chi(t)\big)|\chi(t)|^{p-1}\chi'(t)\,dt = \big[(t+\mathfrak{b})^{sp+1}|\chi(t)|^p\big]_0^d - (sp+1)\int_0^d (t+\mathfrak{b})^{sp}|\chi(t)|^p\,dt.$$

Then, thanks to the Hölder inequality we obtain

$$\begin{aligned}(sp+1)\int_0^d (t+\mathfrak{b})^{sp}|\chi(t)|^p\,dt &= -p\int_0^d (t+\mathfrak{b})^{sp+1} sign\big(\chi(t)\big)|\chi(t)|^{p-1}\chi'(t)\,dt - \mathfrak{b}^{sp+1}|\chi(0)|^p \\ &= -p\int_0^d (t+\mathfrak{b})^{s(p-1)} sign\big(\chi(t)\big)|\chi(t)|^{p-1}(t+\mathfrak{b})^{s+1}\chi'(t)\,dt - \mathfrak{b}^{sp+1}|\chi(0)|^p \\ &\leq p\Big(\int_0^d (t+\mathfrak{b})^{sp}|\chi(t)|^p\,dt\Big)^{1/p'}\Big(\int_0^d (t+\mathfrak{b})^{(s+1)p}|\chi'(t)|^p\,dt\Big)^{1/p}\end{aligned}$$

from which we derive (8.1) for $\chi \in \mathcal{C}^1([0,d])$ satisfying $\chi(d)=0$. A density argument gives (8.1) for any $\chi \in W^{1,p}(0,d)$ satisfying $\chi(d)=0$. □

Lemma 8.2. *Let $(\varphi\,,\chi)$ be in $L^\infty(0,d)\times W^{1,p}(0,d)$, $d>0$, satisfying*

$$m(t+\mathfrak{b}) \leq \varphi(t) \leq M(t+\mathfrak{b}) \quad \textit{for a.e. } t\in(0,d) \qquad \textit{and} \qquad \chi(d)=0$$

where $\mathfrak{b}\geq 0$ and $(m,M)\in(0,+\infty)^2$.
Then

$$\|\varphi^s\chi\|_{L^p(0,d)} \leq C\|\varphi^{s+1}\chi'\|_{L^p(0,d)}, \qquad \forall s\in(-1/p,+\infty) \tag{8.2}$$

where

$$C = \begin{cases} \dfrac{pM^s}{m^{s+1}(sp+1)} & \textit{if } s\in\big[0,+\infty\big), \\ \dfrac{p}{m(sp+1)} & \textit{if } s\in\big(-1/p,0\big]. \end{cases} \tag{8.3}$$

Proof. If $s \geq 0$ then we have

$$\begin{aligned}\int_0^d \varphi(t)^{sp}|\chi(t)|^p\,dt &\leq M^{sp}\int_0^d (t+\mathfrak{b})^{sp}|\chi(t)|^p\,dt\\ &\leq \frac{p^p M^{sp}}{m^{(s+1)p}(sp+1)^p}\int_0^d m^{(s+1)p}(t+\mathfrak{b})^{(s+1)p}|\chi'(t)|^p\,dt\\ &\leq \frac{p^p M^{sp}}{m^{(s+1)p}(sp+1)^p}\int_0^d \varphi(t)^{(s+1)p}|\chi'(t)|^p\,dt.\end{aligned}$$

If $s \in (-1/p, 0]$ then we have $(s+1)p \in (p-1, p] \subset (0, +\infty)$ and

$$\begin{aligned}\int_0^d \varphi(t)^{sp}|\chi(t)|^p\,dt &\leq m^{sp}\int_0^d (t+\mathfrak{b})^{sp}|\chi(t)|^p\,dt\\ &\leq \frac{p^p m^{sp}}{m^{(s+1)p}(sp+1)^p}\int_0^d m^{(s+1)p}(t+\mathfrak{b})^{(s+1)p}|\chi'(t)|^p\,dt\\ &\leq \frac{p^p}{m^p(sp+1)^p}\int_0^d \varphi(t)^{(s+1)p}|\chi'(t)|^p\,dt.\end{aligned}$$

This ends the proof of the estimate (8.2). □

8.2 Preliminary considerations on boundaries

Denote

- $\mathcal{C}(\mathbf{a}, h, \theta)$ the cone with apex O (the origin of $\mathbb{R}^2$), axis directed by the unit vector $\mathbf{a}$, height $h > 0$ and half-angle $\theta \in (0, \pi/2)$

$$\mathcal{C}(\mathbf{a}, h, \theta) \doteq \big\{x' \in \mathbb{R}^2 \;\big|\; |x'|_2 \cos(\theta) < x' \cdot \mathbf{a} < h\big\},$$

- $\mathcal{H}(\mathbf{a}, h) \doteq \big\{x' \in \mathbb{R}^2 \mid x' = t\mathbf{a},\;\; t \in (0, h)\big\}$, $\mathcal{H}(\mathbf{a}, h)$ is the axis of the cone $\mathcal{C}(\mathbf{a}, h, \theta)$,
- $\mathcal{D}(\mathbf{a}, 2h, \theta)$ the diamond

$$\mathcal{D}(\mathbf{a}, 2h, \theta) \doteq \big\{x' \in \mathbb{R}^2 \;\big|\; |x'|_2 \cos(\theta) < x' \cdot \mathbf{a} \leq h\big\} \cup \big\{x' \in \mathbb{R}^2 \;\big|\; |2h\mathbf{a} - x'|_2 \cos(\theta) \leq 2h - x' \cdot \mathbf{a} < h\big\}.$$

First, since $\partial\omega$ is a compact set, there exists $a > 0$ satisfying $a \leq \min\Big\{\frac{A}{3}, \frac{B}{2K_1}\Big\}$ such that

$$\partial\omega = \bigcup_{n=1}^{N} \big(\partial\omega\big)_n^{'}, \quad \big(\partial\omega\big)_n^{'} \doteq \Big\{(x_{1n}, x_{2n}) \in \mathbb{R}^2 \mid x_{2n} = f_n(x_{1n}), \quad x_{1n} \in (a, A-a)\Big\}. \tag{8.4}$$

Set

$$\begin{aligned} &h^{'} = aK_1 \leq \frac{B}{2}, \qquad \theta^{'} \in (0, \pi/3]^4 \quad \text{such that} \quad \cos(\theta^{'}) = \frac{K_1}{\sqrt{1+K_1^2}},\\ &\mathbf{e}_n^{''} = \frac{\mathbf{e}_{1n} + 3K_1\mathbf{e}_{2n}}{\sqrt{1+9K_1^2}}, \quad \mathbf{e}_n^{'''} = \frac{-\mathbf{e}_{1n} + 3K_1\mathbf{e}_{2n}}{\sqrt{1+9K_1^2}}, \quad n \in \{1, \ldots, N\},\\ &\theta^{''} \in (0, \pi/3) \text{ such that } \cos(\theta^{''}) = \frac{1+3K_1^2}{\sqrt{(1+K_1^2)(1+9K_1^2)}}, \quad h^{''} = a\frac{1+3K_1^2}{\sqrt{1+9K_1^2}}\end{aligned} \tag{8.5}$$

[4] Due to the assumption $(3.2)_2$.

and

$$\begin{aligned}\mathcal{B}_n^{'} &\doteq \Big\{(x_{1n}, x_{2n}) \in \mathbb{R}^2 \mid f_n(x_{1n}) < x_{2n} < f_n(x_{1n}) + h^{'}, \quad x_{1n} \in (a, A-a)\Big\} \\ &= \bigcup_{x_n' \in (\partial\omega)_n'} \big(x_n' + \mathcal{H}(\mathbf{e}_{2n}, h^{'})\big) \subset \bigcup_{x_n' \in (\partial\omega)_n'} \big(x_n' + \mathcal{H}(\mathbf{e}_{2n}, 2h^{'})\big) \subset \mathcal{B}_n. \end{aligned} \tag{8.6}$$

Note that by construction of the two cones

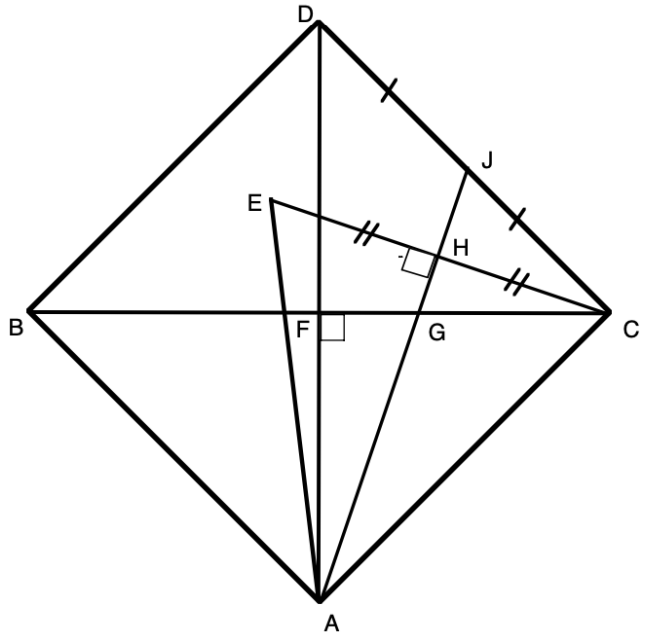

Figure 2: The two cones $CAB = A + \mathcal{C}(\mathbf{e}_{2n}, h^{'}, \theta^{'})$ and $CAE = A + \mathcal{C}(\mathbf{e}_n^{''}, h^{''}, \theta^{''})$ and the diamond $CABD = A + \mathcal{D}(\mathbf{e}_{2n}, 2h^{'}, \theta^{'})$ whose common apex is A.

$$CAB = A + \mathcal{C}(\mathbf{e}_{2n}, h^{'}, \theta^{'}) \quad \text{and} \quad CAE = A + \mathcal{C}(\mathbf{e}_n^{''}, h^{''}, \theta^{''})$$

we have (see Figure 2)

- since A belongs to $\big(\partial\omega\big)_n^{'}$, the diamond $CABD$ is included in $\mathcal{B}_n$,
- $\overrightarrow{AF} = \overrightarrow{FD} = h'\mathbf{e}_{2n}$, $\overrightarrow{BF} = \overrightarrow{FC} = a\mathbf{e}_{1n}$, $\widehat{CAF} = \widehat{FAB} = \theta^{'}$,
- $AC = AE$, $\widehat{CAH} = \widehat{HAE} = \theta^{''}$,
- the line AH intersects the segment $[D, C]$ at its midpoint J,
- $\mathbf{e}_n^{''}$ has the same direction as $\overrightarrow{AJ}$,
- $\overrightarrow{GC} = \dfrac{2}{3}a\mathbf{e}_{1n}, \quad AH = h^{''} = \dfrac{a(1+3K_1^2)}{\sqrt{1+9K_1^2}} > h'$,
- $\text{dist}(G, CABD) = \text{dist}(G, AC) = \dfrac{2}{3}\dfrac{h^{'}}{\sqrt{1+K_1^2}}, \quad \text{dist}(F, CABD) = \text{dist}(F, AC) = \dfrac{h^{'}}{\sqrt{1+K_1^2}}$,

- thanks to assumption (3.2) we have

$$0 < \theta^{''} < \theta' \leq 2\theta^{''} < \frac{\pi}{2}.$$

  As a consequence the segment AF is included in the cone CAE,

- the cone CAE is included in the diamond $CABD$.

So, we have

$$\begin{aligned} &\mathcal{B}_n^{'} \subset \Big(\bigcup_{x_n' \in (\partial\omega)_n^{'}} \big(x_n' + \mathcal{H}(\mathbf{e}_n^{''}, h^{''})\big)\Big) \cup \Big(\bigcup_{x_n' \in (\partial\omega)_n^{'}} \big(x_n' + \mathcal{H}(\mathbf{e}_n^{'''}, h^{''})\big)\Big) \subset \mathcal{B}_n \\ &\text{and} \quad \bigcup_{x_n' \in (\partial\omega)_n^{'}} \big(x_n' + \mathcal{D}(\mathbf{e}_{2n}, 2h^{'}, \theta^{'})\big) \subset \mathcal{B}_n. \end{aligned} \tag{8.7}$$

Since $\bigcup_{n=1}^{N} \big(\partial\omega\big)_n^{'} = \partial\omega$ (see (8.4)), there exists a strictly positive constant $\mathfrak{c}_1 \leq \mathfrak{c}_0/2$ such that

$$\forall \delta \in (0, 2\mathfrak{c}_1], \quad \omega \setminus \overline{\omega_\delta} \subset \omega \setminus \overline{\omega_{2\mathfrak{c}_1}} \subset \omega \setminus \overline{\omega_{\mathfrak{c}_0}} \subset \bigcup_{n=1}^{N} \mathcal{B}_n'. \tag{8.8}$$

Note that $2\mathfrak{c}_1 \leq h^{'}$.

8.3 A weighted Poincaré-Wirtinger inequality

We remind that $W_s^{1,p}(\omega)$ is defined by (6.5).

Lemma 8.3. *For any ϕ in $W_{s+1}^{1,p}(\omega)$ with $s \in \mathbb{R}$ such that $sp + 1 > 0$[5], there exists $\mathbf{a}_s \in \mathbb{R}$ such that*

$$\big\|\rho_\gamma^s(\phi - \mathbf{a}_s)\big\|_{L^p(\omega)} \leq C \|\rho_\gamma^{s+1} \nabla\phi\|_{L^p(\omega)}. \tag{8.9}$$

The constant depends on s, p and $\partial\omega$.

Proof. The proof is based on the result given in Lemma 8.2.
We first prove (8.9) for any $\phi \in \mathcal{C}^1(\overline{\omega})$.
First, observe that for all $x' \in \omega_{\mathfrak{c}_1}$ we have

$$\mathfrak{c}_1 < \rho(x') \leq \rho_\gamma(x') \leq 1.$$

Now, the Poincaré-Wirtinger inequality applied to the function ϕ restricted to $\omega_{\mathfrak{c}_1}$ gives $\widetilde{\mathbf{a}} \in \mathbb{R}^6$[6]such that

$$\begin{aligned} &\|\phi - \widetilde{\mathbf{a}}\|_{L^p(\omega_{\mathfrak{c}_1})} \leq C\|\nabla\phi\|_{L^p(\omega_{\mathfrak{c}_1})} \leq \frac{C}{\mathfrak{c}_1^{s+1}} \big\|\rho_\gamma^{s+1}\nabla\phi\big\|_{L^p(\omega_{\mathfrak{c}_1})} \leq \frac{C}{\mathfrak{c}_1^{s+1}} \big\|\rho_\gamma^{s+1}\nabla\phi\big\|_{L^p(\omega)}, \\ \Longrightarrow \qquad &\|\rho_\gamma^s(\phi - \widetilde{\mathbf{a}})\|_{L^p(\omega_{\mathfrak{c}_1})} \leq \big(1 + \mathfrak{c}_1^{-1/p}\big)\|\phi - \widetilde{\mathbf{a}}\|_{L^p(\omega_{\mathfrak{c}_1})} \leq \frac{C}{\mathfrak{c}_1^{s+1}} \big\|\rho_\gamma^{s+1}\nabla\phi\big\|_{L^p(\omega)}. \end{aligned} \tag{8.10}$$

[5] Observe that $s + 1 > 1/p'$.

[6] $\widetilde{\mathbf{a}}$ is the mean value of ϕ in $\omega_{\mathfrak{c}_1}$, $\widetilde{\mathbf{a}} = \dfrac{1}{|\omega_{\mathfrak{c}_1}|} \displaystyle\int_{\omega_{\mathfrak{c}_1}} \phi \, dx'$

The last constant C depends on $\partial\omega$ and p.
Besides, there exists a function $\psi_0 \in \mathcal{C}_c^\infty(\omega)$ such that

$$\begin{aligned}
&\psi_0(x') \in [0,1] \quad \text{for all } x' \text{ in } \omega,\\
&\psi_0(x') = 0 \quad \text{for all } x' \text{ such that } \rho(x') \le \mathfrak{c}_1,\\
&\psi_0(x') = 1 \quad \text{for all } x' \text{ such that } \rho(x') \ge 2\mathfrak{c}_1.
\end{aligned}$$

We have

$$\nabla\big((1-\psi_0)(\phi-\widetilde{\mathbf{a}})\big) = (1-\psi_0)\nabla\phi - \nabla\psi_0(\phi-\widetilde{\mathbf{a}}).$$

Hence, from (8.10) and the above equality we get

$$\begin{aligned}
\|\rho_\gamma^s\psi_0(\phi-\widetilde{\mathbf{a}})\|_{L^p(\omega)} &\le \|\rho_\gamma^s(\phi-\widetilde{\mathbf{a}})\|_{L^p(\omega_{\mathfrak{c}_1})} \le C\big\|\rho_\gamma^{s+1}\nabla\phi\big\|_{L^p(\omega)},\\
\|\rho_\gamma^{s+1}\,\nabla\big((1-\psi_0)(\phi-\widetilde{\mathbf{a}})\big)\|_{L^p(\omega)} &\le \|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)} + C\|\rho_\gamma^s(\phi-\widetilde{\mathbf{a}})\|_{L^p(\omega_{\mathfrak{c}_1})}\\
&\le C\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)}.
\end{aligned} \tag{8.11}$$

Set $\phi_0 = (1-\psi_0)(\phi-\widetilde{\mathbf{a}})$, this function vanishes in $\omega_{2\mathfrak{c}_1}$. Now, from $(8.11)_2$ we have

$$\|\rho_\gamma^{s+1}\,\nabla\phi_0\|_{L^p(\omega)} \le C\|\rho_\gamma^{s+1}\,\nabla\phi\|_{L^p(\omega)}.$$

Remind that and $\omega \setminus \overline{\omega_{2\mathfrak{c}_1}} \subset \bigcup_{n=1}^{N} \mathcal{B}_n^{'}$ (see (8.6) and (8.8)).
Now, we consider the open set $\mathcal{B}_n^{'}$, $n \in \{1,\dots,N\}$, and we define the function $\varphi \in W^{1,\infty}(0,h')$ by

$$\varphi(t) = \rho_\gamma(x_n^{'} + t\mathbf{e}_{2n}), \qquad x_n^{'} = \big(x_{1n}, f_n(x_{1n})\big), \qquad x_{1n} \in (a, A-a).$$

First case: $\varphi(0) = \rho_\gamma(x_n') = 0$.
In this case, by considering the cone $x_n^{'} + \mathcal{C}\big(\mathbf{e}_{2n}, h^{'}, \theta^{'}\big)$, we obtain (see Figure 2)

$$\frac{t}{\sqrt{1+K_1^2}} \le \varphi(t), \qquad \forall t \in (0, h^{'}).$$

Second case: $\varphi(0) = \rho_\gamma(x_n') \in (0, h^{'}]$.
In this case, simple geometrical arguments lead to

$$\varphi(t) \ge \begin{cases} \dfrac{\varphi(0)}{\sqrt{1+K_1^2}}, & \text{if } t \in (0, \varphi(0)),\\[2ex] \dfrac{t}{\sqrt{1+K_1^2}}, & \text{if } t \in (\varphi(0), h^{'}), \end{cases}$$

Third case: $\varphi(0) = \rho_\gamma(x_n') \ge h^{'}$.
In this case

$$\frac{\varphi(0)}{\sqrt{1+K_1^2}} \le \varphi(t), \qquad \forall t \in (0, h^{'}).$$

Therefore, in these three cases we have

$$m(t+\mathfrak{b}) \le \varphi(t) \le M(t+\mathfrak{b}), \qquad \forall t \in (0, h^{'})$$

where

$$m = \frac{1}{2\sqrt{1+K_1^2}}, \qquad \mathfrak{b} = \phi(0), \qquad M = 1.$$

Lemma 8.2 implies

$$\int_0^{h'} \varphi(t)^{sp}|\psi(t)|^p\,dt \le C^p \int_0^{h'} \varphi(t)^{(s+1)p}|\psi'(t)|^p\,dt,$$

where $\psi(t) = \phi_0(x'_n + t\mathbf{e}_{2n})$ (remind that $\phi_0(x'_n + 2\mathfrak{c}_1\mathbf{e}_{2n}) = 0$ and $2\mathfrak{c}_1 \le h'$).
So, we have

$$\begin{aligned} &\int_0^{h'} \big(\rho_\gamma^{sp}|\phi_0)|^p\big)(x'_n + t\mathbf{e}_{2n})\,dt \le C^p \int_0^{h'} \big(\rho_\gamma^{(s+1)p}|\nabla\phi_0\cdot\mathbf{e}_{2n}|^p\big)(x'_n + t\mathbf{e}_{2n})\,dt, \\ \Longrightarrow\quad &\int_{\mathcal{B}'_n} \rho_\gamma^{sp}|\phi_0)|^p\,dx' \le C^p \int_{\mathcal{B}'_n} \rho_\gamma^{(s+1)p}|\nabla\phi_0|^p\,dx'. \end{aligned} \tag{8.12}$$

Hence, due to (8.6)-(8.8) and the above inequality $(8.12)_2$, we obtain

$$\begin{aligned} \int_\omega \rho_\gamma^{sp}|\phi_0|^p dx' = \int_{\omega\setminus\overline{\omega_{2\mathfrak{c}_1}}} \rho_\gamma^{sp}|\phi_0|^p dx' &\le \sum_{n=1}^N \int_{\mathcal{B}'_n} \rho_\gamma^{sp}|\phi_0|^p dx' \\ &\le \sum_{n=1}^N C^p \int_{\mathcal{B}'_n} \rho_\gamma^{(s+1)p}\big|\nabla\phi_0\big|^p dx' \le C^p \int_\omega \rho_\gamma^{(s+1)p}\big|\nabla\phi_0\big|^p dx'. \end{aligned}$$

Thus

$$\big\|\rho_\gamma^s(1-\psi_0)(\phi-\widetilde{\mathbf{a}})\big\|_{L^p(\omega)} = \|\rho_\gamma^s\phi_0\|_{L^p(\omega)} \le C\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)}.$$

From the above inequality and $(8.11)_1$ we obtain

$$\big\|\rho_\gamma^s(\phi-\widetilde{\mathbf{a}})\big\|_{L^p(\omega)} \le C\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)}$$

for any ϕ in $\mathcal{C}^1(\overline{\omega})$.
Then, taking the mean value of $\rho_\gamma^s(\phi-\widetilde{\mathbf{a}})$ leads to

$$\big|\mathcal{M}_\omega(\rho_\gamma^s\phi) - \widetilde{\mathbf{a}}\mathcal{M}_\omega(\rho_\gamma^s)\big| \le C\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)}.$$

Thus, we can replace $\widetilde{\mathbf{a}}$ by

$$\mathbf{a}_s = \frac{\mathcal{M}_\omega(\rho_\gamma^s\phi)}{\mathcal{M}_\omega(\rho_\gamma^s)}. \tag{8.13}$$

Finally, a density argument gives (8.9) for any $\phi \in W^{1,p}_{\rho_\gamma^{s+1}}(\omega)$. □

Now, since $\gamma_0 \setminus \gamma$ is a set with non-null measure, there exist $n_0 \in \{1,\dots,N\}$, $\mathfrak{c}_3 > 0$ and a closed interval $[c_1, c_2] \subset (0, A)$ such that

$$\gamma_0^* = \Big\{(x_{1n_0}, x_{2n_0}) \in \mathbb{R}^2 \mid x_{2n_0} = f_{n_0}(x_{1n_0}),\ \ x_{1n_0} \in (c_1, c_2)\Big\} \cap \gamma_0$$

has a non-null measure and $\mathrm{dist}(\gamma_0^*, \gamma) \ge 2\mathfrak{c}_3$.

Lemma 8.4. *For any ϕ in $W^{1,p}_{s+1}(\omega)$ vanishing on γ_0^* and any $s \in \mathbb{R}$ such that $sp + 1 > 0$, we have*

$$\big\|\rho_\gamma^s\phi\big\|_{L^p(\omega)} \le C\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)} \tag{8.14}$$

where the constant depends on γ, γ_0^*, $\partial\omega$, s and p.

Proof. As a consequence of the definition of γ_0^*, the distance between γ and

$$\mathcal{B}_{n_0}^* \doteq \Big\{(x_{1n_0}, x_{2n_0}) \in \mathbb{R}^2 \mid f_{n_0}(x_{1n_0}) < x_{2n} < f_{n_0}(x_{1n_0}) + \mathfrak{c}_3, \quad x_{1n_0} \in (c_1, c_2)\Big\}$$

is greater than $\mathfrak{c}_3$.
Then, the Poincaré inequality gives

$$\|\phi\|_{L^p(\mathcal{B}_{n_0}^*)} \leq C\|\nabla\phi\|_{L^p(\mathcal{B}_{n_0}^*)} \leq C\Big(\frac{1}{\mathfrak{c}_3}\Big)^{s+1}\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\mathcal{B}_{n_0}^*)} \leq C\big\|\rho_\gamma^{s+1}\nabla\phi\big\|_{L^p(\omega)}.$$

From Lemma 8.3 there exists $\mathbf{a}_s \in \mathbb{R}$ such that

$$\big\|\rho_\gamma^s(\phi - \mathbf{a}_s)\big\|_{L^p(\omega)} \leq C\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)} \leq C\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)}.$$

So

$$\begin{aligned}\big\|\rho_\gamma^s\mathbf{a}_s\big\|_{L^p(\mathcal{B}_{n_0}^*)} &\leq \big\|\rho_\gamma^s(\phi - \mathbf{a}_s)\big\|_{L^p(\mathcal{B}_{n_0}^*)} + \big\|\rho_\gamma^s\phi\big\|_{L^p(\mathcal{B}_{n_0}^*)} \leq C\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)}\\ &\leq \big\|\rho_\gamma^s(\phi - \mathbf{a}_s)\big\|_{L^p(\omega)} + \mathfrak{c}_3^{1/p}\big\|\phi\big\|_{L^p(\mathcal{B}_{n_0}^*)} \leq C\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)}.\end{aligned}$$

Therefore

$$|\mathbf{a}_s| \leq C\|\rho_\gamma^{s+1}\nabla\phi\|_{L^p(\omega)}.$$

As a consequence (8.14) is proved. □

Corollary 8.5. *[Poincaré-Wirtinger inequality] For any ϕ in $W_s^{1,p}(\omega)$, $s \in [0,1]$, we have*

$$\big\|\phi - \mathcal{M}_\omega(\phi)\big\|_{L^p(\omega)} \leq C\|\rho_\gamma^s\nabla\phi\|_{L^p(\omega)}. \tag{8.15}$$

Moreover, if $\phi = 0$ a.e. on γ_0^* then we have

$$\big\|\phi\big\|_{L^p(\omega)} \leq C\|\rho_\gamma^s\nabla\phi\|_{L^p(\omega)}. \tag{8.16}$$

The constants depend on γ, γ_0^*, $\partial\omega$ and p.

Proof. As a consequence of Lemma 8.3, we have

$$\big\|\phi - \mathcal{M}_\omega(\phi)\big\|_{L^p(\omega)} \leq C\|\rho_\gamma\nabla\phi\|_{L^p(\omega)} \leq C\|\rho_\gamma^s\nabla\phi\|_{L^p(\omega)}.$$

Similarly, if $\phi = 0$ a.e. on γ_0^* then Lemma 8.4 leads to (8.16). □

Lemma 8.6. *For any $s \in [0,1)$ we have*

$$\int_\omega \rho_\gamma^{-s}dx' < +\infty. \tag{8.17}$$

Proof. First, we consider the open set $\mathcal{B}_n'$, $n \in \{1, \dots, N\}$, and the function

$$t \longmapsto \rho_\gamma(x_n^{'} + t\mathbf{e}_{2n}), \qquad x_n^{'} = \big(x_{1n}, f_n(x_{1n})\big), \qquad x_{1n} \in (a, A-a).$$

From the proof of Lemma 8.3, there exists $m > 0$ such that

$$m\big(t + \rho_\gamma(x_n^{'})\big) \leq \rho_\gamma(x_n^{'} + t\mathbf{e}_{2n}) \qquad \forall t \in (0, h^{'}).$$

Then, we have

$$\int_0^{h'} \frac{1}{(\rho_\gamma(x_n' + t\mathbf{e}_{2n}))^s}dt \leq \int_0^{h'} \frac{1}{m^s(t + \rho_\gamma(x_n'))^s}dt \leq \int_0^{h'} \frac{1}{m^s t^s}dt = \frac{(h^{'})^{1-s}}{(1-s)m^s}.$$

Thus

$$\int_\omega \frac{1}{(\rho_\gamma)^s}dx' \leq \sum_{n=1}^N \int_{\mathcal{B}_n'} \frac{1}{(\rho_\gamma)^s}dx' + \int_{\omega_{\mathfrak{c}_1}} \frac{1}{(\rho_\gamma)^s}dx' < +\infty.$$

□

As a consequence we obtain

Lemma 8.7. *For any $p \in (1,\infty)$, $s \in (0, 1/p')$ with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we have the following continuous inclusion:*

$$L^q(\omega) \subset L^p_s(\omega), \qquad \forall q \in \big[1, \frac{p}{ps+1}\big).$$

Proof. Let f be in $L^p_s(\omega)$. The Höldder inequality yields

$$\int_\omega |f|^q dx' = \int_\omega \rho_\gamma^{-sq} (\rho_\gamma^s |f|)^q dx' \leq \Big(\int_\omega \rho_\gamma^{-sqp/(p-q)} dx' \Big)^{1-q/p} \Big(\int_\omega \rho_\gamma^{sp} |f|)^p dx' \Big)^{q/p}.$$

Since $pqs < p - q$ the inclusion is proved. □

8.4 A weighted Korn inequality

Denote ($p \in (1,\infty)$, $s \in \mathbb{R}$)

$$\mathbf{W}^{1,p}_s(\omega) \doteq \Big\{ \Phi \in W^{1,p}_{loc}(\omega)^2 \ \mid \ \|\rho_\gamma^s \, e(\Phi)\|_{L^p(\omega)} < +\infty \Big\}.$$

Lemma 8.8. *For any $s \in \mathbb{R}$ such that $sp + 1 > 0$, we have*

$$\mathbf{W}^{1,p}_{s+1}(\omega) \subset L^p_s(\omega)^2.$$

Moreover, for any displacement $\Phi \in \mathbf{W}^{1,p}_{s+1}(\omega)$ we have

$$\begin{cases} \|\rho_\gamma^s (\Phi - \mathbf{r}_s)\|_{L^p(\omega)} \leq C \|\rho_\gamma^{s+1} \, e(\Phi)\|_{L^p(\omega)}, \\ \|\rho_\gamma^{s+1} \, \nabla(\Phi - \mathbf{r}_s)\|_{L^p(\omega)} \leq C \|\rho_\gamma^{s+1} \, e(\Phi)\|_{L^p(\omega)}, \end{cases} \tag{8.18}$$

where $\mathbf{r}_s$ is a rigid displacement.
In addition, if Φ belongs to $\mathbf{W}^{1,p}_{s+1}(\omega)$ and vanishes on γ_0^* then we can choose $\mathbf{r}_s = 0$.
The constant depends on s, p, γ, $\partial\omega$ and γ_0^*.

Proof. Step 1. We prove $(8.18)_1$.
We start with $\Phi \in \mathcal{C}^1(\overline{\omega})^2$.
We apply the $2D$-Korn inequality in $\omega_{\mathfrak{c}_1}$. There exists a rigid displacement $\widetilde{\mathbf{r}}$ such that

$$\begin{aligned} \|\rho_\gamma^s (\Phi - \widetilde{\mathbf{r}})\|_{L^p(\omega_{\mathfrak{c}_1})} &\leq \big(1 + \mathfrak{c}_1^{-1/p}\big) \|\Phi - \widetilde{\mathbf{r}}\|_{L^p(\omega_{\mathfrak{c}_1})} \leq C \|e(\Phi)\|_{L^p(\omega_{\mathfrak{c}_1})} \leq \frac{C}{\mathfrak{c}_1^{s+1}} \|\rho_\gamma^{s+1} \, e(\Phi)\|_{L^p(\omega_{\mathfrak{c}_1})} \\ &\leq \frac{C}{\mathfrak{c}_1^{s+1}} \|\rho_\gamma^{s+1} \, e(\Phi)\|_{L^p(\omega)}. \end{aligned}$$

We have

$$e_{\alpha\beta}\big((1-\psi_0)(\Phi - \widetilde{\mathbf{r}})\big) = (1-\psi_0) e_{\alpha\beta}(\Phi) - \frac{1}{2}\big(\partial_\alpha \psi_0 (\Phi_\alpha - \widetilde{\mathbf{r}}_\alpha) + \partial_\beta \psi_0 (\Phi_\beta - \widetilde{\mathbf{r}}_\beta)\big).$$

Hence

$$\begin{aligned} \|\rho_\gamma^s \psi_0 (\Phi - \widetilde{\mathbf{r}})\|_{L^p(\omega)} &\leq \|\rho_\gamma^s (\Phi - \widetilde{\mathbf{r}})\|_{L^p(\omega_{\mathfrak{c}_1})} \leq C \|\rho_\gamma^{s+1} \, e(\Phi)\|_{L^p(\omega)}, \\ \|\rho_\gamma^{s+1} \, e\big((1-\psi_0)(\Phi - \widetilde{\mathbf{r}})\big)\|_{L^p(\omega)} &\leq C\big(\|\rho_\gamma^{s+1} e(\Phi)\|_{L^p(\omega)} + \|\rho_\gamma^s (\Phi - \widetilde{\mathbf{r}})\|_{L^p(\omega_{\mathfrak{c}_1})}\big) \\ &\leq C \|\rho_\gamma^{s+1} \, e(\Phi)\|_{L^p(\omega)}. \end{aligned} \tag{8.19}$$

Set $\Phi_0 = (1-\psi_0)(\Phi - \widetilde{\mathbf{r}})$, this displacement vanishes in $\omega_{2\mathfrak{c}_1}$. Moreover, from (8.19) we have

$$\|\rho_\gamma^{s+1} \, e\big(\Phi_0\big)\|_{L^p(\omega)} \leq C \|\rho_\gamma^{s+1} \, e(\Phi)\|_{L^p(\omega)}.$$

Now, proceeding as in Lemma 8.3 and thanks to Lemma 8.2 and the inclusions (8.6)-(8.7) we obtain

$$\int_{(\omega\setminus\overline{\omega_{2\epsilon_1}})\cap\mathcal{B}'_n}\rho_\gamma^{sp}|\Phi_0\cdot\mathbf{e}_{2n}|^p dx' \le C\int_{\mathcal{B}_n}\rho_\gamma^{(s+1)p}|(\nabla\Phi_0\,\mathbf{e}_{2n})\cdot\mathbf{e}_{2n}|^p dx'$$

$$\begin{aligned}\int_{(\omega\setminus\overline{\omega_{2\epsilon_1}})\cap\mathcal{B}'_n}\rho_\gamma^{sp}|\Phi_0\cdot\mathbf{e}_{1n}|^p dx' &\le C\int_{\mathcal{B}_n}\rho_\gamma^{(s+1)p}|(\nabla\Phi_0\,\mathbf{e}_{2n})\cdot\mathbf{e}_{2n}|^p dx'\\ &\quad + C\int_{\mathcal{B}_n}\rho_\gamma^{(s+1)p}\big(|(\nabla\Phi_0\,\mathbf{e}_n^{''})\cdot\mathbf{e}_n^{''}|^p + |(\nabla\Phi_0\,\mathbf{e}_n^{'''})\cdot\mathbf{e}_n^{'''}|^p\big)dx'.\end{aligned}$$

From the above two inequalities we deduce that

$$\int_{(\omega\setminus\overline{\omega_{2\epsilon_1}})\cap\mathcal{B}'_n}\rho_\gamma^{sp}|\Phi_0|^p dx' \le C\int_{\mathcal{B}_n}\rho_\gamma^{(s+1)p}|e(\Phi_0)|^p dx'.$$

Thus

$$\int_{\omega\setminus\overline{\omega_{2\epsilon_1}}}\rho_\gamma^{sp}|\Phi_0|^p dx' \le C\sum_{n=1}^N\int_{(\omega\setminus\overline{\omega_{2\epsilon_1}})\cap\mathcal{B}'_n}\rho_\gamma^{sp}|\Phi_0|^p dx' \le C\sum_{n=1}^N\int_{\mathcal{B}_n}\rho_\gamma^{(s+1)p}|e(\Phi_0)|^p dx' \le C\int_\omega\rho_\gamma^{(s+1)p}|e(\Phi_0)|^p dx'.$$

Hence

$$\|\rho_\gamma^s\Phi_0\|_{L^p(\omega)} = \|\rho_\gamma^s\Phi_0\|_{L^p(\omega\setminus\overline{\omega_{2\epsilon_1}})} \le C\|\rho_\gamma^{s+1}e(\Phi_0)\|_{L^p(\omega)}. \tag{8.20}$$

Finally from $(8.19)_1$ and (8.20) we obtain $(8.18)_1$ for any $\Phi\in\mathcal{C}^1(\overline{\omega})^2$.
Then, we replace $\widetilde{\mathbf{r}}$ by

$$\mathbf{r}_s(x') = \frac{\mathcal{M}_\omega(\rho_\gamma^s\Phi)}{\mathcal{M}_\omega(\rho_\gamma^s)} + b_s\begin{pmatrix}-(x_2-\mathbf{A}_{2s})\\ x_1-\mathbf{A}_{1s}\end{pmatrix}\qquad\forall x'=(x_1,x_2)\in\omega$$

where $\mathbf{A}_s=(\mathbf{A}_{1s},\mathbf{A}_{2s})$ is the "relative center of mass" of ω defined by

$$\int_\omega\rho_\gamma^s(x'-\mathbf{A}_s)dx' = 0$$

and

$$b_s = \frac{1}{\mathbf{I}_s}\int_\omega\rho_\gamma^s\big(\Phi_2(x')(x_1-\mathbf{A}_{1s})-\Phi_1(x')(x_2-\mathbf{A}_{2s})\big)dx',\qquad \mathbf{I}_s=\int_\omega\rho_\gamma^s|x'-\mathbf{A}_s|_2^2dx'.$$

A density argument gives the result for every displacement in $\mathbf{W}^{1,p}_{s+1}(\omega)$.
Step 2. We prove $(8.18)_2$.
Using $(8.18)_1$ and a straightforward calculation, we obtain

$$\big\|e\big(\rho_\gamma^{s+1}(\Phi-\mathbf{r}_s)\big)\big\|_{L^p(\omega)} \le C\|\rho_\gamma^{s+1}e(\Phi-\mathbf{r}_s)\|_{L^p(\omega)} = C\|\rho_\gamma^{s+1}e(\Phi)\|_{L^p(\omega)}.$$

Then, the $2D$-Korn inequality yields

$$\begin{aligned}\big\|\nabla\big(\rho_\gamma^{s+1}(\Phi-\mathbf{r}_s)\big)\big\|_{L^p(\omega)} &\le C\big(\big\|e\big(\rho_\gamma^{s+1}(\Phi-\mathbf{r}_s)\big)\big\|_{L^p(\omega)} + \big\|\rho_\gamma^{s+1}(\Phi-\mathbf{r}_s)\big\|_{L^p(\omega)}\big)\\ &\le C\|\rho_\gamma^{s+1}e(\Phi)\|_{L^p(\omega)}.\end{aligned}$$

Besides we have

$$\nabla\big(\rho_\gamma^{s+1}(\Phi-\mathbf{r}_s)\big) = \rho_\gamma^{s+1}\nabla(\Phi-\mathbf{r}_s) + (s+1)\rho_\gamma^s\nabla\rho_\gamma\cdot(\Phi-\mathbf{r}_s)$$

from which thanks to the estimates $(8.18)_1$ and the above we obtain $(8.18)_2$.
Step 3. We prove the last statement of the proposition.
If Φ belongs to $\mathbf{W}^{1,p}_{s+1}(\omega)$ and vanishes on γ_0^* then proceeding as in the proof of Lemma 8.4 we first obtain

$$\|\rho_\gamma^s\Phi\|_{L^p(\mathcal{B}^*_{n_0})} \le C\|\Phi\|_{L^p(\mathcal{B}^*_{n_0})} \le C\|\rho_\gamma^{s+1}e(\Phi)\|_{L^p(\omega)}$$

which allows to estimate $\mathbf{r}_s$ and then gives (8.18) without rigid displacement. □

Corollary 8.9. *For any Φ in $\mathbf{W}^{1,p}_s(\omega)$, $s \in [0,1]$, we have*

$$\mathbf{W}^{1,p}_s(\omega) \subset L^p(\omega)^2.$$

Moreover, for any displacement $\Phi \in \mathbf{W}^{1,p}_s(\omega)$ we have

$$\|\Phi - \mathbf{r}\|_{L^p(\omega)} \leq C \|\rho^s_\gamma \, e(\Phi)\|_{L^p(\omega)} \tag{8.21}$$

where $\mathbf{r}$ is a rigid displacement.
In addition, if Φ belongs to $\mathbf{W}^{1,p}_s(\omega)$ and vanishes on γ_0^* then we can choose $\mathbf{r} = 0$.
The constant depends on p, γ, $\partial\omega$ and γ_0^*.

Proof. As a consequence of Lemma 8.8, there exists a rigid displacement such that

$$\big\|\Phi - \mathbf{r}\big\|_{L^p(\omega)} \leq C \|\rho_\gamma e(\Phi)\|_{L^p(\omega)} \leq C \|\rho^s_\gamma e(\Phi)\|_{L^p(\omega)}.$$

Similarly, if $\Phi = 0$ a.e. on γ_0^* then, again from Lemma 8.8 we can choose $\mathbf{r} = 0$. □

8.5 Complements

Lemma 8.10. *The domain*

$$\boldsymbol{\mathcal{O}}_{\{s\}} \doteq \big\{x = (x', x_3) \in \omega \times \mathbb{R} \mid |x_3| < \rho^s(x')\big\}, \quad s \in (0,1]$$

has a Lipschitz boundary.

Proof. Step 1. Some recalls.
First, we recall a result shown in [9, Lemma 5.22]: there exists a constant $\mathfrak{c}$, $\mathfrak{c} \in (0, 1/2)$, such that for all $\delta \in [0, \mathfrak{c}]$ the open sets ω_δ are uniformly Lipschitz.
So, there exists an open covering $\big\{\mathcal{V}_i\big\}_{i\in\{1,\dots,M\}}$ of $\partial\omega$ and $2D$-cones $\big\{\mathcal{C}(\mathbf{a}_i, h, \theta)\big\}_{i\in\{1,\dots,M\}}$, $h > 0$, $\theta \in (0, \pi/2)$, $\mathbf{a}_i$ a unit vector, such that for all $\delta \in [0, \mathfrak{c}]$ and $\delta_1 \in (0, \mathfrak{c}]$

- $\Big\{x' \in \omega_\delta \;\mid\; \mathrm{dist}(x', \partial\omega_\delta) < \delta_1\Big\} \subset \bigcup_{i=1}^{M} \big(\omega_\delta \cap \mathcal{V}_i\big)$,
- for every $i \in \{1, \dots, M\}$, $\bigcup_{x' \in \omega_\delta \cap \mathcal{V}_i} \big(x' + \mathcal{C}(\mathbf{a}_i, h, \theta)\big) \subset \omega_\delta$.

For any $i \in \{1, \dots, M\}$, we denote $\mathbf{b}_i \in \mathbb{R}^2$ a unit vector orthogonal to $\mathbf{a}_i$.
Step 2. We prove the claim of the lemma.
We fix $s \in (0, 1]$ and set

$$\mathfrak{c}_s = \min\Big\{\mathfrak{c}, \exp\Big(\frac{\ln(s)}{1-s}\Big)\Big\} < 1.$$

Note that $s\mathfrak{c}_s^{s-1} \geq 1$.
Denote

$$\begin{aligned}
\boldsymbol{\mathcal{O}}_{\{s,i\}} &\doteq \big\{x = (x', x_3) \in \mathcal{V}_i \times \mathbb{R} \;\mid\; |x_3| < \rho^s(x'), \quad \rho(x') < \mathfrak{c}_s\big\}, \qquad i \in \{1, \dots, M\},\\
\boldsymbol{\mathcal{O}}^+_{\{s\}} &\doteq \big\{x = (x', x_3) \in \omega \times \mathbb{R} \;\mid\; 0 < x_3 < \rho^s(x'), \quad \rho(x') > \mathfrak{c}_s/2\big\},\\
\boldsymbol{\mathcal{O}}^-_{\{s\}} &\doteq \big\{x = (x', x_3) \in \omega \times \mathbb{R} \;\mid\; 0 < -x_3 < \rho^s(x'), \quad \rho(x') > \mathfrak{c}_s/2\big\}.
\end{aligned}$$

Step 2. We prove that there exists a $3D$ cone $\mathcal{C}$ such that

$$\bigcup_{x \in \boldsymbol{\mathcal{O}}_{\{s,i\}}} \big(x + \mathcal{C}_i\big) \subset \boldsymbol{\mathcal{O}}_{\{s\}}, \quad \text{for every } i \in \{1, \dots, M\}$$

where $\mathcal{C}_i$ is a cone isometric to a cone $\mathcal{C}$ (given below).
Let $M = \big(x', x_3\big)$ be a point in $\boldsymbol{\mathcal{O}}_{\{s,i\}}$, $x' \in \mathcal{V}_i$ and $|x_3| < \rho^s(x')$. We also have $\rho(x') < \mathfrak{c}_s$.
The $2D$-cone $x' + \mathcal{C}(\mathbf{a}_i, h, \theta)$ is included $\omega_{\rho(x')}$.
Let $\mathcal{P}_i \doteq OP_1P_2P_3P_4$ the right pyramid whose apex is O, height $h = \mathfrak{c}_s/4$ and boundary given by

$$\overrightarrow{OR} = z_1\mathbf{b}_i + z_2\mathbf{a}_i + z_3\mathbf{e}_3, \quad R \in \partial\mathcal{P}_i \quad \text{iff} \quad |z_3| = \sin(\theta)z_2 - \cos(\theta)|z_1|, \qquad 0 \le z_2 \le h.$$

The other four vertices of the pyramid are

$$P_1 = h\mathbf{a}_i + H_3\mathbf{e}_3, \quad P_2 = h\mathbf{a}_i + H_1\mathbf{b}_i, \quad P_3 = h\mathbf{a}_i - H_3\mathbf{e}_3, \quad P_2 = h\mathbf{a}_i - H_1\mathbf{b}_i,$$

where

$$H_1 = h\tan(\theta), \qquad H_3 = h\sin(\theta).$$

The right pyramid $M + \mathcal{P}_i$ is included in $\mathcal{O}_{\{s\}}$.
Indeed, let $\overrightarrow{OQ} = z_1\mathbf{b}_i + z_2\mathbf{a}_i + z_3\mathbf{e}_3$ be a point belonging to the interior of the pyramid $M + \mathcal{P}_i$, $\overrightarrow{OM} = x_1\mathbf{b}_i + x_2\mathbf{a}_i + x_3\mathbf{e}_3$. We have

$$\begin{aligned} & |z_3 - x_3| < \sin(\theta)(z_2 - x_2) - \cos(\theta)|z_1 - x_1| \\ \Longrightarrow \quad & |z_3| < \sin(\theta)(z_2 - x_2) - \cos(\theta)|z_1 - x_1| + |x_3|. \end{aligned}$$

Observe that

$$\rho(x') + \sin(\theta)(z_2 - x_2) - \cos(\theta)|z_1 - x_1| \le \operatorname{dist}(x', \partial\omega) + \operatorname{dist}(z', \partial\omega_{\rho(x')}) \le \operatorname{dist}(z', \partial\omega) = \rho(z').$$

Hence

$$\rho(x') + \mathcal{H} \le \rho(z'), \qquad \mathcal{H} \doteq \sin(\theta)(z_2 - x_2) - \cos(\theta)|z_1 - x_1|.$$

For any $s \in (0, 1]$ we have (remind that $0 < \rho(x') < \mathfrak{c}_s < 1$ and $\rho(z') \le 1$)

$$\rho^s(x') + \mathcal{H} \le \big(\rho(x') + \mathcal{H}\big)^s \quad \text{if} \quad s\rho^{s-1}(x') \ge 1.$$

Therefore, since $s\mathfrak{c}_s^{s-1} \ge 1$ this implies that

$$|z_3| < \rho^s(x') + \mathcal{H} \le \big(\rho(x') + \mathcal{H}\big)^s \le \rho^s(z').$$

The right pyramid $M + \mathcal{P}_i$ contains a cone whose height is h, axis directed by $\mathbf{a}_i$ and base a disc of radius $h\cos(\theta)$. So, we can choose as cone $\mathcal{C}$, the cone whose apex is O (the origin of $\mathbb{R}^3$), height h, axis directed by $\mathbf{e}_3$ and base a disc of radius $h\cos(\theta)$.
Now, for every $(x', y') \in \omega_{\mathfrak{c}_s/4}$ we have

$$|\rho^s(x') - \rho^s(y')| \le K_s|x' - y'|, \qquad K_s \doteq s\Big(\frac{4}{c_s}\Big)^{1-s}.$$

Then, for any $x \in \boldsymbol{\mathcal{O}}^{\pm}_{\{s\}}$ there exists a cone with apex x, half-angle $\theta_s \in (0, \pi/2)$ such that $\cos(\theta_s) = \dfrac{K_s}{\sqrt{1 + K_s^2}}$ and height $h_s = \mathfrak{c}_s/4$ included in $\boldsymbol{\mathcal{O}}_{\{s\}}$.
Then, we can easily give the $N + 2$ cones attached to the cover of the boundary of $\boldsymbol{\mathcal{O}}_{\{s\}}$ and isometric to a same cone.
Step 3. We prove that for all δ in $\Big(0, \Big(\dfrac{\mathfrak{c}_s}{2}\Big)^s \dfrac{1}{\sqrt{1 + K_s^2}}\Big]$ we have

$$\Big\{x \in \boldsymbol{\mathcal{O}}_{\{s\}} \;\mid\; \operatorname{dist}(x, \partial\boldsymbol{\mathcal{O}}_{\{s\}}) < \delta\Big\} \subset \boldsymbol{\mathcal{O}}^+_{\{s\}} \cup \boldsymbol{\mathcal{O}}^-_{\{s\}} \cup \bigcup_{i=1}^{M} \boldsymbol{\mathcal{O}}_{\{s,i\}}.$$

Let x be in $\boldsymbol{\mathcal{O}}_{\{s\}}$ such that $\operatorname{dist}\big(x, \partial\boldsymbol{\mathcal{O}}_{\{s\}}\big) < \delta$. If

- $\rho(x') > \mathfrak{c}_s/2$ and $\rho^s(x') > x_3 > 0$ then, simple geometrical arguments lead to

$$\frac{\rho^s(x') - x_3}{\sqrt{1+K_s^2}} \leq \mathrm{dist}(x, \partial\boldsymbol{\mathcal{O}}^+_{\{s\}}) \leq \rho^s(x') - x_3.$$

  So, $x_3 > \rho^s(x') - \delta\sqrt{1+K_s^2} > 0$ then x belongs to $\boldsymbol{\mathcal{O}}_{\{s\}}$.

- $\rho(x') < \mathfrak{c}_s$ then, there exists $i \in \{1, \ldots, M\}$ such that $x' \in \mathcal{V}_i$, thus $x \in \boldsymbol{\mathcal{O}}_{\{s,i\}}$.

Hence, the claim of this step is proved.
As a consequence of steps 1 and 2, $\boldsymbol{\mathcal{O}}_{\{s\}}$ is a domain with a Lipschitz boundary. □

Remark 8.11. *Denote*

$$(\omega_\gamma)_\delta \doteq \big\{x' \in \omega \mid \rho_\gamma(x') < \delta\big\}.$$

One can construct open subsets γ of $\partial\omega$ (with countable connected components) such that for all $k \in \mathbb{N}^*$ the boundary of $(\omega_\gamma)_{1/k}$ is not Lipschitz. Consequently, the corresponding 3D open sets

$$\boldsymbol{\mathcal{O}}_s = \big\{x = (x', x_3) \in \omega\times\mathbb{R} \mid |x_3| < \rho^s_\gamma(x')\big\}, \quad s \in (0,1]$$

are not Lipschitz.
If ω is a polygonal domain and if γ has a finite number of open connected components then, proceeding as in the proof of Lemma 8.10, we can show that there exists $\mathfrak{c}_4 \in (0,1]$ such that for all $\delta \in (0, \mathfrak{c}_4]$ the sets $\big(\omega_\gamma\big)_\delta$ are uniformly Lipschitz and therefore $\boldsymbol{\mathcal{O}}_s$ has a Lipschitz boundary ($s \in (0,1]$).

We denote ($p \in (1,\infty)$, $s \in \mathbb{R}$)

$$\mathbf{W}^{1,p}_{\{s\}}(\omega) \doteq \Big\{\Phi \in W^{1,p}_{loc}(\omega)^2 \ \mid \ \|\rho^s\, e(\Phi)\|_{L^p(\omega)} < +\infty\Big\},$$
$$L^p_{\{s\}}(\omega) \doteq \Big\{\Phi \in L^p_{loc}(\omega) \ \mid \ \|\rho^s\, \Phi\|_{L^p(\omega)} < +\infty\Big\}.$$

Proposition 8.12. *We have for any $s \in \mathbb{R}$ such that $sp + 1 > 0$*

$$\mathbf{W}^{1,p}_{\{s+1\}}(\omega) \subset L^p_{\{s\}}(\omega)^2.$$

Moreover, for any displacement $\Phi \in \mathbf{W}^{1,p}_{\{s+1\}}(\omega)$ we have

$$\begin{cases} \|\rho^s(\Phi - \mathbf{r}_s)\|_{L^p(\omega)} \leq C\|\rho^{s+1}\, e(\Phi)\|_{L^p(\omega)}, \\ \|\rho^{s+1}\, \nabla(\Phi - \mathbf{r}_s)\|_{L^p(\omega)} \leq C\|\rho^{s+1}\, e(\Phi)\|_{L^p(\omega)}, \end{cases} \tag{8.22}$$

where $\mathbf{r}_s$ is a rigid displacement.
The constants depend on s, p and $\partial\omega$.

Proof. In Lemma 8.8 we replace γ by $\partial\omega$, which gives all the results of the proposition. □

Proposition 8.13. *We have for any $s \in (0,1]$*

$$\mathbf{W}^{1,p}_{\{s\}}(\omega) \subset L^p(\omega)^2.$$

Moreover,

$$\forall \Phi \in \mathbf{W}^{1,p}_{\{s\}}(\omega), \qquad \begin{cases} \|\Phi - \mathbf{r}\|_{L^p(\omega)} \leq C\|\rho^s\, e(\Phi)\|_{L^p(\omega)}, \\ \|\rho^s\, \nabla(\Phi - \mathbf{r})\|_{L^p(\omega)} \leq C\|\rho^s\, e(\Phi)\|_{L^p(\omega)}, \end{cases} \tag{8.23}$$

where $\mathbf{r}$ is a rigid displacement.
The constants depend on s, p and $\partial\omega$.

Proof. For any $s \in (0,1]$ and any $\Phi \in \mathbf{W}^{1,p}_{\{s\}}(\omega)$, we have

$$\|\rho^s\, e(\Phi)\|_{L^p(\omega)} = \|e(\widetilde{\Phi})\|_{L^p(\boldsymbol{\mathcal{O}}_{\{s\}})}$$

where $\widetilde{\Phi} = \Phi_1\mathbf{e}_1 + \Phi_2\mathbf{e}_2 + 0\mathbf{e}_3$. Hence, $\widetilde{\Phi}$ is a displacement belonging to $W^{1,p}(\boldsymbol{\mathcal{O}}_{\{s\}})^3$. The $3D$ Korn inequality gives a rigid displacement $\widetilde{\mathbf{r}}$ such that

$$\|\widetilde{\Phi} - \widetilde{\mathbf{r}}\|_{W^{1,p}(\boldsymbol{\mathcal{O}}_{\{s\}})} \leq C\|e(\widetilde{\Phi})\|_{L^p(\boldsymbol{\mathcal{O}}_{\{s\}})}$$

from which we easily obtain of a $2D$ rigid displacement $\mathbf{r}$ such that

$$\|\rho^s(\Phi - \mathbf{r})\|_{L^p(\omega)} + \|\rho^s\, \nabla(\Phi - \mathbf{r})\|_{L^p(\omega)} \leq C\|\rho^s\, e(\Phi)\|_{L^p(\omega)}.$$

Then, since $\mathbf{W}^{1,p}_{\{s\}}(\omega)$ is included in $\mathbf{W}^{1,p}_{\{1\}}(\omega)$, there exists a rigid displacement $\mathbf{r}'$ such that estimate $(8.22)_1$ holds. So

$$\|\Phi - \mathbf{r}'\|_{L^p(\omega)} \leq C\|\rho\, e(\Phi)\|_{L^p(\omega)} \leq C\|\rho^s\, e(\Phi)\|_{L^p(\omega)}.$$

From the above inequalities we obtain

$$\|\rho^s(\mathbf{r}' - \mathbf{r})\|_{L^p(\omega)} \leq C\|\rho^s\, e(\Phi)\|_{L^p(\omega)}.$$

Thus, we can replace $\mathbf{r}'$ by $\mathbf{r}$. This ends the proof of the proposition. □

Proposition 8.14. *For any Φ in $\mathbf{W}^{1,p}_s(\omega)$, $s \in (0,1]$, we have*

$$\begin{aligned} &\|\Phi - \mathbf{r}\|_{L^p(\omega)} \leq C\|\rho_\gamma^s\, e(\Phi)\|_{L^p(\omega)}, \\ &\|\rho^s\, \nabla(\Phi - \mathbf{r})\|_{L^p(\omega)} \leq C\|\rho_\gamma^s\, e(\Phi)\|_{L^p(\omega)}, \end{aligned} \tag{8.24}$$

where $\mathbf{r}$ is a rigid displacement.
If $\boldsymbol{\mathcal{O}}_s$ has a Lipchitz boundary then we have

$$\forall \Phi \in \mathbf{W}^{1,p}_s(\omega), \qquad \|\rho_\gamma^s\, \nabla(\Phi - \mathbf{r})\|_{L^p(\omega)} \leq C\|\rho_\gamma^s\, e(\Phi)\|_{L^p(\omega)}. \tag{8.25}$$

In addition, if Φ belongs to $\mathbf{W}^{1,p}_s(\omega)$ and vanishes on γ_0^* then in all cases we can choose $\mathbf{r} = 0$.
The constants depend on s, p, $\partial\omega$, γ and γ_0^*.

Proof. First, observe that $\mathbf{W}^{1,p}_s(\omega)$ is included in $\mathbf{W}^{1,p}_{\{s\}}(\omega)$ since we have

$$\|\rho^s\, e(\Phi)\|_{L^p(\omega)} \leq \|\rho_\gamma^s\, e(\Phi)\|_{L^p(\omega)}.$$

As a consequence we obtain (see Proposition 8.13)

$$\|\Phi - \mathbf{r}\|_{L^p(\omega)} + \|\rho^s\, \nabla(\Phi - \mathbf{r})\|_{L^p(\omega)} \leq C\|\rho^s\, e(\Phi)\|_{L^p(\omega)} \leq C\|\rho_\gamma^s\, e(\Phi)\|_{L^p(\omega)}.$$

This gives (8.24).
The proof of (8.25) follows the same lines as the proof of $(8.22)_2$. □

All the results of subsections 8.2, 8.3, 8.4 and 8.5 can be extended to domains ω included in $\mathbb{R}^k$, $k \geq 3$, having a Lipschitz boundary.

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