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# Homogeneous Control Design Using Invariant Ellipsoid Method

Siyuan WANG, Haibin DUAN, Gang ZHENG, Xubin PING, Driss BOUTAT, Andrey POLYAKOV

**Abstract**—The invariant ellipsoid method is aimed at minimization of the smallest invariant and attractive set of a control system operating under bounded external disturbances and parametric uncertainties. This paper extends this technique to a class of the so-called generalized homogeneous system. The generalized homogeneous optimal (in the sense of invariant ellipsoid) controller allows further improvement of the control system providing a faster convergence and smaller overshoots. Theoretical results are supported by numerical simulations and experiments.

**Index Terms**—Homogeneity, Invariant set, LMI

## I. INTRODUCTION

During the formulation of any control issue, there is always a discrepancy between the actual system and the mathematical model used for control design. This mismatch arises from unmodeled dynamics, uncertainties in system parameters, or the approximation of complex plants. However, the engineer needs to guarantee that the designed controller is able to achieve the required performance despite of all these mismatches. This leads to the development of the so-called robust control methods solving this problem. Robust control design is an approach dealing with perturbations of a nominal system. Its objective is to attain the certain level of performance or stability in the system despite the presence of bounded disturbance. Several well-known methodologies of robust control design have been invented such as sliding mode control,  $H_\infty$  approach, attractive/invariant ellipsoid method, and so on. The sliding mode control (SMC) as a robust control design methodology is known since 1960s in Russia. The first survey paper by V. Utkin is published in English in 1977 [1]. Sliding mode methodology has been devised for both linear and nonlinear systems [2], [3] and delivers a good performance in numerous real-world scenarios [4]. The concepts of the  $H_\infty$  control theory to solve the robust stabilization problem for linear system can be found in [5]–[7]. Later the  $H_\infty$  control methodology was extended to various systems [8].

The basic ideas of the attractive/invariant method were introduced in the papers [9]–[11]. One of the main features of

this method is that the states of the robustly stabilized system converge to a minimal (in some sense) ellipsoidal set regardless of perturbations or uncertainties satisfying certain bounds. The attractive/invariant ellipsoid method is widely applied to various control and estimation problems for both linear [12] and nonlinear plants [13]. It deals with linear feedback laws. The key feature of the method is control parameters tuning by means of Linear Matrix Inequalities (LMIs) [14]. The optimal tuning turns out to be equivalent to a Semi-Definite Programming (SDP) problem minimizing the invariant ellipsoid of the closed-loop system. The further improvement of control quality using the same *linear feedback* strategy seems infeasible. This paper develops the attractive/invariant ellipsoid method for a class of generalized homogeneous controllers [15], which may provide faster response, better robustness and smaller overshoots comparing with linear algorithms.

A generalized homogeneity is a crucial characteristic of a homogeneous controller studied in this paper. Mathematically, it represents a symmetry with respect to a dilation, as discussed in [16], [17]. This concept is applied in the field of control theory and experiment for the purpose of system analysis, controller and observer design (e.g., [18]–[24] and references therein). The homogeneity with respect to the standard dilation  $x \mapsto e^s x$  was introduced by Leonhard Euler in the eighteenth century. The homogeneity of dynamical systems with respect to the weighted dilation  $(x_1, x_2, \dots, x_n) \mapsto (e^{r_1 s} x_1, \dots, e^{r_n s} x_n)$  was first studied by Vladimir Zubov in 1958 (see [16]). The generalized concepts of the homogeneity (geometric homogeneity) in  $\mathbb{R}^n$  were considered in [25], [26]. This paper deals with the *linear geometric dilation* (see e.g., [27]) given by  $x \mapsto e^{G_d s} x$ , where  $G_d \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz matrix<sup>1</sup>. The homogeneity as a relaxation of linearity provides an extra degree of freedom for further minimization of the disturbance effects [28].

The key contribution of this article is the extension of the attractive/invariant ellipsoid approach to the generalized homogeneous control systems [29]. Specifically, we introduce a homogeneous invariant ellipsoid using the canonical homogeneous norm [27] and derive its characterization in terms of LMIs. We also show that an optimal tuning a homogeneous controller can be realized by solving an SDP problem minimizing the homogeneous attractive/invariant ellipsoid of the closed-loop system. As an example, the controlled rotary inverted pendulum is studied. We demonstrate that the optimal (in the sense of attractive ellipsoid) homogeneous controller can stabilize the pendulum with a better precision than the

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<sup>1</sup>The matrix  $G_d \in \mathbb{R}^n$  is anti-Hurwitz, if  $(-G_d)$  is Hurwitz.

optimal (in the same sense) linear controller without any degradation of control quality.

The structure of paper is as follows: The problem statement is outlined in Section II. Preliminaries to homogeneity theory are presented in Section III. The main results are discussed in Sections IV and V. Simulation/experiment results and some concluding remarks are given Section VI provides the simulation results to support the proposed theories.

*Notation* :  $\mathbb{R}$ : set of real numbers,  $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ ;  $|x| = \sqrt{x^\top x}$ : norm in  $\mathbb{R}^n$ ;  $\|x\|_P = \sqrt{x^\top P x}$ : weight norm in  $\mathbb{R}^n$ ;  $\text{diag}\{\lambda_i\}_{i=1}^n$ : the diagonal matrix with elements  $\lambda_i$ ;  $P \succ 0$  ( $\prec 0$ ,  $\succeq 0$ ,  $\preceq 0$ ) for  $P \in \mathbb{R}^{n \times n}$  means that the matrix  $P$  is symmetric and positive (negative) definite (semi definite);  $\lambda_{\min}(P)$  and  $\lambda_{\max}(P)$ : the minimal and maximal eigenvalues of matrix  $P = P^\top$ ; for  $P \succeq 0$  the square root of  $P$  is a matrix  $M = P^{\frac{1}{2}}$  such that  $M^2 = P$ ;  $\text{tr}(P)$ : trace of matrix  $P$ ; if  $V : \mathbb{R}^n \mapsto [0, +\infty)$  is a positive definite function and  $\dot{x} = f(t, x)$  is an ODE with  $f$  continuous on  $c$  then we denote the time derivative of  $V$  along all solutions of the ODE as  $\frac{d}{dt}V(x) := \frac{\partial V}{\partial x}f(t, x)$ ; a function  $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be of class  $\mathcal{K}$  if it is continuous strictly increasing and satisfies  $\gamma(0) = 0$ ; a function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be of class  $\mathcal{KL}$  if for each fixed  $t$  the mapping  $\beta(\cdot, t)$  is of class  $\mathcal{K}$  and for each fixed  $s$  it is decreasing to zero on  $t$  as  $t \rightarrow \infty$ ;  $L^\infty$  is the space of Lebesgue measurable essentially bounded function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  with norm defined as  $\|\sigma\|_{L^\infty} := \text{ess sup}_{t \in \mathbb{R}_+} \|\sigma(t)\|_\infty < +\infty$ ;

## II. PROBLEM STATEMENT

In this paper, we deal with the linear time-invariant system with additive perturbations,

$$\dot{x}(t) = Ax(t) + Bu(t) + D\omega(t), \quad t > 0 \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the system state,  $\omega(t) \in \mathbb{R}^p$  is an external bounded disturbance,  $u(t) \in \mathbb{R}^m$  is the control input,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $D \in \mathbb{R}^{n \times p}$  are constant matrix. We assume that the pair  $\{A, B\}$  is controllable.

We study a stabilization problem of the system (1). In the general case, it is not possible to achieve a precise stabilization of this system to zero, but a feedback controller  $u(x)$  can be optimally tuned in order to minimize (in some sense) the effect of external disturbance  $\omega$ . For the case of a linear stabilizing feedback there are several well-known methodologies tackling this problem. For example,  $H_\infty$  and  $H_2$  algorithms (see e.g., [30], [31]) suggest to optimize a norm of a transfer function, while the attractive/invariant ellipsoid method [12] optimizes (in a certain sense) the attractive/invariant set of the system (1) with a linear feedback. This paper is aimed at an extension of the latter approach to a special class of generalized homogeneous [23], [27], [29], [32] feedback laws. The mentioned nonlinear control algorithms are known to be efficient for finite/fixed-time stabilization of linear plants. Our objective is to design a generalized homogeneous controller that minimizes the attractive/invariant ellipsoid of the closed-loop system (1) with

$$\omega^\top(t)Q\omega(t) \leq 1, \quad \omega \in L^\infty(\mathbb{R}_+, \mathbb{R}^p), \quad (2)$$

where  $0 \prec Q = Q^\top \in \mathbb{R}^{p \times p}$  is a known matrix.

## III. PRELIMINARIES: ELEMENTS OF THE HOMOGENEITY THEORY

### A. Linear dilation and monotonicity

The homogeneity is a symmetry of an object (e.g. a function or a set) with respect to a group of transformations called dilation. In this paper we deal only with the linear continuous dilation [15]:  $x \mapsto \mathbf{d}(s)x$ ,  $x \in \mathbb{R}^n$ ,  $s \in \mathbb{R}$ , where  $s \in \mathbb{R}$  is a parameter of the dilation and  $\mathbf{d}(s) = e^{sG_d} = \sum_{i=0}^{+\infty} \frac{s^i G_d^i}{i!}$ . The anti-Hurwitz matrix  $G_d \in \mathbb{R}^{n \times n}$  is known as the *generator of the dilation*  $\mathbf{d}$ .

The monotonicity of dilation is useful for a characterization of homogeneous geometric structures in  $\mathbb{R}^n$  and an analysis of homogeneous control systems.

*Definition 3.1:* [15] *Dilation  $\mathbf{d}(s)$  is strictly monotone with respect to a norm  $\|\cdot\|$  in  $\mathbb{R}^n$ , if there exist  $\beta > 0$  such that*

$$\|\mathbf{d}(s)\| := \sup_{x \neq 0} \frac{\|\mathbf{d}(s)x\|}{\|x\|} \leq e^{\beta s}, \quad \forall s \leq 0. \quad (3)$$

The monotonicity property implies that  $\mathbf{d}(s)$  acts as a strong contraction when  $s < 0$  and a strong expansion when  $s > 0$ .

*Theorem 3.1:* [15, Corollary 6.5] *If  $\mathbf{d}$  is a linear dilation in  $\mathbb{R}^n$ , then the following statement holds :*

- 1)  *$\mathbf{d}$  is strictly monotone with respect to the norm  $\|\cdot\|_P$  if and only if*

$$PG_d + G_d^\top P \succ 0, \quad P \succ 0. \quad (4)$$

- 2) *if  $\mathbf{d}$  is strictly monotone with respect to the norm  $\|x\|_P = \sqrt{x^\top P x}$  then*

$$e^{\alpha s} \leq \|\mathbf{d}(s)\|_P \leq e^{\beta s}, \quad \text{if } s \leq 0 \quad (5)$$

$$e^{\beta s} \leq \|\mathbf{d}(s)\|_P \leq e^{\alpha s}, \quad \text{if } s \geq 0 \quad (6)$$

where  $\alpha = \frac{1}{2}\lambda_{\max}(P^{\frac{1}{2}}G_dP^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_d^\top P^{\frac{1}{2}})$  and  $\beta = \frac{1}{2}\lambda_{\min}(P^{\frac{1}{2}}G_dP^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_d^\top P^{\frac{1}{2}})$ .

### B. Canonical homogeneous norm

In the case of a strictly monotone dilation, the so-called canonical homogeneous norm [32] can be introduced through a homogeneous projection onto the unit sphere.

*Definition 3.2:* [15] *Let a linear dilation  $\mathbf{d}$  in  $\mathbb{R}^n$  be strictly monotone with respect to the norm  $\|\cdot\|_P$ . The  $\mathbf{d}$ -homogeneous function  $\|\cdot\|_{\mathbf{d},P} : \mathbb{R}^n \mapsto [0, +\infty)$  defined as follows*

$$\|x\|_{\mathbf{d},P} := \begin{cases} e^{s_x} : \|\mathbf{d}(-s_x)x\|_P = 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \quad (7)$$

*is called the canonical homogeneous norm.*

As the dilation is strictly monotone then it has been demonstrated in [27] that  $\|\cdot\|_{\mathbf{d},P}$  is single-valued, continuous everywhere on  $\mathbb{R}^n$  and continuously differentiable on  $\mathbb{R}^n \setminus \{0\}$ :

$$\frac{\partial \|x\|_{\mathbf{d},P}}{\partial x} = \|x\|_{\mathbf{d},P} \frac{x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d},P}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d},P})}{x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d},P}) P G_d \mathbf{d} (-\ln \|x\|_{\mathbf{d},P}) x}, \quad x \neq 0. \quad (8)$$

Moreover, for any  $x \in \mathbb{R}^n$  and  $s \in \mathbb{R}$ , we have the following properties:  $\|x\|_{\mathbf{d},P} = \|-x\|_{\mathbf{d},P}$ ,  $\|\mathbf{d}(s)x\|_{\mathbf{d},P} = e^s \|x\|_{\mathbf{d},P}$  and  $\|x\|_{\mathbf{d},P} = 1 \Leftrightarrow \|x\|_P = 1$ .

*Definition 3.3:* [25] *A function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be  $\mathbf{d}$ -homogeneous of degree  $\nu$  if*

$$h(\mathbf{d}(s)x) = e^{\nu s} h(x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \forall s \in \mathbb{R} \quad (9)$$

*Definition 3.4:* [25] A vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be  $\mathbf{d}$ -homogeneous of degree  $\nu \in \mathbb{R}$  if

$$f(\mathbf{d}(s)x) = e^{\nu s} \mathbf{d}(s)f(x), \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}, \forall s \in \mathbb{R} \quad (10)$$

The property of homogeneity of the vector field  $f$  implies that the solutions  $x(t, x_0)$  of the system  $\dot{x} = f(x), t > 0, x(0) = x_0$  are also homogeneous, meaning that  $x(t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(e^{\nu s}t, x_0)$ , where  $\nu \in \mathbb{R}$  represents the homogeneity degree of  $f$ .

### C. Homogeneous control design for linear plants

Homogeneous control systems offer several benefits compared to linear ones, including faster convergence [19], enhanced robustness [33] and reduced overshoots [15, Chapter 1]. The following theorem recalls a procedure of the generalized homogeneous control design for linear plants and summarizes the results of the papers [22], [29], [34].

*Theorem 3.2:* For the time-invariant controllable system

$$\dot{x} = Ax + Bu \quad (11)$$

let a pair  $\{A, B\}$  be controllable. Then

- 1) any solution  $Y_0 \in \mathbb{R}^{m \times n}, G_0 \in \mathbb{R}^{n \times n}$  of the linear algebraic equation

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = 0 \quad (12)$$

is such that the matrix  $G_0 - I_n$  is invertible, the matrix  $G_{\mathbf{d}} = I_n + \mu G_0$  is anti-Hurwitz for any  $\mu \in [-1, \frac{1}{\tilde{n}}]$ , where  $\tilde{n}$  is a minimal natural number such that  $\text{rank}[B, AB, \dots, A^{\tilde{n}-1}B] = n$ , the matrix  $A_0 = A + BY_0(G_0 - I_n)^{-1}$  satisfies the identity

$$A_0G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_n)A_0, \quad G_{\mathbf{d}}B = B \quad (13)$$

- 2) the linear algebraic system

$$A_0X + XA_0^{\top} + BY + Y^{\top}B + \rho(G_{\mathbf{d}}X + XG_{\mathbf{d}}^{\top}) = 0 \quad (14)$$

$$G_{\mathbf{d}}X + XG_{\mathbf{d}}^{\top} \succ 0, \quad X = X^{\top} \succ 0 \quad (15)$$

has a solution  $X \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{m \times n}$  for any  $\rho \in \mathbb{R}_+$

- 3) the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d},P}$  induced by the weighted Euclidean norm  $\|x\|_P = \sqrt{x^{\top}Px}$  with  $P = X^{-1}$  is a Lyapunov function of system (11) with

$$u(x) = K_0x + \|x\|_{\mathbf{d},P}^{\mu+1} K \mathbf{d}(-\ln \|x\|_{\mathbf{d},P})x \quad (16)$$

$$K_0 = Y_0(G_0 - I_n)^{-1}, \quad K = YX^{-1} \quad (17)$$

- 4) the feedback law  $u$  given by (16) is continuously differentiable on  $\mathbb{R}^n \setminus \{0\}$ ,  $u$  is continuous at zero if  $\mu > -1$  and  $u$  is discontinuous at zeros if  $\mu = -1$ ;
- 5) the system (11), (16) is  $\mathbf{d}$ -homogeneous of degree  $\mu$ .

Clearly, when the homogeneous degree  $\mu < 0$ , the closed-loop system (11) with controller (16) is uniformly finite-time stable and if  $\mu > 0$ , it is nearly fixed-time stable. If  $\mu = 0$ , the controller (16) turn out to be a linear controller  $u = K_0x + Kx$ .

## IV. HOMOGENEOUS INVARIANT ELLIPSOID

Let us consider the following MIMO (multiple-inputs multiple-outputs) non-linear system:

$$\dot{x} = f(x, \omega), \quad t > 0 \quad (18)$$

where  $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous vector field,  $x$  and  $\omega$  are defined as before.

To have a bounded solution of (18) for any bounded perturbation  $\omega$ , the system must satisfy a condition like ISS<sup>2</sup>. Recall [12] [36], that the set

$$\varepsilon(P) = \{x \in \mathbb{R}^n : \|x\|_P \leq 1\}, \quad P \succ 0 \quad (19)$$

is an ellipsoid centered at origin and configured with the matrix  $P$ . Similarly, we define the  $\mathbf{d}$ -homogeneous ellipsoid as follows

$$\varepsilon_{\mathbf{d}}(P) = \{x \in \mathbb{R}^n : \|x\|_{\mathbf{d},P} \leq 1\}, \quad P \succ 0 \quad (20)$$

where  $\|\cdot\|_{\mathbf{d},P}$  is the canonical homogeneous norm induced by the weighted Euclidean norm  $\|\cdot\|_P$ .

*Definition 4.1:* [12] For system (18), (2), a  $\mathbf{d}$ -homogeneous ellipsoid  $\varepsilon_{\mathbf{d}}(P)$  with a configuration matrix  $P$  is said to be

- **Invariant** if the condition  $\forall x(0) \in \varepsilon_{\mathbf{d}}(P)$  implies  $x(t) \in \varepsilon_{\mathbf{d}}(P)$  for any  $t \geq 0$ ;
- **Attractive** if  $x(0) \notin \varepsilon_{\mathbf{d}}(P)$  then  $x(t) \rightarrow \varepsilon_{\mathbf{d}}(P)$  as  $t \rightarrow \infty$ .

In other words, an ellipsoid is considered invariant if it retains any trajectory initiated from the interior of the ellipsoid. If the ellipsoid is attractive, any trajectory starting outside the ellipsoid will converges to (or into) it as time goes to infinity. The definition of the conventional invariant ellipsoid can be obtained by replacing  $\varepsilon_{\mathbf{d}}(P)$  with  $\varepsilon(P)$  in the above definition. For any fixed  $G_{\mathbf{d}}$ , the set of the conventional ellipsoids  $\varepsilon(P)$  is more rich than the set of  $\mathbf{d}$ -homogeneous ellipsoids  $\varepsilon_{\mathbf{d}}(P)$ , since, according to the definition of the canonical homogeneous norm, the matrix  $P$  for the  $\mathbf{d}$ -homogeneous ellipsoid must satisfy the restriction  $P \in \mathbb{R}^{n \times n} : PG_{\mathbf{d}} + G_{\mathbf{d}}^{\top}P \succ 0$ , which is required for the existence of the canonical homogeneous norm. However, for any positive definite symmetric matrix  $P$ , a generator  $G_{\mathbf{d}}$  can always be selected such that the latter matrix inequality holds. Indeed, according to the Theorem 3.2 the generator can always be selected as  $G_{\mathbf{d}} = I_n + \mu G_0$ , then for  $\mu$  sufficiently close to zero we have  $PG_{\mathbf{d}} + G_{\mathbf{d}}^{\top}P = 2P + \mu(PG_0 + G_0^{\top}P) \succ 0$  due to the positive definiteness of  $P$ . This means by choosing an appropriate dilation  $\mathbf{d}$ , any conventional ellipsoid can be converted into a  $\mathbf{d}$ -homogeneous one. The conventional and  $\mathbf{d}$ -homogeneous invariant ellipsoids have similar characterization.

*Lemma 4.1:* The following two claims are equivalent :

- 1)  $\varepsilon_{\mathbf{d}}(P)$  is a  $\mathbf{d}$ -homogeneous invariant ellipsoid of the system (18);
- 2)  $PG_{\mathbf{d}} + G_{\mathbf{d}}^{\top}P \succ 0$  and  $x^{\top}Pf(x, \omega) \leq 0, \forall x \in \mathbb{R}^n : \|x\|_P = 1, \forall \omega \in \mathbb{R}^p : \|\omega\|_Q \leq 1$ .

**Proof.** 1)  $\Rightarrow$  2) Suppose inversely that  $\varepsilon_{\mathbf{d}}(P)$  is invariant, but  $\exists x_0 : \|x_0\|_P = 1, \exists \omega_0 \in \mathbb{R}^p : \|\omega_0\|_Q \leq 1$  such that

<sup>2</sup>The system (18) is considered Input-to-State Stable (ISS) [35] if there exist two functions,  $\beta$  from the class  $\mathcal{KL}$  and  $\gamma$  from the class  $\mathcal{K}$ , such that for every bounded control input  $u(\cdot)$  and initial state  $x_0$ , the solution satisfies  $\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|)$ .

$x_0^\top Pf(x_0, \omega_0) > 0$ . Notice that the LMI  $PG_{\mathbf{d}} + G_{\mathbf{d}}P \succ 0$  is fulfilled due to Theorem 3.1. Then, for any solution  $x(t)$  of the system (18) with  $\omega = \omega_0$ , by the formula (8) we have

$$\frac{d\|x(t)\|_{\mathbf{d},P}}{dt} = r(t) \frac{x^\top(t) \mathbf{d}^\top(-\ln r(t)) P \mathbf{d}(-\ln r(t)) f(x(t), \omega_0)}{x^\top(t) \mathbf{d}^\top(-\ln r(t)) P G_{\mathbf{d}} \mathbf{d}(-\ln r(t)) x(t)},$$

where  $r(t) = \|x(t)\|_{\mathbf{d},P}$ . Since  $f$  and  $\|\cdot\|_{\mathbf{d},P}$  are continuous and  $x(t) \rightarrow x_0$  as  $t \rightarrow 0^+$  then  $\|x(t)\|_{\mathbf{d},P} \rightarrow \|x_0\|_{\mathbf{d},P} = 1$  as  $t \rightarrow 0^+$  and  $\left. \frac{d\|x(t)\|_{\mathbf{d},P}}{dt} \right|_{t=0^+} = \frac{x_0^\top P f(x_0, \omega_0)}{x_0^\top P G_{\mathbf{d}} x_0} > 0$ , where the equivalence  $\|x_0\|_{\mathbf{d},P} = 1 \Leftrightarrow \|x_0\|_P = 1$  is utilized. The obtained inequality means that the function  $t \mapsto \|x(t)\|_{\mathbf{d},P}$  is growing on some interval of time  $[0, \varepsilon]$  and  $\|x(t)\|_{\mathbf{d},P} > 1$  for  $t \in (0, \varepsilon)$ . This contradicts to the invariance of the ellipsoid  $\varepsilon_{\mathbf{d}}(P)$ .

2)  $\Rightarrow$  1) Suppose  $\varepsilon_{\mathbf{d}}(P)$  is not invariant ellipsoid, i.e., for some  $\omega$  satisfying (2) there exists a solution  $x(t)$  of the system (18) initiated inside the ellipsoid  $\varepsilon_{\mathbf{d}}(P)$  such that there exists  $T > 0 : \|x(T)\|_P > 1$ . Since the function  $t \mapsto \|x(t)\|_{\mathbf{d}}$  is continuous then there exists  $t_b \in (0, T)$  such that  $\|x(t_b)\|_{\mathbf{d},P} = 1$  and  $1 < \|x(t)\|_{\mathbf{d},P}$  for all  $t \in (t_b, T)$ . In this case, using the continuity of  $f$  we derive  $\limsup_{t \rightarrow t_b} \frac{\|x(t)\|_{\mathbf{d},P} - \|x(t_b)\|_{\mathbf{d},P}}{t - t_b} = \limsup_{t \rightarrow t_b} \frac{x^\top(t_b) P f(x(t_b), \omega(t_b))}{x(t_b)^\top P G_{\mathbf{d}} x(t_b)} > 0$ . Hence, there exists the time instant  $t^*$  (belonging to a neighborhood of  $t_b$ ) such that  $\omega(t^*)^\top Q \omega^*(t^*) \leq 1$  and  $\frac{x^\top(t_b) P f(x(t_b), \omega(t^*))}{x(t_b)^\top P G_{\mathbf{d}} x(t_b)} > 0$ . Therefore, we obtain the contradiction to the condition 2). ■

The following corollary shows that the homogeneity of the system implies the homogeneity of set of invariant ellipsoids.

*Corollary 4.1:* Let  $\mathbf{d}$  and  $\mathbf{d}_\omega$  be dilations in  $\mathbb{R}^n$  and  $\mathbb{R}^p$ , respectively. Let a continuous vector field  $(x, \omega) \mapsto \tilde{f}(x, \omega) = \begin{bmatrix} f(x, \omega) \\ \omega \end{bmatrix}$  be  $\tilde{\mathbf{d}}$ -homogeneous of degree  $\mu \in \mathbb{R}$  with  $\tilde{\mathbf{d}}(s) = \text{diag}(\mathbf{d}(s), \mathbf{d}_\omega(s))$ ,  $s \in \mathbb{R}$ . If  $\varepsilon_{\mathbf{d}}(P)$  is an invariant ellipsoid of the system  $\dot{x} = f(x, \omega)$ , then  $\mathbf{d}(s)\varepsilon_{\mathbf{d}}(P)$  is an invariant ellipsoid of the system  $\dot{x} = f(x, \mathbf{d}_\omega(s)\omega)$ .

This immediately follows from Lemma 4.1 and the  $\tilde{\mathbf{d}}$ -homogeneity of  $\tilde{f}$  implying  $\tilde{\mathbf{d}}(s)\dot{x} = e^{-\mu s} \tilde{f}(\tilde{\mathbf{d}}(s)x, \tilde{\mathbf{d}}_\omega(s)\omega)$ . The invariant/attractive ellipsoid can be used to characterize the effects caused by external perturbations  $\omega(t)$ . For example, a conventional ellipsoid  $\varepsilon(P)$  is invariant for the linear system

$$\dot{x} = Ax + D\omega, \quad A \in \mathbb{R}^{n \times n}, D \in \mathbb{R}^{n \times p}, \quad (21)$$

where  $A$  is a Hurwitz matrix, the pair  $\{A, D\}$  is controllable and  $x, \omega$  are as before, if and only if the LMI holds [36], [37]:  $X^\top A^\top + AX + \beta X + \frac{1}{\beta} DQ^{-1}D^\top \preceq 0$ ,  $X = P^{-1} \succ 0$ .

For the considered linear system, any invariant ellipsoid is attractive [36], [37]. Below we show that the latter conclusion takes a place for the linear plant (1) with any stabilizing homogeneous controller (16).

## V. HOMOGENEOUS CONTROL WITH MINIMAL INVARIANT/ATTRACTIVE ELLIPSOID

### A. LMI-based characterization of invariant/attractive ellipsoid

Below in the proof we use the S-procedure recalled here.

*Lemma 5.1 (Proposition 4.1, [38]):* Let us consider the homogeneous quadratic forms  $f_i(x) = x^\top A_i x$ ,  $i = 0, 1, 2$  with  $x \in \mathbb{R}^n$   $A_i = A_i^\top \in \mathbb{R}^{n \times n}$ , and the numbers  $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ ,  $\alpha_2 \neq 0$ . Let there exist  $\mu_1, \mu_2 \in \mathbb{R}$  and  $x_0 \in \mathbb{R}^n$  such that

$$\mu_1 A_1 + \mu_2 A_2 \succ 0 \text{ and } f_1(x_0) < \alpha_1, f_2(x_0) = \alpha_2.$$

Then the following two claims are equivalent:

- 1)  $\exists \tau_1 \geq 0, \exists \tau_2 \in \mathbb{R} : A_0 \preceq \tau_1 A_1 + \tau_2 A_2, \alpha_0 \geq \tau_1 \alpha_1 + \tau_2 \alpha_2;$
- 2)  $f_0(x) \leq \alpha_0, \forall x \in \mathbb{R}^n : f_1(x) \leq \alpha_1, f_2(x) = \alpha_2.$

Based on Lemma 5.1, a criterion of invariance of the  $\mathbf{d}$ -homogeneous ellipsoid for the system (1) with the homogeneous controller (16) is given by the following theorem.

*Theorem 5.2:* Let  $D \neq 0$  and all parameters of the control (16) be defined as in Theorem 3.2 with  $X \in \mathbb{R}^{n \times n}$ ,  $A_0 = A + BK_0$  and  $G_{\mathbf{d}} = I_n + \mu G_0$ . Let  $\mathbf{d}_\omega$  be a linear continuous dilation in  $\mathbb{R}^p$  such that the vector field

$$(x, \omega) \mapsto \tilde{f}(x, \omega) = \begin{bmatrix} Ax + Bu(x) + D\omega \\ 0 \end{bmatrix} \quad (22)$$

is  $\tilde{\mathbf{d}}$ -homogeneous of degree  $\mu$  with respect to the dilation

$$\tilde{\mathbf{d}}(s) = \begin{bmatrix} \mathbf{d}(s) & 0 \\ 0 & \mathbf{d}_\omega(s) \end{bmatrix}, s \in \mathbb{R} \quad (23)$$

in  $\mathbb{R}^{n+p}$ . The  $\mathbf{d}$ -homogeneous ellipsoid  $\varepsilon_{\mathbf{d}}(X^{-1})$  is **invariant** for the system (1), (16) if and only if there exist  $\beta > 0$ :

$$W := \begin{bmatrix} A_0 X + X A_0^\top + B Y + Y^\top B^\top + \beta X & D \\ D^\top & -\beta Q \end{bmatrix} \preceq 0, \quad (24)$$

$$G_{\mathbf{d}} X + X G_{\mathbf{d}}^\top \succ 0, \quad X \succ 0$$

with  $Y = KX \in \mathbb{R}^{m \times n}$ .

**Proof.** Let us denote  $P = X^{-1}$ . *Sufficiency:* From the first LMI of (24), we derive

$$\begin{bmatrix} x \\ \omega \end{bmatrix}^\top \begin{bmatrix} P A_0 + A_0^\top P + P B K + K^\top B^\top P + \beta P & P D \\ D^\top P & -\beta Q \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} \preceq 0 \quad (25)$$

which can be written as

$$x^\top (P A_0 + A_0^\top P + P B K + K^\top B^\top P) x + x^\top P D \omega + \omega^\top D^\top P x \leq \beta (\omega^\top Q \omega - x^\top P x)$$

Since  $\|x\|_{\mathbf{d},P}$  is continuous and  $x \rightarrow x_0$  as  $t \rightarrow 0$ , then for  $\omega^\top Q \omega \leq 1$  and  $\|x_0\|_{\mathbf{d},P} = 1$  we derive

$$\left. \frac{d\|x(t)\|_{\mathbf{d},P}}{dt} \right|_{t=0^+} = \frac{x_0^\top (P A_0 + A_0^\top P + P B K + K^\top B^\top P) x_0 + x_0^\top P D \omega + \omega^\top D^\top P x_0}{x_0^\top (P G_{\mathbf{d}} + G_{\mathbf{d}}^\top P) x_0} \leq 0,$$

where the equivalence  $\|x_0\|_{\mathbf{d},P} = 1 \Leftrightarrow \|x_0\|_P = 1$  and the second LMI of (24) are utilized. By Lemma 4.1, we conclude that  $\varepsilon_{\mathbf{d}}(P)$  is a  $\mathbf{d}$ -homogeneous **invariant** ellipsoid of the system (1).

*Necessity:* First of all, notice that, by definition of the  $\mathbf{d}$ -homogeneous invariant ellipsoid, the canonical homogeneous norm is well defined, which ensures that the second and the third matrix inequalities in (24) are fulfilled. Therefore, we need to prove the fulfillment of the first one. Let us initially show that if  $\varepsilon_{\mathbf{d}}(P)$  is a  $\mathbf{d}$ -homogeneous **invariant** ellipsoid of the system (1) then

$$\omega^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d},P}) Q \mathbf{d}(-\ln \|x\|_{\mathbf{d},P}) \omega \leq 1 \quad \Rightarrow \quad (26)$$

$$x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d},P}) P \mathbf{d}(-\ln \|x\|_{\mathbf{d},P}) f(x, \omega) \leq 0 \quad (27)$$

Suppose inversely that  $\varepsilon_{\mathbf{d}}(P)$  is a  $\mathbf{d}$ -homogeneous invariant ellipsoid, but  $\exists x_1 \in \mathbb{R}^n, \exists \omega_1 \in \mathbb{R}^p$  such that  $\omega_1^\top \mathbf{d}_\omega^\top(-\ln \|x_1\|_{\mathbf{d},P}) Q \mathbf{d}_\omega(-\ln \|x_1\|_{\mathbf{d},P}) \omega_1 \leq 1$  and  $x_1^\top \mathbf{d}^\top(-\ln \|x_1\|_{\mathbf{d},P}) P \mathbf{d}(-\ln \|x_1\|_{\mathbf{d},P}) f(x_1, \omega_1) > 0$ . Denoting  $x_{s1} = \mathbf{d}(-s)x_1, \omega_{s1} = \mathbf{d}_\omega(-s)\omega_1$  and  $s = \ln \|x_1\|_{\mathbf{d},P}$ , we derive  $\omega_{s1}^\top Q \omega_{s1} \leq 1, e^{\mu s} x_{s1}^\top P f(x_{s1}, \omega_{s1}) > 0$ , where

the homogeneity of  $\tilde{f}(x, \omega)$  is utilized on the last step. Since  $\|x_{s1}\|_P = 1$  then we derive

$$\|x_{s1}\|_P = 1, \quad \omega_{s1}^\top Q \omega_{s1} \leq 1, \quad x_{s1}^\top P f(x_{s1}, \omega_{s1}) > 0.$$

The latter contradicts Lemma 4.1, so the implication (26)-(27) takes a place. Moreover, using the notation  $\pi = \mathbf{d}(-\ln \|x\|_{\mathbf{d},P})x$  and  $\omega_x = \mathbf{d}_\omega(-\ln \|x\|_{\mathbf{d},P})\omega$ , we derive  $\mathbf{d}(-\ln \|x\|_{\mathbf{d},P})f(x, \omega) = e^{\mu s} f(\pi, \omega_x) = e^{\mu s} [(A_0 + BK)\pi + D\omega_x]$ . Since for any  $x \neq 0$  we have  $\pi^\top P \pi = 1$  then the proven implication can be rewritten as follow :

$$\omega_x^\top Q \omega_x \leq 1 \text{ and } \pi^\top P \pi = 1 \Rightarrow \pi^\top P [(A_0 + BK)\pi + D\omega_x] \leq 0$$

Applying Lemma 5.1 we derive  $\exists \tau_1 \geq 0, \exists \tau_2 \in \mathbb{R}: \tau_1 + \tau_2 \leq 0$ ,

$$\begin{bmatrix} P(A+BK)+(A+BK)^\top P & PD \\ D^\top P & 0 \end{bmatrix} \leq \tau_1 \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} + \tau_2 \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}.$$

If the latter inequality holds for  $\tau_2 < -\tau_1$  then it holds for  $\tau_2 = -\tau_1$ . Indeed,  $\tau_2 \leq -\tau_1 \Rightarrow \tau_2 P \preceq -\tau_1 P$ . Taking into account  $P = X^{-1}$ , we conclude that the inequality (24) holds for  $\beta := \tau_1 \geq 0$ . Let us show that  $\beta > 0$ . Indeed, if  $\beta = 0$  then the following inequality must hold :

$$\begin{bmatrix} A_0 X + X A_0^\top + B Y + Y^\top B^\top & D \\ D^\top & 0 \end{bmatrix} \preceq 0. \quad (28)$$

The latter inequality may hold only if  $D = 0$ . Indeed, by the generalized Schur complement [39], it has the equivalent conditions  $A_0 X + X A_0^\top + B Y + Y^\top B^\top \preceq 0$  and  $I \cdot D^\top = 0$ . Therefore,  $\beta$  may be zero only if  $D = 0$ . This contradicts the assumption of the theorem. ■

Similarly to the case of the linear control system [36], [37], the invariant ellipsoids of the  $\mathbf{d}$ -homogeneous control system (1), (16) can be characterized by means of a linear matrix inequality (24) as long as the perturbation  $\omega$  is involved into the system in a generalized homogeneous manner. The only difference in the LMI (24) with respect to the conventional linear case is the condition  $G_{\mathbf{d}} X + X G_{\mathbf{d}}^\top \succ 0$  inspired by the homogeneity of system. Notice that for  $\mu = 0$  the homogeneous controller (16) becomes linear  $u = (K_0 + K)x$  and  $G_{\mathbf{d}} = I_n$ . This means that all linear stabilizing controllers belong to the set of the considered homogeneous stabilizers (16). However, the above theorem introduces a restriction to the structure of the exogenous perturbations. Namely, it requires certain symmetry with respect to a dilation of the vector of disturbances  $\omega$  too. The mentioned condition can be checked as follows.

*Proposition 5.1:* Let  $G_0$  and the homogeneous controller  $u$  be defined as in Theorem 3.2. If there exists a matrix  $G_{0,\omega} \in \mathbb{R}^{p \times p}$  such that

$$G_0 D = D G_{0,\omega} \quad (29)$$

then for any  $\mu \in \mathbb{R}$  satisfying

$$1 + \mu + \mu \operatorname{Re}(\lambda_i(G_{0,\omega})) > 0, \quad \forall i = 1, 2, \dots, n \quad (30)$$

- 1) the matrix  $G_{\mathbf{d},\omega} = I_p + \mu(I_p + G_{0,\omega})$  is anti-Hurwitz and  $\mathbf{d}_\omega(s) = e^{s G_{\mathbf{d},\omega}}, s \in \mathbb{R}$ , is a dilation in  $\mathbb{R}^p$ ;
- 2) the vector field  $\tilde{f}$  defined by (22) is  $\tilde{\mathbf{d}}$ -homogeneous of degree  $\mu$ , where  $\tilde{\mathbf{d}}$  is a dilation in  $\mathbb{R}^{n+p}$  defined by (23).

**Proof.** The inequality (30) immediately implies that  $G_{\mathbf{d},\omega}$  is anti-Hurwitz and  $\mathbf{d}_\omega(s) = e^{s G_{\mathbf{d},\omega}}$  is a dilation in  $\mathbb{R}^p$ . From

equation (29), we derive that  $G_0^i D = D G_{0,\omega}^i$ , where  $i$  is a natural number. Then we have

$$e^{\mu G_0 s} D = \sum_{i=0}^{+\infty} \frac{G_0^i \mu^i s^i}{i!} D = D \sum_{i=0}^{+\infty} \frac{G_{0,\omega}^i \mu^i s^i}{i!} = D e^{\mu G_{0,\omega} s} \quad (31)$$

This implies  $e^{\mu s} \mathbf{d}(s) D = D \mathbf{d}_\omega(s)$ . Finally, since  $G_0$  is defined as in Theorem 3.2, then we have

$$f(\mathbf{d}(s)x, \mathbf{d}_\omega(s)\omega) = A \mathbf{d}(s)x + B u(\mathbf{d}(s)x) + D \mathbf{d}_\omega(s)\omega = e^{\mu s} \mathbf{d}(s)x + e^{\mu s} \mathbf{d}(s) B u(x) + e^{\mu s} \mathbf{d}(s) D \omega = e^{\mu s} \mathbf{d}(s) f(x, \omega),$$

where  $f(x, \omega) = Ax + Bu(x) + D\omega$ . ■

It is well-know [36], [37] that, in many cases, an invariant ellipsoid of the linear control system is attractive as well. The same conclusion can be made for the considered homogeneous control system provided the dilation  $\mathbf{d}_\omega$  in  $\mathbb{R}^p$  is strictly monotone with respect to the norm  $\|\cdot\|_Q$  in  $\mathbb{R}^p$ .

*Corollary 5.1:* Under the conditions of Theorem 5.2, the invariant ellipsoid  $\varepsilon_{\mathbf{d}}(P)$  with  $P = X^{-1}$  is attractive if

$$Q G_{\mathbf{d},\omega} + G_{\mathbf{d},\omega}^\top Q \succ 0. \quad (32)$$

**Proof.** Considering the canonical homogeneous norm as a Lyapunov function of the system (1), (16), we derive

$$\begin{aligned} \frac{d\|x(t)\|_{\mathbf{d},P}}{dt} &= \|x\|_{\mathbf{d},P} \frac{x_s^\top X^{-1} \mathbf{d}(-\ln \|x\|_{\mathbf{d},P}) [Ax + Bu + D\omega]}{x_s^\top P G_{\mathbf{d}} x_s} \\ &= \|x\|_{\mathbf{d},P}^{1+\mu} \frac{x_s^\top X^{-1} [(A_0 + BK)x_s + D\omega_s]}{x_s^\top P G_{\mathbf{d}} x_s} \\ &= \|x\|_{\mathbf{d},P}^{1+\mu} \frac{[(X^{-1} x_s)^\top (A_0 + BK)x_s + (X^{-1} x_s)^\top D\omega_s]}{2x_s^\top P G_{\mathbf{d}} x_s} \\ &\quad + \frac{\|x\|_{\mathbf{d},P}^{1+\mu} [x_s^\top (A_0 + BK)^\top X^{-1} x_s + \omega_s^\top D^\top X^{-1} x_s]}{2x_s^\top P G_{\mathbf{d}} x_s} \\ &\quad + \frac{\|x\|_{\mathbf{d},P}^{1+\mu} [\beta x_s^\top X^{-1} x_s - \beta - \beta \omega_s^\top Q \omega_s + \beta \omega_s^\top Q \omega_s]}{2x_s^\top P G_{\mathbf{d}} x_s} \\ &= \frac{\|x\|_{\mathbf{d},P}^{1+\mu}}{2x_s^\top P G_{\mathbf{d}} x_s} \left( [X^{-1} x_s]^\top W \begin{bmatrix} X^{-1} x_s \\ \omega_s \end{bmatrix} - \beta + \beta \omega_s^\top Q \omega_s \right), \end{aligned}$$

where  $x_s = \mathbf{d}(-\ln \|x\|_{\mathbf{d},P})x$ ,  $\omega_s = \mathbf{d}(-\ln \|x\|_{\mathbf{d},P})\omega$  and the identities  $A_0 \mathbf{d}(s) = e^{\mu s} \mathbf{d}(s) A_0$ ,  $\mathbf{d}(s) B = e^s, \forall s \in \mathbb{R}$  are utilized on the first step. Since

$$\omega^\top \mathbf{d}_\omega^\top(s) Q \mathbf{d}_\omega(s) \omega + \int_s^0 \frac{d}{d\tau} \omega^\top \mathbf{d}_\omega^\top(\tau) Q \mathbf{d}_\omega(\tau) \omega d\tau = \omega^\top Q \omega$$

then, due to (32), for any  $s > 0$  and any  $\omega \neq 0$  we have

$$\int_s^0 \frac{d}{d\tau} \omega^\top \mathbf{d}_\omega^\top(\tau) Q \mathbf{d}_\omega(\tau) \omega d\tau = \int_s^0 \omega^\top \mathbf{d}_\omega^\top(\tau) (Q G_{\mathbf{d},\omega} + G_{\mathbf{d},\omega}^\top Q) \mathbf{d}_\omega(\tau) \omega d\tau > 0$$

and  $\omega^\top \mathbf{d}_\omega^\top(s) Q \mathbf{d}_\omega(s) \omega < \omega^\top Q \omega$ . This implies that  $\omega_s^\top Q \omega_s < \omega^\top Q \omega \leq 1$  for any  $x : \|x\|_{\mathbf{d},P} > 1$  and any  $\omega \neq 0$ . Therefore, using the inequality (24) with  $\beta > 0$ , we conclude that

$$\frac{d}{dt} \|x\|_{\mathbf{d},P} < 0, \quad \forall x : \|x\|_{\mathbf{d},P} > 1, \forall \omega : \|\omega\|_Q \leq 1.$$

Hence, the  $\mathbf{d}$ -homogeneous ellipsoid is attractive. ■

*Corollary 5.2:* If  $G_{0,\omega} \in \mathbb{R}^{p \times p}$  is defined as in Proposition 5.1, then the inequalities (30) and (32) with  $G_{\mathbf{d},\omega} = I_p + \mu(I_p + G_{0,\omega})$  hold for any  $\mu \in \mathbb{R}$  satisfying

$$1 + \mu + \mu \frac{\lambda_i(Q^{\frac{1}{2}} G_{0,\omega} Q^{-\frac{1}{2}} + Q^{-\frac{1}{2}} G_{0,\omega}^\top Q^{\frac{1}{2}})}{2} > 0, \quad \forall i = 1, \dots, n. \quad (33)$$

**Proof.** Since  $1 + \mu + \mu \frac{\lambda_i(Q^{\frac{1}{2}} G_{0,\omega} Q^{-\frac{1}{2}} + Q^{-\frac{1}{2}} G_{0,\omega}^\top Q^{\frac{1}{2}})}{2} = \frac{\lambda_i(Q^{\frac{1}{2}} G_{\mathbf{d},\omega} Q^{-\frac{1}{2}} + Q^{-\frac{1}{2}} G_{\mathbf{d},\omega}^\top Q^{\frac{1}{2}})}{2}$  then the inequality (33) implies the inequality (32). Taking into account that  $Q \succ 0$ , by

Lyapunov inequality, we derive that  $G_{d,\omega}$  is anti-Hurwitz, i.e., the inequality (30) holds as well. ■

Obviously, the inequalities (30) and (33) are always feasible, at least, for  $\mu$  close to 0. So, the existence of solution for the equation (29) is the only critical restriction of an applicability of the homogeneous invariant/attractive ellipsoids method.

### B. Minimal invariant and attractive ellipsoids

Inspired by [36] and Theorem 5.2, the optimal tuning of the homogeneous controller (16) can be formulated in terms of the following Semi-Definite-Programming (SDP) problem:

$$\text{tr}(X) \rightarrow \min \quad (34)$$

subject to the LMI constraints (24).

Note that for any fixed  $\beta$  and any fixed homogeneity degree  $\mu$ , the problem of finding an optimal solution is reduced to minimizing the linear function (34) subject to the LMI constraints (24), which is the classical SDP problem. There exist many MATLAB toolboxes for its numerical solution such as SeDuMi and YALMIP. The optimization with respect to scalar parameters  $\beta$  and  $\mu$  can be realized using some gradient-free algorithms (like `fminsearch` in MATLAB). The considered SDP problem *without* the additional constraint  $G_d X + X G_d^\top \succ 0$  was studied in [36], where its convexity has been proven.

Notice that the linear controller is a particular case of the homogeneous controller with  $\mu = 0$ ,  $\mathbf{d}(s) = e^s I_n$ ,  $\mathbf{d}_\omega(s) = e^s I_p$ , so the condition of the homogeneity of the vector field  $\tilde{f}$  is always fulfilled (see Theorem 5.2) and LMI (24) becomes

$$A_0 X + X A_0^\top + \beta X + B Y + Y^\top B^\top + \frac{1}{\beta} D Q^{-1} D^\top \preceq 0, X \succ 0 \quad (35)$$

which perfectly fits the results of [36], [37]. Moreover, if the system of algebraic equations (12), (29) has a solution then an optimal (in the sense of minimal invariant ellipsoid) linear controller can be upgraded to a (nonlinear) homogeneous one without any degradation of the minimal invariant/attractive ellipsoid. Such an upgrade can be useful in practice, since homogeneous controllers may provide another good properties faster convergence, smaller overshoots and better robustness with respect to unmodelled nonlinearities than linear algorithms (see [15, Chapter 1]).

*Corollary 5.3:* Let the tuple  $(X_{opt}, Y_{opt}, \beta_{opt}) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{m \times n} \times (0, +\infty)$  be a solution of the SDP problem (34), (35) then for any  $\mu \in \mathbb{R}$  satisfying

$$1 + \mu \lambda_{\min} \left( X_{opt}^{-\frac{1}{2}} G_0 X_{opt}^{\frac{1}{2}} + X_{opt}^{\frac{1}{2}} G_0^\top X_{opt}^{-\frac{1}{2}} \right) > 0, \quad (36)$$

the tuple  $(X_{opt}, Y_{opt}, \beta_{opt})$  is a solution of the SDP problem (34), (24), with  $G_d = I_n + \mu G_0 \in \mathbb{R}^{n \times n}$ . Moreover, the  $\mathbf{d}$ -homogeneous ellipsoid  $\varepsilon_d(X^{-1})_{opt}$  is

- **invariant** for the homogeneous control system (1), (16) provided that  $\mu \in \mathbb{R}$  satisfies (36) and (30);
- **attractive** for the homogeneous control system (1), (16) provided that  $\mu \in \mathbb{R}$  satisfies (36) and (33).

**Proof.** The required result directly follows from the equivalence of inequality (36) to  $G_d X + X G_d^\top \succ 0$ . ■

The process of upgrading the optimized linear controller to the optimal homogeneous controller can be summarized in

the following steps: 1) Define the system by matrix  $A, B, D$  and the bounded disturbance characterized by  $Q$ ; 2) Solve the LMIs system (12) (29) to obtain  $G_0$  and  $G_{0,\omega}$ ; 3) Solve the SDP problem (34), (35) to obtain the linear optimal parameters  $(X_{opt}, Y_{opt}, \beta_{opt})$  where  $K_{opt} = Y_{opt} X_{opt}^{-1}$ ; 4) Find the admissible  $\mu$  by solving (36) and (30). If there exists a feasible solution  $\mu$  in the final step, then it means the optimal homogeneous controller (in the sense of minimal ellipsoid) can be implemented in the form of (16) with  $K = K^{opt} - K_0$ ,  $G_d = I_n + \mu G_0$  and  $P = (X^{opt})^{-1}$ .

A transformation of a linear optimal controller to a homogeneous optimal controller can also be useful, for example, if an additional criterion needs to be optimized or an additional condition has to be satisfied. For example, if  $K_0 = 0$  and the selection  $\mu = -1$  is admissible in the latter corollary then the corresponding ("upgraded") homogeneous controller is uniformly bounded. Indeed,

$$u^\top u = x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d},P}) K^\top K \mathbf{d} (-\ln \|x\|_{\mathbf{d},P}) x \leq \lambda_{\max}(P^{-1/2} K^\top K P^{-1/2})$$

where the identity  $x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d},P}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d},P}) x = 1$  is utilized in the last step.

*Corollary 5.4:* If  $K_0 = 0$  and  $\mu = -1$  then the homogeneous control (16) satisfies the restriction  $u^\top(x)u(x) \leq \bar{u}^2, \forall x \in \mathbb{R}^n$ , for some  $\bar{u} > 0$ , if and only if

$$\begin{bmatrix} X & Y^\top \\ Y & \bar{u}^2 I_m \end{bmatrix} \succeq 0 \quad (37)$$

where  $X = P^{-1}, Y = KX$ .

**Proof.** If  $K_0 = 0$  and  $\mu = -1$ , we have

$$u^\top u \leq \lambda_{\max}(P^{-1/2} K^\top K P^{-1/2})$$

The latter inequality is sharp, thus the inequality  $u^\top u \leq \bar{u}^2$  is equivalent to  $\lambda_{\max}(P^{-1/2} K^\top K P^{-1/2}) \leq \bar{u}^2$  and to

$$X^\top K^\top K X \leq \bar{u}^2 X, \quad (38)$$

Apply the Schur complement we derive (37). ■

Therefore, the design of globally bounded controllers that minimize the size of the invariant or attractive ellipsoid can be achieved by using the  $\mathbf{d}$ -homogeneous invariant ellipsoid method. The design procedure in this case is formulated in terms of the SDP problem (34), (24), (37).

## VI. SIMULATION AND EXPERIMENT

### A. Numerical Results

In this subsection, the linearized model of rotary inverted pendulum system Quanser Qube Servo-2 is studied.

$$\dot{x} = Ax + Bu + D\omega \quad (39)$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{l^2 r g m_p^2}{J_t} & -\frac{b_r J_p}{J_t} - \frac{k_m^2}{R_m} \frac{J_p}{J_t} & -\frac{l r m_p b_p}{J_t} \\ 0 & \frac{g l m_p J_r}{J_t} - \frac{l r m_p b_r}{J_t} - \frac{k_m^2}{R_m} \frac{l r m_p}{J_t} & -\frac{J_r b_p}{J_t} & 0 \end{bmatrix},$$

$$B = \frac{k_m}{R_m} \begin{bmatrix} 0 \\ \frac{J_p}{J_t} \\ \frac{l r m_p}{J_t} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix},$$

$\theta$  is the angle position of arm and  $\alpha$  is the angle position of pendulum. The external disturbance is bounded by  $\|\omega\|_Q \leq 1$

TABLE I: Pendulum Model Parameters

Parameter	Description	Value	Units
$R_m$	Motor Resistance	8.4	$\Omega$
$K_m$	Back-emf constant	0.042	$\frac{V \cdot s}{rad}$
$m_r$	Rotary arm mass	0.095	kg
$r$	Rotary arm length	0.085	m
$J_r$	Rotary Inertia Moment	$\frac{r^2 m_r}{3}$	$kg \cdot m^2$
$b_r$	Rotary Damping Coefficient	$10^{-3}$	$\frac{N \cdot M \cdot s}{rad}$
$m_p$	Pendulum Link Mass	0.024	kg
$L_p$	Pendulum Link length	0.129	m
$l$	Pendulum center of mass	$\frac{L_p}{2}$	m
$J_p$	Pendulum inertia moment	$\frac{m_p \cdot L_p^2}{3}$	$kg \cdot m^2$
$b_p$	Pendulum Damping Coefficient	$5 \times 10^{-5}$	$\frac{N \cdot M \cdot s}{rad}$
$g$	Gravity constant	9.81	$\frac{rad}{s^2}$

with

$$Q = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \omega = [0.2 \quad 0.3 \quad 0.3 \quad 0.4]^\top \cdot \sin \frac{t}{2}$$

Using Theorem 3.2 and Proposition 5.1, we find the matrix  $Y_0$ ,  $G_0$  and  $G_{0,\omega}$  such that (12) and (29) hold:

$$Y_0 = [0 \quad 10.65 \quad -0.73 \quad 0.47], \quad (40)$$

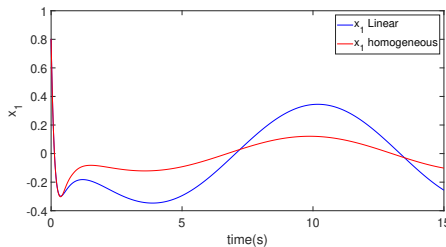
$$G_0 = \begin{bmatrix} -3 & 2.02 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0.38 & -2 & 2.02 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_{0,\omega} = \begin{bmatrix} -3 & 2.02 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0.38 & -2 & 2.02 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (41)$$

We define  $K_0$  and  $G_d$  using  $Y_0$  and  $G_0$  as in Theorem 3.2.

Using (34) (35) we design the optimal linear controller

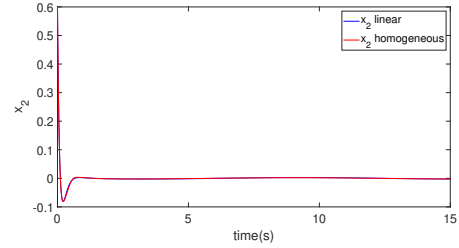
$$K^{opt} = \begin{bmatrix} 27.12 \\ -177.13 \\ 10.91 \\ -17.93 \end{bmatrix}^\top, \quad X = \begin{bmatrix} 1.33 & 0.11 & -0.87 & 0.42 \\ 0.11 & 0.05 & -0.51 & -0.58 \\ -0.87 & -0.51 & 48.52 & 35.47 \\ 0.42 & -0.58 & 35.47 & 30.13 \end{bmatrix} \quad (42)$$

Finally, we verify that for  $\mu = -0.7$ , the LMIs (24) (36) (33) are feasible. We define the gain of homogeneous controller  $K = K^{opt} - K_0$ . The simulation results are shown in Fig.1-2. Theoretically, the optimal linear controller and the "upgraded" (homogeneous) controller has the same invariant ellipsoid. However, according to the Fig. 1, for linear controller the maximum value of  $x_1$  in the invariant set is about 0.1296. The same  $x_1$  by using homogeneous controller is about 0.0494. The precision improvement on  $x_1$  is about 60%. Besides, we can also find that  $x_1$  has a faster response by using the homogeneous controller.


 Fig. 1:  $x_1$  by linear and homogeneous controller

### B. Experiment results

This experiment is based on the rotary inverted pendulum Quanser Qube Servo-2, where the same model (39) is applied locally (close-to the upper unstable position of the pendulum). The system parameters are presented in Table


 Fig. 2:  $x_2$  by linear and homogeneous controller

I. Here we consider more physical model of disturbances  $D = [0 \quad 0 \quad 2.53 \quad 2.50]^\top$  and  $Q = 1$ . We use the well tuned linear feedback gain  $K = [2 \quad -35 \quad 1.5 \quad -3]^\top$  provided by Quanser to design the homogeneous controller by upgrading the linear one. We obtain

$$X = \begin{bmatrix} 0.00159 & 0.00002 & -0.00275 & -0.00018 \\ 0.00002 & 0.00002 & 0.00011 & -0.00004 \\ -0.00275 & 0.00011 & 0.02492 & 0.01377 \\ -0.00018 & -0.00004 & 0.01377 & 0.01210 \end{bmatrix} \quad (43)$$

for  $\mu = -0.7$  satisfying (36) (33) and LMI (24).

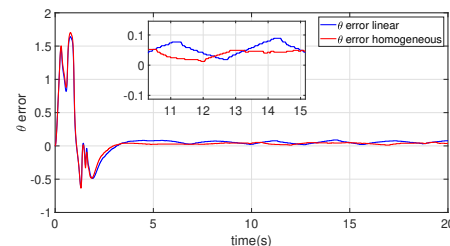
The experiment is repeated five times to guarantee a fair comparison. The results are presented in the Table II and Table III. It is clear to see that the stabilization precision of  $\theta$  and  $\alpha$  in  $L^2$ -norm is improved by using the homogeneous controller about 20.22% and 8.3%, respectively. The stabilization precision in the  $L^\infty$ -norm is improved about 11.62% and 15.4%, respectively. The energy consumption increases about 2.41%. Fig. 3 and 4 provide the error trajectory of  $\theta$  and  $\alpha$  in Test 2 of experiment.

TABLE II: Pendulum stabilization with linear controller

Linear	$\ \theta\ _{L^\infty}$	$\ \alpha\ _{L^\infty}$	$\ \theta\ _{L^2}$	$\ \alpha\ _{L^2}$	$\ u\ _{L^2}$
Test 1	0.0767	0.0092	0.21333	0.01347	0.79136
Test 2	0.07363	0.00613	0.20377	0.01271	0.7717
Test 3	0.07363	0.00613	0.21414	0.01282	0.67597
Test 4	0.0859	0.0092	0.2379	0.01423	0.74302
Test 5	0.0859	0.0092	0.24353	0.01332	0.60481
Average	0.0792	0.00797	0.2225	0.0133	0.7174

TABLE III: Pendulum stabilization with homogeneous controller

Homogeneous	$\ \theta\ _{L^\infty}$	$\ \alpha\ _{L^\infty}$	$\ \theta\ _{L^2}$	$\ \alpha\ _{L^2}$	$\ u\ _{L^2}$
Test 1	0.0675	0.00613	0.18762	0.01268	0.6292
Test 2	0.0675	0.00613	0.18695	0.01277	0.77291
Test 3	0.0675	0.00613	0.16906	0.01126	0.71843
Test 4	0.0767	0.0092	0.17481	0.01245	0.76564
Test 5	0.07056	0.00613	0.16912	0.01202	0.78741
Average	0.0700	0.00674	0.1775	0.0122	0.7347


 Fig. 3:  $\theta$  error trajectory by linear and homogeneous controller



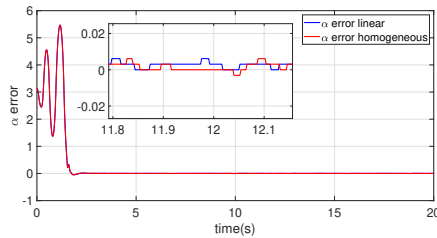


Fig. 4:  $\alpha$  error trajectory by linear and homogeneous controller

## VII. CONCLUSION

This article extends the invariant/attractive ellipsoid method [12], [36] to a class of the generalized homogeneous system. The LMI-based characterization of  $\mathbf{d}$ -homogeneous invariant/attractive ellipsoid for a linear plant is obtained by the homogeneous control. Linear stabilizers belong to the set of the considered homogeneous controllers assuming that the homogeneity degree is a tuning parameter. Under certain restriction to the structure of perturbations, an optimal (in the sense of the minimal invariant ellipsoid)  $\mathbf{d}$ -homogeneous controller can be obtained from the optimal linear one by a proper selection of dilation group  $\mathbf{d}$ . Despite that, in this case, optimal invariant ellipsoids are theoretically the same for linear and homogeneous controllers, the homogeneous controller still may improve the control quality by faster response, better robustness to unmodelled uncertainties and reduction of overshoots. Numerical and practical experiments support this theoretical conclusion.

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