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Generalized Homogeneous Rigid-Body Attitude Control

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Abstract

The attitude tracking problem for a full-actuated rigid body in 3D is studied using a system model based on Lie algebra $\mathfrak{so}(3)$. A nonlinear homogeneous controller is designed to track a smooth attitude trajectory in a finite/fixed time. A global settling time estimate is obtained, which is easily adjustable by tuning the homogeneity degree. The input-to-state stability of the control system with respect to measurement noises and additive perturbations is studied. Simulations illustrating the performance of the proposed algorithm are presented.

Key words: homogeneous control; attitude tracking; impulsive system.

1 Introduction

The problem of rigid body attitude control is well-known in a variety of applications such as satellites, robots, and aerial/underwater vehicles. There are three commonly-used global mathematical descriptions of the rigid body's attitude (orientation): Special Orthogonal Group $SO(3)$, quaternion, Lie algebra $\mathfrak{so}(3)$ (also known as exponential coordinates or Chevalley's coordinates, see [Chaturvedi et al. \(2011\)](#)). Indeed, the manifold $SO(3)$ consists of all possible rotation matrices in 3D space. The quaternion also represents the global orientation of a rigid body but with an ambiguity, which may cause an unwinding phenomenon in attitude control [Wen and Kreutz-Delgado \(1991\)](#). Lie algebra $\mathfrak{so}(3)$ is a minimal (three parameters) representation that contains ambiguities, as it was stated in [Stuelpnagel \(1964\)](#) that any representation with three parameters cannot uniquely represent the global orientation of a rigid body.

Based on the above mentioned three representations, different attitude control techniques have been developed in the literature. Notice that a topological structure of the group precludes the existence of continuous time-invariant feedback control for a global asymptotic attitude stabilization [Bullo and Murray \(1995\)](#); [Chaturvedi et al. \(2011\)](#); [Mayhew and Teel \(2013\)](#).

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Several switching-based algorithms of a global asymptotic/exponential attitude control design have been developed for the system described by quaternions (see, *e.g.*, [Mayhew et al. \(2011\)](#)); $SO(3)$ group (see, *e.g.*, [Mayhew and Teel \(2013\)](#), [Lee \(2015\)](#), [Berkane et al. \(2017\)](#)) and the Lie algebra $\mathfrak{so}(3)$ (see, *e.g.*, [Bharadwaj et al. \(1998\)](#) and [Yu and Ding \(2016\)](#)).

A non-asymptotic (i.e. finite/fixed-time) attitude control of rigid body has also been intensively studied in the last decade. However, due to the topological constraint, non-asymptotic stabilization and robustness analysis on the manifold $SO(3)$ are not trivial. Thus, most of non-asymptotic attitude control algorithms are currently based on quaternion and Lie algebra $\mathfrak{so}(3)$. There are two approaches to non-asymptotic attitude control design: direct Lyapunov methods (see, *e.g.*, [Zou \(2013\)](#), [Huang et al. \(2018\)](#), [Chen et al. \(2018\)](#), [Zou et al. \(2020\)](#)) and homogeneity-based methods. The direct Lyapunov methods provide an estimate of the settling time but it involves a complicated Lyapunov construction. Homogeneity-based methods simplify the design of finite-time control because an asymptotically stable homogeneous system with a negative degree is finite-time stable. Several attitude controllers inspired by the ideas of homogeneity can be found in the literature. They provide local ([Guo et al. \(2019\)](#), [Gui et al. \(2015\)](#)) and global ([Du and Li \(2012\)](#), [Gui and Vukovich \(2016\)](#), [Shi et al. \(2018\)](#)) stabilization, and combined with sliding mode algorithms in ([Zhu et al. \(2011\)](#), [Tiwari et al. \(2015\)](#), [Shi et al. \(2017\)](#), [Guo et al. \(2019\)](#)).

The mentioned homogeneity-based finite-time global attitude control design follows the idea of combining

a negative-degree locally homogeneous feedback with globally asymptotically stabilizing algorithm [Hong et al. \(2001\)](#). The main drawbacks of this approach are the absence of a settling time estimate, the sophisticated Lyapunov analysis, and the impossibility of fixed-time stabilization. The aforementioned observation motivates us to develop a new homogeneity-based attitude control.

Continuous homogeneous control systems in \mathbb{R}^n have the following advantages: finite-time and fixed-time convergence rates, [Bhat and Bernstein \(2005\)](#), [Andrieu et al. \(2008\)](#); local stability is equivalent to global one even in a nonlinear case, [Zubov \(1958\)](#), [Rosier \(1992\)](#); homogeneity ensures robustness with respect to a larger class of uncertainties [Ryan \(1995\)](#), [Hong \(2001\)](#), [Andrieu et al. \(2008\)](#); an elimination of an unbounded peaking effect, ([Polyakov, 2020](#), Chapter 1) and a reduction of overshoots ([Polyakov and Krstic \(2023\)](#)).

The aim of this paper is to develop a homogeneous attitude control on Lie algebra. However, the global model on Lie algebra $\mathfrak{so}(3)$ is a specific *impulsive* system ([Bharadwaj et al. \(1998\)](#)). The jump behavior may destroy the above mentioned advantages. There are few studies on the impulsive homogeneous system and its stability (*e.g.*, [Tuna and Teel \(2006\)](#), [Goebel and Teel \(2008\)](#), [Forni and Teel \(2010\)](#)), but they require the scaling invariance of jump set (*i.e.*, if the state in the jump set, then the arbitrarily scaled state must remain in the jump set). This condition is not satisfied in the case of attitude control. This motivates us to study a specific class of impulsive system that admits an analysis based on the homogeneous implicit Lyapunov function method [Polyakov et al. \(2015\)](#). To the best of authors' knowledge, no research has been done for this class of systems before.

For homogeneous control design, the implicit Lyapunov-function-based methods [Adamy and Flemming \(2004\)](#), [Korobov \(1979\)](#) are well-developed for robust homogeneous finite/fixed-time stabilization of continuous-time linear control systems ([Polyakov et al. \(2015\)](#), [Polyakov \(2020\)](#), [Zimenko et al. \(2020\)](#)), this paper extends the mentioned control technique to a nonlinear (impulsive) system and applies it to an attitude control design. The main contributions of our work are as follows.

- The *homogeneous* implicit Lyapunov function method is extended to a specific class of impulsive systems such that various convergence rates (finite-time, exponential or nearly fixed-time) can be assessed by a proper selection of the homogeneity degree. An estimate of a convergence time is given.
- A homogeneous attitude control for global attitude tracking is developed using the implicit Lyapunov function technique. The design procedure repeats the linear control case but allows the tracking error to converge to zero in a finite/fixed time or exponentially

independently of the homogeneity degree. The convergence time is estimated. The homogeneous algorithm becomes the classical linear PD controller (see *e.g.*, [Bullo and Murray \(1995\)](#), [Yu and Ding \(2016\)](#)) in a particular case.

- Globally fixed-time stable attitude controller is designed by a commutation of two homogeneous controllers as in [Polyakov et al. \(2015\)](#). Robustness (Input-to-State Stability) of the closed-loop system with respect to various perturbations is studied.

A preliminary version of this work has been presented at European Control Conference, [Zhou et al. \(2022\)](#), where local homogeneous attitude control has been developed. The present paper extends this local result to a global homogeneous attitude controller. The paper is organized as follows. The problem of attitude tracking is stated in Section 2. In Section 3, some preliminaries about generalized homogeneous stabilization and new results about homogeneous stabilization of linear impulsive systems are given. A homogeneous attitude tracking control is designed in Section 4. The performance and robustness of the developed control algorithm are demonstrated in Section 5 in numerical simulations. Some properties of $SO(3)$ group and $\mathfrak{so}(3)$ algebra are given in Appendix.

Notation

\mathbb{R} is the field of reals, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$; $|\cdot|$ is the Euclidean norm in \mathbb{R}^n ; $\mathbf{0}$ denotes the zero element of a vector space; $\text{diag}\{\lambda_i\}_{i=1}^n$ is the diagonal matrix with elements λ_i ; $P \succ 0$ ($\prec 0$, $\succeq 0$, $\preceq 0$) for $P \in \mathbb{R}^{n \times n}$ means that the matrix P is symmetric and positive (negative) definite (semi-definite); $C(X, Y)$ denotes the space of continuous functions $X \mapsto Y$, where X, Y are subsets of normed vector spaces; $C^p(X, Y)$ is the space of functions continuously differentiable at least up to the order p ; $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ represent the minimal and maximal eigenvalue of a matrix P ; for $P \succeq 0$ the square root of P is a matrix $M = P^{\frac{1}{2}}$ such that $M^2 = P$; $SO(3) \subset \mathbb{R}^{3 \times 3}$ is a special orthogonal group (see Appendix) and $\mathfrak{so}(3)$ is the corresponding Lie algebra consisting of the skew-symmetric 3×3 matrices; the mapping $(\cdot)^\wedge : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ is given by $x^\wedge = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$, where $x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$, and $(\cdot)^\vee : \mathfrak{so}(3) \mapsto \mathbb{R}^3$ is the inverse mapping to $(\cdot)^\wedge$; $\text{Exp}(\cdot) = \exp((\cdot)^\wedge)$ defines a mapping from \mathbb{R}^3 to $SO(3)$, $\text{Log}(\cdot) = (\log(\cdot))^\vee$ defines a mapping from $SO(3)$ to \mathbb{R}^3 , where \exp and \log are the matrix exponential and the matrix logarithm, respectively; \mathcal{K} denotes a class of continuous strictly increasing positive definite functions $\mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\alpha(0) = 0$. \mathcal{KL} denotes a continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$, if, for each fixed s , $\beta(\cdot, s) \in \mathcal{K}$ and for each fixed r , $\beta(r, \cdot)$ is strictly decreasing to zero. $\mathbf{B}^n[r] = \{x \in \mathbb{R}^n : |x| \leq r\}$ and $\mathbf{B}^n(r) = \{x \in \mathbb{R}^n : |x| < r\}$ denote the closed ball and open ball of the radius r on \mathbb{R}^n . $\partial\mathcal{C}$ and $\bar{\mathcal{C}}$ denote the boundary and closure of an open set $\mathcal{C} \in \mathbb{R}^n$. L^∞

denotes the function space that its elements are the essentially bounded measurable function. For $f \in L^\infty$, $\|f\|_\infty = \inf \{c \in \mathbb{R}_{\geq 0} : |f(x)| \leq c \text{ for almost every } x\}$.

2 Problem statement

The dynamics of a fully-actuated rigid body rotation motion is governed by the following nonlinear differential equation (see, e.g. [Bharadwaj et al. \(1998\)](#)):

$$\dot{R} = R\omega^\wedge, \quad t > 0, \quad R(0) = R_0, \quad (1a)$$

$$\dot{\omega} = J^{-1}(-\omega \times J\omega + M), \quad \omega(0) = \omega_0, \quad (1b)$$

where $R(t) \in SO(3)$ is the rotation matrix with respect to the inertial frame, $\omega \in \mathbb{R}^3$ is the angular velocity expressed in body frame, $J \in \mathbb{R}^{3 \times 3}$ is the inertia matrix with respect to rigid body's mass center, $M(t) \in \mathbb{R}^3$ is the torque expressed in body frame (control input).

This paper investigates how to design a feedback controller M that can ensure a *global uniform finite/fixed-time attitude tracking*, i.e.,

$$\forall R_0 \in SO(3), \forall \omega_0 \in \mathbb{R}^3, \exists T \in \mathbb{R}_+ : R(t) = R_d(t), \forall t \geq T,$$

where T is independent of R_0, ω_0 in the fixed-time case. For this purpose, this paper adopts the so-called implicit generalized homogeneous algorithms ([Polyakov \(2020\)](#), [Zimenko et al. \(2020\)](#)) to treat the global finite/fixed time attitude trajectory tracking problem, since they can ensure faster (finite/fixed-time) convergence, better robustness, less overshoots.

3 Homogeneous Stabilization

In this section, we briefly recall some fundamental concepts of generalized homogeneous control design.

3.1 Linear dilations in \mathbb{R}^n

By definition, the homogeneity is a dilation symmetry [Zubov \(1958\)](#), [Kawski \(1991\)](#), [Fischer and Ruzhansky \(2016\)](#), [Polyakov \(2020\)](#), and the dilation (see [Husch \(1970\)](#); [Kawski \(1991\)](#)) in a normed vector space is a one-parameter group $\mathbf{d}(s)$, $s \in \mathbb{R}$ of transformations satisfying the limit property $\lim_{s \rightarrow \pm\infty} \|\mathbf{d}(s)x\| = e^{\pm\infty}$, $\forall x \neq \mathbf{0}$, where $\|\cdot\|$ is a norm in \mathbb{R}^n . In the general case, the dilation may not be a Lie (continuous) group. In this paper we deal only with the linear (geometric) dilation in \mathbb{R}^n which is defined as follows

$$\mathbf{d}(s) = e^{sG_d} := \sum_{i=0}^{+\infty} \frac{s^i G_d^i}{i!}, \quad s \in \mathbb{R}, \quad (2)$$

where $G_d \in \mathbb{R}^{n \times n}$ is an anti-Hurwitz¹ matrix being the generator of the dilation \mathbf{d} . A dilation \mathbf{d} is monotone if

¹ A matrix $G_d \in \mathbb{R}^{n \times n}$ is anti-Hurwitz if $-G_d$ is Hurwitz.

$s \mapsto \|\mathbf{d}(s)x\|$ is a strictly increasing function for any $x \neq \mathbf{0}$. It is worth noting that monotonicity of the dilation may depend of the norm $\|\cdot\|$ in \mathbb{R}^n . Also, any linear dilation in \mathbb{R}^n is monotone [Polyakov \(2020\)](#) provided that the norm in \mathbb{R}^n is defined as follows

$$\|x\| = \sqrt{x^\top P x}, \quad P G_d + G_d^\top P \succ 0, P \succ 0. \quad (3)$$

The linear dilation introduces an alternative norm topology in \mathbb{R}^n by means of a homogeneous norm (see, e.g., [Grüne \(2000\)](#)).

Definition 1 *The scalar-valued function $\|\cdot\|_d : \mathbb{R} \mapsto [0, \infty)$ defined as $\|\mathbf{0}\|_d = 0$ and*

$$\|u\|_d = e^{s_u}, \text{ where } s_u \in \mathbb{R} : \|\mathbf{d}(-s_u)x\| = 1,$$

is called the canonical \mathbf{d} -homogeneous norm in \mathbb{R}^n , where \mathbf{d} is a monotone dilation in \mathbb{R}^n .

For any linear monotone dilation in \mathbb{R}^n , the canonical homogeneous norm is continuous on \mathbb{R}^n and locally Lipschitz continuous on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Moreover, it is differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$:

$$\frac{\partial \|x\|_d}{\partial x} = \|x\|_d \frac{x^\top \mathbf{d}^\top (-\ln \|x\|_d) P \mathbf{d} (-\ln \|x\|_d)}{x^\top \mathbf{d}^\top (-\ln \|x\|_d) P G_d \mathbf{d} (-\ln \|x\|_d) x} \quad (4)$$

provided that the canonical homogeneous norm is induced by the Euclidean norm (3). Below we use the canonical homogeneous norm as an implicit Lyapunov function for an analysis impulsive locally homogeneous systems and define the \mathbf{d} -homogeneous projector on the unit sphere ([Polyakov, 2020](#), p. 159) as follows

$$\pi_d(x) = \mathbf{d}(-\ln \|x\|_d)x, \quad x \neq \mathbf{0}. \quad (5)$$

Indeed, for any $x \neq \mathbf{0}$, by definition of the canonical homogeneous norm, we have $\|\pi_d(x)\| = 1$.

3.2 Homogeneous Systems

Definition 2 *Kawski (1991)* A vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (resp. a function $h : \mathbb{R}^n \rightarrow \mathbb{R}$) is said to be \mathbf{d} -homogeneous of degree $\mu \in \mathbb{R}$ if

$$f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s)f(x), \text{ (resp. } h(\mathbf{d}(s)x) = e^{\mu s} h(x)),$$

$\forall x \in \mathbb{R}^n, \forall s \in \mathbb{R}$, where \mathbf{d} is a linear dilation in \mathbb{R}^n .

Any \mathbf{d} -homogeneous system

$$\dot{x} = f(x), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}^n \quad (6)$$

is diffeomorphic on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and homeomorphic on \mathbb{R}^n to a standard homogeneous system ([Polyakov \(2020\)](#)). The following result is the straightforward corollary of the Zubov-Rosier Theorem ([Zubov \(1958\)](#), [Rosier \(1992\)](#)).

Theorem 1 *Let a continuous vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a \mathbf{d} -homogeneous of degree $\mu \in \mathbb{R}$. The system (6) is globally uniformly asymptotically stable if and only if there exists a positive definite \mathbf{d} -homogeneous function $V : \mathbb{R}^n \mapsto [0, +\infty)$ such that $V \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{\mathbf{0}\})$,*

$\dot{V}(x) \leq -\rho V^{1+\mu}(x)$, $\forall x \neq 0, \rho > 0$. Moreover, the system (6) is globally uniformly finite-time stable for $\mu < 0$; globally uniformly exponentially stable for $\mu = 0$; globally uniformly nearly fixed-time stable for $\mu > 0$.

More details about finite/nearly fixed-time stability and their differences with respect to other stability concepts are given in Definition 3 in Appendix.

4 Main results

In this section we design a global homogeneous attitude control. First, we study the problem of homogeneous stabilization of a class of impulsive systems. Then we apply the obtained results to design global homogeneous controller for an attitude control of rigid body, based on the fact that system (1) can be re-formulated as an impulsive system on Lie algebra $\mathfrak{so}(3)$.

4.1 Homogeneous Lyapunov function for Impulsive Nonlinear Systems

Let us consider the following impulsive system

$$\begin{cases} \dot{x} = f(t, x), & x^- \in \mathcal{C}, \\ x = \Pi x^-, & x^- \in \mathcal{D}, \end{cases} \quad (7)$$

where $t > t_0$, $x \in \mathbb{R}^n$, $f \in C(\mathbb{R}^{n+1}, \mathbb{R}^n)$, the matrix $\Pi \in \mathbb{R}^{n \times n}$ defines a linear reset map, $x^- = \lim_{h \rightarrow 0^-} x(t+h)$,

$$\mathcal{C} = \{x \in \mathbb{R}^n : g_1(x) < 0\}, \mathcal{D} = \left\{x \in \mathbb{R}^n : \begin{matrix} g_1(x) = 0, \\ g_2(x) > 0 \end{matrix} \right\} \quad (8)$$

are subsets of \mathbb{R}^n , $g_1, g_2 \in C^1(\mathbb{R}^n, \mathbb{R})$. Inspired by [Hespanha et al. \(2008\)](#), a solution of (7) is a right-continuous function x satisfying the differential equation almost everywhere, belonging to \mathcal{C} almost everywhere and having left limits at each $t > t_0$. The following theorem provides a sufficient condition for the existence of solutions and a finite/fixed-time stability analysis of the impulsive system (7) by means of implicit homogeneous Lyapunov function designed for the impulse-free system.

Theorem 2 *Let \mathcal{C} given by (8) be an open connected set such that $\mathbf{0} \in \mathcal{C}$, $\Pi \mathcal{D} \subset \partial \mathcal{C} \setminus \overline{\mathcal{D}}$,*

$$\frac{\partial g_1}{\partial x} f(t, x) < 0, \quad \forall x \in \partial \mathcal{C} \setminus \overline{\mathcal{D}}, \quad \forall t \in \mathbb{R} \quad (9)$$

and for any $r > 0$ there exists $\gamma > 0$ such that

$$\frac{\partial g_1}{\partial x} f(t, x) \leq 0, \quad \frac{\partial g_2}{\partial x} f(t, x) \leq -\gamma, \quad \forall x \in \partial \mathcal{D} \cap B^n[r], \quad \forall t \in \mathbb{R}. \quad (10)$$

Let the canonical homogeneous norm induced by the weighted Euclidean norm (3) be a local Lyapunov function $V(x) = \|x\|_{\mathbf{d}}$ of the impulse-free system $\dot{x} = f(t, x)$ satisfying

$$\frac{\partial V}{\partial x} f(t, x) \leq -\rho V^{1+\mu}(x), \quad \forall x \in \mathcal{C} \setminus \{\mathbf{0}\}, \quad \forall t \in \mathbb{R}, \quad (11)$$

where $\mu \in \mathbb{R}$ and $\rho > 0$. If

$$G_{\mathbf{d}} \Pi = \Pi G_{\mathbf{d}}, \quad (12)$$

$$y^\top \Pi^\top P \Pi y \leq 1, \quad \forall y \in \pi_{\mathbf{d}}(\mathcal{D}) \quad (13)$$

then the closed-loop impulsive system (7) is

- uniformly finite-time stable for $\mu < 0$ and

$$x(t) = \mathbf{0}, \quad \forall t \geq t_0 + \frac{\|x(t_0)\|_{\mathbf{d}}^{-\mu}}{-\rho\mu}, \quad \forall x(t_0) \in \overline{\mathcal{C}}, \quad \forall t_0 \in \mathbb{R};$$

- uniformly exponentially stable for $\mu = 0$ and

$$\|x(t)\|_{\mathbf{d}} \leq \frac{\|x(t_0)\|_{\mathbf{d}}}{e^{\rho(t-t_0)}}, \quad \forall t \geq t_0, \quad \forall x(t_0) \in \overline{\mathcal{C}}, \quad \forall t_0 \in \mathbb{R};$$

- uniformly nearly fixed-time stable for $\mu > 0$,

$$\|x(t)\|_{\mathbf{d}} \leq r, \quad \forall t \geq t_0 + \frac{1}{\rho r^\mu}, \quad \forall r > 0, \quad \forall x(t_0) \in \overline{\mathcal{C}}, \quad \forall t_0 \in \mathbb{R}.$$

Proof: Since $f \in C(\mathbb{R}^{n+1}, \mathbb{R}^n)$, \mathcal{C} is an open connected set and $\mathcal{D} \subset \partial \mathcal{C}$, then for any initial value $x(t_0) \in \mathcal{C}$, $t_0 \in \mathbb{R}$ the impulsive system (7) has a solution defined as long as $x^- \notin \partial \mathcal{C}$ and $|x^-| < +\infty$. The condition (9) implies that for any $t \in \mathbb{R}$ and any $y \in \partial \mathcal{C} \setminus \overline{\mathcal{D}}$, the vector field $f(t, y)$ is oriented inside the open connected set \mathcal{C} , so $\partial \mathcal{C} \setminus \overline{\mathcal{D}}$ is a repelling set of the system (7). Hence, for any initial value $x(t_0) \in \overline{\mathcal{C}} \setminus \overline{\mathcal{D}}$ the system \mathcal{H} has a solution $x(t)$ (belonging to \mathcal{C} for $t > t_0$) as long as $x^- \notin \overline{\mathcal{D}}$ and $|x^-| < +\infty$. Similarly, the condition (10) guarantees that for any $y \in \partial \mathcal{D}$ the vector field $f(t, y)$ is oriented inside the set $\overline{\mathcal{C}} \setminus \{\mathcal{D}\}$. So, for any initial value $x(t_0) \in \overline{\mathcal{C}} \setminus \mathcal{D}$ the system (7) has a solution $x(t)$ (belonging to \mathcal{C} for $t > t_0$) as long as $x^- \notin \mathcal{D}$ and $|x^-| < +\infty$. Since $\Pi \mathcal{D} \subset \partial \mathcal{C} \setminus \overline{\mathcal{D}}$ then $\Pi \mathcal{D} \cap \mathcal{D} = \emptyset$ and there are no multiple values at the instant of jump and the solution enters in \mathcal{C} after the jump, so it can be continuously prolonged till the next jump. Let t_k with $k = 1, 2, \dots$ denote time instances such that $x^-(t_k) \in \mathcal{D}$. Let us prove that the sequence t_k does not have an accumulation point. Suppose the contrary, $t_k \rightarrow t^* \neq +\infty$ as $k \rightarrow +\infty$ provided that $x(t)$ is uniformly bounded $\|x(t)\| < r$ on $[t_0, t^*]$. Since the solution is continuous and uniformly bounded between jumps, then due to continuity of f , its derivative is uniformly bounded $\|\dot{x}(t)\| \leq p, \forall t \in (t_k, t_{k+1})$, where $p = \max_{t \in [t_0, t^*], x \in B^n[r]} \|f(t, x)\|$. Thus $\delta_k = x^-(t_{k+1}) - x(t_k) = \int_{t_k}^{t_{k+1}} \dot{x}(\tau) d\tau \rightarrow \mathbf{0}$ as $k \rightarrow +\infty$. By Bolzano-Weierstrass Theorem there always exists a subsequence t_{k_i} such that $x^-(t_{k_i}) \rightarrow x^* \in \overline{\mathcal{D}} \cap B^n[r]$ as $i \rightarrow +\infty$. In this case, $\Pi x^-(t_{k_i}) \rightarrow \Pi x^*$ as $i \rightarrow +\infty$. Since $\Pi x^-(t_{k_i}) = x^-(t_{k_i+1}) - \delta_{k_i}$ and $\delta_{k_i} \rightarrow \mathbf{0}$ as $i \rightarrow +\infty$ then $\Pi x^* = x^{**} := \lim_{i \rightarrow +\infty} x^-(t_{k_i}^-) \in \overline{\mathcal{D}} \cap B^n[r]$. Notice that $x^* \in \partial \mathcal{D} \cap B^n[r]$, otherwise $\Pi x^* = x^{**} \in \partial \mathcal{C} \setminus \overline{\mathcal{D}}$, but this contradicts to $x^{**} \in \overline{\mathcal{D}}$. The condition (10) together with the continuity of f and the smoothness of g_2 imply that for a sufficiently small $\delta > 0$ we have $\frac{\partial g_2}{\partial x} f(t, x) \leq -\gamma/2$ for all $x \in (\partial \mathcal{D} \cap B^n[r]) \dot{+} B^n[\delta]$, where $\dot{+}$ denotes the geometric sum of sets. Moreover, there exists $i_r \in \mathbb{N}$ such that for all $i \geq i_r$ we have $x(t) \in x^* \dot{+} B^n[\delta], \forall t \in (t_{k_i}, t_{k_i+1})$. Taking into account $g_2(x(t_{k_i})) < 0$ we derive $g_2(x^-(t_{k_i+1})) = g_2(x(t_{k_i})) + \int_{t_{k_i}}^{t_{k_i+1}} \frac{\partial g_2}{\partial x} \dot{x} dt < 0$. This contradicts to $x^-(t_{k_i+1}) \in \mathcal{D}$. Therefore, any initial value $x(t_0) \in \overline{\mathcal{C}}, t_0 \in \mathbb{R}$ the system (7) has a solution defined as long as $\|x^-\| < +\infty$.

For any $x(t_0) \in \mathcal{C}$, due to the condition (11) and the above explanations, the function $t \mapsto V(x(t))$ decreases for $t > t_0$ as long as $x^- \notin \mathcal{D}$. On the one hand, since $\mathbf{d}(s) = e^{G_{\mathbf{d}} s} = \sum_{k=0}^{\infty} \frac{s^k G_{\mathbf{d}}^k}{k!}$, then the identity (12) implies that:

$$\mathbf{d}(s)\Pi = \Pi\mathbf{d}(s), \quad \forall s \in \mathbb{R}. \quad (14)$$

On the other hand, the Lyapunov function $V(x) = \|x\|_{\mathbf{d}}$ decreases during the jump if $\|x\|_{\mathbf{d}} \leq \|x^-\|_{\mathbf{d}}$. The latter is equivalent to $\frac{\|x\|_{\mathbf{d}}}{\|x^-\|_{\mathbf{d}}} = \|\mathbf{d}(-\ln \|x^-\|_{\mathbf{d}})x\|_{\mathbf{d}} \leq 1$ and to

$$x^\top \mathbf{d}^\top (-\ln \|x^-\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x^-\|_{\mathbf{d}}) x \leq 1. \quad (15)$$

Denoting $s^- = \ln \|x^-\|_{\mathbf{d}}$ and using (14), we derive

$$\begin{aligned} x^\top \mathbf{d}^\top (-s^-) P \mathbf{d} (-s^-) x &= (\Pi x^-)^\top \mathbf{d}^\top (-s^-) P \mathbf{d} (-s^-) \Pi x^- \\ &= (x^-)^\top \mathbf{d}^\top (-s^-) \Pi^\top P \Pi \mathbf{d} (-s^-) x^-, \quad x^- \in \mathcal{D}. \end{aligned} \quad (16)$$

Since $\|\mathbf{d}(-s^-)x^-\|_{\mathbf{d}} = 1$ then $\mathbf{d}(-s^-)x^- \in \pi_{\mathbf{d}}(\mathcal{D})$, where $\pi_{\mathbf{d}}(\mathcal{D})$ is the \mathbf{d} -homogeneous projection of the set \mathcal{D} on the unit sphere. It is well defined since $\mathbf{0} \notin \mathcal{D}$. Therefore, the condition (13) and the identity (16) imply the inequality (15), or equivalently, $\|x\|_{\mathbf{d}} \leq \|x^-\|_{\mathbf{d}}$ for any $x^- \in \mathcal{D}$. The latter means that the function $t \mapsto V(x(t))$ is monotone decreasing along any trajectory $x(t)$ of the system and the sequence t_k of time instances (of jumps) has no accumulation points (see the above explanations). In this case, the inequality (11) holds almost everywhere along the trajectory of the system and the inequality $V(x(t)) \leq V(0) - \rho \int_0^t V^{1+\mu}(x(\tau)) d\tau$ is fulfilled due to monotonicity of $t \mapsto V(x(t))$. Hence, we conclude that $(-\mu)^{-1}(V^{-\mu}(x(t)) - V^{-\mu}(x_0)) \leq -\rho t$ for $\mu \neq 0$ and $V(x(t)) \leq e^{-\rho t} V(x_0)$ for $\mu = 0$. The proof is complete. \square

Notice that, the existence of a Lyapunov function in the form of a canonical homogeneous norm is not a conservative condition for the class of stable \mathbf{d} -homogeneous vector fields f (see, Polyakov (2020) for more details). Theorem 2 can also be utilized for non-homogeneous vector fields, which are close to homogeneous in some sense. Below, we investigate the stability of a specific impulsive system, which is common in practice of attitude control.

Remark 1 If $x \in \mathbb{R}^{2n}$, $u \in \mathbb{R}^n$, $G_{\mathbf{d}} = \text{diag}(g_1 I_n, g_2 I_n)$, $g_1 > 0, g_2 > 0$, $P = \begin{bmatrix} p_{11} I_n & p_{12} I_n \\ p_{12} I_n & p_{22} I_n \end{bmatrix}$, $p_{i,j} > 0, \forall i, j \in \mathbb{N}_{<2}$, $\Pi = \begin{bmatrix} -I_n & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix}$, $\mathcal{C} = \{x \in \mathbb{R}^{2n} : x^\top H^\top H x < r\}$, $\mathcal{D} = \{x \in \mathbb{R}^{2n} : x^\top H^\top H x = r, x^\top Q x > 0\}$, where $r > 0$ is a constant, and $Q = \begin{bmatrix} \mathbf{0} & I_n \\ I_n & \mathbf{0} \end{bmatrix}$ and $H \in \mathbb{R}^n$ is an arbitrary non-zero matrix, then $\Pi \mathcal{D} \subset \partial \mathcal{C} \setminus \overline{\mathcal{D}}$ and the conditions (12) and (13) are fulfilled.

Proof: The identity $\Pi^\top Q \Pi = -Q$ implies that $\Pi \mathcal{D} \subset \partial \mathcal{C} \setminus \overline{\mathcal{D}}$. On the one hand, for $G_{\mathbf{d}} = \text{diag}(g_1 I_n, g_2 I_n)$ we have $G_{\mathbf{d}} \Pi = \Pi G_{\mathbf{d}}$. On the other hand, if $x = [x_1^\top, x_2^\top]^\top \in \mathcal{D}$, then $x^\top Q x = 2x_2^\top x_1 = 2x_1^\top x_2 > 0$. In this case, for $p_{11} > 0, p_{12} > 0, p_{22} > 0$, we derive

$$\begin{aligned} x^\top \Pi^\top P \Pi x &= p_{11} x_1^\top x_1 - 2p_{12} x_1^\top x_2 + p_{22} x_2^\top x_2 \leq \\ p_{11} x_1^\top x_1 + 2p_{12} x_1^\top x_2 + p_{22} x_2^\top x_2 &= x^\top P x. \end{aligned}$$

Since $y \in \{\pi_{\mathbf{d}}(x), \forall x \in \mathcal{D}\}$, according to the latter inequality, we have $y^\top \Pi^\top P \Pi y \leq y^\top P y$. Then, the definition of canonical homogeneous norm indicates that $\|y\| = \|\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x\| = 1$, we have $y^\top \Pi^\top P \Pi y \leq 1$. Thus, we conclude that the condition (13) is fulfilled as well. \square

4.2 Global Homogeneous Attitude Control

The linear dilations, as well as the generalized homogeneity, are introduced for normed vector spaces, therefore the $SO(3)$ group does not admit a dilation in the above sense. So, the homogeneity-based control cannot be introduced directly for the model (1a). However, the Lie algebra $\mathfrak{so}(3)$ is equivalent to a vector space, so the homogeneous control can be designed for a model expressed in terms of the exponential coordinates, which will be discussed below.

4.2.1 Global Attitude Dynamics on Lie Algebra $\mathfrak{so}(3)$

Let us introduce an attitude tracking error

$$R_e = R R_d^\top = e^{\theta_e^\wedge} = \text{Exp}(\theta_e) \quad (17)$$

where $R_d \in C^2(\mathbb{R}_+, SO(3))$ represents a desired smooth attitude trajectory and $\theta_e \in \mathbb{R}^3$. For a given smooth trajectory, the attitude error dynamics can be derived in the exponential coordinates, and the relation between θ_e and ω_e is (see, e.g., Chirikjian (2012))

$$\dot{\theta}_e = J_r^{-1} \omega_e, \quad (18)$$

where $J_r = J_r(\theta_e)$ is the right Jacobian defined in (37), $\omega_e = R_d(\omega - \omega_d)$ and $\omega_d = (R_d^\top \dot{R}_d)^\vee$ is the desired angular velocity in body frame. The error $\theta_e \in B^3[\pi]$ can capture all elements of R_e , but the ambiguity appears on the sphere $\{\theta_e \in \mathbb{R}^3 : |\theta_e| = \pi\}$. The map from $SO(3)$ to $\mathfrak{so}(3)$ is a bijection when $\theta_e \in B^3(\pi)$, to complete the dynamics on $B^3[\pi] \times \mathbb{R}^3$, a jump mechanism $\theta_e = -\theta_e^-$ is introduced for $|\theta_e^-| = \pi$ and $\frac{d}{dt}|\theta_e^-| > 0$ (see Appendix or Bharadwaj et al. (1998) for more details).

Given a smooth attitude trajectory R_d , the global error dynamics of (17) can be then formulated as an impulsive system (Bharadwaj et al. (1998))

$$\begin{cases} \dot{\xi} = \begin{bmatrix} \mathbf{0} & J_r^{-1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \xi + B u, & \xi^- \in \mathcal{C}, \\ \xi = \Pi \xi^-, & \xi^- \in \mathcal{D}, \end{cases} \quad (19)$$

with $\xi = \begin{bmatrix} \theta_e \\ \omega_e \end{bmatrix}$, $\mathcal{D} = \{\xi \in B^3[\pi] \times \mathbb{R}^3 : \xi^\top H^\top H \xi = \pi^2 \text{ and } \xi^\top Q \xi > 0\}$, $\mathcal{C} = B^3(\pi) \times \mathbb{R}^3$, where $H = \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $B = [\mathbf{0}, I_3]^\top$, the matrices Π and Q are as described in Corollary 1 with $n = 3$. Then $u = \dot{\omega}_e$ denotes the virtual

stabilizing control that needs to be designed. Since (1) is equivalent to (19), so the asymptotic stability of (19) implies $R \rightarrow R_d$ and $\omega \rightarrow \omega_d$ as $\xi \rightarrow \mathbf{0}$.

4.3 Homogeneous attitude control on $\mathfrak{so}(3)$

As stated in the introduction, there is no continuous time-invariant feedback control for global asymptotic attitude stabilization. The jump behavior of (19) reveals this statement. A continuous feedback may generate a discontinuous signal due to the state jump.

Theorem 3 Let

$$u_{hom} = \|\xi\|_{\mathbf{d}}^{1+\mu} K \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) \xi, \quad \xi = [\theta_e^\top, \omega_e^\top]^\top \quad (20)$$

be a \mathbf{d} -homogeneous controller with $\mu \in [-1, 1)$, $K = [-K_1 \quad -k_2 I_3]$, $0 < K_1 \in \mathbb{R}^{3 \times 3}$, $k_2 > 0$, the dilation $\mathbf{d}(s) = e^{sG_d}$, $s \in \mathbb{R}$, the canonical homogeneous norm induced by the Euclidean norm $\sqrt{\xi^\top P \xi}$ and

$$G_d = \begin{bmatrix} (1-\mu)I_3 & \mathbf{0} \\ \mathbf{0} & I_3 \end{bmatrix}, P = \begin{bmatrix} I_3 & \varepsilon I_3 \\ \varepsilon I_3 & K_1^{-1} \end{bmatrix}, 0 < \varepsilon < \min\{\varepsilon_\mu, \tilde{\varepsilon}\}, \quad (21)$$

$\varepsilon_\mu = \frac{2(1-\mu)^{\frac{1}{2}}}{(2-\mu)\lambda_{\max}^{\frac{1}{2}}(K_1)}$, $\tilde{\varepsilon} = \frac{4k_2}{2c\lambda_{\max}(K_1)+k_2^2}$, $c = \max_{|\theta_e| \leq \pi} \lambda_{\max}(J_r^{-1+} J_r^{-\top})$. Then the impulsive system (19) with $u = u_{hom}$ is:

- globally uniformly finite-time stable for $\mu < 0$:
 $\xi(t) = \mathbf{0}, \quad \forall t \geq \|\xi(0)\|_{\mathbf{d}}^{-\mu} / (-\mu\tilde{\rho})$;
- globally uniformly exponentially stable for $\mu = 0$:
 $\|\xi(t)\|_{\mathbf{d}} \leq e^{-\tilde{\rho}t} \|\xi(0)\|_{\mathbf{d}}, \quad \forall t \geq 0$;
- globally uniformly nearly fixed-time stable for $\mu > 0$:
 $\|\xi(t)\|_{\mathbf{d}} \leq r, \quad \forall t \geq \frac{1}{\tilde{\rho}\mu r^\mu}, \forall r > 0$;

$$\tilde{\rho} = \frac{\lambda_{\min} \left(P^{-\frac{1}{2}} \begin{bmatrix} 2\varepsilon K_1 & \varepsilon k_2 \\ \varepsilon k_2 & 2k_2 K_1^{-1} - \varepsilon c I_3 \end{bmatrix} P^{-\frac{1}{2}} \right)}{\lambda_{\max}(P^{1/2} G_d P^{-1/2} + P^{-1/2} G_d^\top P^{1/2})}. \quad (22)$$

Moreover, the selection

$$M = J [R_d^\top u_{hom} - \omega_d^\wedge \omega + \dot{\omega}_d] + \omega \times J \omega \quad (23)$$

where $\dot{\omega}_d = \frac{d}{dt} (R_d^\top \dot{R}_d)^\vee = (R_d^\top \ddot{R}_d - (\dot{\omega}_d^\wedge)^2)^\vee \in C(\mathbb{R}_+, \mathbb{R}^3)$, solves the finite-time attitude tracking problem for the system (1a) and (1b) if $\mu < 0$.

Proof: For $\omega_e = R_d(\omega - \omega_d)$ we derive

$$\dot{\omega}_e = R_d \omega_d^\wedge (\omega - \omega_d) + R_d (\dot{\omega} - \dot{\omega}_d). \quad (24)$$

Since $\omega_d^\wedge \omega_d = \mathbf{0}$, then using (1b) and (23) we obtain $\dot{\omega}_e = u_{hom}$ and the dynamics of the closed-loop system becomes (19) with $u = u_{hom}$. The dilation \mathbf{d} is monotone since $PG_d + G_d^\top P > 0 \Leftrightarrow 2(1-\mu)I_3 - \frac{1}{2}\varepsilon^2(2-\mu)^2 K_1 > 0$, $P > 0 \Leftrightarrow K_1^{-1} > \varepsilon^2$. The latter inequalities hold for $\varepsilon < \varepsilon_\mu$. For the Lyapunov function candidate $V(\xi) = \|\xi\|_{\mathbf{d}}$, we have

$$\frac{d}{dt} \|\xi\|_{\mathbf{d}} = \|\xi\|_{\mathbf{d}} \frac{\xi^\top \mathbf{d}^\top (-s_\xi) P \mathbf{d} (-s_\xi) \dot{\xi}}{\xi^\top \mathbf{d}^\top (-s_\xi) P G_d \mathbf{d} (-s_\xi) \xi}, \quad \forall \xi^- \in \mathcal{C}. \quad (25)$$

Hence, using the representation

$$\begin{aligned} \xi^\top \mathbf{d}^\top (-s_\xi) P \mathbf{d} (-s_\xi) \dot{\xi} &= \left[\frac{\|\xi\|_{\mathbf{d}}^{\mu-1} \theta_e}{\|\xi\|_{\mathbf{d}}^{\mu-1} \omega_e} \right]^\top P \left[\frac{\|\xi\|_{\mathbf{d}}^{\mu-1} \dot{\theta}_e}{\|\xi\|_{\mathbf{d}}^{\mu-1} \dot{\omega}_e} \right] \\ &= \|\xi\|_{\mathbf{d}}^{2(\mu-1)} \theta_e^\top J_r^{-1} \omega_e + \varepsilon \|\xi\|_{\mathbf{d}}^{\mu-2} \omega_e^\top J_r^{-1} \omega_e \\ &\quad + \varepsilon \theta_e^\top \left(-\|\xi\|_{\mathbf{d}}^{\mu-2} K_1 \theta_e - \|\xi\|_{\mathbf{d}}^{2(\mu-1)} k_2 \omega_e \right) \\ &\quad + \omega_e^\top K_1^{-1} \left(-\|\xi\|_{\mathbf{d}}^{2\mu-2} K_1 \theta_e - \|\xi\|_{\mathbf{d}}^{\mu-2} k_2 \omega_e \right) \\ &= -\varepsilon \|\xi\|_{\mathbf{d}}^{3\mu-2} \theta_e^\top K_1 \theta_e + \|\xi\|_{\mathbf{d}}^{\mu-2} \omega_e^\top (\varepsilon J_r^{-1} - k_2 K_1^{-1}) \omega_e \\ &\quad + \|\xi\|_{\mathbf{d}}^{2(\mu-1)} \theta_e^\top (J_r^{-1} - (\varepsilon k_2 + 1) I_3) \omega_e \end{aligned}$$

where $s_\xi = \ln \|\xi\|_{\mathbf{d}}$, we derive

$$\frac{d}{dt} \|\xi\|_{\mathbf{d}} = \|\xi\|_{\mathbf{d}}^{1+\mu} \frac{\xi^\top \mathbf{d}^\top (-s_\xi) W_0 \mathbf{d} (-s_\xi) \xi}{\xi^\top \mathbf{d}^\top (-s_\xi) P G_d \mathbf{d} (-s_\xi) \xi}, \quad \xi^- \in \mathcal{C} \quad (26)$$

where $W_0 = \begin{bmatrix} -\varepsilon K_1 & \frac{1}{2}(J_r^{-1} - (\varepsilon k_2 + 1) I_3) \\ * & \varepsilon J_r^{-1} + J_r^{-\top} - k_2 K_1^{-1} \end{bmatrix}$. Taking into account the form of J_r^{-1} given by (37) we conclude

$$\theta_e^\top J_r^{-1} \omega_e = \theta_e^\top \omega_e. \quad (27)$$

Since J_r^{-1} is bounded for $|\theta_e| \leq \pi$, then

$$\frac{d}{dt} \|\xi\|_{\mathbf{d}} \leq \frac{\|\xi\|_{\mathbf{d}}^{1+\mu} \xi^\top \mathbf{d}^\top (-s_\xi) W \mathbf{d} (-s_\xi) \xi}{\xi^\top \mathbf{d}^\top (-s_\xi) (P G_d + G_d^\top P) \mathbf{d} (-s_\xi) \xi}, \quad \xi^- \in \mathcal{C}, \quad (28)$$

where $W = \begin{bmatrix} -2\varepsilon K_1 & -\varepsilon k_2 I_3 \\ -\varepsilon k_2 I_3 & \varepsilon c I_3 - 2k_2 K_1^{-1} \end{bmatrix}$ and c is defined in the statement of Theorem 3. Using Schur Complement we derive $W \prec 0$ if $\varepsilon < \tilde{\varepsilon}$. Therefore, for $\varepsilon < \min\{\varepsilon_\mu, \tilde{\varepsilon}\}$, we have $P \succ 0$, $W \prec 0$. Using the definition of the canonical homogeneous norm, we obtain $\xi^\top \mathbf{d}^\top (-s_\xi) P \mathbf{d} (-s_\xi) \xi = 1$ and

$$\dot{V} \leq -\tilde{\rho} \|\xi\|_{\mathbf{d}}^{1+\mu}, \quad \xi^- \in \mathcal{C} \setminus \{\mathbf{0}\}. \quad (29)$$

In the view of Remark 1, to complete the proof by applying Theorem 2, we just need to show that conditions (9) and (10) are fulfilled for $g_1(\xi) = \theta_e^\top \theta_e - \pi^2$ and $g_2(\xi) = \omega_e^\top \theta_e$. Indeed, for $\xi \in \partial \mathcal{C} \setminus \overline{\mathcal{D}}$ we have $g_2(\xi) < 0$ and

$$\frac{\partial g_1}{\partial \xi} f(\xi, t) = 2\theta_e^\top J_r^{-1} \omega_e = 2\theta_e^\top \omega_e < 0, \quad \xi \in \partial \mathcal{C} \setminus \overline{\mathcal{D}}$$

but for $\xi \in \partial \mathcal{D}$ we have $g_1(\xi) = \theta_e^\top \theta_e - \pi^2 = 0$, $g_2(\xi) = \omega_e^\top \theta_e = 0$, and

$$\frac{\partial g_1}{\partial \xi} f(\xi, t) = 2\theta_e^\top J_r^{-1} \omega_e = 2\theta_e^\top \omega_e = 0, \quad \xi \in \partial \mathcal{D},$$

$$\frac{\partial g_2}{\partial \xi} f(\xi, t) = 2\omega_e^\top J_r^{-1} \omega_e + 2\|\xi\|_{\mathbf{d}}^{\mu+1} \theta_e^\top K \mathbf{d} (-s_\xi) \xi, \quad \xi \in \partial \mathcal{D}.$$

For $\theta_e^\top \omega_e = 0$, using (37), we derive $\omega_e^\top J_r^{-1} \omega_e = |\omega_e|^2 + \omega_e^\top \left(\frac{\theta_e^\wedge}{|\theta_e|} \right)^2 \omega_e = \omega_e^\top \frac{\theta_e \theta_e^\top}{|\theta_e|^2} \omega_e = 0$. Since $\theta_e^\top K \mathbf{d} (-s_\xi) \xi = -\|\xi\|_{\mathbf{d}}^{\mu-1} \theta_e^\top K_1 \theta_e - \|\xi\|_{\mathbf{d}}^{-1} k_2 \theta_e^\top \omega_e$, then

$$\frac{\partial g_2}{\partial \xi} f(\xi, t) = -\|\xi\|_{\mathbf{d}}^{2\mu} \theta_e^\top K_1 \theta_e < 0, \quad \xi \in \partial \mathcal{D}.$$

Taking into account the equivalence of Euclidean norm and the canonical homogeneous norm, we conclude that for any $\forall r \in \mathbb{R}_+$ there exists $\gamma > 0$ such that

$$\frac{\partial g_2}{\partial \xi} f(\xi, t) < -\gamma, \quad \xi \in \partial \mathcal{D} \cap B^n[r].$$

The proof is complete. \square

Due to the monotone decay of the Lyapunov function, the states of the closed-loop system (23) and (19) will eventually enter an invariant set (a neighborhood of the origin). For any bounded initial condition, the system trajectory state will have a finite number of jumps (see the proof of Theorem 2). Clearly, for $\mu < 0$, finite-time convergence to the origin is guaranteed but the settling time depends on the initial states; for $\mu > 0$ the algorithm ensures a fixed-time convergence to a neighborhood of the origin independently of the initial state of the system. Using a commutation of the homogeneous controller (as in Polyakov et al. (2015)), a global fixed-time stabilization of tracking error at zero can be guaranteed. The nearly fixed-time controller in Theorem 3 is a supporting result for the following fixed-time control design.

Corollary 1 (Fixed-time stabilization) *Let $-1 < \mu_2 < 0 < \mu_1 < 1$, $K = [-K_1 \quad -k_2 I_3]$, $0 \prec K_1 \in \mathbb{R}^{3 \times 3}$, $k_2 \in \mathbb{R}_+$ and*

$$u_{fxt} = \begin{cases} \|\xi\|_{\mathbf{d}_1}^{1+\mu_1} K \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}_1}) \xi & \text{if } \xi^\top P \xi \geq 1, \\ \|\xi\|_{\mathbf{d}_2}^{1+\mu_2} K \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}_2}) \xi & \text{if } \xi^\top P \xi \leq 1, \end{cases} \quad (30)$$

where the dilations $\mathbf{d}_1, \mathbf{d}_2$ and the canonical homogeneous norms $\|\cdot\|_{\mathbf{d}_1}, \|\cdot\|_{\mathbf{d}_2}$ are defined as in Theorem 3 for $\mu = \mu_1$ and $\mu = \mu_2$, respectively, but the matrix P is defined as in Theorem 3 with $\varepsilon \in (0, \min\{\epsilon_{\mu_1}, \epsilon_{\mu_2}, \tilde{\epsilon}\})$. Then the impulsive system (19) with $u = u_{fxt}$ is globally uniformly fixed-time stable:

$$\forall \xi(t) = \mathbf{0}, \quad \forall t \geq \frac{1}{-\tilde{\rho}_2 \mu_2} + \frac{1}{\tilde{\rho}_1 \mu_1},$$

where $\tilde{\rho}_1 > 0$ and $\tilde{\rho}_2 > 0$ are defined by the formula (22) with $\mu = \mu_1$ and $\mu = \mu_2$, respectively. Moreover, the selection of M in the form (23) replacing u_{hom} with u_{fxt} solves the fixed-time attitude tracking problem for the system (1a) and (1b).

Proof: Since $\xi^\top P \xi = 1$ (resp., $\leq 1, \geq 1$) is equivalent to $\|\xi\|_{\mathbf{d}_1} = 1$ (resp., $\leq 1, \geq 1$) and it is equivalent to $\|\xi\|_{\mathbf{d}_2} = 1$ (resp., $\leq 1, \geq 1$), then, in both cases $\mu = \mu_1$ and $\mu = \mu_2$, the unit ball $\{\xi : \xi^\top P \xi \leq 1\}$ is a strictly positively invariant and globally uniformly attractive set of the system (19) with the homogeneous control (20). So, when $\xi^\top(t) P \xi(t) \leq 1$ and $\mu = \mu_1 > 0$, due to the estimate $\dot{V} \leq -\tilde{\rho}_1 V^{1+\mu}$ (see the proof of Theorem 3) and the nearly fixed-time stability, any trajectory of the closed-loop system (19) with (30) converges to the unit ball in a fixed time independent of the initial time. Namely, $\|\xi(t)\|_{\mathbf{d}_2} \leq 1, \forall t \geq T_1 = \frac{1}{\tilde{\rho}_1 \mu_1}$. The latter means that $\xi^\top(t) P \xi(t) \leq 1$ for all $t \geq T_1$. When the state lies inside the unit ball, we have $\mu = \mu_2 < 0$. According to Theorem 3, the closed-loop system (19) with (30) converges to zero in finite time. Thus, the system converges to zero in a fixed time, and $\xi(t) = \mathbf{0}$ for all $t \geq T_1 + \frac{1}{-\tilde{\rho}_2 \mu_2}$ independently of the initial state $\xi(0)$. \square

The switching control (30) combines two homogeneous

controllers Polyakov et al. (2015) to guarantee a global fixed-time tracking. Since $\|\xi\|_{\mathbf{d}_1} = \|\xi\|_{\mathbf{d}_2} = 1$ for $\xi^\top P \xi = 1$ and $\mathbf{d}_1(0) = \mathbf{d}_2(0) = I_6$, then the control u_{fxt} is continuous on $\{\xi : \xi^\top P \xi = 1\}$.

4.4 Robustness analysis of homogeneous controller

Any control system is subjected to perturbations and measurement noises in practice. In this case, it is important to investigate its robustness in the sense of Input-to-State Stability (ISS), Sontag (1989), Hespanha et al. (2008). Below we also prove the so-called finite-time and fixed-time ISS developed in Hong et al. (2010), Lopez-Ramirez et al. (2020), Aleksandrov et al. (2022). Notice that the noised measurement of $R \in SO(3)$ can be modeled as follows

$$\tilde{R} = d_R R,$$

where $\tilde{R} \in SO(3)$ is a measured/estimated rotation matrix, and $d_R \in SO(3)$ models measurement noise on $SO(3)$. In this case, we derive

$$\tilde{\theta}_e^\wedge = \log(d_R R R_d^\top) = \log(R_e) + \log(d_R) = \theta_e^\wedge + \log(d_R),$$

i.e. the model of measurements in exponential coordinates (on the Lie algebra $\mathfrak{so}(3)$) includes the noise in the additive manner. So, the perturbed closed-loop attitude control system can be modeled as follows

$$\begin{cases} \dot{\xi} = \tilde{A} \xi + B u(\xi + \delta_1) + \delta_2, & \xi^- \in \mathcal{C}, \\ \xi = H \xi^-, & \xi^- \in \mathcal{D}, \end{cases} \quad (31)$$

where $\tilde{A} = \begin{bmatrix} \mathbf{0} & J_r^{-1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $\xi, u, \mathcal{C}, \mathcal{D}$ are as before, $\delta_1 = [(\log(d_R)^\vee)^\top, d_\omega^\top]^\top \in \mathbb{R}^6$ is a measurement noise, $\delta_2 \in \mathbb{R}^6$ is an exogenous perturbation and $d_\omega \in \mathbb{R}^3$ is the measurement noise of the angular velocity.

Corollary 2 *Under conditions of Theorem 3 (Corollary 1), the impulsive system (31) with $u = u_{hom}$ ($u = u_{fxt}$) is*

- locally ISS with respect to $\delta = [\delta_1^\top, \delta_2^\top]^\top \in L^\infty(\mathbb{R}, \mathbb{R}^6)$,
- (fixed-time) ISS with respect to $\delta = [\delta_1^\top, \delta_2^\top]^\top \in \Delta$,

for $\mu > -1$, where $\Delta \subset L^\infty(\mathbb{R}, \mathbb{R}^6)$ is the set of perturbations such that the system (31) is forward complete².

Proof: Let the vector field $f : \mathbb{R}^{18} \mapsto \mathbb{R}^6$ be defined as

$$f(\xi, \delta) = \dot{\xi} = \tilde{A} \xi + B u_{hom}(\xi + \delta_1) + \delta_2,$$

where $\xi \in \mathbb{R}^6, \delta_1, \delta_2 \in \mathbb{R}^6, \delta = (\delta_1^\top, \delta_2^\top)^\top \in \mathbb{R}^{12}$. For $V = \|\xi\|_{\mathbf{d}}$ we have

$$\frac{\partial V(z)}{\partial z} \Big|_{z=\mathbf{d}(s)\xi} \mathbf{d}(s) = e^s \frac{\partial V}{\partial \xi}, \quad \forall \xi \neq \mathbf{0}, \forall s \in \mathbb{R}. \quad (32)$$

In this case, using the identity $\mathbf{d}(s)B = e^s B$ we derive

² The system is forward complete if all solutions of the system exist globally in the forward time.

$$\begin{aligned}\frac{\partial V}{\partial \xi} f(\xi, \delta) &= \frac{\partial V}{\partial \xi} f(\xi, 0) + \frac{\partial V}{\partial \xi} (f(\xi, \delta) - f(\xi, 0)) = \\ \frac{\partial V}{\partial \xi} f(\xi, 0) &+ \frac{\partial V}{\partial \xi} (Bu_{hom}(\xi + \delta_1) - Bu_{hom}(\xi) + \delta_2) = \\ \frac{\partial V}{\partial \xi} f(\xi, 0) &+ \frac{\partial V(z)}{\partial z} (Bu_{hom}(\xi + \delta_1) - u_{hom}(\xi)) + \mathbf{d}(s)\delta_2,\end{aligned}$$

where $z = \mathbf{d}(s)\xi$. The \mathbf{d} -homogeneity of u_{hom} gives

$$u_{hom}(y) = e^{-(1+\mu)s} u_{hom}(\mathbf{d}(s)y), \forall y \in \mathbb{R}^6, \forall s \in \mathbb{R}.$$

Taking $s = -\ln \|\xi\|_{\mathbf{d}}$ we derive $\|Bu_{hom}(\xi + \delta_1) - Bu_{hom}(\xi)\| = \|\xi\|_{\mathbf{d}}^{1+\mu} \|Bu_{hom}(z + \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}})\delta_1) - Bu_{hom}(z)\|$, where $z = \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}})\xi$ belongs to the unit sphere $\|z\| = 1$. Since $u_{hom} \in C^1(\mathbb{R}^6 \setminus \{\mathbf{0}\}, \mathbb{R}^3)$, then for $\|\mathbf{d}(-\ln \|\xi\|_{\mathbf{d}})\delta_1\| \leq 0.5$, by using mean value inequality we derive

$$\begin{aligned}\|Bu_{hom}(z + \mathbf{d}(s)\delta_1) - Bu_{hom}(z)\| &\leq \\ \|B \int_0^1 \frac{\partial u_{hom}(y)}{\partial y} \Big|_{y=z+\lambda \mathbf{d}(s)\delta_1} \mathbf{d}(s)\delta_1 d\lambda\| &\leq C_1 \|\mathbf{d}(s)\delta_1\|,\end{aligned}$$

where $s = -\ln \|\xi\|_{\mathbf{d}}$, $C_1 = \sup_{0.5 \leq \|y\| \leq 1.5} \|B \frac{\partial u_{hom}(y)}{\partial y}\|$. Since $C_2 = \sup_{\|z\|=1} \|\frac{\partial V}{\partial z}\| < +\infty$ then

$$\frac{\partial V}{\partial x} f(\xi, \delta) \leq \frac{\partial V}{\partial x} f(\xi, 0) + V^{1+\mu} \tilde{C} \sum_{i=1}^2 \|\mathbf{d}(-\ln \|\xi\|_{\mathbf{d}})\delta_i\|,$$

where $\tilde{C} = C_2 \max\{C_1, 1\}$. In Theorem 3, it is shown that $\frac{\partial V}{\partial x} f(\xi, 0) \leq -\tilde{\rho}V$ for all $\xi \in \mathcal{C}$. If $\xi \in \mathcal{C}$, then $\|\mathbf{d}(-\ln \|\xi\|_{\mathbf{d}})\delta_i\| \leq \frac{\tilde{\rho}}{4\tilde{C}}, i = 1, 2$ imply $\frac{\partial V}{\partial x} f(\xi, \delta) \leq -\frac{\tilde{\rho}}{2}V^{1+\mu}$. Since $\|\mathbf{d}(-\ln \|\xi\|_{\mathbf{d}})\delta_i\| \leq \frac{\tilde{\rho}}{4\tilde{C}}$ is equivalent to $\|4\tilde{C}\delta_i/\tilde{\rho}\| \leq \|\xi\|_{\mathbf{d}}$, then for $\xi \in \mathcal{C}$ we have

$$V(\xi) \geq \|4\tilde{C}\delta_i/\tilde{\rho}\| \Rightarrow \dot{V}(\xi) \leq -\frac{\tilde{\rho}}{2}V^{1+\mu}(\xi). \quad (33)$$

In the case of the impulse-free continuous time system, the latter implication guarantees that V is an ISS Lyapunov function (see, [Sontag and Wang \(1995\)](#)). Since \mathcal{C} is an open connected set and $0 \in \mathcal{C}$, then for sufficiently small perturbations and initial conditions, the system (31) has no impulse, so it is locally ISS. Using the continuity of f and smoothness of V we expand (33) to $\xi \in \bar{\mathcal{C}} \setminus \mathcal{D}$. In Theorem 3, it is shown that $V(\Pi\xi) \leq V(\xi)$ for any $\xi \in \mathcal{D}$. Hence, if the impulsive system (31) has solutions (in the sense of [Hespanha et al. \(2008\)](#)) belonging to \mathcal{C} almost everywhere, then the function $t \mapsto V(\xi(t))$ with $t \geq t_0$ is strictly decreasing as long as $V(\xi(t)) \geq \frac{4\tilde{C}\|\delta_i\|_{L^\infty(t_0, t)}}{\tilde{\rho}}$. Using conventional arguments (see, [Sontag and Wang \(1995\)](#), [Hespanha et al. \(2008\)](#)) we derive ISS of the impulsive system. Moreover, from (33) with $\mu < 0$ it follows the finite-time ISS ([Hong et al. \(2010\)](#)), but the fixed-time ISS ([Lopez-Ramirez et al. \(2020\)](#), [Aleksandrov et al. \(2022\)](#)) of the system (31) with $u = u_{fxt}$ follows from (33) with $\mu_2 < 0$ for $V(\xi) \leq 1$ and from (33) with $\mu = \mu_1 > 0$ for $V(\xi) > 1$. The proof is complete. \square

5 Numerical Simulation

A simulation of a quadrotor attitude tracking is created in Simulink with the step size 0.001 sec. The model corre-

sponds to QDrone platform manufactured by Quanser: $J = 10^{-2} \text{diag}(1.0, 0.82, 1.48) \text{ kg} \cdot \text{m}^2$. The initial states are given by $\omega(0) = [0, -1, 0]^\top \text{ rad} \cdot \text{sec}^{-1}$ and $R(0) = I$.

The parameters of the finite-time homogeneous controller (20) are selected as follows: $\mu = -1/3$ and $K_{fnt} = [-9I_3, -5I_3]$, $G_{\mathbf{d}} = \text{diag}(\frac{4}{3}I_3, I_3)$, $\varepsilon = 0.05$, $P = \begin{bmatrix} I_3 & 0.05I_3 \\ 0.05I_3 & 0.083I_3 \end{bmatrix}$. Hence, we have $\tilde{\rho} = 0.2985$. The parameters of the fixed-time homogeneous controller (30) are selected as follows: $\mu_1 = 0.1$, $\mu_2 = -1/3$ and $K_{fxt} = [-9I_3, -5I_3]$, $G_{\mathbf{d}_1} = \text{diag}(0.9I_3, I_3)$, $G_{\mathbf{d}_2} = \text{diag}(\frac{4}{3}I_3, I_3)$, $\varepsilon = 0.05$, $P = \begin{bmatrix} I_3 & 0.05I_3 \\ 0.05I_3 & 0.1I_3 \end{bmatrix}$, $\tilde{\rho}_1 = 0.3983$, $\tilde{\rho}_2 = 0.2985$. By u_{fnt} and u_{fxt} we denote the proposed finite-time and fixed-time homogeneous controllers, respectively.

For comparison, we consider the finite-time controller proposed in [Shi et al. \(2018\)](#), [Shi et al. \(2017\)](#)

$$u_1 = J_r^{-1} k_p \text{sign}(\theta_e)^{\alpha_1} + k_d \text{sign}(\omega_e)^{\alpha_2}, \quad (34)$$

where $\alpha_1 = \alpha$, $\alpha_2 = 2\alpha/(1 + \alpha)$, $\alpha \in (0, 1)$; $x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$, $\text{sign}(x)^\alpha = [|x_1|^\alpha \text{sign}(x_1), |x_2|^\alpha \text{sign}(x_2), |x_3|^\alpha \text{sign}(x_3)]^\top$. Since the tracking error system with u_1 is \mathbf{d} -homogeneous with degree $\frac{\alpha-1}{\alpha+1}$, so, for a fair comparison, we select $\alpha = 0.5 \Rightarrow \frac{\alpha-1}{\alpha+1} = \mu = -1/3$ to have the same homogeneity degree for both controllers. The linear controller proposed in [Yu and Ding \(2016\)](#):

$$u_2 = k_p^l \theta_e + k_d^l \omega_e \quad (35)$$

is also utilized for comparison. The controller u_2 corresponds to our homogeneous controller if $\mu = 0$. We use the same gains for all controllers: $K_{fnt} = K_{fxt} = [k_p, k_d] = [k_p^l, k_d^l] = [-9I_3, -5I_3]$ (the gain selection takes into consideration that each element of torque is bounded by $[-0.5, 0.5]$). In all cases, the torque M is given by (23).

The desired trajectory is generated by $\omega_d = [-0.2t, -0.2t + 3, t]^\top$ and R_d obtained by integration of $\dot{R}_d = R_d \omega_d^\wedge$ with the initial condition $R_d(0) = e^{([-0.1\pi, 0.95\pi, 0]^\top)^\wedge}$. Two cases are studied by numerical simulations:

Case 1: no measurement noises or perturbations;

Case 2: the measurement noise δ_1 is modeled by the uniformly distributed random variables d_R and d_ω satisfying $\|d_R\|_\infty \leq 0.05$ and $\|d_\omega\|_\infty \leq 0.05$, but the additive perturbation is deterministic, state-dependent and defined as $\delta_2 = [\mathbf{0}, 0.01 \text{diag}(2, 2, 1)\omega]^\top$.

The simulation results are shown in Figs. 1, 2, and 3. The control energy $E = (\int_0^{t_f} \|M\|^2 d\tau)^{1/2}$ on the time interval $[0, t_f]$ is compared in Table 1. In the disturbance-free case, as shown in Fig. 1, the system with finite/fixed-time controls exhibits faster convergence than the linear one. The proposed finite/fixed-time homogeneous controllers u_{fnt} , u_{fxt} are faster than u_1 . Moreover, u_{fxt} demonstrates fastest convergence but u_{fnt} requires less control energy (see Table 1). In Fig 1, the chattering of u_1 is caused by numerical errors

of calculation of $J_r^{-1} = J_r^{-1}(\theta_e)$ (see the formula (37)). Fig. 2 illustrates the state-dependent jump and the decay of the canonical homogeneous norm during the jump with finite/fixed-time controllers. Measurement noises and perturbations destroy the ideal asymptotic and finite/fixed-time stability properties. However, in this specific case, the finite/fixed-time controllers u_{fnt} , u_{fxt} and u_1 ensure higher accuracy than the linear controller (see Fig. 3). Similarly to the disturbance-free case, the homogeneous controllers u_{fnt} , u_{fxt} respond faster than the finite-time controller u_1 . Additionally, as shown in Table 1. the proposed homogeneous controller u_{fnt} consumes less energy providing better control quality (faster response and/or better precision) than u_1 . The proposed fixed-time controller can achieve fastest robust tracking, but more energy consumption. For the finite-time homogeneous controller, the settling time estimate obtained in Theorem 3 gives 13.99 sec. This estimate is rather conservative, but it is obtained for the first time for the global homogeneous attitude control system.

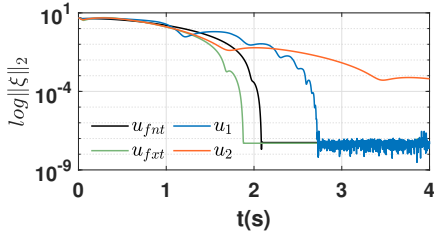


Fig. 1. The norm of the tracking error ξ (Case 1).

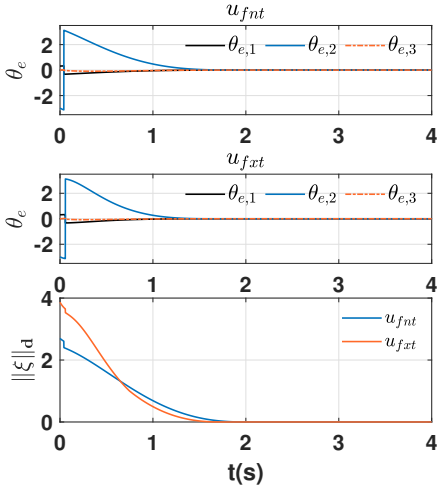


Fig. 2. The jump behavior under proposed homogeneous controller (Case 1).

Table 1
The control energy E for $t_f = 4$.

	u_{fnt}	u_{fxt}	u_1	u_2
$E(case1)$	0.095	0.146	0.117	0.127
$E(case2)$	0.141	0.185	0.197	0.173

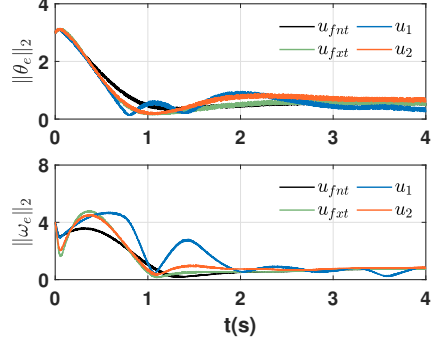


Fig. 3. The norm of the tracking error (Case 2).

6 Conclusion

In this paper, a global attitude tracking control for a fully-actuated rigid body is developed using various symmetries of the system. First, a rotation symmetry of the system is utilized in order to derive an impulsive linear system, which describes the rigid-body dynamics on the Lie algebra $\mathfrak{so}(3)$. Next, a dilation symmetry (generalized homogeneity) is applied for finite/fixed-time controller design. The control law is formulated in terms of the canonical homogeneous norm being the strict Lyapunov function of the closed-loop error system. For homogeneity-based methods, a global estimate of the settling time is provided for the first time. A global fixed-time attitude control approach is designed using a switching strategy, which offers global boundedness of the settling time independently of initial conditions, simplifying control tuning, providing better robustness. Future research may focus on deriving of a better settling time estimate and its tuning as well as on exploring the issues of the digital implementation of the proposed homogeneous controllers.

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Appendix

Let us recall briefly some results (see, e.g., Chirikjian (2012) for more details) about $SO(3)$ and $\mathfrak{so}(3)$. $SO(3) : \{R \in \mathbb{R}^{3 \times 3} : RR^T = I_3, \det(R) = 1\}$, $\mathfrak{so}(3) : \{X \in \mathbb{R}^{3 \times 3} : X^T = -X\}$. The $\mathfrak{so}(3)$ consists of 3×3 skew-matrices with the Lie bracket given by the commutator $[X_1, X_2] = X_1X_2 - X_2X_1$, where $X_1, X_2 \in \mathfrak{so}(3)$. Since any $X \in \mathfrak{so}(3)$ can be uniquely represented by $X = x^\wedge$ with $x \in \mathbb{R}^3$, then $\mathfrak{so}(3)$ is isomorphic to \mathbb{R}^3 . Notice also that for $x \in \mathbb{R}^3$ and $R \in SO(3)$ one holds $Rx^\wedge R^T = (Rx)^\wedge$. The exponential map $\exp(\cdot)$ is a surjective map from $\mathfrak{so}(3)$ to $SO(3)$. For $X \in \mathfrak{so}(3)$ the exponential

map is given by $\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} (X)^n \in SO(3)$. Any $X \in \mathfrak{so}(3)$ (resp. any vector $x \in \mathbb{R}^3 : X = x^\wedge$) uniquely defines a $R = \exp(X) \in SO(3)$. For $R \in SO(3) : \text{trace}(R) \neq -1$, the exponential map $\exp(\cdot)$ and its inverse map $\log(\cdot)$ can be defined as follows:

$$\begin{aligned} R &= \exp(X) = I_3 + \frac{\sin|x|}{|x|} X + \frac{1-\cos|x|}{|x|^2} X^2 \\ X &= \log(R) = \frac{\phi}{2\sin\phi} (R - R^\top) \end{aligned} \quad (36)$$

where $\phi \in (-\pi, \pi)$ is a solution of the equation $1 + 2\cos\phi = \text{trace}(R)$. For $\text{trace}(R) = -1$ then there is an ambiguity in the definition of X , since in this case $X = (\pm\pi y)^\wedge$, where $y \in \mathbb{R}^3$ is a unique unit vector ($y^\top y = 1$) satisfying $Ry = y$. In fact, $SO(3)$ group in the exponential coordinates has a correspondence with the closed ball $B^3[\pi]$. The transformation $\text{Log}(\cdot) = \log(\cdot)^\wedge : \mathbb{R}^3 \rightarrow SO(3)$ is a bijection on $B^3(\pi)$. The ambiguity appear on the sphere $S^3(\pi)$. The left Jacobian of the group $SO(3)$ is

$$\begin{aligned} J_r(x^\wedge) &= I - \frac{1-\cos|x|}{|x|^2} x^\wedge + \frac{|x|-\sin|x|}{|x|^3} (x^\wedge)^2, \\ J_r^{-1}(x^\wedge) &= I + \frac{1}{2} x^\wedge + \left(\frac{1}{|x|^2} - \frac{1+\cos|x|}{2|x|\sin|x|} \right) (x^\wedge)^2. \end{aligned} \quad (37)$$

Hence, J_r^{-1} bounded for $|\theta_e| \in [0, \pi]$.

Definition 3 *Efimov and Polyakov (2021)* The origin of the system $\dot{x}(t) = f(x(t)), t \geq 0, x(0) = x_0$, is said to be

- globally uniformly **Lyapunov stable** if $\exists \alpha \in \mathcal{K}$ such that $\|x(t)\| \leq \alpha(\|x_0\|), \forall t > 0$;
- globally uniformly **asymptotically stable** if it is globally Lyapunov stable and $\forall r > 0, \forall R > r, \exists T_{r,R} > 0 : \|x_0\| \leq R \Rightarrow \|x(t)\| \leq r, \forall t \geq T_{r,R}$;
- globally uniformly **nearly fixed-time stable** if it is globally Lyapunov stable and $\forall r > 0, \exists T_r > 0$, such that $\|x(t)\| \leq r \forall t \geq T_r, \forall x_0$;
- globally uniformly **finite-time stable** if it is globally Lyapunov stable and $\forall x_0, \exists T_{x_0} \in (0, +\infty)$, such that $x(t) = 0, \forall t \geq T_{x_0}$;
- globally uniformly **fixed-time stable** if it is globally Lyapunov stable and $\exists T_{\max} > 0$, such that $x(t) = 0, \forall t \geq T_{\max}, \forall x_0$.

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