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Abstract

Let k and N be positive integers with $k \geq 2$ even. In this paper we give general explicit upper-bounds in terms of k and N from which all the residual representations $\overline{\rho}_{f,\lambda}$ attached to non-CM newforms of weight k and level $\Gamma_0(N)$ with λ of residue characteristic greater than these bounds are "as large as possible". The results split into different cases according to the possible types for the residual images and each of them is illustrated on some numerical examples.

Introduction

Let f be a newform of weight $k \geq 2$, level $N \geq 1$ and trivial Nebentypus whose Fourier expansion at infinity is given by $f(\tau) = q + \sum_{n\geq 2} a_n q^n$, with $q = e^{2i\pi\tau}$ and τ in the complex upper half-plane. We denote by K the number field generated by the coefficients a_n and by $\mathcal{O}$ its ring of integers. Given a prime ℓ, we shall denote by $\rho_{f,\ell}$ the ℓ-adic representation attached to f by Deligne :

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}(2, \mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_\ell).$$

The decomposition $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_\ell = \prod_{\lambda|\ell} \mathcal{O}_\lambda$ where the product runs over prime ideals in $\mathcal{O}$ of residue characteristic ℓ, in turn produces for each such λ a representation $\rho_{f,\lambda}$ with values in $\mathrm{GL}(2, \mathcal{O}_\lambda)$ where $\mathcal{O}_\lambda$ is the completion of $\mathcal{O}$ at λ. Composing it with the reduction map $\mathrm{GL}(2, \mathcal{O}_\lambda) \to \mathrm{GL}(2, \mathbf{F}_\lambda)$, where $\mathbf{F}_\lambda$ is the residue field of λ, finally gives rise to a representation $\overline{\rho}_{f,\lambda}$ which is unique up to semi-simplification.

Let us denote by $\overline{G}_\lambda$ the image of $\overline{\rho}_{f,\lambda}$. Using results of Carayol ([Car86]), Ribet proved in [Rib85, th. 2.1] the following theorem (for a definition of forms with complex multiplication see [Rib77] or Def. 3.1).

Theorem (Ribet, 1985)**.** *Assume that f is not a form with complex multiplication. Then for almost all λ (i.e. all but a finite number) the following assertions hold :*

1. *the representation $\overline{\rho}_{f,\lambda}$ is irreducible;*
2. *the order of the group $\overline{G}_\lambda$ is divisible by the residue characteristic of λ.*

As explained in [Rib85, §3], this theorem implies that for almost all primes ℓ, the image G_ℓ of $\rho_{f,\ell}$ is as "large" as possible. Namely, if for simplicity f does not have any inner twist (see [Rib77] for a definition), the following equality holds for all but finitely many ℓ :

$$G_\ell = \left\{ x \in \mathrm{GL}(2, \mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_\ell) \mid \det(x) \in \mathbf{Z}_\ell^{*(k-1)} \right\},$$

where $\mathbf{Z}_\ell^{*(k-1)}$ denotes the group of $(k-1)$-th powers in $\mathbf{Z}_\ell^*$.

This theorem is a generalization of [Rib75] on the case $N = 1$, which itself extends pioneer results of Serre ([Ser73]) and Swinnerton-Dyer ([SD73]) on the the case $N = 1$ and $K = \mathbf{Q}$. Although these latter results provide a precise characterization of the prime ideals for which one of the assertions above fails, the general theorem of Ribet is however non-effective.

The main goal of this paper is to give an effective version of Ribet's theorem, that is a general explicit set of prime numbers depending the weight k and the level N such that each representation $\overline{\rho}_{f,\lambda}$ with f newform in $\mathcal{S}_k(\Gamma_0(N))$ and λ of residue characteristic away from this set satisfies the conclusion of Ribet's theorem.

Before describing our main results, we mention that among the special cases covered are a generalization to arbitrary square-free levels of a result of Mazur ([Maz77]) on the so-called Eisenstein primes for weight 2 and prime level modular forms and an explicit version of Serre's theorem on the surjectivity of Galois representations attached to elliptic curves over $\mathbf{Q}$ due to Kraus ([Kra95]) and Cojocaru ([Coj05]).

Let us denote by $\mathbf{P}(\overline{\rho}_{f,\lambda})$ the projectivization of $\overline{\rho}_{f,\lambda}$ and by $\mathbf{P}(\overline{G}_\lambda)$ its image in $\mathrm{PGL}(2,\mathbf{F}_\lambda)$. For simplicity, we shall say that λ is exceptional if it belongs to the finite set of prime ideals for which one of the assertions of Ribet's theorem does not hold. According to Dickson's classification of subgroups of $\mathrm{PGL}(2,\mathbf{F}_\lambda)$ ([Ser72, Prop. 16]), if λ is exceptional (we warn the reader that in the literature, the term "exceptional" sometimes refers to the last situation below only), then we have :

(i) either $\overline{\rho}_{f,\lambda}$ is reducible;

(ii) or the image $\mathbf{P}(\overline{G}_\lambda)$ in $\mathrm{PGL}(2,\mathbf{F}_\lambda)$ is dihedral;

(iii) or $\mathbf{P}(\overline{G}_\lambda)$ is isomorphic to A_4, S_4 or A_5.

In each case, we thus provide a divisibility relation or an upper-bound in terms of k and N satisfied by the residue characteristic ℓ of λ. A general bound can therefore be obtained by combining the results of the three situations. The last case is the simplest one. Namely we prove :

Theorem (Thm. 4.1)**.** *If $\mathbf{P}(\overline{G}_\lambda)$ is isomorphic to A_4, S_4 or A_5, then either $\ell \mid N$ or $\ell \leq 4k-3$.*

In the second case we give a general upper-bound together with a much finer result in the square-free level case that imply the following :

Theorem (Thm. 3.1)**.** *Assume $\mathbf{P}(\overline{G}_\lambda)$ to be dihedral. If f does not have complex multiplication, then we have*

$$\ell \leq \left(2\left(4.8kN^2(1+\log\log N)\right)^{\frac{k-1}{2}}\right)^{g_0^\sharp(k,N)},$$

where $g_0^\sharp(k,N)$ is the number of newforms of weight k and level $\Gamma_0(N)$. Besides, if N is square-free, then either $\ell \mid N$, or $\ell \leq k$, or $\ell = 2k-1$.

The first case is by far the most complicated one and we refer the reader to Theorems 2.1, 2.2, 2.3 and 2.4 for precise and complete statements. Nevertheless, these results combined with those mentioned in this introduction yield to (slightly stronger versions of) the following theorems in the particular but important cases where N is square-free and N is a square respectively.

Theorem (Square-free level case)**.** *Assume that $N = p_1 \cdots p_t$ where $p_1, \ldots, p_t$ are $t \geq 1$ distinct primes, is square-free, and λ is exceptional. Then, we have :*

1. *either* $\ell \in \{p_1, \ldots, p_t\}$*;cc*
2. *or* $\ell \leq 4k-3$*;*
3. *or* ℓ *divides* $\begin{cases} \gcd_{1\leq i\leq t}\left(\mathrm{lcm}\left(p_i^k-1, p_i^{k-2}-1\right)\right) & \text{if } k>2 \\ \mathrm{lcm}_{1\leq i\leq t}\left(p_i^2-1\right) & \text{if } k=2 \end{cases}$.

Theorem (Square level case)**.** *Assume that* $N=c^2$ *is a square,* f *does not have complex multiplication and* λ *is exceptional. Then, we have :*

1. *either* $\ell \mid N$
2. *or* $\ell \leq \left(2\left(4.8kN^2(1+\log\log N)\right)^{\frac{k-1}{2}}\right)^{g_0^\sharp(k,N)}$*, where* $g_0^\sharp(k,N)$ *is the number of newforms of weight* k *and level* $\Gamma_0(N)$*;*
3. *or there exists a primitive Dirichlet character* $\nu : (\mathbf{Z}/c\mathbf{Z})^\times \to \mathbf{C}^\times$ *such that :*
   (a) *either* ℓ *divides the norm of* $p^k - \epsilon^{-1}(p)$ *for some prime* $p \mid c$*;*
   (b) *or* ℓ *divides the numerator of the norm of* $B_{k,\epsilon}/2k$

   where c_0 *divides* c*,* $\epsilon : (\mathbf{Z}/c_0\mathbf{Z})^\times \to \mathbf{C}^\times$ *is the inverse of the primitive Dirichlet character attached to* ν^2 *and* $B_{k,\epsilon}$ *is the* k*-th Bernoulli number attached to* ϵ*.*

Apart from (iii) which is slightly different, the main idea in proving the results of the paper is to interpret situations (i) and (ii) above in terms of congruences between modular forms. In the case of reducible representations $\overline{\rho}_{f,\lambda}$, the original form f is then shown to be congruent modulo ℓ to a suitable Eisenstein series whose construction depends on the weight and level. The theory of modular forms modulo ℓ of Serre and Katz enables us to interpret this congruence as an equality. The desired bound then follows from a careful study of the constant term of these Eisenstein series at various cusps. Besides, in the case of dihedral projective image, the congruent modular form is a specific twist of the original form f. In that case, the upper-bound follows from those of Sturm and Deligne.

Ghate and Parent recently addressed the question of whether the residual Galois representations attached to rational simple non-CM modular abelian varieties have "uniform" large images (see [GP, Question 1.2] for a precise statement). A positive answer to their question would follow from the existence of an upper-bound for exceptional primes in the weight 2 case of Ribet's theorem depending only on the degree $[K : \mathbf{Q}]$ (and not on the level N). While we are in contrary working with a fixed level, their work is still quite relevant for us.

The first section of the paper is devoted to classical facts about modular Galois representations and their local behaviors. The next three sections deal with cases (i), (ii) and (iii) above respectively. Finally some numerical examples illustrating our results are presented in the last section.

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1 Preliminaries

For simplicity, we shall write ρ and $\overline{\rho}$ for $\rho_{f,\lambda}$ and $\overline{\rho}_{f,\lambda}$ respectively. We further denote by $\overline{\rho}^{ss}$ the semi-simplification of $\overline{\rho}$. In this section, we also assume $\ell \nmid N$.

1.1 Local decomposition at Steinberg primes

Let p be a prime dividing N exactly once. We shall write $p\|N$. Under this assumption, the ℓ-adic representation ρ has a unique one-dimensional subspace unramified at p and the action of a Frobenius at p on it is given by multiplication by the Fourier coefficient a_p.

We now give a description of $\overline{\rho}_p$ which is defined to be the restriction of $\overline{\rho}$ to a decomposition group G_p at p. For any $x \in \mathcal{O}$, let us denote by $\lambda(x)$ the unramified character of G_p that maps a Frobenius element to $x \pmod{\lambda}$. Langlands has proved that ([LW12, Prop. 2.8])

$$\overline{\rho}_p \simeq \begin{pmatrix} \mu\overline{\chi}_\ell^{k/2-1} & \star \\ 0 & \mu\overline{\chi}_\ell^{k/2} \end{pmatrix} \tag{1}$$

where $\mu = \lambda(a_p/p^{k/2-1})$ is quadratic since $a_p = \pm p^{k/2-1}$ ([Miy06, Th. 4.6.17]). In particular, if $\mathrm{Frob}_p \in G_p$ is a Frobenius element at p, then the roots of the characteristic polynomial of $\overline{\rho}(\mathrm{Frob}_p)$ are $a_p \pmod{\lambda}$ and $pa_p \pmod{\lambda}$.

1.2 Classification of degeneration cases

Let $N(\overline{\rho}^{ss})$ be the Artin conductor of $\overline{\rho}^{ss}$. It was proved by Carayol that $N(\overline{\rho}^{ss})$ is a divisor of N ([Car86]). Moreover Carayol ([Car89]) and Livné ([Liv89]) have (independently) classified the so-called degeneration cases, that is when $e_p \overset{\text{def}}{=} v_p(N) - v_p(N(\overline{\rho}^{ss})) > 0$ for some prime p. They proved that when $e_p > 0$, we are in one of the situations described in the table below.

$v_p(N)$	$b+1 \geq 2$	1	2
$v_p(N(\overline{\rho}^{ss}))$	$b \geq 1$	0	0
e_p	1	1	2

Table 1: Classification of the degeneration cases

It moreover follows from their classification that in the first and third cases, p satisfies certain congruences modulo ℓ. Namely we have the following proposition.

Proposition 1.1 (Carayol-Livné)**.** *Assume $e_p > 0$ and $v_p(N) \geq 2$. Then we have $p \equiv \pm 1 \pmod{\ell}$.*

1.3 Local description at ℓ

Assume $2 \leq k \leq \ell+1$. Let G_ℓ be a decomposition group at ℓ and I_ℓ its inertia subgroup. Then Deligne and Fontaine ([Edi92]) have respectively proved that

- if f is ordinary at λ (that is if $a_\ell \not\equiv 0 \pmod{\lambda}$), then $\overline{\rho}_{|G_\ell}$ is reducible and

$$\overline{\rho}_{|I_\ell} \simeq \begin{pmatrix} \overline{\chi}_\ell^{k-1} & \star \\ 0 & 1 \end{pmatrix};$$

- if f is not ordinary at λ, then $\overline{\rho}_{|G_\ell}$ is irreducible and

$$\overline{\rho}_{|I_\ell} \simeq \begin{pmatrix} \psi^{k-1} & 0 \\ 0 & \psi'^{k-1} \end{pmatrix}$$

  where $\{\psi, \psi'\} = \{\psi, \psi^\ell\}$ is the set of fundamental characters of level 2 (*loc. cit.*, §2.4).

The following lemma is immediate.

Lemma 1.1. *Assume $\ell > k$.*

1. *The image of $\overline{\chi}_\ell^{k-1}$ is cyclic of order $n = (\ell-1)/\gcd(\ell-1, k-1) \geq 2$. In particular, we have $n = 2$ if and only if $\ell = 2k-1$. Moreover, if $\ell > 4k-3$, then $n > 5$.*
2. *The image of $\psi^{(\ell-1)(k-1)}$ is cyclic of order $m = (\ell+1)/\gcd(\ell+1, k-1) \geq 2$. In particular, we have $m = 2$ if and only if $\ell = 2k-3$. Moreover, if $\ell > 4k-5$, then $m > 5$.*

2 Reducible representations

2.1 Preliminaries: Gauss sums and Bernoulli numbers

Let $\psi : (\mathbf{Z}/f\mathbf{Z})^\times \to \mathbf{C}^\times$ be a primitive Dirichlet character of modulus $f \geq 1$. The Gauss sum attached to ψ is defined by

$$W(\psi) = \sum_{n=1}^{f} \psi(n) e^{2i\pi n/f}.$$

Lemma 2.1. *We have $|W(\psi)| = \sqrt{f}$. Moreover, as an algebraic integer, the norm of $W(\psi)$ is a power of f.*

Proof. The first part of the lemma is [Miy06, Lem. 3.1.1]. Let σ be a $\overline{\mathbf{Q}}$-automorphism and $m \in \mathbf{Z}$ such that $\sigma(e^{2i\pi/f}) = e^{2i\pi m/f}$. Then, by *loc. cit.*, we have :

$$\sigma(W(\psi)) = \sum_{n=1}^{f} \psi^\sigma(n) e^{2i\pi nm/f} = \overline{\psi^\sigma}(m) W(\psi^\sigma)$$

and thus $|\sigma(W(\psi))| = |W(\psi^\sigma)| = \sqrt{f}$. This completes the proof of the lemma. □

The Bernoulli numbers attached to ψ are defined by :

$$\sum_{n=1}^{f} \psi(n) \frac{te^{nt}}{e^{ft}-1} = \sum_{m \geq 0} B_{m,\psi} \frac{t^m}{m!}.$$

In particular, if ψ is the trivial character, $B_{m,\psi}$ is the classical Bernoulli number B_m, except when $m = 1$ in which case $B_{1,\psi} = -B_1 = 1/2$. The following proposition is a well-known result of van Staudt-Clausen.

Proposition 2.1 (van Staudt-Clausen)**.** *Let $m \geq 2$ be an even integer. The denominator of B_m is $\prod_{p-1|m} p$ where the product runs over the primes p such that $p-1$ divides m.*

The Bernoulli numbers are also related to certain special values of the L-function $L(s, \psi)$ attached to ψ. More precisely, we have the following proposition ([Was97, Ch. 4]).

Proposition 2.2. *Assume ψ to be even. Let $m \geq 2$ be an even integer. Then, we have*

$$L(m, \psi) = -W(\psi) \frac{C_m}{f^m} \cdot \frac{B_{m,\psi^{-1}}}{2m} \neq 0, \quad \text{where } C_m = \frac{(2i\pi)^m}{(m-1)!}.$$

2.2 Statement of the results

Theorem 2.1. *Assume $\overline{\rho}_{f,\lambda}$ to be reducible. If $v_2(N) = 2$ or $v_2(N) \geq 3$ is odd, then either ℓ divides N, or $\ell < k-1$, or $\ell = 3$.*

Put $c = \max\{d \geq 1;\ d^2 \mid N\}$. The following result is a generalization of Ribet's [Rib75, Lem. 5.2] on the level 1 case to higher levels.

Theorem 2.2 (main result)**.** *Assume $\overline{\rho}_{f,\lambda}$ to be reducible. Then one of the following assertions holds :*

1. *the prime ℓ divides N or $\ell < k-1$;*
2. *the level N is a not square and there exists an even Dirichlet character $\eta : (\mathbf{Z}/c\mathbf{Z})^\times \to \mathbf{C}^\times$ such that for every prime p dividing N with odd valuation $v_p(N)$, we have*
   (a) *either $v_p(N) \geq 3$ and $p \equiv \pm 1 \pmod{\ell}$;*
   (b) *or $v_p(N) = 1$ and ℓ divides the norm of either $p^k - \eta(p)$, or $p^{k-2} - \eta(p)$.*
3. *the level N is a square (i.e. $N = c^2$) and one of the following holds :*
   (a) *either there exists a prime p such that $v_p(N) = 2$ and $p \equiv \pm 1 \pmod{\ell}$;*
   (b) *or there exists a primitive Dirichlet character $\nu : (\mathbf{Z}/c\mathbf{Z})^\times \to \mathbf{C}^\times$ such that for $\ell > k+1$ we have :*
      i. *either ℓ divides the norm of $p^k - \epsilon^{-1}(p)$ for some prime $p \mid c$;*
      ii. *or ℓ divides the numerator of the norm of $B_{k,\epsilon}/2k$*

   where c_0 divides c and $\epsilon : (\mathbf{Z}/c_0\mathbf{Z})^\times \to \mathbf{C}^\times$ is the inverse of the primitive Dirichlet character attached to ν^2.

Note that these two results give an effective bound for ℓ in terms of N and k unless $k = 2$ and $N = p_1 \cdots p_t c^2$ where $p_1, \ldots, p_t$ are $t \geq 1$ distinct primes not dividing c, and c is odd or divisible by 4. In the square-free level case (namely when $c = 1$), we however have the following theorem whose first part is an immediate corollary of Thm. 2.2 and whose second part follows from a generalization of a result of Mazur on the weight 2 and prime level case (cf. [Maz77] and [MS76, Prop. 1]).

Theorem 2.3 (square-free level case)**.** *Assume $\overline{\rho}_{f,\lambda}$ reducible and $N = p_1 \cdots p_t$ where $p_1, \ldots, p_t$ are $t \geq 1$ distinct primes.*

1. *If $k > 2$, then one of the following assertions holds :*
   (a) *either ℓ divides N or $\ell < k-1$;*
   (b) *ℓ divides the following non-zero integer*

$$\gcd\left(\mathrm{lcm}\left(p_i^k - 1, p_i^{k-2} - 1\right), 1 \leq i \leq t\right).$$

2. *If $k = 2$ and $\ell \nmid 6N$, then the following assertions hold :*
   (a) *for any $1 \leq i \leq t$ with $a_{p_i} = -1$, we have $p_i \equiv -1 \pmod{\ell}$;*
   (b) *we have $(a_{p_1}, \ldots, a_{p_t}) \neq (-1, \ldots, -1)$;*
   (c) *if $(a_{p_1}, \ldots, a_{p_t}) = (+1, \ldots, +1)$, then ℓ divides the non-zero integer $\prod_{i=1}^t (p_i - 1)$.*

We point out that Ribet already proved (but did not publish) the second part of this theorem as well as "converse results" (see the notes [Rib10] on his homepage).

The last theorem of this section deals with the cases not covered by the previous results.

Theorem 2.4. *Assume $\overline{\rho}_{f,\lambda}$ reducible. If $k = 2$ and N is of the form $N = p_1 \cdots p_t c^2$, where $c \neq 1$, $p_1, \ldots, p_t$ are $t \geq 1$ distinct primes not dividing c, and c is odd or divisible by 4, then :*

1. *either $\ell \mid N$;*
2. *or $\ell < k - 1$;*
3. *or there exists a prime p such that $v_p(N) = 2$ and $p \equiv \pm 1 \pmod{\ell}$;*
4. *or there exists a primitive Dirichlet character $\nu : (\mathbf{Z}/c\mathbf{Z})^\times \to \mathbf{C}^\times$ such that for $\ell > 3$ we have :*
   (a) *either ℓ divides the norm of $p_i^2 - \nu^2(p_i)$ for some $1 \leq i \leq t$;*
   (b) *or ℓ divides the norm of $p^2 - \epsilon^{-1}(p)$ for some prime $p \mid c$;*
   (c) *or ℓ divides $p_i - 1$ for some $1 \leq i \leq t$;*
   (d) *or ℓ divides the numerator of the norm of $B_{2,\epsilon}/4$*

   where $c_0 \mid c$ and $\epsilon : (\mathbf{Z}/c_0\mathbf{Z})^\times \to \mathbf{C}^\times$ is the inverse of the primitive Dirichlet character attached to ν^2.

2.3 The Eisenstein series E

Assume $\ell \nmid N$. For simplicity, let us denote $\overline{\rho}$ for $\overline{\rho}_{f,\lambda}$ and assume $\overline{\rho}$ to be reducible. The semi-simplification $\overline{\rho}^{ss}$ of $\overline{\rho}$ is the direct sum of two characters ϵ_1 and ϵ_2. Each of them may be decomposed as a product $\overline{\nu}_i \overline{\chi}_\ell^{\alpha_i}$ with $\overline{\nu}_i$ is unramified at ℓ and $0 \leq \alpha_i < \ell - 1$ $(i = 1, 2)$. Using that $\overline{\rho}^{ss}$ has determinant $\overline{\chi}_\ell^{k-1}$, we get $\alpha_1 + \alpha_2 \equiv k - 1 \pmod{\ell - 1}$ and $\overline{\nu}_2 = \overline{\nu}_1^{-1}$.

Let us further assume that $\ell + 1 \geq k$. Using the results of §1.3, one sees that $\{\alpha_1, \alpha_2\} = \{0, k - 1\}$ and thus

$$\overline{\rho}^{ss} \simeq \overline{\nu} \oplus \overline{\nu}^{-1} \overline{\chi}_\ell^{k-1}, \tag{2}$$

with $\overline{\nu} \in \{\overline{\nu}_1, \overline{\nu}_2\}$. Moreover, according to Carayol's theorem of §1.2, the conductor $\mathfrak{c}$ of $\overline{\nu}$ satisfies :

$$N(\overline{\rho}^{ss}) = \mathfrak{c}^2 \mid N. \tag{3}$$

In particular, $N(\overline{\rho}^{ss})$ is a square dividing N.

Let ν be the Teichmüller lift of $\overline{\nu}$. We may identify it with a primitive Dirichlet character modulo $\mathfrak{c}$. From now on, assume that :

1. either $k > 2$;
2. or, $k = 2$ and $\mathfrak{c} \neq 1$.

Under this assumption, we may consider the Eisenstein series in $\mathcal{M}_k(\Gamma_0(\mathfrak{c}^2))$ whose Fourier expansion is given by :

$$E(\tau) = -\vartheta(\mathfrak{c}) \frac{B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}^{\nu}(n) q^n,$$

where

$$\vartheta(\mathfrak{c}) = \begin{cases} 1 & \text{if } \mathfrak{c} = 1 \\ 0 & \text{otherwise} \end{cases}, \quad \sigma_{k-1}^{\nu}(n) = \sum_{0<m|n} \nu(n/m)\nu^{-1}(m)m^{k-1}$$

and B_k is the k-th Bernoulli number. Note also that our notation E differs from the notation $E_k^{\nu,\nu^{-1}}$ of [DS05, Ch. 4] by a factor 2 : $E_k^{\nu,\nu^{-1}} = 2E$.

The following proposition gives the constant term of the Fourier expansion of E at the various cusps of $\Gamma_0(\mathfrak{c}^2)$.

Proposition 2.3. *The Eisenstein series E is defined over $\mathcal{O}_L$ where L is the field generated by the values of ν, unless $\mathfrak{c} = 1$ (and $k > 2$) in which case E is the classical Eisenstein series $E_k(\tau) = -B_k/2k + \sum_{n\geq 1}\sigma_{k-1}(n)q^n$ of weight k and level 1. Let $s = u/v$ with $\gcd(u, v) = 1$, $v \mid \mathfrak{c}^2$ and $u \pmod{\gcd(v, \mathfrak{c}^2/v)}$ be a cusp of $\Gamma_0(\mathfrak{c}^2)$ and let $\gamma \in \mathrm{SL}(2, \mathbf{Z})$ such that $\gamma\infty = s$. Then the constant term Υ of $E|_k\gamma$ is independent of the choice of such a γ and satisfies :*

$$\Upsilon \neq 0 \Leftrightarrow v = \mathfrak{c}.$$

In that case, we have :

$$\Upsilon = -\nu(-u)\left(\frac{\mathfrak{c}}{c_0}\right)^k \frac{W((\nu^2)_0)}{W(\nu)} \frac{B_{k,(\nu^2)_0^{-1}}}{2k} \prod_{p|\mathfrak{c}} \left(1 - (\nu^2)_0(p)p^{-k}\right),$$

where $(\nu^2)_0$ is the primitive character associated to ν^2 of modulus $c_0 \mid \mathfrak{c}$. Moreover, if $\mathfrak{c} > 1$, then $E|_k\gamma \in \mathcal{O}_L\left[\frac{1}{\mathfrak{c}^2}\right](\mu_{\mathfrak{c}^2})[[q^{1/\mathfrak{c}^2}]]$ where $\mu_{\mathfrak{c}^2}$ is the group of $\mathfrak{c}^2$-th roots of unity.

Proof. The proposition is immediate when $\mathfrak{c} = 1$. Assume therefore $\mathfrak{c} > 1$. Then by construction the Fourier expansion of E has coefficients in $\mathcal{O}_L$ and therefore E is defined over $\mathcal{O}_L[1/\mathfrak{c}^2](\mu_{\mathfrak{c}^2})$ ([Kat73, §1.6]).

Let $s = u/v$ as in the proposition be a cusp of $\Gamma_0(\mathfrak{c}^2)$ (for the description of a set of representatives of the cusps of $\Gamma_0(\mathfrak{c}^2)$, see [Iwa97, Prop. 2.6]) and $\gamma \in \mathrm{SL}(2, \mathbf{Z})$ such that $\gamma\infty = s$. The last assertion follows from the q-expansion principle and the fact that the Fourier of E at ∞ has coefficients in $\mathcal{O}_L$ ([Kat73, Cor. 1.6.2.]).

Since k is even, the constant term of E at s is well-defined (i.e. does not depend of the choice of such a γ). Put

$$\gamma = \begin{pmatrix} u & \beta \\ v & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}) \quad \text{and} \quad G = \frac{C_k W(\nu)}{\mathfrak{c}^k} E, \quad \text{where } C_k = \frac{(2i\pi)^k}{(k-1)!}. \tag{4}$$

The constant part of $G|_k\gamma$ is then given by the following sum (see [DS05, Ch. 4] and [Sch74, § VII.3] for a justification in the weight 2 case; the factor 1/2 comes from our normalization for E) :

$$\Upsilon_0 = \frac{1}{2}\sum_{i,j,l=0}^{\mathfrak{c}-1} \nu(ij)\vartheta\left(\overline{i\mathfrak{c}u + v(j + l\mathfrak{c})}\right)\zeta^{\overline{\mathfrak{c}i\beta+(j+l\mathfrak{c})\delta}}(k),$$

where the bar means reduction modulo $\mathfrak{c}^2$,

$$\vartheta(\overline{n}) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{\mathfrak{c}^2} \\ 0 & \text{otherwise} \end{cases}, \quad \zeta^{\overline{n}}(k) = \sum_{m\equiv n \pmod{\mathfrak{c}^2}}{}' \frac{1}{m^k},$$

and the primed summation notation means to sum over non-zero integers.

Assume Υ_0 to be non-zero. Then, there exist $i, j, l \in \{0, \ldots, \mathfrak{c}-1\}$ such that

$$\nu(ij)\vartheta\left(\overline{i\mathfrak{c}u + v(j+l\mathfrak{c})}\right) \neq 0.$$

In other words, $\gcd(ij, \mathfrak{c}) = 1$ and $i\mathfrak{c}u + v(j+l\mathfrak{c}) \equiv 0 \pmod{\mathfrak{c}^2}$. It follows that $vj \equiv 0 \pmod{\mathfrak{c}}$. But j is co-prime to $\mathfrak{c}$ by assumption. So, $v \equiv 0 \pmod{\mathfrak{c}}$ and u is invertible modulo $\mathfrak{c}$. The congruence $i \equiv -(j/u)(v/\mathfrak{c}) \pmod{\mathfrak{c}}$ follows easily and therefore, we have :

$$\nu(ij) = \nu\left(-\frac{vj^2}{u\mathfrak{c}}\right) = \nu\left(-\frac{j^2}{u}\right)\nu\left(\frac{v}{\mathfrak{c}}\right) \neq 0.$$

So, $\gcd(v/\mathfrak{c}, \mathfrak{c}) = 1$ and since $\mathfrak{c} \mid v$ and $v \mid \mathfrak{c}^2$, we get $v = \mathfrak{c}$.

Conversely, assume $v = \mathfrak{c} > 1$. Then, $\gcd(u, \mathfrak{c}) = 1$ and on one hand, we have :

$$i\mathfrak{c}u + v(j+l\mathfrak{c}) \equiv 0 \pmod{\mathfrak{c}^2} \Longleftrightarrow i \equiv -j/u \pmod{\mathfrak{c}}$$

and on the other hand :

$$\begin{aligned}
\mathfrak{c}i\beta + (j+l\mathfrak{c})\delta &= \frac{1}{u}\left(u\mathfrak{c}i\beta + (j+l\mathfrak{c})u\delta\right) \\
&\equiv \frac{1}{u}\left(-vj\beta + (j+l\mathfrak{c})(1+\beta v) \pmod{\mathfrak{c}^2}\right. \\
&\equiv \frac{1}{u}\left(j+l\mathfrak{c}\right) \pmod{\mathfrak{c}^2}.
\end{aligned}$$

Combining these two facts, we find that :

$$\begin{aligned}
2\Upsilon_0 &= \sum_{l=0}^{\mathfrak{c}-1} \sum_{\substack{j=0 \\ \gcd(j,\mathfrak{c})=1}}^{\mathfrak{c}-1} \nu(-j^2/u)\zeta^{\overline{\frac{j+l\mathfrak{c}}{u}}}(k) \\
&= \nu(-u)\sum_{l=0}^{c-1} \sum_{\substack{j=0 \\ \gcd(j,\mathfrak{c})=1}}^{\mathfrak{c}-1} \nu\left(j^2/u^2\right) \sum_{m \equiv (j+l\mathfrak{c})/u \pmod{\mathfrak{c}^2}}{}' \frac{1}{m^k} \qquad (5) \\
&= \nu(-u) \sum_{\substack{j=0 \\ \gcd(j,\mathfrak{c})=1}}^{\mathfrak{c}-1} \sum_{m \equiv j/u \pmod{\mathfrak{c}}}{}' \frac{\nu^2(m)}{m^k} \\
&= 2\nu(-u)\sum_{m \geq 1} \frac{\nu^2(m)}{m^k} = 2\nu(-u)L(k, \nu^2),
\end{aligned}$$

where ν^2 is viewed as a character modulo $\mathfrak{c}$. Let $(\nu^2)_0$ be the primitive Dirichlet character attached to ν^2. It is an even character modulo $c_0 \mid \mathfrak{c}$ and we have :

$$L(k, \nu^2) = L\left(k, (\nu^2)_0\right) \prod_{p \mid \mathfrak{c}} \left(1 - (\nu^2)_0(p)p^{-k}\right). \qquad (6)$$

Applying Prop. 2.2 to $\psi = (\nu^2)_0$ and $m = k$, we get :

$$L(k, (\nu^2)_0) = -W\left((\nu^2)_0\right)\frac{C_k}{c_0^k}\frac{B_{k,(\nu^2)_0^{-1}}}{2k} \neq 0. \qquad (7)$$

According to Eq. (5)-(7) together with (4), when $v = \mathfrak{c}$, the constant term of the Fourier expansion of E at s is thus the non-zero algebraic number :

$$\Upsilon = \frac{\mathfrak{c}^k}{C_k W(\nu)} \Upsilon_0 = -\nu(-u) \left(\frac{\mathfrak{c}}{c_0}\right)^k \frac{W\left((\nu^2)_0\right)}{W(\nu)} \frac{B_{k,(\nu^2)_0^{-1}}}{2k} \prod_{p|\mathfrak{c}} \left(1 - (\nu^2)_0(p) p^{-k}\right),$$

as claimed. □

2.4 Proof of Theorems 2.1 and 2.2

Assume $\overline{\rho}$ reducible with $\ell \nmid N$ and $\ell + 1 \geq k$. We keep the notation of §2.3. In particular, we have (cf. (2) and (3))

$$\overline{\rho}^{ss} \simeq \overline{\nu} \oplus \overline{\nu}^{-1} \overline{\chi}_\ell^{k-1}, \tag{8}$$

where $\overline{\nu}$ is a character of conductor $\mathfrak{c}$ such that $\mathfrak{c}^2 \mid N$. So, in particular, we have $\mathfrak{c} \mid c$.

Assume that $v_2(N) = 2$. Then, $v_2(c) = 1$ and $\mathfrak{c}$ is odd since there is no primitive Dirichlet character modulo twice an odd integer. Therefore, we are in a degeneracy case at $p = 2$ as described in §1.2. By Prop. 1.1, we have $2 \equiv \pm 1 \pmod{\ell}$, namely $\ell = 3$.

If N is not a square, let us consider a prime p dividing N with odd valuation $v_p(N)$. Once again, we necessarily are in one of the degeneration cases. If $v_p(N) \geq 3$, then by Prop. 1.1, we get $p \equiv \pm 1 \pmod{\ell}$. This completes the proof of Thm. 2.1.

Assume now that for some prime p, we have $v_p(N) = 1$ and let us denote by η the Teichmüller lift of $\overline{\nu}^2$. Since $\mathfrak{c}$ is a divisor of c, we may identify η with an even Dirichlet character modulo c. Comparing the restriction to a decomposition group at p of $\overline{\rho}^{ss}$ given by (2) with the local representation given by (1) we get the following equality between sets of characters of G_p :

$$\left\{\overline{\nu}, \overline{\nu}^{-1} \overline{\chi}_\ell^{k-1}\right\} = \left\{\mu \overline{\chi}_\ell^{k/2}, \mu \overline{\chi}_\ell^{k/2-1}\right\},$$

where $\mu = \lambda(a_p / p^{k/2-1})$ is the quadratic character defined in §1.1. We thus are in one of the following situations :

1. Either $\overline{\nu} = \mu \overline{\chi}_\ell^{k/2}$ and then $\overline{\nu}^2 = \overline{\chi}_\ell^k$. Applying this equality to a Frobenius element at p, we get that $\overline{\nu}^2(\mathrm{Frob}_p) = p^k \pmod{\ell}$ and therefore ℓ divides the norm of $p^k - \eta(p)$.

2. Or $\overline{\nu} = \mu \overline{\chi}_\ell^{k/2-1}$ and then $\overline{\nu}^2 = \overline{\chi}_\ell^{k-2}$. Again we have $\overline{\nu}^2(\mathrm{Frob}_p) = p^{k-2} \pmod{\ell}$ and we conclude as before that ℓ divides the norm of $p^{k-2} - \eta(p)$.

It remains to prove Thm. 2.2 when N is a square, namely when $N = c^2$. Assume first that $\mathfrak{c} \neq c$. Then we are in a degeneracy case as described in §1.2 for some prime number p. Moreover, $N(\overline{\rho}^{ss}) = \mathfrak{c}^2$ is a square and therefore we have $v_p(N) = 2$ and $v_p(N(\overline{\rho}^{ss})) = 0$. By Prop. 1.1, it follows that $p \equiv \pm 1 \pmod{\ell}$.

In other words, if for every prime p dividing N with valuation 2, we have $p \not\equiv \pm 1 \pmod{\ell}$, then $\mathfrak{c} = c$, $N = \mathfrak{c}^2$ and there is no degeneration at all. Assume now that we are in this situation. Since the space of weight 2 and level 1 modular forms is trivial, it follows that either $k > 2$, or $k = 2$ and $\mathfrak{c} \neq 1$. Therefore we may consider the Eisenstein series E of §2.3. Let M denote the compositum of K and L (the field generated by the values of ν).

Lemma 2.2. *The Eisenstein series E is a normalized eigenform for all the Hecke operators at level $\Gamma_0(N)$. Moreover, there exists a prime ideal $\mathcal{L}$ above ℓ in the integer ring of M such that :*

$$a_r \equiv a_r(E) \pmod{\mathcal{L}}, \quad \textit{for all primes } r \neq \ell.$$

Proof. The fact that E is a normalized eigenform for all the Hecke operators at level $\Gamma_0(N)$ follows for instance from [DS05, Prop. 5.2.3]. Moreover by isomorphism (8) there exists a prime ideal $\mathcal{L}$ above ℓ in the integer ring of M such that :

$$a_r \equiv a_r(E) \pmod{\mathcal{L}}, \quad \text{for all primes } r \nmid \ell N.$$

If now r is a prime dividing N, then $r^2 \mid N$ and $a_r = 0$ ([Miy06, Th. 4.6.17]). Besides, $\nu(r) + \nu^{-1}(r)r^{k-1} = 0$. Hence $a_r = 0 = a_r(E)$. This proves the lemma. □

Let now Θ be the Katz' operator on modular forms over $\overline{\mathbf{F}}_\ell$ whose action on q-expansions is given by $q\frac{d}{dq}$ (denoted $A\theta$ in [Kat77]). Assume $\ell > k+1$. Then the constant term of E at ∞ is non-zero only if $\mathfrak{c} = 1$ and $k > 2$. In that case it is $-B_k/2k$ which is ℓ-integral by Prop. 2.1. We denote by $\overline{f}$ and $\overline{E}$ the modular forms over $\overline{\mathbf{F}}_\ell$ obtained by reduction modulo $\mathcal{L}$ of f and E respectively. Lemma 2.2 implies that $\Theta(\overline{f}) = \Theta(\overline{E})$. Moreover Katz has proved that if $\ell > k+1$, then Θ is injective ([Kat77, Cor. (3)]). Under this assumption, it thus follows that the Eisenstein series E becomes cuspidal after reduction.

If $\mathfrak{c} = 1$ we immediately get that ℓ divides the numerator of $B_k/2k$ as stated in the theorem. Assume therefore that $\mathfrak{c} > 1$. Then ℓ divides the numerator of the norm of the constant term of E at each cusp of $\Gamma_0(\mathfrak{c}^2)$, namely by Prop. 2.3 :

$$\Upsilon = \pm \left(\frac{\mathfrak{c}}{c_0}\right)^k \frac{W\left(\epsilon^{-1}\right)}{W(\nu)} \frac{B_{k,\epsilon}}{2k} \prod_{p \mid \mathfrak{c}} \left(1 - \epsilon^{-1}(p)p^{-k}\right),$$

where $\epsilon : (\mathbf{Z}/c_0\mathbf{Z})^\times \to \mathbf{C}^\times$ is the inverse of the primitive Dirichlet character attached to ν^2. By Lemma 2.1, the prime divisors of the norm of $W\left(\epsilon^{-1}\right)/W(\nu)$ divide N and therefore are co-prime to ℓ. The same obviously holds for $\mathfrak{c}/c_0$. Therefore we eventually get that ℓ divides the norm of either $p^k - \epsilon^{-1}(p)$ for some p dividing $\mathfrak{c}$ (and thus c) or the norm of the numerator of $B_{k,\epsilon}/2k$. This completes the proof of Thm. 2.2.

2.5 Proof of Theorem 2.3

As already mentioned, the first part of Thm. 2.3 is a direct corollary of Thm. 2.2. So, let us assume $k = 2$ and $\ell \nmid 6N$. By the reasoning at the beginning of §2.3, we may write :

$$\overline{\rho}^{ss} \simeq \mathbf{1} \oplus \overline{\chi}_\ell, \tag{9}$$

where $\mathbf{1}$ is the trivial character of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. In particular, we have $\mathfrak{c}^2 = N(\overline{\rho}^{ss}) = 1$, hence $\mathfrak{c} = 1$. Let now $p \in \{p_1, \ldots, p_t\}$ be a prime dividing N. By §1.1, the local representation $\overline{\rho}_p$ at p semi-simplifies to :

$$\lambda(a_p) \oplus \lambda(a_p)\overline{\chi}_\ell. \tag{10}$$

Comparing (9) and (10) we get the following equality between sets of characters of G_p :

$$\{\mathbf{1}, \overline{\chi}_\ell\} = \{\lambda(a_p)\overline{\chi}_\ell, \lambda(a_p)\}.$$

If moreover $a_p = -1$, then the character $\lambda(a_p)$ is non-trivial and therefore, we must have $\lambda(a_p) = \overline{\chi}_\ell$ as characters of G_p. In other words, $p \equiv -1 \pmod{\ell}$. This proves assertion (2a) of Thm. 2.3.

Before proving the next two assertions, note that we precisely are in the excluded situation of §2.3, namely $k = 2$ and $\mathfrak{c} = 1$. For that reason, we cannot use the Eisenstein series E as in the proof of Thm. 2.2 (cf. §2.4).

To circumvent the lack of weight 2 level 1 Eisenstein series, it will be more convenient to directly work with modular forms over $\overline{\mathbf{F}}_\ell$. Let $\overline{E}_2$ be the reduction modulo ℓ (recall that $\ell \geq 5$) of the classical series E_2 in characteristic 0 defined by :

$$E_2(\tau) = -\frac{1}{24} + \sum_{n\geq 1} \sigma_1(n) q^n.$$

Viewed as a modular form over $\overline{\mathbf{F}}_\ell$ of level N (which is co-prime to ℓ by assumption), $\overline{E}_2$ has filtration $\ell + 1$ ([Ser73]). Put :

$$E' = \left[\prod_{p|N} (a_p \mathcal{U}_p - p\mathrm{Id}) \right] \overline{E}_2.$$

The following proposition summarizes the main properties of E'.

Proposition 2.4. *As a modular form over $\overline{\mathbf{F}}_\ell$, E' is a well-defined normalized eigenform for all the Hecke operators at level $\Gamma_0(N)$ such that :*

$$\begin{cases} T_r E' &= (1+r)E' & \text{for all prime } r \nmid N \\ \mathcal{U}_p E' &= a_p E' & \text{for any prime } p \mid N. \end{cases}$$

Moreover E' has filtration 2 unless $(a_{p_1}(f), \ldots, a_{p_t}(f)) = (-1, \ldots, -1)$ when it has filtration $\ell + 1$. The constant term of its Fourier expansion at infinity is given by :

$$a_0(E') = \begin{cases} (-1)^{t+1} \frac{(p_1-1)\cdots(p_t-1)}{24} & \text{if } (a_{p_1}(f), \ldots, a_{p_t}(f)) = (+1, \ldots, +1) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the commutativity of the Hecke algebra, E' is a well-defined modular form over $\overline{\mathbf{F}}_\ell$ of level N. Let r be a prime not dividing N. Since $T_r \overline{E}_2 = (1+r)\overline{E}_2$, we get that $T_r E' = (1+r)E'$, as claimed.

Let $u \neq 1$ be an integer dividing N. We denote by $\overline{E}_{2,u}$ the reduction modulo ℓ of the classical characteristic-0 Eisenstein series $E_{2,u} \in \mathcal{M}_2(\Gamma_0(u))$ defined by :

$$E_{2,u}(\tau) = E_2(\tau) - uE_2(u\tau) = \frac{u-1}{24} + \sum_{n\geq 1} \Big(\sum_{\substack{0<m|n \\ u\nmid m}} m \Big) q^n. \tag{11}$$

If p is a prime divisor of N, recall that we have :

$$\begin{aligned} \mathcal{U}_p \overline{E}_2 &= \overline{E}_{2,p} + p\overline{E}_2; \\ \mathcal{U}_p \overline{E}_{2,u} &= \begin{cases} \overline{E}_{2,p} + (1+p)\overline{E}_{2,u} - \overline{E}_{2,pu} & \text{if } p \nmid u \\ \overline{E}_{2,p} + p\overline{E}_{2,u/p} & \text{if } p \mid u \text{ and } p \neq u \\ \overline{E}_{2,p} & \text{if } p = u. \end{cases} \end{aligned}$$

So, let p be a prime divisor of N. We have :

$$\begin{aligned} (a_p \mathcal{U}_p - p\mathrm{Id})\mathcal{U}_p \overline{E}_2 &= ((a_p \mathcal{U}_p - p\mathrm{Id}))(\overline{E}_{2,p} + p\overline{E}_2) \\ &= p^2(a_p - 1)\overline{E}_2 + (a_p - p + pa_p)\overline{E}_{2,p}. \end{aligned}$$

If $a_p = +1$, then we get $(a_p\mathcal{U}_p - p\mathrm{Id})\mathcal{U}_p\overline{E}_2 = \overline{E}_{2,p} = (a_p\mathcal{U}_p - p\mathrm{Id})\overline{E}_2$ which is the desired result. On the other hand, if $a_p = -1$, then, by the assertion (2a) proved above, we have $p \equiv -1 \pmod{\ell}$ and the previous equality between forms over $\overline{\mathbf{F}}_\ell$ thus gives :

$$(a_p\mathcal{U}_p - p\mathrm{Id})\mathcal{U}_p\overline{E}_2 = -2\overline{E}_2 + \overline{E}_{2,p} = -(a_p\mathcal{U}_p - p\mathrm{Id})\overline{E}_2.$$

To finish the proof, it now remains to compute the filtration of E' and the first two terms of its Fourier expansion at infinity. Let $s = \sharp\{1 \leq i \leq t \mid a_{p_i}(f) = +1\}$. If $0 < s < t$, we may assume without loss of generality that :

$$N = p_1 \cdots p_s \cdot p_{s+1} \cdots p_t \quad \text{with} \quad \left\{ \begin{array}{lcll} \mathcal{U}_{p_i} f & = & f & \text{for all } 1 \leq i \leq s \\ \mathcal{U}_{p_i} f & = & -f & \text{for all } s+1 \leq i \leq t. \end{array} \right.$$

By induction on t, we prove that :

$$E' = \delta_{(s=0)} 2^t \overline{E}_2 + \sum_{\substack{(k,l) \in \{0,\ldots,s\} \times \{0,\ldots,t-s\} \\ (k,l) \neq (0,0)}} (-1)^{k+1} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq s \\ s+1 \leq j_1 < \cdots < j_l \leq t}} \overline{E}_{2, p_{i_1} \cdots p_{i_k} \cdot p_{j_1} \cdots p_{j_l}}$$

where

$$\delta_{(s=0)} = \left\{ \begin{array}{ll} 1 & \text{if } s = 0 \\ 0 & \text{otherwise} \end{array} \right.$$

and the condition $1 \leq i_1 < \cdots < i_k \leq s$ or $s+1 \leq j_1 < \cdots < j_l \leq t$ is empty if $s = 0$ or $s = t$ respectively. From this equality it follows the assertion on the filtration. Moreover an easy computation using Newton's binomial theorem and (11) proves the assertions on the first two Fourier coefficients. □

Let us now finish the proof of Thm. 2.3. According to (9) and the previous proposition, we have :

$$a_n(\overline{f}) = a_n(E') \quad \text{for all prime-to-}\ell \text{ integers } n,$$

where $\overline{f}$ denotes the modular form over $\overline{\mathbf{F}}_\ell$ obtained by reduction of f modulo λ. Since $\ell \geq 5 > k+1 = 3$, Katz' theory ([Kat77, Cor. (3)]) actually shows that $\overline{f} = E'$. Thus E' has filtration 2 and we cannot have $(a_{p_1}(f), \ldots, a_{p_t}(f)) = (-1, \ldots, -1)$. Moreover, the constant term of E' at infinity must vanish and when $(a_{p_1}(f), \ldots, a_{p_t}(f)) = (+1, \ldots, +1)$, this gives the congruence stated in the theorem.

2.6 Proof of Theorem 2.4

Assume $\overline{\rho}$ reducible with $\ell \nmid N$ and $\ell + 1 \geq k$. As in § 2.4, we have

$$\overline{\rho}^{ss} \simeq \overline{\nu} \oplus \overline{\nu}^{-1}\overline{\chi}_\ell \tag{12}$$

where $\overline{\nu}$ is a character of conductor $\mathfrak{c}$ such that $N(\overline{\rho}^{ss}) = \mathfrak{c}^2 \mid N$. So, in particular, we have $\mathfrak{c} \mid c$.

If $\mathfrak{c} \neq c$, then we necessarily are in a degeneracy case as described in §1.2, with $e_p = 2$ at some prime divisor p of c. Therefore, $v_p(N) = 2$ and by Prop. 1.1, we have $p \equiv \pm 1 \pmod{\ell}$.

We can thus assume, from now on, that $\mathfrak{c} = c$. Let us denote by ν the Teichmüller lift of $\overline{\nu}$, viewed as a primitive Dirichlet character modulo c.

Let $1 \leq i \leq t$. Comparing the restriction to a decomposition group at p_i of $\overline{\rho}^{ss}$ with the local representation given by (1) we get the following equality between sets of characters of G_{p_i} :

$$\left\{\overline{\nu}, \overline{\nu}^{-1}\overline{\chi}_\ell\right\} = \left\{\lambda(a_{p_i})\overline{\chi}_\ell, \lambda(a_{p_i})\right\},$$

where $\lambda(a_{p_i})$ is the quadratic character defined in §1.1.

Assume that for some $1 \leq i \leq t$, we have $\overline{\nu} = \lambda(a_{p_i})\overline{\chi}_\ell$ (again, as characters of G_{p_i}). Since $a_{p_i} = \pm 1$, it then follows that ℓ divides the norm of $\nu(p_i)^2 - p_i^2$.

From now on, we will therefore assume that $\overline{\nu} = \lambda(a_{p_i})$ for every $1 \leq i \leq t$. It then follows that $\overline{\nu}(p_i) = a_{p_i} \pmod{\ell}$. Since $\mathfrak{c} > 1$, we may consider the Eisenstein series

$$E(\tau) = \sum_{n \geq 1} \sigma_1^\nu(n) q^n \in \mathcal{M}_2\left(\Gamma_0\left(\mathfrak{c}^2\right)\right)$$

introduced in §2.3. This is an eigenform for all the Hecke operators at level $\Gamma_0(\mathfrak{c}^2)$.

2.6.1 The Eisenstein series E'

Put

$$E'(\tau) = \left[\prod_{i=1}^{t} (\mathcal{U}_{p_i} - p_i \nu^{-1}(p_i)\mathrm{Id})\right] E(p_1 \cdots p_t \tau) \in \mathcal{M}_2\left(\Gamma_0\left(N\right)\right),$$

where $\mathcal{U}_{p_i}$ denotes the p_i-th Hecke operator acting on $\mathcal{M}_2\left(\Gamma_0\left(N\right)\right)$. In expanded form, we have :

$$E'(\tau) = E + \sum_{j=1}^{t} (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq t} p_{i_1} \cdots p_{i_j} \nu^{-1}(p_{i_1} \cdots p_{i_j}) E(p_{i_1} \cdots p_{i_j} \tau). \tag{13}$$

As before let us denote by L the field generated by the values of ν and by M the compositum of L and K. The following lemma is crucial.

Lemma 2.3. *The Eisenstein series E' is a normalized eigenform for all the Hecke operators at level $\Gamma_0(N)$. Moreover, there exists a prime ideal $\mathcal{L}$ above ℓ in the integer ring of M such that :*

$$a_r \equiv a_r(E') \pmod{\mathcal{L}}, \quad \textit{for all primes } r \neq \ell.$$

Proof. The Eisenstein series E' is clearly normalized and since ℓ is co-prime to N, this is an eigenfunction for the T_ℓ-operator acting on $\mathcal{M}_2\left(\Gamma_0\left(N\right)\right)$. By isomorphism (12) and assumption $\overline{\nu}(p_i) = a_{p_i} \pmod{\ell}$, $1 \leq i \leq t$, there exists a prime ideal $\mathcal{L}$ above ℓ in the integer ring of M such that :

$$\nu(r) + \nu^{-1}(r) r \equiv a_r \pmod{\mathcal{L}}, \quad \text{for every prime } r \nmid \ell N$$

and $\nu(p_i) \equiv a_{p_i} \pmod{\mathcal{L}}$ for any $1 \leq i \leq t$. Let r be a prime. If r does not divide ℓN, then E' is a T_r-eigenfunction with eigenvalue $a_r(E') = \nu(r) + \nu^{-1}(r) r$ which is congruent to a_r modulo $\mathcal{L}$. If else r divides c (and thus N), then E' is a $\mathcal{U}_r$-eigenfunction with corresponding eigenvalue $0 = a_r$. Finally, if $r = p_j \in \{p_1, \ldots, p_t\}$, then we have

$$\left(\mathcal{U}_{p_j} E'\right)(\tau) = \left(\prod_{\substack{i=1 \\ i \neq j}}^{t} (\mathcal{U}_{p_i} - p_i \nu^{-1}(p_i)\mathrm{Id})\right) \cdot \left(\mathcal{U}_{p_j}^2 - p_j \nu^{-1}(p_j) \mathcal{U}_{p_j}\right) E(p_1 \cdots p_t \tau).$$

Besides, according to [Shi94, Rk. 3.59], we have :

$$\begin{aligned}
&\left(\mathcal{U}_{p_j}^2 - p_j \nu^{-1}(p_j) \mathcal{U}_{p_j}\right) E(p_1 \cdots p_t \tau) \\
&= (\nu(p_j) + \nu^{-1}(p_j) p_j) E(\widehat{p_1 \cdots p_t} \tau) - p_j E(p_1 \cdots p_t \tau) - p_j \nu^{-1}(p_j) E(\widehat{p_1 \cdots p_t} \tau) \\
&= \nu(p_j) \left(\mathcal{U}_{p_j} - p_j \nu^{-1}(p_j) \mathrm{Id}\right) E(p_1 \cdots p_t \tau),
\end{aligned}$$

where $\widehat{p_1 \cdots p_t} = \prod_{\substack{i=1 \\ i \neq j}}^{t} p_i$. This equality proves that E' is a $\mathcal{U}_{p_j}$-eigenfunction with corresponding eigenvalue $\nu(p_j)$ and the congruence $\nu(p_j) \equiv a_{p_j} \pmod{\mathcal{L}}$ eventually completes the proof of the lemma. □

2.6.2 Constant term at $1/\mathfrak{c}$ and end of the proof of Theorem 2.4

Since E' vanishes at ∞, we compute its constant term at another specific cusp, where it is non-vanishing, namely $1/\mathfrak{c}$. Put

$$\gamma = \begin{pmatrix} 1 & 0 \\ \mathfrak{c} & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}).$$

We postpone the proof of the following proposition to §2.6.3.

Proposition 2.5. *The constant term of the Fourier expansion of $E'|_2\gamma$ is the non-zero algebraic number in $\mathcal{O}_L[1/\mathfrak{c}^2](\mu_{\mathfrak{c}^2})$:*

$$\Upsilon' = -\nu(-1)\left(\frac{\mathfrak{c}}{c_0}\right)^2 \frac{W\left((\nu^2)_0\right)}{W(\nu)} \frac{B_{2,(\nu^2)_0^{-1}}}{4} \left(\prod_{i=1}^{t}\left(1 - p_i^{-1}\right)\right) \cdot \left(\prod_{p|\mathfrak{c}}\left(1 - (\nu^2)_0(p)p^{-2}\right)\right),$$

where the second product runs over the primes and $(\nu^2)_0$ is the primitive Dirichlet character associated to ν^2 of modulus $c_0 \mid \mathfrak{c}$.

Using this proposition, we now complete the proof of Thm. 2.4. Let Θ be the Katz' operator on modular forms over $\overline{\mathbf{F}}_\ell$ whose action on q-expansions is given by $q\frac{d}{dq}$ (denoted $A\theta$ in [Kat77]). Assume $\ell > k+1 = 3$. Lemma 2.3 implies that $\Theta(\overline{f}) = \Theta(\overline{E})$ where $\overline{f}$ and $\overline{E'}$ are the modular forms over $\overline{\mathbf{F}}_\ell$ obtained by reduction modulo $\mathcal{L}$ of f and E' respectively. Moreover Katz has proved that if $\ell > 3$, then Θ is injective ([Kat77, Cor. (3)]). Under this assumption, it thus follows that the Eisenstein series E' becomes cuspidal after reduction.

Put $\epsilon = (\nu^2)_0^{-1}$. By Prop. 2.5 and using the assumption $\mathfrak{c} = c$, we therefore have that ℓ divides the numerator of the norm of :

$$\Upsilon' = \pm\left(\frac{c}{c_0}\right)^2 \frac{W\left(\epsilon^{-1}\right)}{W(\nu)} \frac{B_{2,\epsilon}}{4} \left(\prod_{i=1}^{t}\left(1 - p_i^{-1}\right)\right) \cdot \left(\prod_{p|c}\left(1 - \epsilon^{-1}(p)p^{-2}\right)\right).$$

By Lemma 2.1, the prime divisors of the norm of $W\left(\epsilon^{-1}\right)/W(\nu)$ divide N and therefore are co-prime to ℓ. The same obviously holds for c/c_0. It thus follows that either $p_i \equiv 1 \pmod{\ell}$ for some $1 \leq i \leq t$ or ℓ divides the norm of either $p^2 - \epsilon^{-1}(p)$ for some p dividing c or the norm of the numerator of $B_{2,\epsilon}/4$. This completes the proof of Thm. 2.4.

2.6.3 Proof of Proposition 2.5

Let us first introduce notation as in the proof of Prop. 2.3. Put :

$$G = \frac{C_2 W(\nu)}{\mathfrak{c}^2} E, \quad \text{where } C_2 = -4\pi^2$$

and similarly

$$G' = \frac{C_2 W(\nu)}{\mathfrak{c}^2} E'.$$

For simplicity, we shall denote by $\underline{i}$ the elements of

$$\mathcal{N} = \{(i_1, \ldots, i_j) \text{ such that } j \in \{1, \ldots, t\} \text{ and } 1 \leq i_1 < \cdots < i_j \leq t\}.$$

If $\underline{i} = (i_1, \ldots, i_j) \in \mathcal{N}$, we put :

$$p_{\underline{i}} = p_{i_1} \cdots p_{i_j} \quad \text{and} \quad a_{\underline{i}} = a_{p_{i_1}} \cdots a_{p_{i_j}}.$$

Let $v = \overline{(c_v, d_v)} \in (\mathbf{Z}/\mathfrak{c}^2\mathbf{Z})^2$ of order $\mathfrak{c}^2$. Following [DS05, §4.6], define

$$G_2^v(\tau) = \frac{1}{(c_v\tau + d_v)^2} + \frac{1}{\mathfrak{c}^4} \sum_{d \in \mathbf{Z}}{}' \frac{1}{\left(\frac{c_v\tau + d_v}{\mathfrak{c}^2} - d\right)^2} + \frac{1}{\mathfrak{c}^4} \sum_{c \neq 0} \sum_{d \in \mathbf{Z}} \frac{1}{\left(\frac{c_v\tau + d_v}{\mathfrak{c}^2} - c\tau - d\right)^2} \tag{14}$$

where the primed summation notation means to sum over non-zero integers. For any $\underline{i} \in \mathcal{N}$ and any $v \in (\mathbf{Z}/\mathfrak{c}^2\mathbf{Z})^2$ of order $\mathfrak{c}^2$, put

$$G_2^{v,p_{\underline{i}}}(\tau) = G_2^v(p_{\underline{i}}\tau) \quad \text{and} \quad G^{p_{\underline{i}}}(\tau) = G(p_{\underline{i}}\tau).$$

According to [DS05, §4.2] and the definition of E (cf. §2.3), we have

$$G = \frac{1}{2} \sum_{i,j,l=0}^{\mathfrak{c}-1} \nu(ij) G_2^{\overline{(i\mathfrak{c}, j+l\mathfrak{c})}}$$

and therefore

$$G^{p_{\underline{i}}} = \frac{1}{2} \sum_{i,j,l=0}^{\mathfrak{c}-1} \nu(ij) G_2^{\overline{(i\mathfrak{c}, j+l\mathfrak{c})}, p_{\underline{i}}}. \tag{15}$$

Lemma 2.4. *Let $v = \overline{(c_v, d_v)} \in (\mathbf{Z}/\mathfrak{c}^2\mathbf{Z})^2$ of order $\mathfrak{c}^2$. The constant term of $G_2^{v,p_{\underline{i}}}|_2\gamma$ is*

$$\Upsilon_{v,\underline{i}} = \vartheta(\overline{c_v p_{\underline{i}} + d_v \mathfrak{c}}) \left(\frac{1}{p_{\underline{i}}}\right)^2 \zeta^{\overline{d_v/p_{\underline{i}}}}(2)$$

where the bar means reduction modulo $\mathfrak{c}^2$,

$$\vartheta(\overline{n}) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{\mathfrak{c}^2} \\ 0 & \text{otherwise} \end{cases}, \quad \zeta^{\overline{n}}(2) = \sum_{m \equiv n \pmod{\mathfrak{c}^2}}{}' \frac{1}{m^2},$$

and the primed summation notation means to sum over non-zero integers.

Proof. We first compute $G_2^{v,p_{\underline{i}}}|_2\gamma$ using (14). We find :

$$\begin{aligned}\left(G_2^{v,p_{\underline{i}}}|_2\gamma\right)(\tau) = \frac{1}{(c_v p_{\underline{i}}\tau + d_v(\mathfrak{c}\tau+1))^2} + \sum_{d \in \mathbf{Z}}{}' \frac{1}{(c_v p_{\underline{i}}\tau + d_v(\mathfrak{c}\tau+1) - \mathfrak{c}^2 d(\mathfrak{c}\tau+1))^2} \\ + \sum_{c \neq 0} \sum_{d \in \mathbf{Z}} \frac{1}{(c_v p_{\underline{i}}\tau + d_v(\mathfrak{c}\tau+1) - \mathfrak{c}^2(cp_{\underline{i}}\tau + d(\mathfrak{c}\tau+1)))^2}.\end{aligned}$$

In other words, we have $\left(G_2^{v,p_{\underline{i}}}|_2\gamma\right)(\tau) = A + B$, where

$$A = \frac{1}{((c_v p_{\underline{i}} + d_v\mathfrak{c})\tau + d_v)^2} + \sum_{d \in \mathbf{Z}}{}' \frac{1}{((c_v p_{\underline{i}} + d_v\mathfrak{c} - \mathfrak{c}^2 d\mathfrak{c})\tau + d_v - \mathfrak{c}^2 d)^2}$$

and

$$B = \sum_{c\neq 0}\sum_{d\in\mathbf{Z}} \frac{1}{((c_v p_{\underline{i}} + d_v\mathfrak{c} - \mathfrak{c}^2(cp_{\underline{i}} + d\mathfrak{c}))\tau + d_v - \mathfrak{c}^2 d)^2}.$$

Since $\gcd(p_{\underline{i}}, \mathfrak{c}) = 1$, we may assume without loss of generality that $0 \leq c_v p_{\underline{i}} + d_v\mathfrak{c} < \mathfrak{c}^2$. Therefore the constant term of A is given by :

$$\vartheta(\overline{c_v p_{\underline{i}} + d_v\mathfrak{c}})\frac{1}{d_v^2}$$

and the one of B by :

$$\vartheta(\overline{c_v p_{\underline{i}} + d_v\mathfrak{c}})\sum_{c\neq 0}\sum_{\substack{d\in\mathbf{Z}\\ cp_{\underline{i}}+d\mathfrak{c}=0}} \frac{1}{(d_v - \mathfrak{c}^2 d)^2}.$$

Therefore, the constant term of $G_2^{v,p_{\underline{i}}}|_2\gamma$ is :

$$\Upsilon_{v,\underline{i}} = \vartheta(\overline{c_v p_{\underline{i}} + d_v\mathfrak{c}})\sum_{c\in\mathbf{Z}}\sum_{\substack{d\in\mathbf{Z}\\ cp_{\underline{i}}+d\mathfrak{c}=0}} \frac{1}{\overline{(d_v - \mathfrak{c}^2 d)^2}}.$$

Note that if $\vartheta(\overline{c_v p_{\underline{i}} + d_v\mathfrak{c}}) = 1$, then $d_v \not\equiv 0 \pmod{\mathfrak{c}^2}$ since v is of order $\mathfrak{c}^2$. A change of variable yields :

$$\Upsilon_{v,\underline{i}} = \vartheta(\overline{c_v p_{\underline{i}} + d_v\mathfrak{c}})\sum_{c\in\mathbf{Z}}\sum_{\substack{d\in\mathbf{Z}\\ cp_{\underline{i}}+d\mathfrak{c}=0\\ (c,d)\equiv v\ (\mathfrak{c}^2)}} \frac{1}{d^2}$$

and thus

$$\Upsilon_{v,\underline{i}} = \vartheta(\overline{c_v p_{\underline{i}} + d_v\mathfrak{c}})\sum_{\substack{d\neq 0\\ d\equiv d_v\ (\mathfrak{c}^2)\\ p_{\underline{i}}|d}} \frac{1}{d^2} = \vartheta(\overline{c_v p_{\underline{i}} + d_v\mathfrak{c}})\sum_{\substack{m\neq 0\\ m\equiv d_v/p_{\underline{i}}\ (\mathfrak{c}^2)}} \frac{1}{(p_{\underline{i}} m)^2}.$$

Finally we get $\Upsilon_{v,\underline{i}} = \vartheta(\overline{c_v p_{\underline{i}} + d_v\mathfrak{c}})/p_{\underline{i}}^2 \cdot \zeta^{\overline{d_v/p_{\underline{i}}}}(2)$ as asserted. □

Using this lemma and formula (15), we are now able to compute the constant term of $G^{p_{\underline{i}}}|_2\gamma$.

Lemma 2.5. *The constant term of $G^{p_{\underline{i}}}|_2\gamma$ is*

$$\Upsilon_{\underline{i}} = \nu(p_{\underline{i}})\frac{1}{p_{\underline{i}}^2}\cdot\Upsilon_0, \quad \textit{with } \Upsilon_0 = -\nu(-1)W\left((\nu^2)_0\right)\frac{C_2}{c_0^2}\frac{B_{2,(\nu^2)_0^{-1}}}{4}\prod_{p|\mathfrak{c}}\left(1 - (\nu^2)_0(p)p^{-2}\right),$$

where $(\nu^2)_0$ is the primitive Dirichlet character associated to ν^2 of modulus $c_0 \mid \mathfrak{c}$.

Proof. The proof of this lemma is quite similar to the proof of Prop. 2.3. According to (15), we have :

$$\Upsilon_{\underline{i}} = \frac{1}{2}\sum_{i,j,l=0}^{\mathfrak{c}-1}\nu(ij)\Upsilon_{\overline{(i\mathfrak{c},j+l\mathfrak{c})},\underline{i}}$$

and thus by Lemma 2.4 :

$$\Upsilon_{\underline{i}} = \frac{1}{2}\cdot\frac{1}{p_{\underline{i}}^2}\sum_{i,j,l=0}^{\mathfrak{c}-1}\nu(ij)\vartheta(\overline{i\mathfrak{c}p_{\underline{i}} + \mathfrak{c}(j+l\mathfrak{c})})\zeta^{\overline{d_v/p_{\underline{i}}}}(2).$$

This yields to :

$$\begin{aligned}\Upsilon_{\underline{i}} &= \frac{1}{2}\cdot\frac{1}{p_{\underline{i}}^2}\sum_{l=0}^{\mathfrak{c}-1}\sum_{\substack{j=0\\ \gcd(j,\mathfrak{c})=1}}^{\mathfrak{c}-1}\nu\left(-\frac{j^2}{p_{\underline{i}}}\right)\zeta^{\overline{d_v/p_{\underline{i}}}}(2)\\ &= \frac{1}{2}\cdot\frac{1}{p_{\underline{i}}^2}\nu(p_{\underline{i}})\nu(-1)\sum_{l=0}^{\mathfrak{c}-1}\sum_{\substack{j=0\\ \gcd(j,\mathfrak{c})=1}}^{\mathfrak{c}-1}\nu\left((j^2/p_{\underline{i}})^2\right)\sum_{m\equiv(j+l\mathfrak{c})/p_{\underline{i}}\ (\mathfrak{c}^2)}{}'\frac{1}{m^2}\\ &= \frac{1}{p_{\underline{i}}^2}\nu(p_{\underline{i}})\nu(-1)L(2,\nu^2).\end{aligned}$$

Let $(\nu^2)_0$ be the primitive character associated to ν^2 of modulus $c_0 \mid \mathfrak{c}$. We have :

$$L(2,\nu^2) = L\left(2,(\nu^2)_0\right)\prod_{p\mid\mathfrak{c}}\left(1-(\nu^2)_0(p)p^{-2}\right).$$

Applying Prop. 2.2 to $\psi=(\nu^2)_0$ and $m=k$, we get :

$$L(2,(\nu^2)_0) = -W\left((\nu^2)_0\right)\frac{C_2}{c_0^2}\frac{B_{2,(\nu^2)_0^{-1}}}{4}\neq 0$$

and thus

$$\Upsilon_{\underline{i}} = -\frac{1}{p_{\underline{i}}^2}\nu(p_{\underline{i}})\nu(-1)W\left((\nu^2)_0\right)\frac{C_2}{c_0^2}\frac{B_{2,(\nu^2)_0^{-1}}}{4}\prod_{p\mid\mathfrak{c}}\left(1-(\nu^2)_0(p)p^{-2}\right),$$

as claimed. □

Let us now complete the proof of Prop. 2.5. With the notation introduced at the beginning of this paragraph and Eq. (13), we have :

$$G'|_2\gamma = G|_2\gamma + \sum_{\underline{i}\in\mathcal{N}}(-1)^{\sharp\underline{i}}p_{\underline{i}}\nu^{-1}(p_{\underline{i}})G^{p_{\underline{i}}}|_2\gamma.$$

Therefore, according to Prop. 2.3 and Lem. 2.5, the constant term of $G'|_2\gamma$ is :

$$\Upsilon_0+\sum_{\underline{i}\in\mathcal{N}}(-1)^{\sharp\underline{i}}p_{\underline{i}}\nu^{-1}(p_{\underline{i}})\Upsilon_{\underline{i}} = \Upsilon_0\left(1+\sum_{\underline{i}\in\mathcal{N}}(-1)^{\sharp\underline{i}}p_{\underline{i}}\nu^{-1}(p_{\underline{i}})\nu(p_{\underline{i}})\frac{1}{p_{\underline{i}}^2}\right) = \Upsilon_0\prod_{i=1}^{t}\left(1-p_i^{-1}\right)$$

where $(\nu^2)_0$ is the primitive character associated to ν^2 of modulus $c_0 \mid \mathfrak{c}$. Prop. 2.5 now follows from the normalization $E' = (\mathfrak{c}^2/(C_2W(\nu)))G'$.

3 Dihedral representations

3.1 Preliminaries: twisting and CM forms

Let M be an integer, $F(\tau)=\sum_{n\geq 1}a_n(F)q^n \in \mathcal{S}_k\left(\Gamma_0\left(M\right)\right)$ and ψ be a Dirichlet character of modulus $f\geq 1$. Define :

$$(F\otimes\psi)(\tau) = \sum_{n\geq 1}a_n(F)\psi(n)q^n.$$

The following result is a special case of [Shi94, Prop. 3.64].

Lemma 3.1. *With the notations above, assume ψ to be a quadratic primitive Dirichlet character. Then $F \otimes \psi$ belongs to $\mathcal{S}_k\left(\Gamma_0\left(\mathrm{lcm}(M, f^2)\right)\right)$. Moreover, if F is a normalized Hecke eigenform for the Hecke operators $\{T_p\}_{p\nmid M}$, then $F \otimes \psi$ is a normalized Hecke eigenform for the Hecke operators $\{T_p\}_{p\nmid fM}$ with corresponding eigenvalues $\{a_p(F)\psi(p)\}_{p\nmid fM}$.*

We take the following definition for CM forms ([Rib77]).

Definition 3.1 (CM forms)**.** *Assume that ψ is not the trivial character. The form F has complex multiplication (or, F is a CM form) by ψ if $a_p(F) = a_p(F)\psi(p)$ for all p in a set of primes of density* 1.

3.2 Statement of the result

Recall that

$$\mathbf{P}(\overline{\rho}_{f,\lambda}) : G_{\mathbf{Q}} \xrightarrow{\overline{\rho}_{f,\lambda}} \mathrm{GL}(2, \mathbf{F}_\lambda) \longrightarrow \mathrm{PGL}(2, \mathbf{F}_\lambda),$$

where $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and put $\mathbf{P}(\overline{G}_\lambda) = \mathbf{P}(\overline{\rho}_{f,\lambda})(G_{\mathbf{Q}})$.

The following result is a generalization to arbitrary weights and fields of coefficients of a theorem on the surjectivity of Galois representations attached to elliptic curves over $\mathbf{Q}$ independently proved by Kraus ([Kra95]) and Cojocaru ([Coj05]). In particular, it implies that in the case of dihedral projective image, ℓ is explicitly bounded in terms of k and N.

Theorem 3.1. *Assume that Assume $\mathbf{P}(\overline{G}_\lambda)$ dihedral. If f does not have complex multiplication, then we have*

$$\ell \leq \left(2\left(4.8kN^2(1+\log\log N)\right)^{\frac{k-1}{2}}\right)^{[K:\mathbf{Q}]}.$$

Besides, if N is square-free, then either $\ell \mid N$, or $\ell \leq k$, or $\ell = 2k-1$.

Remarks.

1. The integer $[K : \mathbf{Q}]$ is bounded from above by the dimension $g_0^\sharp(k, N)$ of the new subspace of $\mathcal{S}_k\left(\Gamma_0\left(N\right)\right)$. A closed formula in terms of k and N for $g_0^\sharp(k, N)$ as well as asymptotic estimates can be found in [Mar05].

2. When $N = 1$, the result goes back to Ribet (see the proof of (ii) p. 264 and the remark after Cor. 4.5 of [Rib75]). Moreover, our argument for the case of arbitrary square-free level is a combination of tricks from [Rib85] and [Rib97].

3. A newform of square-free level and trivial Nebentypus is automatically non-CM (see e.g. [Tsa12, §4]).

3.3 Proof of Theorem 3.1

Assume $\ell \nmid N$ and $\mathbf{P}(\overline{G}_\lambda)$ dihedral. Then $\mathbf{P}(\overline{G}_\lambda)$ is an extension of $\{\pm 1\}$ by a cyclic group C and every element of $\overline{G}_\lambda$ which does not map to C has trace 0. Hence, we may consider the following quadratic character :

$$\epsilon_\lambda : G_{\mathbf{Q}} \xrightarrow{\mathbf{P}(\overline{\rho}_{f,\lambda})} \mathbf{P}(\overline{G}_\lambda) \to \{\pm 1\}.$$

Let L_λ be the number field cut out by $\mathbf{P}(\overline{\rho}_{f,\lambda})$ and $K_\lambda/\mathbf{Q}$ its quadratic sub-extension fixed by the kernel of ϵ_λ. The extension $L_\lambda/\mathbf{Q}$ has Galois group isomorphic to $\mathbf{P}(\overline{G}_\lambda)$ while $C \simeq \mathrm{Gal}(L_\lambda/K_\lambda)$. Clearly, ϵ_λ is unramified outside ℓN. The following proposition describes more precisely the ramification set of ϵ_λ.

Proposition 3.1. *Assume $\ell \nmid N$.*

1. *Let $p \neq \ell$ be a ramified prime for ϵ_λ. Then $p^2 \mid N$.*
2. *Assume $\ell > k$ and*
   (a) *either f is ordinary at λ and $\ell \neq 2k-1$;*
   (b) *or f is not ordinary at λ and $\ell \neq 2k-3$.*

   Then, ϵ_λ is unramified at ℓ.

Proof. Let p be a prime dividing N exactly once. By §1.1, we know that the inertia subgroup I_p at p acts unipotently in $\overline{\rho}$. Since $\overline{G}_\lambda$ has prime-to-ℓ order, it follows that I_p acts trivially. So $\overline{\rho}$ and, hence, ϵ_λ are unramified at p. This proves the first part of the proposition.

Assume now $\ell > k$. Let I_ℓ be the inertia group of a decomposition subgroup at ℓ and recall that $\ell \nmid N$. We prove that ϵ_λ is unramified at ℓ under conditions (a) and (b) in turn.

(a) Assume that f is ordinary at λ and $\ell \neq 2k-1$. By §1.3, we have

$$\overline{\rho}_{|I_\ell} \simeq \begin{pmatrix} \overline{\chi}_\ell^{k-1} & \star \\ 0 & 1 \end{pmatrix}.$$

But $\overline{G}_\lambda$ has prime-to-ℓ order and therefore $\star = 0$. In particular, $\mathbf{P}(\overline{\rho}_{f,\lambda})(I_\ell)$ is isomorphic to the image of $\overline{\chi}_\ell^{k-1}$ which is, by Lemma 1.1, cyclic of order $(\ell-1)/\gcd(\ell-1, k-1) > 2$. Therefore, it has to be included in C and hence ϵ_λ is unramified at ℓ.

(b) Assume that f is not ordinary at λ and $\ell \neq 2k-3$. By §1.3, $\mathbf{P}(\overline{\rho}_{f,\lambda})(I_\ell)$ is isomorphic to the image of I_ℓ under $\psi^{(\ell-1)(k-1)}$ where ψ is a fundamental character of level 2. By the assumption $\ell \neq 2k-3$ and Lemma 1.1, it is therefore cyclic of order $(\ell+1)/\gcd(\ell+1, k-1) > 2$. We conclude as before.

□

Assume N to be square-free and $\ell > k$. Then, by the above proposition, K_λ is the unique quadratic extension of $\mathbf{Q}$ ramified at ℓ only and $\ell \in \{2k-1, 2k-3\}$. The case $\ell = 2k-3$ however does not occur. This is proved in [Die12, Lem. 3.2]. Hence Thm. 3.1 in the square-free level case.

Assume now that N is any integer not divisible by ℓ, and that $\ell > k$ satisfies $\ell \neq 2k-1$ and $\ell \neq 2k-3$. We may identify ϵ_λ with a Dirichlet character. Let us denote by $\mathfrak{c}$ its conductor. It is co-prime to ℓ by the above proposition. We then have $\mathfrak{c} = |D_{K_\lambda}|$ where D_{K_λ} is the fundamental discriminant of the quadratic field K_λ fixed by the kernel of ϵ_λ ([Neu99, VII. §11]). In particular, if $K_\lambda = \mathbf{Q}(\sqrt{D_0})$ with D_0 square-free, then $\mathfrak{c} = D_0$ or $4D_0$ depending on whether $D_0 \equiv 1 \pmod 4$ or not. If moreover, $\ell > 2k-1$, then by the proposition above, $\mathfrak{c}^2 \mid 2^4 N$. Put $g = f \otimes \epsilon_\lambda$. By the Lemma 3.1, $g \in \mathcal{S}_k(\Gamma_0(2^4 N))$ and for any prime $p \nmid 2N$, g is an eigenform for the T_p Hecke operator with corresponding eigenvalue $a_p(g) = a_p \epsilon_\lambda(p)$. Let $D_0' = \varepsilon \prod_{3 \leq p \mid N} p$ be the product of all odd primes dividing N with a sign $\varepsilon \in \{\pm 1\}$ chosen so that $D_0' \equiv 3 \pmod 4$. Then $4D_0'$ is a fundamental discriminant and the Kronecker symbol $\psi = (4D_0'/\cdot)$ is a primitive quadratic Dirichlet character of modulus $4D_0'$ ([Coh07, Th. 2.2.15]) precisely ramified at the primes dividing $2N$. Put :

$$\widetilde{f} = f \otimes \psi \quad \text{and} \quad \widetilde{g} = g \otimes \psi.$$

Since $(4D_0')^2 \mid 2^4N^2$, it follows from Lemma 3.1 that $\widetilde{f}, \widetilde{g} \in \mathcal{S}_k\left(\Gamma_0\left(2^4N^2\right)\right)$ and for any integer n, we have :

$$\begin{cases} a_n(\widetilde{f}) &= a_n\psi(n) \\ a_n(\widetilde{g}) &= a_n\epsilon_\lambda(n)\psi(n). \end{cases} \tag{16}$$

Since f is assumed to be non-CM (in the sense of Def. 3.1), we have $\widetilde{f} \neq \widetilde{g}$ and by [Mur97, Th. 1], there exists an integer

$$n \leq \frac{4k}{3}N^2 \prod_{p\mid 2N}\left(1+\frac{1}{p}\right) \leq 2kN^2 \prod_{p\mid N}\left(1+\frac{1}{p}\right) \tag{17}$$

such that $a_n(\widetilde{f}) \neq a_n(\widetilde{g})$. According to (16), it follows that we have :

$$\psi(n) \neq 0, \quad a_n \neq 0 \quad \text{and} \quad \epsilon_\lambda(n) = -1.$$

From the condition $\epsilon_\lambda(n) = -1$, we deduce that there exists a prime divisor q of n together with an odd integer t such that $q^t \mid n$ but $q^{t+1} \nmid n$ and $\epsilon_\lambda(q) = -1$. If $q = \ell$, we are done in bounding ℓ in terms of k and N. Assume therefore $q \neq \ell$. The multiplicativity of the Fourier coefficients of f gives that $a_{q^t} \mid a_n$ and hence (since t is odd) that $a_q \neq 0$. Besides, since $\epsilon_\lambda(q) = -1$, the image under $\overline{\rho}_{f,\lambda}$ of a Frobenius at q has trace 0 modulo λ. In other words, ℓ divides the norm of the non-zero algebraic integer a_q. Applying Deligne's estimate on the Fourier coefficients of f and its Galois conjugates by $\overline{\mathbf{Q}}$-automorphisms, we get that :

$$\ell \leq \mathrm{N}_{K/\mathbf{Q}}\left(a_q\right) = \prod_{\sigma:K\hookrightarrow\mathbf{C}} |\sigma(a_q)| \leq \left(2q^{(k-1)/2}\right)^{[K:\mathbf{Q}]}. \tag{18}$$

Besides, using [RS62, (3.27)] and inequality (17), we get the following estimate for q :

$$q \leq 4.8kN^2(1+\log\log N). \tag{19}$$

The theorem follows from (18) and (19).

4 Projective image isomorphic to A_4, S_4 or A_5

The following result is proved in a different way in [Rib85].

Theorem 4.1. *If $\mathbf{P}(\overline{G}_\lambda)$ is isomorphic to A_4, S_4 or A_5, then either $\ell \mid N$ or $\ell \leq 4k-3$.*

Proof. Assume that $\ell \nmid N$ and $\ell > k$. Then, by §1.3, $\mathbf{P}(\overline{G}_\lambda)$ has a cyclic subgroup given the image of inertia at ℓ. In the case of ordinariness, this cyclic subgroup is isomorphic to the image of $\overline{\chi}_\ell^{k-1}$ which has order > 5 if $\ell > 4k-3$ by Lemma 1.1. If else f is not ordinary at λ, then it has order $(\ell+1)/\gcd(\ell+1, k-1)$ which is also > 5 if $\ell > 4k-3$.

In any case, if $\ell > 4k-3$, then $\mathbf{P}(\overline{G}_\lambda)$ has an element of order > 5. This rules out the possibility for $\mathbf{P}(\overline{G}_\lambda)$ to be isomorphic to A_4, S_4 or A_5. □

5 Numerical examples

In this section we give some examples illustrating the theorems of the paper. All the computations were performed on SAGE ([S$^+$12]).

5.1 Reducible representations

Before dealing with examples, let us first recall that for the representations $\overline{\rho}_{f,\lambda}$, irreducibility is equivalent to absolute irreducibility.

5.1.1 Square level case

Fix $(k, N) = (6, 81)$. The new subspace in $\mathcal{S}_6(\Gamma_0(81))$ is 18-dimensional and splits into 5 Galois conjugacy classes labeled 81.6a,...,81.6e in SAGE ([S$^+$12]). According to Theorem 2.2, the prime ideals λ such that $\overline{\rho}_{f,\lambda}$ is reducible for some newform $f \in \mathcal{S}_6(\Gamma_0(81))$ have residue characteristic ℓ in $\{2, 3, 5, 7, 43, 1171\}$. Let us first show that 2, 3, 7, 43 and 1171 are indeed the residue characteristics of some prime ideals λ for which $\overline{\rho}_{f,\lambda}$ is reducible for the specific (up to Galois conjugacy) modular form f labeled 81.6c. We have :

$$f(\tau) = q + \alpha q^2 + (\alpha^2 - 32)q^4 + \left(-\frac{1}{4}\alpha^3 - \frac{9}{4}\alpha^2 + \frac{25}{2}\alpha + 54\right)q^5 + O(q^5),$$

where α is a root of $X^4 + 3X^3 - 84X^2 - 72X + 792$.

Let us denote by K the number field generated by α. We call ν the primitive Dirichlet character modulo 9 sending 2 on ζ_3, where ζ_3 is a primitive third root of unity and $L = \mathbf{Q}(\zeta_3)$. Since ν has order 3, we have $\epsilon = \nu$ with the notation of Thm. 2.2. Moreover we have $B_{6,\nu}/12 = (751\zeta_3 + 1172)/3$ which has norm $3^{-1} \cdot 7 \cdot 43 \cdot 1171$.

Then we more precisely show that for each $\ell \in \{2, 3, 7, 43, 1171\}$ there are prime ideals λ_ℓ and $\mathfrak{p}_\ell$ above ℓ in $\mathcal{O}$ and $\mathbf{Z}[\zeta_3]$ respectively such that $\overline{\rho}^{ss}_{f,\lambda_\ell} \simeq \overline{\rho}_{E,\mathfrak{p}_\ell}$ where E is the following Eisenstein series

$$E(\tau) = \sum_{n\geq 1} \sigma_5^\nu(n)q^n = q - (31\zeta_3 + 32)q^2 + (1023\zeta_3 + 31)q^4 + (3124\zeta_3 - 1)q^5 + O(q^5).$$

Such an isomorphism is proved to hold by checking that for all integers n up to the Sturm bound (which, here, equals 54) we have a congruence

$$a_n \equiv a_n(E) \pmod{\mathcal{L}_\ell},$$

for some prime ideal $\mathcal{L}_\ell$ above ℓ in the integer ring of the compositum KL. For instance, if $\ell = 43$, we can take

$$\mathcal{L}_{43} = (43, \alpha + \zeta_3 - 6).$$

Therefore we have $\overline{\rho}^{ss}_{f,\lambda_\ell} \simeq \overline{\nu}_\ell \oplus \overline{\nu}_\ell^{-1}\overline{\chi}_\ell^5$ where

$$\overline{\nu}_\ell : G_{\mathbf{Q}} \twoheadrightarrow (\mathbf{Z}/9\mathbf{Z})^\times \xrightarrow{\nu} \mathbf{Z}[\zeta_3] \twoheadrightarrow \mathbf{Z}[\zeta_3]/\mathfrak{p}_\ell$$

is ν modulo $\mathfrak{p}_\ell$ viewed as a character of $G_{\mathbf{Q}}$. For each ℓ as above the corresponding ideals λ_ℓ and $\mathfrak{p}_\ell$ are listed in Table 2 (as given in SAGE).

Let us now see what happens for the remaining prime, namely $\ell = 5$. For the specific newform above with coefficients field K, we have $5\mathcal{O} = \lambda_5\lambda_5'$ where $\lambda_5 = (5, \alpha + 4)$ and $\lambda_5' = (5, \alpha^3 + 4\alpha^2 + 3)$. Then λ_5 and λ_5' have inertia degree 1 and 3 respectively. Besides, if Frob_2 denotes a Frobenius at 2, the characteristic polynomial of $\overline{\rho}_{f,\lambda_5}(\mathrm{Frob}_2)$ and $\overline{\rho}_{f,\lambda_5'}(\mathrm{Frob}_2)$ is $X^2 - \alpha X + 2^5$. Such a polynomial being irreducible modulo λ_5 and λ_5' as one checks, we get that $\overline{\rho}_{f,\lambda_5}$ and $\overline{\rho}_{f,\lambda_5'}$ are both irreducible.

For each pair (f, λ) where f is a newform in $\mathcal{S}_6(\Gamma_0(81))$ and λ is a prime ideal in $\mathcal{O}$ above 5 we give in Table 3 the smallest prime number $p \neq 3, 5$ and ≤ 100 for which the characteristic

ℓ	λ_ℓ	$\mathfrak{p}_\ell$
2	$(2, \alpha^3/36 + \alpha^2/4 - 7\alpha/6 - 7)$	(2)
3	$(3, -\alpha^3/36 + \alpha^2/12 + 7\alpha/6 - 7)$	$(2\zeta_3 + 1)$
7	$(7, \alpha^3/36 + \alpha^2/12 - 5\alpha/3 + 2)$	$(3\zeta_3 + 1)$
43	$(43, \alpha^3/36 + \alpha^2/12 - 5\alpha/3 - 20)$	$(7\zeta_3 + 6)$
1171	$(1171, \alpha^3/36 + \alpha^2/12 - 5\alpha/3 - 586)$	$(39\zeta_3 + 25)$

Table 2: Congruence primes for f and E

f	$K = \mathbf{Q}(\alpha)$	λ	p
81.6a	$\alpha^2 + 3\alpha - 30 = 0$	$(-6\alpha + 25)$	2
		$(-6\alpha - 43)$	7
81.6b	$\alpha^2 - 3\alpha - 30 = 0$	$(-6\alpha - 25)$	2
		$(-6\alpha + 43)$	7
81.6c	$\alpha^4 + 3\alpha^3 - 84\alpha^2 - 72\alpha + 792 = 0$	$(5, \alpha + 4)$	2
		$(5, \alpha^3 + 4\alpha^2 + 3)$	2
81.6d	$\alpha^4 - 3\alpha^3 - 84\alpha^2 + 72\alpha + 792 = 0$	$(5, \alpha + 1)$	2
		$(5, \alpha^3 + \alpha^2 + 2)$	2
81.6e	$\alpha^6 - 171\alpha^4 + 7128\alpha^2 - 432 = 0$	$(5, \alpha^2 + 1)$	$\emptyset$
		$(5, \alpha^2 + 3\alpha + 3)$	7
		$(5, \alpha^2 + 2\alpha + 3)$	7

Table 3: Smallest prime $p \neq 3, 5$ and ≤ 100 such that $\overline{\rho}_{f,\lambda}(\mathrm{Frob}_p)$ acts irreducibly

polynomial of $\overline{\rho}_{f,\lambda}(\mathrm{Frob}_p)$ is irreducible. Therefore all the representations $\overline{\rho}_{f,\lambda}$ are irreducible unless perhaps if f is the form 81.6e and $\lambda = (5, \alpha^2 + 1)$. But this latter representation is also proved to be irreducible by noticing that the eigenvalues of $\overline{\rho}_{f,\lambda}(\mathrm{Frob}_2)$ and $\overline{\rho}_{f,\lambda}(\mathrm{Frob}_{19})$ in $\mathbf{F}_\lambda$ are $\{3\beta, 3\beta\}$ and $\{2\beta + 1, 3\beta + 1\}$ respectively where β is the image of α in $\mathbf{F}_\lambda$ (since if it were reducible, we would have $\overline{\rho}^{ss}_{f,\lambda} \simeq \epsilon_1 \oplus \epsilon_2$ where both ϵ_1 and ϵ_2 factor through $(\mathbf{Z}/45\mathbf{Z})^\times$). This eventually proves the following proposition.

Proposition 5.1. *Let $(k, N) = (6, 81)$. Then there exists a newform $f \in \mathcal{S}_6(\Gamma_0(81))$ together with a prime ideal λ in $\mathcal{O}$ such that $\overline{\rho}_{f,\lambda}$ is reducible if and only if ℓ belongs to $\{2, 3, 7, 43, 1171\}$.*

5.1.2 Square-free level case

Fix $(k, N) = (4, 11)$. The new subspace in $\mathcal{S}_4(\Gamma_0(11))$ is 2-dimensional and generated by one Galois orbit labeled 11.4a in SAGE ([S$^+$12]). Let f be a representative of this Galois orbit. We have

$$f(\tau) = q + \alpha q^2 + (-4\alpha + 3)q^3 + (2\alpha - 6)q^4 + (8\alpha - 7)q^5 + O(q^5),$$

where α is a root of $X^2 - 2X - 2$. The field $K = \mathbf{Q}(\alpha)$ is the coefficients field of f. According to Theorem 2.3, if $\overline{\rho}_{f,\lambda}$ is reducible then λ has residue characteristic ℓ in the set $\{2, 3, 5, 11, 61\}$. For each prime ℓ in $\{2, 3, 5, 11, 61\}$ we give in Table 4 the smallest prime $p \neq 11, \ell$ and $p \leq 100$ such that the characteristic polynomial of $\overline{\rho}_{f,\lambda}(\mathrm{Frob}_p)$ is irreducible.

Therefore all such Galois representations are irreducible except perhaps $\overline{\rho}_{f,(2\alpha-1)}$ and $\overline{\rho}_{f,(\alpha-9)}$. These latter representations turn out to be reducible and we have

$$\overline{\rho}^{ss}_{f,(2\alpha-1)} \simeq \overline{\chi}_{11} \oplus \overline{\chi}^2_{11} \quad \text{and} \quad \overline{\rho}^{ss}_{f,(\alpha-9)} \simeq \mathbf{1} \oplus \overline{\chi}^3_{61} \simeq \overline{\rho}_{E_4,61}.$$

This eventually proves the following proposition.

ℓ	2	3	5	11		61	
λ	(α)	$(\alpha-1)$	(5)	$(2\alpha-3)$	$(2\alpha-1)$	$(\alpha-9)$	$(\alpha+7)$
p	3	2	2	2	$\emptyset$	$\emptyset$	2

Table 4: Smallest prime $p \neq 11, \ell$ and ≤ 100 such that $\overline{\rho}_{f,\lambda}(\mathrm{Frob}_p)$ acts irreducibly

Proposition 5.2. *Let $(k, N) = (4, 11)$. Then there exists a newform $f \in \mathcal{S}_4(\Gamma_0(11))$ together with a prime ideal λ in $\mathcal{O}$ such that $\overline{\rho}_{f,\lambda}$ is reducible if and only if $\ell = 11$ or $\ell = 61$.*

5.2 Dihedral representation

In this section we discuss an example of dihedral projective representation attached to some specific newform. The new subspace in $\mathcal{S}_2(\Gamma_0(1888))$ has dimension 58 and is split into 16 Galois orbits. Among them let us consider the newform f (up to Galois conjugacy) labeled 1888.10a whose first terms in its Fourier expansion at infinity are

$$f(\tau) = q + \frac{1}{2}\alpha q^3 + \left(-\frac{1}{16}\alpha^4 + \frac{3}{2}\alpha^2 - \alpha - 2\right) q^5 + O(q^6)$$

where α is a root of $X^5+6X^4-20X^3-128X^2+48X+320$. The prime 5 is definitely smaller than the bound given in Thm. 3.1 (namely 3476092007703911714679 in this case) and one proves that there is mod. 5 representation attached to f which has dihedral projective image. Namely, let us consider the prime ideal $\lambda = (5, \alpha/2)$ above 5 in $\mathcal{O}$. Then one checks that the representation $\overline{\rho}_{f,\lambda}$ is isomorphic to $\overline{\rho}_{\mathcal{E},5}$ where $\mathcal{E}$ is the rational CM elliptic curve of conductor 32 given by the equation $y^2 = x^3 - x$. Since $5 \equiv 1 \pmod 4$, one knows by the theory of complex multiplication that $\overline{\rho}_{\mathcal{E},5}$ has image included in the normalizer of a split Cartan subgroup of $\mathrm{GL}(2, \mathbf{F}_5)$. The same conclusion for $\overline{\rho}_{f,\lambda}$ thus follows.

5.3 Projective image isomorphic to A_4, S_4 or A_5

As an illustration of Thm. 4.1, we report here on an example due to Ribet ([Rib97, Rk. 2, p. 283]) and recalled in [KV05, Ex. 3.2, p. 244] (we warn the reader that the term "exceptional" therein refers to a modular representation with projective image isomorphic to A_4, S_4 or A_5). The new subspace in $\mathcal{S}_2(\Gamma_0(23))$ is 2-dimensional and generated by one Galois orbit labeled 23.4a in SAGE with coefficients field $K = \mathbf{Q}(\alpha)$ where α is a root of $X^2 + X - 1$. Let λ be the unique prime ideal above 3 in $\mathcal{O}$. It is shown in *loc. cit.* that the corresponding projective representation has image isomorphic to A_5 and that the field cut out by its kernel is the A_5-extension of $\mathbf{Q}$ given as the splitting field of the polynomial $X^5 + 3X^3 + 6X^2 + 9$.

Several other examples may also be found in *loc. cit.* such as a mod. 19 representation of projective image isomorphic to S_4 attached to the unique cusp form of weight 6, level 4 and trivial Nebentypus. The authors also discuss an effective procedure that given a newform f and a prime ℓ determines whether some mod. ℓ representation attached to f has projective image isomorphic to A_4, S_4 or A_5.

References

[Car86] Henri Carayol. Sur les représentations l-adiques associées aux formes modulaires de Hilbert. *Ann. Sci. École Norm. Sup. (4)*, 19(3):409–468, 1986.

[Car89] Henri Carayol. Sur les représentations galoisiennes modulo l attachées aux formes modulaires. *Duke Math. J.*, 59(3):785–801, 1989.

[Coh07] Henri Cohen. *Number theory. Vol. I. Tools and Diophantine equations*, volume 239 of *Graduate Texts in Mathematics*. Springer, New York, 2007.

[Coj05] Alina Carmen Cojocaru. On the surjectivity of the Galois representations associated to non-CM elliptic curves. *Canad. Math. Bull.*, 48(1):16–31, 2005. With an appendix by Ernst Kani.

[DS05] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[Die12] Luis V. Dieulefait. Automorphy of $\mathrm{Symm}^5(\mathrm{GL}(2))$ and base change. *arXiv:1208.3946*, 2012.

[Edi92] Bas Edixhoven. The weight in Serre's conjectures on modular forms. *Invent. Math.*, 109(3):563–594, 1992.

[GP] Eknath Ghate and Pierre Parent. On uniform large Galois images for modular abelian varieties. *Bull. London Math. Soc. (to appear).*

[Iwa97] Henryk Iwaniec. *Topics in classical automorphic forms*, volume 17 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.

[Kat73] Nicholas M. Katz. p-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 69–190. Lecture Notes in Mathematics, Vol. 350. Springer, Berlin, 1973.

[Kat77] Nicholas M. Katz. A result on modular forms in characteristic p. In *Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, pages 53–61. Lecture Notes in Math., Vol. 601. Springer, Berlin, 1977.

[KV05] Ian Kiming and Helena A. Verrill. On modular mod l Galois representations with exceptional images. *J. Number Theory*, 110(2):236–266, 2005.

[Kra95] Alain Kraus. Une remarque sur les points de torsion des courbes elliptiques. *C. R. Acad. Sci. Paris Sér. I Math.*, 321(9):1143–1146, 1995.

[Liv89] Ron Livné. On the conductors of mod l Galois representations coming from modular forms. *J. Number Theory*, 31(2):133–141, 1989.

[LW12] David Loeffler and Jared Weinstein. On the computation of local components of a newform. *Math. Comp.*, 81(278):1179–1200, 2012.

[Mar05] Greg Martin. Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$. *J. Number Theory*, 112(2):298–331, 2005.

[Maz77] Barry Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, (47):33–186 (1978), 1977.

[MS76] Barry Mazur and Jean-Pierre Serre. Points rationnels des courbes modulaires $X_0(N)$ (d'après A. Ogg). In *Séminaire Bourbaki (1974/1975), Exp. No. 469*, pages 238–255. Lecture Notes in Math., Vol. 514. Springer, Berlin, 1976.

[Miy06] Toshitsune Miyake. *Modular forms*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2006. Translated from the 1976 Japanese original by Yoshitaka Maeda.

[Mur97] M. Ram Murty. Congruences between modular forms. In *Analytic number theory (Kyoto, 1996)*, volume 247 of *London Math. Soc. Lecture Note Ser.*, pages 309–320. Cambridge Univ. Press, Cambridge, 1997.

[Neu99] Jürgen Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[Rib75] Kenneth A. Ribet. On l-adic representations attached to modular forms. *Invent. Math.*, 28:245–275, 1975.

[Rib77] Kenneth A. Ribet. Galois representations attached to eigenforms with Nebentypus. In *Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, pages 17–51. Lecture Notes in Math., Vol. 601. Springer, Berlin, 1977.

[Rib85] Kenneth A. Ribet. On l-adic representations attached to modular forms. II. *Glasgow Math. J.*, 27:185–194, 1985.

[Rib97] Kenneth A. Ribet. Images of semistable Galois representations. *Pacific J. Math.*, (Special Issue):277–297, 1997. Olga Taussky-Todd: in memoriam.

[Rib10] Kenneth A. Ribet. Non-optimal levels of mod l reducible Galois representations or Modularity of residually reducible representations. July 9, 2010. Notes of a talk given at the Centre de Recerca Matemàtica (Barcelona).

[RS62] J. Barkley Rosser and Lowell Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6:64–94, 1962.

[Sch74] Bruno Schoeneberg. *Elliptic modular functions: an introduction*. Springer-Verlag, 1974. Translated from the German by J. R. Smart and E. A. Schwandt; Die Grundlehren der mathematischen Wissenschaften, Band 203.

[Ser72] Jean-Pierre Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972.

[Ser73] Jean-Pierre Serre. Congruences et formes modulaires [d'après H. P. F. Swinnerton-Dyer]. In *Séminaire Bourbaki, 24e année (1971/1972), Exp. No. 416*, pages 319–338. Lecture Notes in Math., Vol. 317. Springer, Berlin, 1973.

[Shi94] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1.

[S+12] W. A. Stein et al. *Sage Mathematics Software (Version 5.2)*. The Sage Development Team, 2012. `http://www.sagemath.org`.

[SD73] H. P. F. Swinnerton-Dyer. On l-adic representations and congruences for coefficients of modular forms. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972)*, pages 1–55. Lecture Notes in Math., Vol. 350. Springer, Berlin, 1973.

[Tsa12] Panagiotis Tsaknias. A possible generalization of Maeda's conjecture. *preprint, arXiv:1205.3420*, 2012.

[Was97] Lawrence C. Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.