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HEAT FLOW IN A PERIODICALLY FORCED, UNPINNED THERMOSTATED CHAIN

TOMASZ KOMOROWSKI, STEFANO OLLA, AND MARIELLE SIMON

ABSTRACT. We prove the hydrodynamic limit for a one-dimensional harmonic chain of interacting atoms with a random flip of the momentum sign. The system is open: at the left boundary it is attached to a heat bath at temperature T_- , while at the right endpoint it is subject to an action of a force which reads as $\bar{F} + \frac{1}{\sqrt{n}}\tilde{\mathcal{F}}(n^2t)$, where $\bar{F} \geq 0$ and $\tilde{\mathcal{F}}(t)$ is a periodic function. Here n is the size of the microscopic system. Under a diffusive scaling of space-time, we prove that the empirical profiles of the two locally conserved quantities – the volume stretch and the energy – converge, as $n \rightarrow +\infty$, to the solution of a non-linear diffusive system of conservative partial differential equations with a Dirichlet type and Neumann boundary conditions on the left and the right endpoints, respectively.

1. INTRODUCTION

In the thermodynamics theory of energy transport there is an important distinction between the *mechanical work* done into the system and the *heat*, *i.e.* the thermal energy exchanged with the external heat bath. Our goal is to investigate the emergence of such a difference from a microscopic dynamics where the energy transport is diffusive. On the microscopic scale we make a distinction between the *mechanical energy*, that is concentrated on the long waves (low modes), and the thermal energy, distributed on the short waves (high modes). We consider a one-dimensional finite system where the bulk of the dynamics has a chaotic mechanism (in the present case a random flip of the sign of the velocities) which transforms the energy from the low modes, corresponding to the mechanical energy transport, to the higher ones, transporting the thermal energy. At the two boundaries of the system the energy is exchanged with various mechanisms. On the left endpoint there is a heat bath at temperature T_- , modelled by a Langevin random dynamics. On the right, a time dependent deterministic force is applied. This forcing is periodic and contains different frequencies: *low* frequencies whose corresponding work contributes to the exchange of the mechanical energy, while the higher frequencies exchange the thermal energy, but in a very different way with respect to the action of a heat bath.

More precisely, the bulk dynamics is given by an *unpinned* chain of harmonic oscillators perturbed by the *velocity-flip* stochastic mechanism: each particle flips the velocity sign at independent exponential times. The energy is conserved in the bulk, but since the chain is unpinned and the random mechanism acts only on the velocities, also the volume is conserved. After a diffusing scaling of space and time, the macroscopic energy density and the volume stretch at the point $u \in [0, 1]$ at time $t \geq 0$, denoted by $e(t, u)$ and $r(t, u)$, respectively, evolve in the bulk according

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to the diffusive system:

$$\begin{aligned}\partial_t r(t, u) &= \frac{1}{2\gamma} \partial_{uu} r(t, u), \\ \partial_t e(t, u) &= \frac{1}{4\gamma} \partial_{uu} \left(e(t, u) + \frac{1}{2} r^2(t, u) \right).\end{aligned}\tag{1.1}$$

To the solution of (1.1) one can assign the corresponding evolution of the temperature profile (or twice the thermal energy) defined by $T(t, u) = e(t, u) - \frac{1}{2} r^2(t, u)$, which is given by

$$\partial_t T(t, u) = \frac{1}{4\gamma} \partial_{uu} T(t, u) + \frac{1}{2\gamma} \left(\partial_u r(t, u) \right)^2.\tag{1.2}$$

The diffusive system (1.1) or (1.2) is a special case of the general diffusive evolution for systems with multiple conserved quantities (see [Oll19]).

The heat bath (at the left endpoint) and the work performed upon the system (at the right endpoint) affect the macroscopic boundary conditions for the equations (1.1) and (1.2). Since the chain is unpinned, we have $r(t, 0) = 0$. Besides, the heat bath fixes the temperature at $u = 0$, resulting in the Dirichlet boundary condition: $T(t, 0) = e(t, 0) = T_-$. The effect of the forcing on the other boundary is more complex, and constitutes the main result of the paper. As we have already mentioned the forcing acting on the right endpoint is time dependent and periodic. It has a slow part varying on the macroscopic time scale, and a fast part evolving on the microscopic scale, namely it reads:

$$\mathcal{F}_n(t) = \overline{F}(t) + \frac{1}{\sqrt{n}} \tilde{\mathcal{F}}(n^2 t).\tag{1.3}$$

Here $\tilde{\mathcal{F}}(t)$ is a smooth periodic function of period θ and null average. The factor $1/\sqrt{n}$ is a necessary scaling in order to have a finite effect on the macroscopic time scale. In what follows we shall assume, with no loss of generality, that \overline{F} is constant in time. This establishes the boundary condition for the stretch: $r(t, 1) = \overline{F}$.

Moreover, the total work done by the force at the macroscopic time t is given by

$$W(t) := \frac{\overline{F}}{2\gamma} \int_0^t (\partial_u r)(s, 1) ds + \mathbb{W}^Q t,\tag{1.4}$$

where $r(t, u)$ is the solution of (1.1) with the boundary conditions $r(t, 0) = 0$, $r(t, 1) = \overline{F}$, and \mathbb{W}^Q is the contribution coming from the fast fluctuating part $\tilde{\mathcal{F}}$ and it is expressed by (2.26). This yields the boundary conditions for the energy evolution, or the temperature profile $T(t, u)$, at $u = 1$:

$$\partial_u e(t, 1) = W'(t) = \frac{\overline{F}}{2\gamma} \partial_u r(t, 1) + \mathbb{W}^Q, \quad \text{or} \quad \partial_u T(t, 1) = \mathbb{W}^Q.\tag{1.5}$$

In other words, $\frac{\overline{F}}{2\gamma} \partial_u r(t, 1)$ is the contribution to the mechanical energy, while \mathbb{W}^Q is the contribution to the thermal energy (or heat), due to the work performed by the fluctuating part of the force. According to (1.2), the mechanical energy is transformed, in the bulk, into the thermal one at the rate $\frac{1}{2\gamma} (\partial_u r(t, u))^2$.

The result discussed above has been announced (without proof) in [KLOS23]. The present paper contains its rigorous proof, under certain condition on the initial distribution of the system. The usual condition on the initial probability distribution is that the relative entropy with respect to the equilibrium distribution is bounded by the size n of the system, see Assumption 2.7. We need to supplement it with an assumption about the distribution of the thermal component of the potential energy,

which states that the latter should not concentrate too much on the lower modes, see Assumption 2.8. This hypothesis does not follow from the usual entropy bound, see Remark 2.9. It is not clear to us whether the assumption is indeed necessary for the proof of the diffusive hydrodynamic limit in case of more conserved quantities, besides the energy. On the other hand, the hypothesis is satisfied by *local Gibbs measures*, which are natural initial conditions, see Remark 2.10.

Dynamics of harmonic oscillators perturbed by a conservative noise has recently been a subject of intense study by many authors, see [BBJ+16] and the references therein. In particular, a chain of unpinned harmonic oscillators with a stochastic perturbation not conserving momentum has *two* conserved quantities evolving in the same diffusive time scale (see [Oll19] and the references therein). The hydrodynamic limit has been studied in [Ber07] and [KOS18] in the case of *periodic* conditions, with two different types of noises. Open system with Langevin heat baths attached to the boundaries have been studied in [BO05, KOS20, KOS21]. In [KOS21] we have considered the case when heat baths at different temperatures are attached at the boundaries, while only a constant forcing \overline{F} is present at the right endpoint. In that situation, the presence of the heat bath at temperature T_+ at the same right point imposes a local equilibrium, with temperature T_+ . Consequently, in [KOS21] the boundary condition is of the *Dirichlet type*, precisely $e(t, 1) = \frac{\overline{F}^2}{2\gamma} + T_+$. The situation considered here is quite different. The temperature at the right extremity of the chain is not fixed and this results in the emergence of the inhomogeneous *Neumann* boundary condition given in (1.5). The case of a *pinned* chain in presence of the periodic forcing has been studied in [KLO23a] (stationary state) and [KLO23b] (nonstationary initial condition). The pinning destroys the translation invariance of the system and only the energy keeps conserved by the dynamics. Consequently, there is no mechanical component of the energy. The only work that affects the system is performed by the fluctuating force with the microscopic time period.

We now outline the contents of the paper. In Section 2 we formulate the model and present the main results, that are shown in the following sections. In Sections 3–5 we prove the results concerning the evolution of the mechanical energy and the work performed by the forcing. In Sections 6–9 we consider the evolution of the thermal energy. Similarly to [KLO23b], in this part our strategy is based on the full resolution of the covariance matrix of the momenta and stretches of all particles of the chain. It should be noted that this method has been used for the first time in [RLL04] to describe the energy distribution in the non-equilibrium stationary state in a harmonic crystal. Due to the fact that in that case there is no randomness in the bulk present, the total energy grows proportionally to the size of the system. Contrary to the case considered in [KLO23b], the spectrum of the unpinned chain does not have a spectral gap. As a result the resolution of the covariance matrix is technically much more challenging here. It results in the already mentioned additional Assumption 2.7 on the initial distribution. In particular we need to show that this hypothesis is maintained at any positive macroscopic time, see Corollary 8.9. Finally, in the Appendix section we present some technical results concerning the spectral analysis of discrete gradient and divergence operators, as well as some auxiliary facts dealing with the resolution of the aforementioned covariance matrix.

2. DEFINITION OF THE DYNAMICS AND RESULTS

2.1. Description of the model. The configuration of the system is described by

$$(\mathbf{r}, \mathbf{p}) = (r_1, \dots, r_n, p_0, \dots, p_n) \in \Omega_n := \mathbb{R}^n \times \mathbb{R}^{n+1}, \quad (2.1)$$

where $\mathbf{r} = (r_1, \dots, r_n)$ and $\mathbf{p} = (p_0, \dots, p_n)$ correspond to the inter-particle stretches and particle momenta. The total energy of the chain is given by the Hamiltonian:

$$\mathcal{H}_n(\mathbf{r}, \mathbf{p}) := \sum_{x=0}^n \mathcal{E}_x(\mathbf{r}, \mathbf{p}), \quad (2.2)$$

where the microscopic energy per atom at any $x \in \{0, \dots, n\}$ is given by

$$\mathcal{E}_x(\mathbf{r}, \mathbf{p}) := \frac{p_x^2}{2} + \frac{r_x^2}{2} \quad (2.3)$$

with the convention here and in the following that $r_0 := 0, r_{n+1} := 0$.

The microscopic dynamics of the stretch/momenta process $\{(\mathbf{r}(t), \mathbf{p}(t))\}_{t \geq 0}$ is given in the bulk by:

$$\begin{aligned} dr_x(t) &= (p_x(t) - p_{x-1}(t))dt, \\ dp_x(t) &= (r_{x+1}(t) - r_x(t))dt - 2p_x(t^-)dN_x(\gamma t), \quad x = 1, \dots, n-1. \end{aligned} \quad (2.4)$$

The atom labelled $x = 0$ is in contact with a Langevin thermostat at temperature T_- , while atom $x = n$ is subject to a time-dependent force. Therefore at both boundaries we have

$$\begin{aligned} dp_0(t) &= r_1(t)dt - 2\gamma p_0(t)dt + \sqrt{4\gamma T_-}dw_-(t) \\ dr_n(t) &= (p_n(t) - p_{n-1}(t))dt, \\ dp_n(t) &= (\mathcal{F}_n(t) - r_n(t))dt - 2p_n(t^-)dN_n(\gamma t). \end{aligned} \quad (2.5)$$

Hereinabove, $\{N_x(t); x = 1, \dots, n\}_{t \geq 0}$ are independent Poisson processes of intensity 1, while $\{w_-(t)\}_{t \geq 0}$ is a standard one dimensional Wiener process, independent of the Poisson processes. These processes are defined over a certain probability space $(\Xi, \mathfrak{F}, \mathbb{P})$. Moreover, the parameter $\gamma > 0$ regulates the intensity of *both* the random perturbations and the Langevin thermostat¹, while $T_- > 0$ is the thermostat temperature. Finally, the time-dependent force $\mathcal{F}_n(t)$ is assumed to be a smooth, θ -periodic, with $\theta > 0$, and it is of the form

$$\mathcal{F}_n(t) = \bar{F} + \tilde{\mathcal{F}}_n(t), \quad \text{where} \quad \tilde{\mathcal{F}}_n(t) := \frac{1}{\sqrt{n}} \sum_{\substack{\ell \in \mathbb{Z} \\ \ell \neq 0}} \hat{\mathcal{F}}(\ell) e^{i\ell\omega t}, \quad (2.6)$$

with $\omega = \frac{2\pi}{\theta}$. We suppose moreover that $\mathcal{F}_n(t)$ is real valued, so $\hat{\mathcal{F}}(-\ell) = \hat{\mathcal{F}}^*(\ell)$, and that the decay of the Fourier coefficient is sufficiently fast (*e.g.* exponential) so that \mathcal{F}_n is continuous and also

$$C_{\mathcal{F}} := \sum_{\ell \neq 0} |\hat{\mathcal{F}}(\ell)| < +\infty. \quad (2.7)$$

Throughout the paper, we will shorten notation by introducing the discrete *gradient* and *divergence* operators $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ and $\nabla^* : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ acting as follows (see Section A.2 of Appendix for some properties of these operators)

$$\begin{aligned} \nabla g_x &:= g_{x+1} - g_x, & \text{for } g = (g_1, \dots, g_n) \in \mathbb{R}^n, \\ \nabla^* f_x &:= f_x - f_{x-1}, & \text{for } f = (f_0, \dots, f_n) \in \mathbb{R}^{n+1}, \end{aligned} \quad (2.8)$$

with the convention $g_0 = g_{n+1} = 0$ and $f_{-1} = f_0$.

¹We have chosen the same parameter in order to simplify notations, but it does not affect the results concerning the macroscopic properties of the dynamics.

Note that $\{(\mathbf{r}(t), \mathbf{p}(t))\}_{t \geq 0}$ is a Markov process on Ω_n with the time-dependent generator \mathcal{G}_t which we decompose in three parts as follows:

$$\mathcal{G}_t := \mathcal{A}_t + \gamma S_{\text{flip}} + 2\gamma S_-, \quad (2.9)$$

where

$$\mathcal{A}_t := \sum_{x=1}^n \nabla^* p_x \partial_{r_x} + \sum_{x=0}^n \nabla r_x \partial_{p_x} + \mathcal{F}_n(t) \partial_{p_n}. \quad (2.10)$$

In addition, for any $f : \Omega_n \rightarrow \mathbb{R}$ bounded and measurable function,

$$S_{\text{flip}} f(\mathbf{r}, \mathbf{p}) := \sum_{x=1}^n \left(f(\mathbf{r}, \mathbf{p}^x) - f(\mathbf{r}, \mathbf{p}) \right), \quad (2.11)$$

where \mathbf{p}^x is the velocity configuration with sign flipped at the x component, *i.e.* $\mathbf{p}^x = (p'_0, \dots, p'_n)$, with $p'_y = p_y$, for any $y \neq x$ and $p'_x = -p_x$. Furthermore,

$$S_- := T_- \partial_{p_0}^2 - p_0 \partial_{p_0}. \quad (2.12)$$

One can easily verify that the microscopic *energy currents* $\{j_{x,x+1}; x = -1, \dots, n\}$ which satisfy:

$$\mathcal{G}_t \mathcal{E}_x = j_{x-1,x} - j_{x,x+1}, \quad \text{for any } x = 0, \dots, n, \quad (2.13)$$

are given by

$$j_{x,x+1}(t) = \begin{cases} -p_x(t) r_{x+1}(t) & \text{if } x = 0, \dots, n-1, \\ 2\gamma (T_- - p_0^2(t)) & \text{if } x = -1, \\ -\mathcal{F}_n(t) p_n(t) & \text{if } x = n. \end{cases} \quad (2.14)$$

We assume that the initial distribution of stretches and momenta $(\mathbf{r}(0), \mathbf{p}(0))$ in Ω_n is random and distributed according to a probability measure μ_n . We then denote by $\mu_n(t)$ the probability measure on Ω_n of the configuration $(\mathbf{r}(t), \mathbf{p}(t))$ evolving according to (2.4)–(2.5). Finally we denote by \mathbb{E}_{μ_n} the expectation with respect to the probability measure $\mathbb{P}_{\mu_n} := \mu_n \otimes \mathbb{P}$.

We decompose the configurations as

$$r_x(t) := r'_x(t) + \bar{r}_x(t), \quad p_x(t) := p'_x(t) + \bar{p}_x(t), \quad (2.15)$$

where

$$\begin{aligned} \bar{\mathbf{r}}(t) &= (\bar{r}_1(t), \dots, \bar{r}_n(t)) := \mathbb{E}_{\mu_n}[\mathbf{r}(t)], \\ \bar{\mathbf{p}}(t) &= (\bar{p}_0(t), \dots, \bar{p}_n(t)) := \mathbb{E}_{\mu_n}[\mathbf{p}(t)], \end{aligned} \quad (2.16)$$

while $\mathbf{r}'(t), \mathbf{p}'(t)$ corresponds to the *fluctuating parts* of the dynamics. We adopt the convention that $r_0(t) = \bar{r}_0(t) = r'_0(t) \equiv 0$. Finally, for any measurable $f : \Omega_n \rightarrow \mathbb{R}$ and $t > 0$, we introduce the following time average in the diffusive scale:

$$\langle\langle f \rangle\rangle_t := \frac{1}{t} \int_0^t \mathbb{E}_{\mu_n}[f(\mathbf{r}(n^2 s), \mathbf{p}(n^2 s))] ds, \quad (2.17)$$

provided that the integral on the right hand side makes sense.

2.2. Formulation of results. Let us first formulate two main assumptions.

Assumption 2.1 (Initial bound on the averages). *We assume that there exists $\bar{\mathcal{H}} > 0$ such that*

$$\frac{1}{n} \sum_{x=0}^n (\bar{r}_x^2(0) + \bar{p}_x^2(0)) \leq \bar{\mathcal{H}}, \quad n = 1, 2, \dots \quad (2.18)$$

Furthermore, initially, the average profile of stretches approximates a continuous function – the *initial stretch profile* $r_0(\cdot)$. Namely:

Assumption 2.2 (Initial stretch profile). *We assume that there exists a continuous function $r_0 : [0, 1] \rightarrow \mathbb{R}$ such that*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=1}^n \left(\bar{r}_x(0) - r_0\left(\frac{x}{n}\right) \right)^2 = 0. \quad (2.19)$$

2.2.1. *Macroscopic evolution of stretch and mechanical energy.* Let us define the microscopic mechanical energy as

$$\mathcal{E}_x^{\text{mech}}(t) := \frac{1}{2} \left(\bar{p}_x^2(t) + \bar{r}_x^2(t) \right), \quad x = 0, \dots, n. \quad (2.20)$$

Under the assumptions formulated in the foregoing we obtain the following macroscopic limits:

Theorem 2.3 (Limit of stretch and mechanical energy). *Assume Assumptions 2.1 and 2.2. Then, for any continuous test function $\varphi : [0, 1] \rightarrow \mathbb{R}$,*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=1}^n \varphi\left(\frac{x}{n}\right) \bar{r}_x(n^2 t) = \int_0^1 r(t, u) \varphi(u) \, du \quad (2.21)$$

and,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=0}^n \varphi\left(\frac{x}{n}\right) \mathcal{E}_x^{\text{mech}}(n^2 t) = \int_0^1 \varphi(t, u) \frac{1}{2} r^2(t, u) \, du, \quad (2.22)$$

where $r(t, u)$ is the solution of the following heat equation with the Cauchy-Dirichlet boundary condition:

$$\begin{aligned} \partial_t r(t, u) &= \frac{1}{2\gamma} \partial_{uu} r(t, u), & (t, u) &\in \mathbb{R}_+ \times (0, 1), \\ r(t, 0) &= 0, & r(t, 1) &= \bar{F}, & r(0, u) &= r_0(u). \end{aligned} \quad (2.23)$$

Theorem 2.3 is proved in Section 4.

2.2.2. *Macroscopic work.* Let $W_n(t)$ be the average work done by the force in the diffusive time scale, namely:

$$W_n(t) = \frac{1}{n} \int_0^{tn^2} \mathcal{F}_n(s) \bar{p}_n(s) \, ds. \quad (2.24)$$

Observe that, with our notation (2.17), we have $W_n(t) = -nt \langle j_{n, n+1} \rangle_t$.

Theorem 2.4. *Under Assumptions 2.1 and 2.2 we have, for any $t > 0$,*

$$\lim_{n \rightarrow +\infty} W_n(t) = W(t) := \frac{\bar{F}}{2\gamma} \int_0^t (\partial_{ur})(s, 1) \, ds + \mathbb{W}^Q t, \quad (2.25)$$

with

$$\mathbb{W}^Q = \sum_{\ell=1}^{+\infty} |\widehat{\mathcal{F}}(\ell)|^2 (\ell\omega) \operatorname{Re} \sqrt{\frac{4}{(\ell\omega)^2 - i2\gamma\ell\omega} - 1}. \quad (2.26)$$

Here $w \mapsto \sqrt{w}$ is the branch of the inverse of $z \mapsto z^2$ such that $\operatorname{Re} \sqrt{w} > 0$, when $w \in \mathbb{C} \setminus (-\infty, 0]$.

The proof of Theorem 2.4 is given in Section 5.

Remark 2.5. *Since $(\partial_{ur})(s, 1) \rightarrow \bar{F}$, as $s \rightarrow \infty$, the macroscopic work satisfies*

$$\lim_{t \rightarrow +\infty} \frac{W(t)}{t} = \frac{\bar{F}^2}{2\gamma} + \mathbb{W}^Q.$$

Remark 2.6. Notice that when $\gamma \rightarrow 0$ we have

$$\mathbb{W}^Q \longrightarrow \sum_{\ell=1}^{\lfloor \frac{2}{\omega} \rfloor} |\widehat{\mathcal{F}}(\ell)|^2 \sqrt{4 - (\ell\omega)^2},$$

in agreement with the calculations for the deterministic dynamics (i.e. $\gamma = 0$) in [GKLO24].

2.2.3. The macroscopic limit of the energy functional. In order to obtain the macroscopic energy profile, we need two additional assumptions: 1) on the initial *entropy* and 2) on the initial *stretch fluctuations*.

In order to introduce them, let us define $\nu_{T_-}(\mathbf{dr}, \mathbf{dp})$ as the product Gaussian measure on Ω_n of zero average and variance $T_- > 0$ given by

$$\nu_{T_-}(\mathbf{dr}, \mathbf{dp}) := g_{T_-}(\mathbf{r}, \mathbf{p}) \mathbf{dr} \mathbf{dp} \quad \text{where } g_{T_-}(\mathbf{r}, \mathbf{p}) = \frac{e^{-\mathcal{E}_0/T_-}}{\sqrt{2\pi T_-}} \prod_{x=1}^n \frac{e^{-\mathcal{E}_x/T_-}}{2\pi T_-}. \quad (2.27)$$

Let $f_n(t, \mathbf{r}, \mathbf{p})$ be the density of $\mu_n(t)$ with respect to ν_{T_-} . We can now define the relative entropy of $\mu_n(t)$ with respect to ν_{T_-} as

$$\mathbf{H}_n(t) := \int_{\Omega_n} f_n(t) \log f_n(t) d\nu_{T_-}. \quad (2.28)$$

Assumption 2.7. We assume that the initial measure μ_n is such that $f_n(0)$ is of the C^2 class of regularity on Ω_n , and there exists $C > 0$ such that,

$$\mathbf{H}_n(0) \leq Cn, \quad n = 1, 2, \dots \quad (2.29)$$

Assumption 2.7 implies the following initial bound on the energy: there exists $C > 0$ such that

$$\mathbb{E}_{\mu_n}[\mathcal{H}_n(0)] \leq Cn, \quad n = 1, 2, \dots \quad (2.30)$$

Here $\mathcal{H}_n(t) = \mathcal{H}_n(\mathbf{r}(t), \mathbf{p}(t))$ (recall (2.2)). Indeed, the entropy inequality, see e.g. [KL13, p. 338], gives

$$\begin{aligned} \mathbb{E}_{\mu_n}[\mathcal{H}_n(0)] &= \int_{\Omega_n} \left(\sum_{x=0}^n \mathcal{E}_x \right) f_n(0) d\nu_{T_-} \\ &\leq \frac{1}{\alpha} \left\{ \log \left(\int_{\Omega_n} \exp \left\{ \frac{\alpha}{2} \sum_{x=0}^n (p_x^2 + r_x^2) \right\} d\nu_{T_-} \right) + \mathbf{H}_n(0) \right\} \end{aligned} \quad (2.31)$$

for any $t \geq 0$ and $\alpha > 0$. Hence, we obtain (2.30) choosing $\alpha \in (0, T_-^{-1})$.

We need a further assumption on the initial condition, that involves a weak decorrelations of the initial stretches $r_x(0)$. Define (recall (2.15))

$$q'_x(t) = \sum_{y=1}^x r'_y(t), \quad x = 1, \dots, n \quad \text{and} \quad q'_0(t) \equiv 0. \quad (2.32)$$

Assumption 2.8. We assume that

$$\lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{x=1}^n \mathbb{E}_{\mu_n} \left[(q'_x(0))^2 \right] = 0. \quad (2.33)$$

Remark 2.9. Assumption 2.8 implies that the potential energy should not concentrate too much on the lower thermal modes (this will become more precise in the estimate (8.42) below). The entropy bound (2.29) by itself does not prevent such concentration. In fact finiteness of the initial energy (that follows from (2.29)) implies only that

$$\sup_{n \geq 1} \frac{1}{n^3} \sum_{x=1}^n \mathbb{E}_{\mu_n} \left[(q'_x(0))^2 \right] < +\infty.$$

We will need to prove that (2.33) is maintained by the dynamics at all times. This is the goal of Proposition 8.8 below.

Remark 2.10. Note that Assumption 2.8 is satisfied by the local Gibbs measures, whose density (with respect to the Lebesgue measure) takes the form

$$\frac{e^{-\mathcal{E}_0/T_-}}{\sqrt{2\pi T_-}} \prod_{x=1}^n \frac{e^{-\mathcal{E}_x/T_x}}{2\pi T_x}, \quad (2.34)$$

where $(T_x)_{x \in \mathbb{Z}}$ is a bounded sequence of positive numbers. Indeed, under this distribution $r'_1(0), \dots, r'_n(0)$ are independent, centered Gaussian random variables, and $\mathbb{E}[q'_x(0)^2] = \sum_{y=1}^x T_y \leq n \max_x T_x$.

The macroscopic behavior of the energy profile is given by the following:

Theorem 2.11 (Limit of total energy). *Suppose that Assumptions 2.1, 2.2, 2.7, and 2.8 are in force. Assume furthermore that there exists a continuous function (the initial energy profile) $e_0 : [0, 1] \rightarrow (0, +\infty)$ such that, for any continuous test function $\varphi : [0, 1] \rightarrow \mathbb{R}$,*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=0}^n \varphi\left(\frac{x}{n}\right) \mathbb{E}_{\mu_n}[\mathcal{E}_x(0)] = \int_0^1 e_0(u) \varphi(u) du. \quad (2.35)$$

Then for any continuous $\varphi : [0, 1] \rightarrow \mathbb{R}$ and any $t \geq 0$,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=0}^n \varphi\left(\frac{x}{n}\right) \mathbb{E}_{\mu_n}[\mathcal{E}_x(n^2 t)] = \int_0^1 e(t, u) \varphi(u) du, \quad (2.36)$$

where $e(t, u)$ is the solution of the initial-boundary value problem

$$\begin{aligned} \partial_t e(t, u) &= \frac{1}{4\gamma} \partial_{uu} \left(e(t, u) + \frac{1}{2} r^2(t, u) \right), & (t, u) \in \mathbb{R}_+ \times (0, 1), \\ e(t, 0) &= T_-, & \partial_u e(t, 1) = W'(t) = \frac{\overline{F}}{2\gamma} \partial_u r(t, 1) + \mathbb{W}^Q, & e(0, u) = e_0(u), \end{aligned} \quad (2.37)$$

and $W(t)$ is the macroscopic work given in Theorem 2.4, while $r(t, u)$ is the macroscopic stretch profile as in the statement of Theorem 2.3.

We conclude this section by stating a last result on the macroscopic *thermal energy* behavior. Let us define the macroscopic *temperature* (or thermal energy) profile $T(t, u) := e(t, u) - \frac{1}{2} r^2(t, u)$. From (2.23) and (2.37) it satisfies the following initial-boundary value problem

$$\begin{aligned} \partial_t T(t, u) &= \frac{1}{4\gamma} \partial_{uu} T(t, u) + \frac{1}{2\gamma} (\partial_u r(t, u))^2, & (t, u) \in \mathbb{R}_+ \times (0, 1), \\ T(t, 0) &= T_-, & \partial_u T(t, 1) = \mathbb{W}^Q, & T(0, u) = T_0(u). \end{aligned} \quad (2.38)$$

Remark 2.12. The quantity $\frac{1}{2\gamma} (\partial_u r(t, u))^2$ appearing in (2.38) represents the local rate of conversion of the mechanical energy into the thermal one, i.e. a gradient of the volume stretch generates a local increase of the thermal component of the energy.

Let us now define, for any $t \geq 0$ and $x = 0, \dots, n$,

$$\mathcal{E}'_x(t) := \frac{(r'_x(t))^2}{2} + \frac{(p'_x(t))^2}{2}. \quad (2.39)$$

Theorem 2.13 (The limit of thermal energy and equipartition). *For any continuous test function $\varphi : [0, 1] \rightarrow \mathbb{R}$ and any $t \geq 0$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=0}^n \varphi\left(\frac{x}{n}\right) \mathbb{E}_{\mu_n}[\mathcal{E}'_x(n^2t)] = \int_0^1 T(t, u) \varphi(u) du. \quad (2.40)$$

In addition, for any compactly supported, continuous function $\Phi : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=0}^n \int_{\mathbb{R}_+} \Phi\left(t, \frac{x}{n}\right) \mathbb{E}_{\mu_n}[p_x^2(n^2t)] dt \\ = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=0}^n \int_{\mathbb{R}_+} \Phi\left(t, \frac{x}{n}\right) \mathbb{E}_{\mu_n}[\mathcal{E}'_x(n^2t)] dt. \end{aligned} \quad (2.41)$$

Both Theorems 2.11 and 2.13 will be proved in Section 7 *modulo* several technical results which will be shown in Sections 8 and 9.

3. EVOLUTION OF THE AVERAGES IN DIFFUSIVE SCALING

This section contains some results on the *averages* $(\bar{r}(t), \bar{p}(t))$. They are obtained thanks to an explicit resolution of the system of equations satisfied by the averages. We start with the following:

Proposition 3.1. *We assume Assumption 2.1. There exists a constant $C = C(\gamma) > 0$ such that, for any $n \geq 1$, any $t \geq 0$, we have*

(i) (Control of ℓ^2 norms)

$$\sum_{x=0}^n \bar{p}_x^2(t) \leq Cn, \quad \text{and} \quad \sum_{x=0}^n \int_0^t \bar{p}_x^2(n^2s) ds \leq \frac{C(t+1)}{n}, \quad (3.1)$$

and

$$\sum_{x=1}^n \bar{r}_x^2(t) \leq Cn. \quad (3.2)$$

(ii) (Boundary behavior)

$$\left| \int_0^t \bar{p}_0(n^2s) ds \right| \leq \frac{C(t+1)}{n}, \quad \left| \int_0^t \bar{p}_n(n^2s) ds \right| \leq \frac{C(t+1)}{n}, \quad (3.3)$$

and furthermore, for any $t > 0$

$$\lim_{n \rightarrow \infty} \langle r_n \rangle_t = \lim_{n \rightarrow \infty} \frac{1}{t} \int_0^t \bar{r}_n(n^2s) ds = \bar{F}. \quad (3.4)$$

The proofs of (3.1) and (3.2) are presented in Sections 3.2 and 3.3, respectively. The proof of (3.3) and (3.4) is given in Section 3.4. We start with some preliminaries. The strategy of the proof consists in solving explicitly the dynamics satisfied by the averages in terms of Fourier transforms. Indeed, from (2.4)–(2.5) we get

$$\begin{aligned} \frac{d\bar{r}_x(t)}{dt} &= \nabla^* \bar{p}_x(t), \quad x = 1, \dots, n, \\ \frac{d\bar{p}_x(t)}{dt} &= \nabla \bar{r}_x(t) - 2\gamma \bar{p}_x(t) + \delta_{x,n} \mathcal{F}_n(t), \quad x = 0, \dots, n, \end{aligned} \quad (3.5)$$

with the convention $\bar{r}_0 \equiv 0$ and $\bar{r}_{n+1} \equiv 0$. We denote by $\delta_{x,y}$ the Kronecker delta function, namely $\delta_{x,y} = 0$, if $x \neq y$ and $\delta_{x,x} = 1$. Its generator can be written in the form

$$\bar{\mathcal{G}}_t := \bar{\mathcal{A}}_t + \gamma \bar{\mathcal{S}} \quad (3.6)$$

where

$$\bar{A}_t := \sum_{x=1}^n \nabla^* \bar{p}_x \partial_{\bar{r}_x} + \sum_{x=0}^n \nabla \bar{r}_x \partial_{\bar{p}_x} + \mathcal{F}_n(t) \partial_{\bar{p}_n}, \quad \bar{S} := -2 \sum_{x=0}^n \bar{p}_x \partial_{\bar{p}_x}.$$

System (3.5) can be rewritten in the matrix form

$$\frac{d}{dt} \begin{pmatrix} \bar{\mathbf{r}}(t) \\ \bar{\mathbf{p}}(t) \end{pmatrix} = -A \begin{pmatrix} \bar{\mathbf{r}}(t) \\ \bar{\mathbf{p}}(t) \end{pmatrix} + \mathcal{F}_n(t) \mathbf{e}_{p,n}. \quad (3.7)$$

Here A is the block matrix of the form

$$A := \begin{bmatrix} 0_n & -\nabla^* \\ -\nabla & 2\gamma \text{Id}_{n+1} \end{bmatrix}, \quad (3.8)$$

where ∇ and ∇^* are the matrices corresponding to the discrete divergence and gradient operators, see (2.8), defined as:

$$\nabla^* = \underbrace{\begin{pmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix}}_{n+1} \Bigg\} n, \quad \nabla = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 1 \\ 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}}_n \Bigg\} n+1, \quad (3.9)$$

and $0_n, \text{Id}_{n+1}$ are the $n \times n$ -null and $(n+1) \times (n+1)$ -identity matrices respectively. In (3.7) the column vector $\mathbf{e}_{p,n}$ of size $2n+1$ is given by

$$(\mathbf{e}_{p,n})_x = \delta_{2n+1,x}, \quad x = 1, \dots, 2n+1. \quad (3.10)$$

The solution of (3.7) is therefore

$$\begin{pmatrix} \bar{\mathbf{r}}(t) \\ \bar{\mathbf{p}}(t) \end{pmatrix} = e^{-At} \begin{pmatrix} \bar{\mathbf{r}}(0) \\ \bar{\mathbf{p}}(0) \end{pmatrix} + \int_0^t \mathcal{F}_n(s) e^{-A(t-s)} \mathbf{e}_{p,n} ds. \quad (3.11)$$

3.1. Homogeneous evolution and key lemma. Let us define the homogeneous term as

$$\begin{pmatrix} \bar{\mathbf{y}}(t) \\ \bar{\mathbf{z}}(t) \end{pmatrix} := e^{-At} \begin{pmatrix} \bar{\mathbf{r}}(0) \\ \bar{\mathbf{p}}(0) \end{pmatrix}, \quad \text{with } \bar{\mathbf{y}}(t) \in \mathbb{R}^n, \bar{\mathbf{z}}(t) \in \mathbb{R}^{n+1}. \quad (3.12)$$

We first obtain an immediate bound on the ℓ^2 -norm of the averages for the homogeneous term:

Lemma 3.2 (ℓ^2 -norm bound).

$$\sum_{x=0}^n \bar{z}_x^2(t) + \sum_{x=1}^n \bar{y}_x^2(t) \leq \left(\sum_{x=0}^n \bar{p}_x^2(0) + \sum_{x=1}^n \bar{r}_x^2(0) \right). \quad (3.13)$$

Proof. This follows immediately by taking the time derivative of the mechanical energy:

$$\frac{d}{dt} \left(\sum_{x=0}^n \bar{z}_x^2(t) + \sum_{x=1}^n \bar{y}_x^2(t) \right) = -4\gamma \sum_{x=0}^n \bar{z}_x^2(t) \leq 0.$$

□

Now, let us introduce the respective Fourier transforms of $\bar{\mathbf{y}}$ and $\bar{\mathbf{z}}$ in the two orthonormal basis $\{\psi_j\}_{j=0,\dots,n}$ and $\{\phi_j\}_{j=1,\dots,n}$ defined in (A.1) and (A.2), namely:

$$\tilde{\mathbf{y}}(t) = \begin{pmatrix} \tilde{y}_1(t) \\ \vdots \\ \tilde{y}_n(t) \end{pmatrix}, \quad \tilde{\mathbf{z}}(t) = \begin{pmatrix} \tilde{z}_0(t) \\ \vdots \\ \tilde{z}_n(t) \end{pmatrix},$$

where

$$\tilde{y}_j(t) := \sum_{x=1}^n \bar{y}_x(t) \phi_j(x), \quad \text{and} \quad \tilde{z}_j(t) := \sum_{x=0}^n \bar{z}_x(t) \psi_j(x),$$

Solving the systems of equations for the Fourier transforms, that arises from (3.5), a direct computation detailed in Appendix A.2 (see (A.9) and (A.10) in particular), leads to the following system: for $j = 0, \dots, n$

$$\frac{d^2 \tilde{z}_j(t)}{dt^2} + 2\gamma \frac{d\tilde{z}_j(t)}{dt} + \lambda_j \tilde{z}_j(t) = 0, \quad (3.14)$$

with the initial conditions

$$\frac{d\tilde{z}_j}{dt}(0) = \lambda_j^{1/2} \tilde{r}_j(0) - 2\gamma \tilde{p}_j(0), \quad \tilde{z}_j(0) = \tilde{p}_j(0), \quad (3.15)$$

where λ_j is the square eigenvalue (see (A.11))

$$\lambda_j := 4 \sin^2 \left(\frac{j\pi}{2(n+1)} \right). \quad (3.16)$$

The characteristic equation of (3.14) takes the form $\lambda^2 + 2\gamma\lambda + \lambda_j = 0$, which yields two solutions $-\lambda_{j,\pm}$, where

$$\lambda_{j,\pm} := \gamma \pm \sqrt{\gamma^2 - \lambda_j}.$$

Note the following important relations:

$$|\lambda_{j,+} \lambda_{j,-}| = \lambda_j, \quad \lambda_{j,+} + \lambda_{j,-} = 2\gamma, \quad \Delta\lambda_j = \lambda_{j,+} - \lambda_{j,-} = 2\sqrt{\gamma^2 - \lambda_j}, \quad (3.17)$$

$$\operatorname{Re} \lambda_{j,-} \geq 0, \quad \operatorname{Re} \lambda_{j,+} \geq \gamma, \quad |\lambda_{j,\pm}| \leq \gamma + \sqrt{\gamma^2 + 4}. \quad (3.18)$$

Solving (3.14) we easily obtain: first, for $j = 0, \dots, n$ such that $\lambda_{j,+} \neq \lambda_{j,-}$ (i.e. $\lambda_j \neq \gamma^2$)

$$\begin{aligned} \tilde{z}_j(t) &= \tilde{p}_j(0) \frac{\lambda_{j,+} e^{-\lambda_{j,+}t} - \lambda_{j,-} e^{-\lambda_{j,-}t}}{\Delta\lambda_j} + \tilde{r}_j(0) \lambda_j^{1/2} \frac{e^{-\lambda_{j,-}t} - e^{-\lambda_{j,+}t}}{\Delta\lambda_j} \\ \tilde{y}_j(t) &= \tilde{p}_j(0) \lambda_j^{1/2} \frac{e^{-\lambda_{j,+}t} - e^{-\lambda_{j,-}t}}{\Delta\lambda_j} + \tilde{r}_j(0) \frac{\lambda_{j,+} e^{-\lambda_{j,-}t} - \lambda_{j,-} e^{-\lambda_{j,+}t}}{\Delta\lambda_j}. \end{aligned} \quad (3.19)$$

Next, if $\gamma^2 = \lambda_j$ for some j , then $\lambda_{j,+} = \lambda_{j,-} = \gamma$. To avoid a complicated notation, by convention, we shall interpret the above formulas as $0/0$ symbols, remembering that $\lambda_{j,+} = \lambda_{j,-} + \Delta\lambda_j$ and $\Delta\lambda_j \rightarrow 0$. This leads to

$$\begin{aligned} \tilde{z}_j(t) &= \left((1 - \gamma t) \tilde{p}_j(0) + \gamma t \tilde{r}_j(0) \right) e^{-\gamma t}, \\ \tilde{y}_j(t) &= \left((1 + \gamma t) \tilde{r}_j(0) - \gamma t \tilde{p}_j(0) \right) e^{-\gamma t}. \end{aligned} \quad (3.20)$$

In what follows we shall repeatedly make use of the following estimate, which we call the *key lemma*. It enables us to understand the behavior of the quotients appearing in (3.19) when n goes to infinity.

Lemma 3.3 (Key lemma). *There exist constants $c, C > 0$, depending only on γ , such that, for any $n = 1, 2, \dots$,*

$$c \left(\frac{j}{n} \right)^2 \leq \operatorname{Re} \lambda_{j,-} \leq |\lambda_{j,-}| \leq C \left(\frac{j}{n} \right)^2, \quad j = 0, \dots, n. \quad (3.21)$$

Therefore, for any $q \in [0, +\infty)$, there exists constants $C, c > 0$, depending only q and γ , such that, for any $t > 0$ and any $j = 0, \dots, n$,

$$|\lambda_{j,-}|^q \left| e^{-\lambda_{j,-}t} \frac{1 - e^{-\Delta\lambda_j t}}{\Delta\lambda_j} \right| \leq C \left(\frac{j}{n} \right)^{2q} e^{-ctj^2/n^2}. \quad (3.22)$$

Moreover, since from (3.18) we have $\lambda_j = |\lambda_{j,+}\lambda_{j,-}| \leq C(\gamma)\lambda_{j,-}$, an analogous estimate holds also when $|\lambda_{j,-}|^q$ is replaced by λ_j^q ,

Proof. Recall first the standard inequality $2x/\pi \leq \sin x \leq x$ that holds for $x \in [0, \pi/2]$. Note that

$$|\lambda_{j,-}| = \frac{\lambda_j}{|\lambda_{j,+}|} \leq \frac{\lambda_j}{\operatorname{Re} \lambda_{j,+}} \leq \frac{4 \sin^2(\frac{j\pi}{2(n+1)})}{\gamma} \leq C(\gamma) \left(\frac{j}{n}\right)^2,$$

where we used (3.18) in the second inequality. Moreover,

$$\operatorname{Re} \lambda_{j,-} = \operatorname{Re} \frac{\lambda_j}{\lambda_{j,+}} = \frac{4 \sin^2(\frac{j\pi}{2(n+1)})}{|\lambda_{j,+}|^2} \operatorname{Re} \lambda_{j,+} \geq c(\gamma) \left(\frac{j}{n}\right)^2,$$

where we used once again (3.18). This proves (3.21).

Now, let us prove (3.22). Note that $\Delta\lambda_j$ is either real non-negative, or purely imaginary, therefore $\operatorname{Re} \Delta\lambda_j \geq 0$. We distinguish two cases:

- If $\operatorname{Re} \Delta\lambda_j \geq \gamma/2$, then, using the crude estimate $|1 - e^{-\Delta\lambda_j t}| \leq 2$ and $\Delta\lambda_j \geq \gamma/2$ we may bound the left hand side of (3.22) by

$$\gamma^{-1} |\lambda_{j,-}|^q e^{-\operatorname{Re} \lambda_{j,-} t}$$

and from (3.21) we conclude that (3.22) holds.

- If $0 \leq \operatorname{Re} \Delta\lambda_j < \gamma/2$, then in that case we get a better lower estimate than (3.21), more precisely we can write $\operatorname{Re} \lambda_{j,-} = \gamma - \frac{1}{2} \operatorname{Re} \Delta\lambda_j \geq \gamma/2$.

We now use the estimate $|1 - e^{-z}| \leq |z|$ valid for $\operatorname{Re} z \geq 0$, and we note that $|te^{-\lambda_{j,-} t}| = te^{-\operatorname{Re} \lambda_{j,-} t}$. We therefore obtain that the expression on the left hand side of (3.22) can be estimated by

$$|\lambda_{j,-}|^q te^{-\gamma t/2} \leq C_* |\lambda_{j,-}|^q e^{-\gamma t/4}, \quad \text{with } C_* := \sup_{t \geq 0} \{te^{-\gamma t/4}\}. \quad (3.23)$$

This together with (3.21) implies (3.22) (since $j/n \leq 1$ and taking $c = \gamma/4$). \square

From now on, the constants $C > 0$ which appear in the statements and proofs may depend on $\gamma, \bar{F}, C_{\mathcal{F}}, T_-$, namely the parameters of the model. Unless explicitly stated, they do not depend on $t > 0$ nor on n .

The rest of the section is dedicated to the proof of Proposition 3.1. Therefore, we will always assume that Assumption 2.1 holds, which implies that the right hand side of (3.13) in Lemma 3.2 is of order $\mathcal{O}(n)$.

3.2. Control of the ℓ^2 -norm of momenta averages. Here we prove (3.1), for the momenta. Let us compute explicitly the momenta averages.

We use the first equation in (3.19) to write the decomposition

$$\tilde{z}_j(t) = \text{I}_j(t) + \text{II}_j(t) + \text{III}_j(t), \quad (3.24)$$

where $\text{I}_j(t) := \tilde{p}_j(0) e^{-\lambda_{j,+} t}$ and

$$\text{II}_j(t) := -\lambda_{j,-} \tilde{p}_j(0) e^{-\lambda_{j,-} t} \frac{1 - e^{-\Delta\lambda_j t}}{\Delta\lambda_j}, \quad \text{III}_j(t) := \lambda_j^{1/2} \tilde{r}_j(0) e^{-\lambda_{j,-} t} \frac{1 - e^{-\Delta\lambda_j t}}{\Delta\lambda_j}.$$

Then, using (3.11) and (3.24) (or (3.19)) in order to express e^{-At} , we obtain, after a direct calculation,

$$\bar{p}_x(t) = \bar{z}_x(t) + \text{I}_x^{(p)}(t) + \text{II}_{x,1}^{(p)}(t) + \text{II}_{x,2}^{(p)}(t). \quad (3.25)$$

where $\bar{z}_x(t)$ can be explicitly written by taking the inverse Fourier transform of expression (3.19), and besides

$$\begin{aligned} \mathbb{I}_x^{(p)}(t) &:= \bar{F} \sum_{j=0}^n \psi_j(n) \psi_j(x) e^{-\lambda_{j,-} t} \frac{1 - e^{-\Delta \lambda_j t}}{\Delta \lambda_j}, \\ \mathbb{II}_{x,1}^{(p)}(t) &:= \frac{1}{\sqrt{n}} \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) \sum_{j=0}^n \psi_j(n) \psi_j(x) \frac{(e^{i\ell \omega t} - e^{-\lambda_{j,-} t}) i \ell \omega}{\lambda_j - (\ell \omega)^2 + 2i\ell \omega \gamma}, \\ \mathbb{II}_{x,2}^{(p)}(t) &:= \frac{1}{\sqrt{n}} \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) \sum_{j=0}^n \psi_j(n) \psi_j(x) \frac{\lambda_{j,+} e^{-\lambda_{j,-} t} (1 - e^{-\Delta \lambda_j t})}{\Delta \lambda_j (i\ell \omega + \lambda_{j,+})}. \end{aligned}$$

We recall the notation $\ell \omega = \frac{2\pi \ell}{\theta}$.

Let us now start the proof of Proposition 3.1, beginning with (3.1).

Proof of (3.1). Recall (3.25), and let us start with the first contribution, namely let us estimate $\sum_{x=0}^n |\bar{z}_x(n^2 s)|^2$. Note that we cannot use the previous bound obtained in Lemma 3.2, instead we need to improve it. Recall the decomposition of $\tilde{z}_j(t)$ given in (3.24). From the Parseval identity, we are going to bound in a more refined way the following three terms:

$$\sum_{j=0}^n |\mathbb{I}_j(n^2 s)|^2, \quad \sum_{j=0}^n |\mathbb{II}_j(n^2 s)|^2, \quad \sum_{j=0}^n |\mathbb{III}_j(n^2 s)|^2. \quad (3.26)$$

First of all, using the fact that $\operatorname{Re} \lambda_{j,+} \geq \gamma$, and Parseval identity, we can write

$$\sum_{j=0}^n |\mathbb{I}_j(n^2 s)|^2 = \sum_{j=0}^n (\tilde{p}_j(0))^2 e^{-2\operatorname{Re} \lambda_{j,+} n^2 s} \leq \sum_{j=0}^n (\tilde{p}_j(0))^2 e^{-\gamma n^2 s} = \sum_{x=0}^n \bar{p}_x^2(0) e^{-\gamma n^2 s}. \quad (3.27)$$

Therefore, from Assumption 2.1, we get

$$\sum_{j=0}^n |\mathbb{I}_j(n^2 s)|^2 \leq \bar{\mathcal{H}} n, \quad s \geq 0 \quad (3.28)$$

and, for $t \geq 0$,

$$\int_0^t \sum_{j=0}^n |\mathbb{I}_j(n^2 s)|^2 ds \leq \int_0^t \sum_{x=0}^n \bar{p}_x^2(0) e^{-\gamma n^2 s} ds = \frac{1}{\gamma n^2} \sum_{x=0}^n \bar{p}_x^2(0) (1 - e^{-\gamma n^2 t}) \leq \frac{\bar{\mathcal{H}}}{\gamma n}. \quad (3.29)$$

Note that both estimates (3.28) and (3.29) are exactly of the needed order to prove (3.1). We will now proceed in the same way for all the other contributions. We have

$$\begin{aligned} \sum_{j=0}^n |\mathbb{II}_j(n^2 s)|^2 &= 4 \sum_{j=0}^n (\tilde{p}_j(0))^2 |\lambda_{j,-}|^2 \left| e^{-\lambda_{j,-} n^2 s} \frac{1 - e^{-\Delta \lambda_j n^2 s}}{\Delta \lambda_j} \right|^2 \\ &\leq C \sum_{j=0}^n (\tilde{p}_j(0))^2 \left(\frac{j}{n}\right)^4 e^{-c j^2 s} \quad \text{from the key Lemma 3.3, (3.22)}. \end{aligned} \quad (3.30)$$

Therefore, from Assumption 2.1 and the fact that $\left(\frac{j}{n}\right)^4 e^{-c j^2 s} \leq 1$, we obtain again that $\sum_{j=0}^n |\mathbb{II}_j(n^2 s)|^2 \leq C n$ and, integrating in time

$$\int_0^t \sum_{j=0}^n |\mathbb{II}_j(n^2 s)|^2 ds \leq \frac{C}{n^2} \sum_{j=0}^n (\tilde{p}_j(0))^2 \left(\frac{j}{n}\right)^2 (1 - e^{-c j^2 t}) \leq \frac{C}{n}.$$

Similarly, invoking again Lemma 3.3, namely (3.22) with λ_j instead of $\lambda_{j,-}$, we write

$$\sum_{j=0}^n |\mathbb{III}_j(n^2 s)|^2 \leq C \sum_{j=1}^n (\tilde{r}_j(0))^2 \left(\frac{j}{n}\right)^2 e^{-c j^2 s} \quad (3.31)$$

and from this point we estimate as before obtaining

$$\sum_{j=0}^n |\text{III}_j(n^2s)|^2 \leq Cn \quad \text{and} \quad \int_0^t \sum_{j=0}^n |\text{III}_j(n^2s)|^2 ds \leq \frac{C}{n}.$$

The remaining contributions coming from the right hand side of (3.25) are treated in a similar way: first, using once again the Plancherel identity and the key Lemma 3.3 (with $q = 0$), we get

$$\begin{aligned} \sum_{x=0}^n |\text{I}_x^{(p)}(n^2s)|^2 &= \overline{F}^2 \sum_{j=0}^n \psi_j^2(n) \left| e^{-\lambda_j, -n^2s} \frac{1 - e^{-\Delta\lambda_j n^2s}}{\Delta\lambda_j} \right|^2 \leq C\overline{F}^2 \sum_{j=0}^n \psi_j^2(n) e^{-cj^2s} \\ &\leq \frac{C\overline{F}^2}{n} \sum_{j=0}^n e^{-cj^2s} \end{aligned} \quad (3.32)$$

where we have used the fact that $\psi_j^2(n) \leq C/n$. The bound $\sum_{x=0}^n |\text{I}_x^{(p)}(n^2s)|^2 \leq Cn$ comes easily. Integrating in time, we obtain

$$\int_0^t \sum_{x=0}^n |\text{I}_x^{(p)}(n^2s)|^2 ds \leq \frac{C}{n} \left\{ t + \sum_{j=1}^n \frac{1}{cj^2} (1 - e^{-cj^2t}) \right\} \leq \frac{C(t+1)}{n}.$$

Then, we have

$$\sum_{x=0}^n |\text{II}_{x,1}^{(p)}(n^2s)|^2 = \frac{1}{n} \sum_{j=0}^n \psi_j^2(n) \left| \sum_{\ell \neq 0} \frac{\widehat{\mathcal{F}}(\ell) (e^{i\ell\omega n^2s} - e^{-\lambda_j, -n^2s}) i\omega(\ell)}{\lambda_j - (\ell\omega)^2 + 2i\omega(\ell)\gamma} \right|^2. \quad (3.33)$$

Note that $|\lambda_j - (\ell\omega)^2 + 2i\omega(\ell)\gamma|^2 \geq 4\gamma^2(\ell\omega)^2$, and $|e^{i\ell\omega n^2s} - e^{-\lambda_j, -n^2s}| \leq 2$, and recall $\psi_j^2(n) \leq C/n$ from (A.6). Therefore we may write

$$\sum_{x=0}^n |\text{II}_{x,1}^{(p)}(n^2s)|^2 \leq \frac{C}{n} \left(\sum_{\ell \neq 0} |\widehat{\mathcal{F}}(\ell)| \right)^2 = \frac{CC_{\mathcal{F}}^2}{n} \quad (3.34)$$

where $C_{\mathcal{F}}$ was defined in (2.7), and the bounds in this case become trivial. Finally, we use once again (3.22) with $q = 0$, together with the fact that $|\lambda_{j,+}| \leq C$ and $|i\omega(\ell) + \lambda_{j,+}| \geq \text{Re } \lambda_{j,+} \geq \gamma$, and we obtain

$$\begin{aligned} \sum_{x=0}^n |\text{II}_{x,2}^{(p)}(n^2s)|^2 &= \frac{1}{n} \sum_{j=0}^n \psi_j^2(n) \left| \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) \frac{\lambda_{j,+} e^{-\lambda_j, -n^2s} (1 - e^{-\Delta\lambda_j n^2s})}{(i\ell\omega + \lambda_{j,+}) \Delta\lambda_j} \right|^2 \\ &\leq \frac{C}{n} \left(\sum_{\ell \neq 0} |\widehat{\mathcal{F}}(\ell)| \right)^2 \sum_{j=0}^n \psi_j^2(n) e^{-cj^2s} \leq \frac{CC_{\mathcal{F}}^2}{n} \sum_{j=0}^n \psi_j^2(n) e^{-cj^2s} \end{aligned} \quad (3.35)$$

and it is treated similarly to (3.32). Hence, summing all the contributions, the proof of (3.1) easily follows. \square

3.3. Control of the ℓ^2 -norm of the stretches averages. Now, let us do the same procedure for the stretch averages, and prove (3.2). From (3.11) and (3.19), we get

$$\bar{r}_x(t) = \bar{y}_x(t) + \sum_{m=1}^2 \left(\text{I}_{x,m}^{(r)}(t) + \text{II}_{x,m}^{(r)}(t) \right). \quad (3.36)$$

We can write $\bar{y}_x(t)$ explicitly taking the inverse Fourier transform of the second identity in (3.19). Furthermore,

$$\begin{aligned} \mathbb{I}_{x,1}^{(r)}(t) &:= \bar{F} \sum_{j=1}^n \psi_j(n) \phi_j(x) \frac{e^{-\lambda_{j,-}t} - 1}{\lambda_j^{1/2}}, \\ \mathbb{I}_{x,2}^{(r)}(t) &:= \bar{F} \sum_{j=1}^n \frac{\psi_j(n) \phi_j(x)}{\lambda_{j,+}^{1/2}} \lambda_{j,-}^{1/2} e^{-\lambda_{j,-}t} \frac{1 - e^{-\Delta\lambda_j t}}{\Delta\lambda_j} \end{aligned}$$

and

$$\begin{aligned} \mathbb{II}_{x,1}^{(r)}(t) &:= \frac{1}{\sqrt{n}} \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) \sum_{j=1}^n \psi_j(n) \phi_j(x) \frac{\lambda_j^{1/2} (e^{-\lambda_{j,-}t} - e^{i\ell\omega t})}{\lambda_j - (\ell\omega)^2 + 2i\ell\omega\gamma}, \\ \mathbb{II}_{x,2}^{(r)}(t) &:= \frac{1}{\sqrt{n}} \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) \sum_{j=1}^n \psi_j(n) \phi_j(x) \frac{\lambda_j^{1/2} e^{-\lambda_{j,-}t} (1 - e^{-\Delta\lambda_j t})}{(i\ell\omega + \lambda_{j,+}) \Delta\lambda_j}. \end{aligned}$$

We claim the following: there exists a constant $C > 0$ such that, for any $n = 1, 2, \dots$ and $t \geq 0$,

$$\sum_{x=1}^n \bar{r}_x^2(t) \leq C \left(\sum_{x=0}^n (\bar{p}_x^2(0) + \bar{r}_x^2(0)) + n\bar{F}^2 \right). \quad (3.37)$$

Then, from Assumption 2.1, this concludes the proof of (3.2).

It remains to prove (3.37), using (3.36). The first contribution coming from $\bar{y}_x^2(t)$ can be directly estimated using (3.13), and this gives the first term in the right hand side of (3.37). Then, by the Plancherel identity we have

$$\sum_{x=1}^n |\mathbb{I}_{x,1}^{(r)}(t)|^2 = \bar{F}^2 \sum_{j=1}^n \frac{\psi_j^2(n)}{\lambda_j} |e^{-\lambda_{j,-}t} - 1|^2 \leq \frac{\bar{F}^2}{2(n+1)} \sum_{j=1}^n \cot^2\left(\frac{\pi j}{2(n+1)}\right) \leq C\bar{F}^2 n,$$

where we used the expressions of λ_j given in (3.16) and $\psi_j(n)$ given in (A.1), together with the crude bound $|e^{-\lambda_{j,-}t} - 1| \leq 2$. Likewise, using the Plancherel identity and then the key Lemma, estimate (3.22), together with $|\lambda_{j,+}| \geq \operatorname{Re} \lambda_{j,+} \geq \gamma$, we get

$$\sum_{x=1}^n |\mathbb{I}_{x,2}^{(r)}(t)|^2 = \bar{F}^2 \sum_{j=1}^n \frac{\psi_j^2(n)}{|\lambda_{j,+}|} \left| \lambda_{j,-}^{1/2} e^{-\lambda_{j,-}t} \frac{1 - e^{-\Delta\lambda_j t}}{\Delta\lambda_j} \right|^2 \leq C\bar{F}^2 \sum_{j=1}^n \psi_j^2(n) \left(\frac{j}{n}\right)^2 e^{-cj^2 t} \leq C\bar{F}^2,$$

where in the last estimate we used the fact that $\psi_j^2(x) \leq C/n$ from (A.6) and we have simply bounded $\frac{j}{n}$ and $e^{-cj^2 t}$ by 1. Therefore this contribution is smaller than the previous one. The last two estimates are very similar to the ones conducted for $\mathbb{II}_{x,1}^{(p)}(t)$ and $\mathbb{II}_{x,2}^{(p)}(t)$ (see (3.33) and (3.35) in particular) in the previous section. More precisely, we easily get (using the bounds $|\lambda_j - (\ell\omega)^2 + 2i\ell\omega\gamma|^2 \geq 4\gamma^2\omega^2(\ell)$ for any $\ell \neq 0$ and $\lambda_j \leq 4$):

$$\begin{aligned} \sum_{x=1}^n |\mathbb{II}_{x,1}^{(r)}(t)|^2 &= \frac{1}{n} \sum_{j=1}^n \lambda_j \psi_j^2(n) \left| \sum_{\ell \neq 0} \frac{\widehat{\mathcal{F}}(\ell) (e^{-\lambda_{j,-}t} - e^{i\ell\omega t})}{\lambda_j - (\ell\omega)^2 + 2i\ell\omega\gamma} \right|^2 \\ &\leq \frac{C}{n} \sum_{j=1}^n \psi_j^2(n) \left(\sum_{\ell \neq 0} |\widehat{\mathcal{F}}(\ell)| \right)^2 \leq \frac{C}{n} \end{aligned} \quad (3.38)$$

and from the key lemma, namely (3.22) but bounded crudely by C , plus (3.18), we obtain similarly

$$\sum_{x=1}^n |\Pi_{x,2}^{(r)}(t)|^2 = \frac{1}{n} \sum_{j=1}^n \psi_j^2(n) \left| \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) \frac{\lambda_{j,+}^{1/2}}{i\ell\omega + \lambda_{j,+}} \frac{\lambda_{j,-}^{1/2} e^{-\lambda_{j,-}t} (1 - e^{-\Delta\lambda_j t})}{\Delta\lambda_j} \right|^2 \leq \frac{C}{n}.$$

Those last two estimates are smaller than the previous ones, in particular smaller than $C\overline{F}^2 n$, therefore we conclude the proof of (3.37). \square

3.4. Estimates at the boundaries. Finally, we investigate the behavior at both boundaries, and we prove in this section the second point (ii) of Proposition 3.1, namely (3.3) and (3.4).

We first prove (3.3) for $x = n$, the proof for $x = 0$ is analogous. Using formula (3.25) we can write

$$\overline{p}_n(t) = P_0(t) + P_{\overline{F}}(t) + P_{fl}(t) + P_{dp}(t), \quad (3.39)$$

where

$$P_0(t) = \sum_{j=0}^n \psi_j(n) \tilde{z}_j(t) = \tilde{z}_n(t) \quad (3.40)$$

$$P_{\overline{F}}(t) = \overline{F} \sum_{j=0}^n \psi_j^2(n) e^{-\lambda_{j,-}t} \frac{1 - e^{-\Delta\lambda_j t}}{\Delta\lambda_j} \quad (3.41)$$

$$P_{fl}(t) = \frac{1}{\sqrt{n}} \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) i\ell\omega \sum_{j=0}^n \psi_j^2(n) \frac{(e^{i\ell\omega t} - e^{-\lambda_{j,-}t})}{\lambda_j - (\ell\omega)^2 + 2i\ell\omega\gamma} \quad (3.42)$$

$$P_{dp}(t) = \frac{1}{\sqrt{n}} \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) \sum_{j=0}^n \psi_j^2(n) \frac{\lambda_{j,+} e^{-\lambda_{j,-}t} (1 - e^{-\Delta\lambda_j t})}{(i\ell\omega + \lambda_{j,+}) \Delta\lambda_j}. \quad (3.43)$$

It is quite clear that the contributions coming from P_0 and $P_{\overline{F}}$ are the biggest ones: in fact, thanks to the extra term $1/\sqrt{n}$ in front of the sums in (3.42) and (3.43), and using the fact that $\sum |\widehat{\mathcal{F}}(\ell)| < \infty$, we will see that the last two terms always have a smaller order. Let us start with the contribution coming from P_0 , and use the decomposition of $\tilde{z}_j(t)$ given in (3.24). We can bound by the triangle inequality,

$$\begin{aligned} \left| \int_0^t P_0(n^2 s) ds \right| &\leq \sum_{j=0}^n |\psi_j(n)| |\tilde{p}_j(0)| \int_0^t \left\{ |e^{-\lambda_{j,+}n^2 s}| + |\lambda_{j,-}| \left| e^{-\lambda_{j,-}n^2 s} \frac{e^{-\Delta\lambda_j n^2 s} - 1}{\Delta\lambda_j} \right| \right\} ds \\ &\quad + \sum_{j=1}^n |\psi_j(n)| |\tilde{r}_j(0)| \int_0^t \lambda_j^{1/2} \left| e^{-\lambda_{j,-}n^2 s} \frac{e^{-\Delta\lambda_j n^2 s} - 1}{\Delta\lambda_j} \right| ds. \end{aligned}$$

We use the key Lemma 3.3 in both terms containing quotients, and then we integrate over s . For the first term in the first integral we use the fact that $\operatorname{Re} \lambda_{j,+} \geq \gamma$ and then integrate over s . Finally, the right hand side can be estimated by

$$\begin{aligned} &C \sum_{j=0}^n |\psi_j(n)| |\tilde{p}_j(0)| \left(\frac{1 - e^{-\gamma n^2 t}}{n^2} + \left(\frac{j}{n} \right)^2 \cdot \frac{1 - e^{-cj^2 t}}{j^2} \right) + C \sum_{j=1}^n |\psi_j(n)| |\tilde{r}_j(0)| \frac{j}{n} \cdot \frac{1 - e^{-cj^2 t}}{j^2} \\ &\leq \frac{C}{n^2} \left(\sum_{j=0}^n \psi_j^2(n) \right)^{1/2} \left(\sum_{j=0}^n (\tilde{p}_j(0))^2 \right)^{1/2} + \frac{C}{n} \left(\sum_{j=1}^n \frac{\psi_j^2(n)}{j^2} \right)^{1/2} \left(\sum_{j=1}^n (\tilde{r}_j(0))^2 \right)^{1/2} \quad (3.44) \end{aligned}$$

thanks to the Cauchy-Schwarz inequality. Using the initial bound (2.18), and also $\psi_j^2(x) \leq C/n$, we can estimate the whole right hand side of (3.44) by C/n (with C independent of t).

The other estimates are quite similar. In order to bound $|\int_0^t P_{\bar{F}}(n^2s)ds|$, we use the key Lemma 3.3 and integrate over s (and also recall that $\lambda_{0,-} = 0$), then we get it is less than, or equal to

$$\frac{|\bar{F}|t}{2\gamma n} + |\bar{F}| \sum_{j=1}^n \psi_j^2(n) \frac{1 - e^{-cj^{2t}}}{j^2} \leq \frac{C(t_*)}{n} \quad \text{for any } t \in [0, t_*].$$

Finally, the last two estimates are even smaller, thanks to the fact that $\psi_j^2(n) \leq C/n$: for instance, integrating over s we can write

$$\left| \int_0^t P_{fl}(n^2s)ds \right| \leq \frac{1}{\sqrt{n}} \sum_{\ell \neq 0} |\widehat{\mathcal{F}}(\ell)| \left| \frac{e^{i\ell\omega n^2t} - 1}{n^2\ell\omega} \right| \leq \frac{C}{n^{3/2}}$$

and similarly for $|\int_0^t P_{dp}(n^2s)ds|$. Estimate (3.3) then follows.

Now let us show (3.4). We integrate both sides of (3.5) for $x = n$ and obtain

$$\frac{1}{n^2} (\bar{p}_n(n^2t) - \bar{p}_n(0)) = \frac{1}{n^2} \int_0^{n^2t} \mathcal{F}_n(s)ds - \frac{1}{n^2} \int_0^{n^2t} (\bar{r}_n(s) + 2\gamma\bar{p}_n(s)) ds. \quad (3.45)$$

We have

$$\frac{1}{n^2} \int_0^{n^2t} \mathcal{F}_n(s)ds \xrightarrow[n \rightarrow \infty]{} t\bar{F}. \quad (3.46)$$

Note that, from (3.1)

$$|\bar{p}_n(n^2t)| \leq \left(\sum_{x=0}^n \bar{p}_x^2(n^2t) \right)^{1/2} \leq C\sqrt{n}.$$

Therefore, formula (3.4) follows directly from (3.3), (3.45), (3.46). \square

4. STRETCH AND MECHANICAL ENERGY: PROOF OF THEOREM 2.3

4.1. Macroscopic evolution of the stretch. Now we have all the ingredients at hand to prove the first convergence result, namely (2.21).

Let $\mathcal{M}_K[0, 1]$ be the space of signed measures m on $[0, 1]$, whose total variation $|m|[0, 1]$ is bounded by a constant $K > 0$, endowed with the topology of weak convergence. The space is metrizable and compact. Given $t_* > 0$, we consider the space $\mathcal{C}([0, t_*], \mathcal{M}_K[0, 1])$ endowed with the corresponding uniform topology. Define

$$\xi_n^r(t, \varphi) = \frac{1}{n} \sum_{x=1}^n \varphi\left(\frac{x}{n}\right) \bar{r}_x(n^2t). \quad (4.1)$$

By Assumption 2.1 there exists a finite $K > 0$ such that $\xi_n^r \in \mathcal{C}([0, t_*], \mathcal{M}_K[0, 1])$ – the space of all continuous functions from $[0, t_*]$ into $\mathcal{M}_K[0, 1]$. Since the latter is compact under the weak topology, the compactness of the sequence $\{\xi_n^r\}$ in $\mathcal{C}([0, t_*], \mathcal{M}_K[0, 1])$ follows from a bound on the modulus of continuity in time, by an extension of the Ascoli-Arzelà Theorem, see *e.g.* [Kel75, p. 234]. This in particular implies compactness of the sequence of the empirical measures corresponding to $\{\bar{r}_x(n^2t); t \in [0, t_*], x = 0, \dots, n\}$. In what follows we identify the limit.

Let $\mathcal{C}[0, 1]$, resp. $\mathcal{C}^k[0, 1]$ for a positive integer k (or $k = \infty$), be the space of all continuous, resp. k (or infinitely many) times continuously differentiable, functions on $[0, 1]$. Let $\varphi \in \mathcal{C}^2[0, 1]$ be such that $\varphi(0) = 0 = \varphi(1)$. Denote in the following

$\varphi_x := \varphi\left(\frac{x}{n}\right)$. Note that $\varphi_0 = 0 = \varphi_n$. Then, by the integration by parts,

$$\begin{aligned}\xi_n^r(t, \varphi) - \xi_n^r(0, \varphi) &= n \int_0^t \sum_{x=1}^n \varphi_x (\bar{p}_x(n^2s) - \bar{p}_{x-1}(n^2s)) ds \\ &= -n \int_0^t \sum_{x=0}^{n-1} (\varphi_{x+1} - \varphi_x) \bar{p}_x(n^2s) ds + n\varphi_n \int_0^t \bar{p}_n(n^2s) ds.\end{aligned}\quad (4.2)$$

Since $\varphi_n = 0$, the last term equals 0. Recall the evolution equations (3.5). Then, for $x = 0, \dots, n-1$ we can write

$$\int_0^t \bar{p}_x(n^2s) ds = \frac{1}{2\gamma} \int_0^t (\bar{r}_{x+1}(n^2s) - \bar{r}_x(n^2s)) ds - \frac{1}{2\gamma n^2} (\bar{p}_x(n^2t) - \bar{p}_x(0)). \quad (4.3)$$

Substituting into (4.2) we obtain

$$\begin{aligned}\xi_n^r(t, \varphi) - \xi_n^r(0, \varphi) &= -\frac{n}{2\gamma} \int_0^t \sum_{x=0}^{n-1} \nabla \varphi_x (\bar{r}_{x+1}(n^2s) - \bar{r}_x(n^2s)) ds \\ &\quad + \frac{1}{2\gamma n} \sum_{x=0}^{n-1} \nabla \varphi_x (\bar{p}_x(n^2t) - \bar{p}_x(0)) + o_n(t),\end{aligned}\quad (4.4)$$

where the expression $o_n(t)$ means that $\sup_{t \in [0, t_*]} |o_n(t)| \rightarrow 0$, as $n \rightarrow +\infty$ for any $t_* > 0$. Since

$$\sum_{x=0}^{n-1} \nabla \varphi_x (\bar{r}_{x+1} - \bar{r}_x) = -\sum_{x=1}^{n-1} \Delta \varphi_x \bar{r}_x + \nabla \varphi_{n-1} \bar{r}_n, \quad (4.5)$$

we have

$$\begin{aligned}\xi_n^r(t, \varphi) - \xi_n^r(0, \varphi) &= \frac{1}{2\gamma n} \int_0^t \sum_{x=1}^{n-1} n^2 \Delta \varphi_x \bar{r}_x(n^2s) ds - \frac{n \nabla \varphi_{n-1}}{2\gamma} \int_0^t \bar{r}_n(n^2s) ds \\ &\quad + \frac{1}{2\gamma n^2} \sum_{x=0}^{n-1} n \nabla \varphi_x (\bar{p}_x(n^2t) - \bar{p}_x(0)) + o_n(t).\end{aligned}\quad (4.6)$$

By approximating $n \nabla \varphi_x \sim \varphi'\left(\frac{x}{n}\right)$ and $n^2 \Delta \varphi_x \sim \varphi''\left(\frac{x}{n}\right)$, with all errors controlled uniformly by (3.1) and (3.2) from Proposition 3.1. Using the limit obtained in (3.4) for the boundary term we conclude that

$$\lim_{n \rightarrow \infty} \left[\xi_n^r(t, \varphi) - \xi_n^r(0, \varphi) - \frac{1}{2\gamma} \int_0^t \xi_n^r(s, \varphi'') ds \right] + \frac{1}{2\gamma} \varphi'(1) \bar{F}t = 0. \quad (4.7)$$

This corresponds exactly to the weak formulation of (2.23).

The bound of the modulus of continuity in time follows by a similar argument.

4.2. Macroscopic evolution of the mechanical energy. Now we prove the convergence of the mechanical energy density stated in (2.22). Let $\Phi \in \mathcal{C}([0, 1])$ be continuous. By Proposition 3.1, (3.1), we already know that for any $t > 0$

$$\frac{1}{n} \sum_{x=0}^n \Phi\left(\frac{x}{n}\right) \bar{p}_x^2(n^2t) \xrightarrow{n \rightarrow \infty} 0, \quad (4.8)$$

so we have only to prove that

$$\frac{1}{n} \sum_{x=1}^n \Phi\left(\frac{x}{n}\right) \bar{r}_x^2(n^2t) \xrightarrow{n \rightarrow \infty} \int_0^1 \Phi(u) r^2(t, u) du. \quad (4.9)$$

We start with the following:

Lemma 4.1. *There exists a constant C such that, for any $n = 1, 2, \dots$ and $t \geq 0$,*

$$n \sum_{x=0}^{n-1} \int_0^t (\nabla \bar{r}_x)^2(n^2 s) ds \leq C. \quad (4.10)$$

Proof. From (3.5) we obtain

$$\frac{1}{2} \frac{d}{dt} \bar{p}_x^2(n^2 t) = n^2 \nabla \bar{r}_x(n^2 t) \bar{p}_x(n^2 t) - 2\gamma n^2 \bar{p}_x^2(n^2 t) \quad (4.11)$$

for $x = 0, \dots, n-1$. In addition we also have

$$\bar{p}_x(n^2 t) = \frac{1}{2\gamma} \nabla \bar{r}_x(n^2 t) - \frac{1}{2\gamma n^2} \frac{d}{dt} \bar{p}_x(n^2 t). \quad (4.12)$$

Substituting from (4.12) for $\bar{p}_x(n^2 t)$ into the first expression on the right hand side of (4.11) we obtain: for $x = 0, \dots, n-1$

$$\frac{n^2}{2\gamma} (\nabla \bar{r}_x(n^2 t))^2 - \frac{1}{2\gamma} \nabla \bar{r}_x(n^2 t) \frac{d}{dt} \bar{p}_x(n^2 t) - 2\gamma n^2 \bar{p}_x^2(n^2 t) = \frac{1}{2} \frac{d}{dt} \bar{p}_x^2(n^2 t). \quad (4.13)$$

Summing up over x and integrating in time we obtain

$$\begin{aligned} n \sum_{x=0}^{n-1} \int_0^t (\nabla \bar{r}_x(n^2 s))^2 ds &= \frac{1}{n} \sum_{x=0}^{n-1} \int_0^t \nabla \bar{r}_x(n^2 s) \frac{d}{ds} \bar{p}_x(n^2 s) ds \\ &\quad + 4\gamma^2 n \sum_{x=0}^{n-1} \int_0^t \bar{p}_x^2(n^2 s) ds + \frac{\gamma}{n} \sum_{x=0}^{n-1} (\bar{p}_x^2(n^2 t) - \bar{p}_x^2(0)). \end{aligned} \quad (4.14)$$

By Proposition 3.1, see (3.1), the second and third terms on the right hand side of (4.14) are bounded. The only term in question is the first one.

After the integration by parts in time and using (3.5) we conclude that it equals

$$-n \sum_{x=0}^{n-1} \int_0^t \bar{p}_x(n^2 s) \nabla \nabla^* \bar{p}_x(n^2 s) ds + \frac{1}{n} \sum_{x=0}^{n-1} (\nabla \bar{r}_x(n^2 t) \bar{p}_x(n^2 t) - \nabla \bar{r}_x(0) \bar{p}_x(0)). \quad (4.15)$$

Using first the Cauchy-Schwarz inequality and then (3.1)–(3.2) from Proposition 3.1, both terms in (4.15) are bounded uniformly in t and n . Hence (4.10) follows. \square

Define $\bar{r}_{\text{int}}^{(n)} : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ as the function obtained by the piecewise linear interpolation between the nodal points

$$\left(\frac{x}{n}, \bar{r}_x(n^2 t) \right), \quad x = 0, \dots, n,$$

of the piecewise constant function $\bar{r}^{(n)} : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$, given by

$$\bar{r}^{(n)}(t, u) = \begin{cases} \bar{r}_x(n^2 t), & \text{for any } u \in \left[\frac{x-1}{n}, \frac{x}{n} \right), x = 1, \dots, n-1 \\ \bar{r}_n(n^2 t) & \text{for any } u \in \left[\frac{n-1}{n}, 1 \right]. \end{cases}$$

By convention we let $\bar{r}_0(n^2 t) = 0$. Let $H^1[0, 1]$ be the completion of $\mathcal{C}_c^\infty(0, 1)$ – the space of smooth and compactly supported functions – in the norm

$$\|\varphi\|_{H^1[0,1]}^2 := \|\varphi\|_{L^2[0,1]}^2 + \|\varphi'\|_{L^2[0,1]}^2, \quad \varphi \in \mathcal{C}_c^\infty(0, 1).$$

As a consequence of Lemma 4.1 above we obtain the following.

Corollary 4.2. *For any $t \geq 0$ we have*

$$\sup_{n \geq 1} \int_0^t \|\bar{r}_{\text{int}}^{(n)}(s, \cdot)\|_{H^1[0,1]}^2 ds < +\infty. \quad (4.16)$$

Moreover,

$$\lim_{n \rightarrow +\infty} \sup_{u \in [0,1]} \left| \int_0^t \bar{r}_{\text{int}}^{(n)}(s, u) ds - \int_0^t r(s, u) ds \right| = 0 \quad (4.17)$$

and there exists a constant $C > 0$ such that, for any $n \geq 1$, $u \in [0, 1]$, $t \geq 0$,

$$\int_0^t (\bar{r}_{\text{int}}^{(n)}(s, u))^2 ds \leq Cu. \quad (4.18)$$

Proof. It is easy to see that

$$\|\bar{r}_{\text{int}}^{(n)}(t, \cdot) - \bar{r}^{(n)}(t, \cdot)\|_{L^2[0,1]}^2 = \frac{1}{3(n+1)} \sum_{x=0}^{n-1} (\bar{r}_{x+1}(t) - \bar{r}_x(t))^2, \quad n \geq 1. \quad (4.19)$$

Estimate (4.16) is a direct consequence of (4.10). Using (4.19) we also get

$$\lim_{n \rightarrow +\infty} \int_0^t \|\bar{r}_{\text{int}}^{(n)}(s, \cdot) - \bar{r}^{(n)}(s, \cdot)\|_{L^2[0,1]}^2 ds = 0, \quad t > 0. \quad (4.20)$$

From the convergence (2.21) proved in Section 4.1 above we know that the sequence $\int_0^t \bar{r}_{\text{int}}^{(n)}(s, u) ds$ weakly converges in $L^2[0, 1]$ to $\int_0^t r(s, u) ds$ for each $t > 0$. From (4.16) and the compactness of Sobolev embedding of $H^1[0, 1]$ into $C[0, 1]$ in dimension 1 we conclude (4.17).

Besides, by the Cauchy-Schwarz inequality we get

$$(\bar{r}_x)^2(n^2s) = \left(\sum_{y=0}^{x-1} \nabla \bar{r}_y(n^2s) \right)^2 \leq x \sum_{y=0}^{x-1} (\nabla \bar{r}_y(n^2s))^2$$

and (4.18) follows from (4.10). \square

We are now ready to prove (4.9). In light of (4.19) it suffices only to show that

$$\int_0^1 \Phi(u) (\bar{r}_{\text{int}}^{(n)}(t, u))^2 du \xrightarrow{n \rightarrow \infty} \int_0^1 \Phi(u) r^2(t, u) du \quad (4.21)$$

for any $\Phi \in \mathcal{C}([0, 1])$. For $\delta > 0$ let us define

$$r_{\text{int}}^{(n, \delta)}(t, u) = \delta^{-1} \int_0^\delta \bar{r}_{\text{int}}^{(n)}(t + s, u) ds.$$

We have that $r_{\text{int}}^{(n, \delta)}(t, \cdot)$ converges weakly to $r^{(\delta)}(t, \cdot) = \delta^{-1} \int_0^\delta r(t + s, \cdot) ds$, when $n \rightarrow +\infty$, for any $t \geq 0$ and $\delta > 0$.

In fact, using (4.10) and the Cauchy-Schwarz inequality, together with (4.16), we have that for any $t \geq 0$, $n = 1, 2, \dots$

$$\|\bar{r}_{\text{int}}^{(n, \delta)}(t, \cdot)\|_{H^1[0,1]}^2 \leq \frac{C}{\delta}, \quad (4.22)$$

which implies that $r_{\text{int}}^{(n, \delta)}(t, u)$ converges to $r^{(\delta)}(t, \cdot)$ uniformly in $u \in [0, 1]$, as $n \rightarrow +\infty$, for each $t \geq 0$ and $\delta > 0$. This in particular entails the strong convergence of $(\bar{r}_{\text{int}}^{(n, \delta)}(t, \cdot))^2$ to $(r^{(\delta)}(t, \cdot))^2$ in $L^1[0, 1]$. We can write therefore

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \left| \int_0^1 \Phi(u) (\bar{r}_{\text{int}}^{(n)}(t, u))^2 du - \int_0^1 \Phi(u) r^2(t, u) du \right| \\ & \leq \limsup_{n \rightarrow +\infty} \left| \int_0^1 \Phi(u) (\bar{r}_{\text{int}}^{(n, \delta)}(t, u))^2 du - \int_0^1 \Phi(u) (r^{(\delta)}(t, u))^2 du \right| \\ & \quad + \left| \int_0^1 \Phi(u) (r^{(\delta)}(t, u))^2 du - \int_0^1 \Phi(u) r^2(t, u) du \right|, \end{aligned}$$

the first term on the right hand side being equal to 0.

Since $r(t, u)$ is the strong solution of (2.23), it is regular and $r^{(\delta)}(t, u)$ converges to $r(t, u)$ pointwise, and strongly in $L^2[0, 1]$, as $\delta \rightarrow 0$. Since $\delta > 0$ is arbitrary, therefore (4.21), thus also (4.9), follows. \square

5. ASYMPTOTICS OF THE WORK FUNCTIONAL: PROOF OF THEOREM 2.4

In this section we prove Theorem 2.4. The total work done in the macroscopic time scale is defined by (2.24), and we decompose it into

$$W_n(t) = W_n^{\text{mech}}(t) + W_n^{\text{th}}(t), \quad (5.1)$$

where

$$W_n^{\text{mech}}(t) = \frac{1}{n} \int_0^{n^2 t} \bar{F} \bar{p}_n(s) ds, \quad W_n^{\text{th}}(t) = \frac{1}{n} \int_0^{n^2 t} \tilde{\mathcal{F}}_n(s) \bar{p}_n(s) ds. \quad (5.2)$$

We compute the limits of both terms in (5.2) in the following two propositions, which straightforwardly imply Theorem 2.4.

Proposition 5.1 (Macroscopic mechanical work). *For any $t \geq 0$,*

$$\lim_{n \rightarrow \infty} W_n^{\text{mech}}(t) = \frac{\bar{F}}{2\gamma} \int_0^t (\partial_u r)(s, 1) ds. \quad (5.3)$$

Proof. By the same calculation done for (4.3) and (4.4) (using (3.5)), we have

$$\begin{aligned} n \int_0^t \bar{p}_n(n^2 s) ds &= \frac{1}{n} \sum_{x=0}^n \frac{x}{n} (\bar{r}_x(n^2 t) - \bar{r}_x(0)) + \frac{1}{2\gamma} \int_0^t r_n(n^2 s) ds \\ &\quad - \frac{1}{2\gamma n^2} \sum_{x=0}^{n-1} (\bar{p}_x(n^2 t) - \bar{p}_x(0)). \end{aligned} \quad (5.4)$$

By (3.1) the last term is negligible. Using (3.4) and Theorem 2.3 with $\varphi(u) = u$, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} n \int_0^t \bar{p}_n(n^2 s) ds &= \int_0^1 u (r(t, u) - r(0, u)) du + \frac{\bar{F}}{2\gamma} t \\ &= \frac{1}{2\gamma} \int_0^t ds \int_0^1 u \partial_u^2 r(s, u) du + \frac{\bar{F}}{2\gamma} t \\ &= -\frac{1}{2\gamma} \int_0^t ds \int_0^1 \partial_u r(s, u) du + \frac{1}{2\gamma} \int_0^t ds \partial_u r(s, 1) + \frac{\bar{F}}{2\gamma} t \\ &= -\frac{1}{2\gamma} \int_0^t r(s, 1) ds + \frac{1}{2\gamma} \int_0^t \partial_u r(s, 1) ds + \frac{\bar{F}}{2\gamma} t \\ &= \frac{1}{2\gamma} \int_0^t \partial_u r(s, 1) ds, \end{aligned} \quad (5.5)$$

since $r(s, 1) \equiv \bar{F}$ (recall (2.23)). \square

Proposition 5.2 (Macroscopic thermal work). *Recall the decomposition (3.39) for the average momentum \bar{p}_n . We have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{n^2 t} \tilde{\mathcal{F}}_n(s) \bar{z}_n(s) ds = 0 \quad (5.6)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{n^2 t} \tilde{\mathcal{F}}_n(s) P_{\bar{F}}(s) ds = 0 \quad (5.7)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{n^2 t} \tilde{\mathcal{F}}_n(s) P_{dp}(s) ds = 0 \quad (5.8)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{n^2 t} \tilde{\mathcal{F}}_n(s) P_{fl}(s) ds = t \mathbb{W}^Q, \quad (5.9)$$

where \mathbb{W}^Q has been defined in (2.26). These four limits imply that

$$\lim_{n \rightarrow \infty} W_n^{\text{th}}(t) = t \mathbb{W}^Q. \quad (5.10)$$

Proof. First of all, (5.6) follows easily from (3.20). Let us go on with (5.7). This is equal to

$$\frac{\bar{F}}{n^{3/2}} \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) \sum_{j=0}^n \frac{\psi_j^2(n)}{\Delta \lambda_j} \left(\frac{1 - e^{(i\omega(\ell) - \lambda_{j,-})n^2 t}}{\lambda_{j,-} - i\omega(\ell)} - \frac{1 - e^{(i\omega(\ell) - \lambda_{j,+})n^2 t}}{\lambda_{j,+} - i\omega(\ell)} \right). \quad (5.11)$$

Let us define, for any $\alpha \in \mathbb{R}$, and $A, B \in \mathbb{C}$,

$$\Phi_\alpha(A, B) := \frac{1 - e^{-A\alpha}}{A} - \frac{1 - e^{-B\alpha}}{B}. \quad (5.12)$$

Invoke the following straightforward identity:

$$\frac{\Phi_\alpha(A, B)}{B - A} = \frac{1 - e^{-A\alpha}}{AB} + \frac{e^{-A\alpha}(e^{-(B-A)\alpha} - 1)}{B(B - A)}. \quad (5.13)$$

We use it with $A = \lambda_{j,-} - i\ell\omega$, $B = \lambda_{j,+} - i\ell\omega$ and $\alpha = n^2 t$. The first contribution on the right hand side reads

$$\frac{\bar{F}}{n^{3/2}} \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) \sum_{j=0}^n \psi_j^2(n) \frac{1 - e^{(i\omega(\ell) - \lambda_{j,-})n^2 t}}{\lambda_j - (\ell\omega)^2 - 2i\ell\omega\gamma}$$

and from the exact same estimates that we already used in the previous proofs, see for instance (3.38), we easily get that this contribution vanishes, as $n \rightarrow +\infty$. The remaining part is treated similarly invoking the key Lemma 3.3, and we conclude that this term vanishes as $n \rightarrow \infty$. This ends the proof of (5.7).

We now proceed with (5.8). Recall formula (3.43) for $P_{dp}(t)$. We have

$$\frac{1}{n} \int_0^{n^2 t} \mathcal{F}_n(s) P_{dp}(s) ds = \mathbb{I}_{n,1}^{(dp)}(t) + \mathbb{I}_{n,2}^{(dp)}(t)$$

where

$$\mathbb{I}_{n,1}^{(dp)}(t) := \frac{\bar{F}}{n^{3/2}} \sum_{\ell \neq 0} \widehat{\mathcal{F}}(\ell) \sum_{j=0}^n \frac{\psi_j^2(n)}{\Delta \lambda_j} \left\{ \frac{\lambda_{j,+} [1 - e^{-\lambda_{j,+} n^2 t}]}{i\ell\omega - \lambda_{j,+}} - \frac{\lambda_{j,-} [1 - e^{-\lambda_{j,-} n^2 t}]}{i\ell\omega - \lambda_{j,-}} \right\} \quad (5.14)$$

and

$$\mathbb{I}_{n,2}^{(dp)}(t) = \frac{1}{n^2} \sum_{\ell, \ell' \neq 0} \widehat{\mathcal{F}}(\ell') \widehat{\mathcal{F}}(\ell) \sum_{j=0}^n \frac{\psi_j^2(n)}{\Delta \lambda_j} \left\{ \frac{\lambda_{j,+} [1 - e^{(i\omega(\ell') - \lambda_{j,+})n^2 t}]}{(i\ell\omega - \lambda_{j,+})(i\omega(\ell') - \lambda_{j,+})} - \frac{\lambda_{j,-} [1 - e^{(i\omega(\ell') - \lambda_{j,-})n^2 t}]}{(i\ell\omega - \lambda_{j,-})(i\omega(\ell') - \lambda_{j,-})} \right\}. \quad (5.15)$$

The proof proceeds along the same lines as previously: there is a way to decompose the terms which appear inside the brackets on the right hand sides of (5.14) and (5.15), so that to make the quotient $(1 - e^{-\Delta\lambda_j n^2 t})/\Delta\lambda_j$ appears. Then we apply the key Lemma 3.3. All the estimates are standard ones, reminiscent to the proof of estimate (3.3). In the end one concludes that both $I_{n,1}^{(dp)}(t)$ and $I_{n,2}^{(dp)}(t)$ vanish as $n \rightarrow \infty$. This concludes the proof of (5.8).

It remains to prove (5.9). Recall (3.42) for the definition of $P_{fl}(t)$. We have, similarly

$$\frac{1}{n} \int_0^{n^2 t} \tilde{\mathcal{F}}_n(s) P_{fl}(s) ds = I_{n,2}^{(fl)}(t) + I_{n,3}^{(fl)}(t),$$

where

$$I_{n,2}^{(fl)}(t) := t \sum_{\ell \neq 0} |\widehat{\mathcal{F}}(\ell)|^2 i\ell\omega \sum_{j=0}^n \frac{\psi_j^2(n)}{\lambda_j - (\ell\omega)^2 + 2i\ell\omega\gamma},$$

$$I_{n,3}^{(fl)}(t) := \frac{1}{n^2} \sum_{\substack{\ell, \ell' \neq 0 \\ \ell \neq -\ell'}} \widehat{\mathcal{F}}(\ell) \widehat{\mathcal{F}}(\ell') \frac{\ell\omega(1 - e^{i(\ell\omega + \ell'\omega)n^2 t})}{\ell\omega - \ell'\omega} \sum_{j=0}^n \frac{\psi_j^2(n)}{\lambda_j - (\ell\omega)^2 + 2i\ell\omega\gamma}.$$

It is straightforward to argue that $I_{n,3}^{(fl)}(t)$ vanish as $n \rightarrow \infty$. Concerning $I_{n,2}^{(fl)}(t)$, thanks to the fact that $\widehat{\mathcal{F}}(-\ell) = \widehat{\mathcal{F}}^*(\ell)$, we have

$$I_{n,2}^{(fl)}(t) = 4\gamma t \sum_{\ell=1}^{+\infty} (\ell\omega)^2 |\widehat{\mathcal{F}}(\ell)|^2 \frac{1}{n+1} \sum_{j=0}^n \frac{(2 - \delta_{j,0}) \cos^2\left(\frac{\pi j}{2(n+1)}\right)}{\left(4 \sin^2\left(\frac{\pi j}{2(n+1)}\right) - (\ell\omega)^2\right)^2 + 4(\ell\omega)^2 \gamma^2}.$$

Therefore

$$\lim_{n \rightarrow +\infty} I_{n,2}^{(fl)}(t) = 8\gamma t \sum_{\ell=1}^{+\infty} (\ell\omega)^2 |\widehat{\mathcal{F}}(\ell)|^2 \int_0^1 \frac{\cos^2\left(\frac{\pi u}{2}\right) du}{\left(4 \sin^2\left(\frac{\pi u}{2}\right) - (\ell\omega)^2\right)^2 + 4(\ell\omega)^2 \gamma^2}.$$

We now need to prove that last expression equals (5.9). Observe that

$$\frac{1}{\left(4 \sin^2\left(\frac{\pi u}{2}\right) - (\ell\omega)^2\right)^2 + 4\gamma^2(\ell\omega)^2} = -\frac{1}{2\gamma\omega\ell} \operatorname{Im} \left[\frac{1}{4 \sin^2\left(\frac{\pi u}{2}\right) - (\ell\omega)^2 + i2\gamma\ell\omega} \right].$$

Using contour integration one can show that

$$\int_0^1 \frac{\cos^2\left(\frac{\pi u}{2}\right) du}{4 \sin^2\left(\frac{\pi u}{2}\right) + \lambda} = \frac{1}{4} \left(1 - i \sqrt{-\frac{4+\lambda}{\lambda}} \right),$$

for any complex valued λ such that $\operatorname{Im} \lambda > 0$. Using the above formula for $\lambda = -(\ell\omega)^2 + 2\gamma i\ell\omega$, when ℓ is a positive integer, we get

$$\mathbb{W}^Q = \sum_{\ell=1}^{+\infty} |\widehat{\mathcal{F}}(\ell)|^2 \ell\omega \operatorname{Re} \sqrt{\frac{4}{(\ell\omega)^2 - i2\gamma\ell\omega} - 1}. \quad (5.16)$$

which is the expression (2.26) for \mathbb{W}^Q . \square

6. ENERGY BOUNDS FROM ENTROPY PRODUCTION

Before turning to the proof of the convergence for the total energy profile, in this section we provide a few important preliminary results. Recall that μ_n is the initial distribution of the momenta and stretches. We show that the macroscopic energy functional stays bounded in the macroscopic time. Recall the notation \mathcal{H}_n for the total microscopic energy introduced in (2.2) and the initial bound (2.29).

Theorem 6.1 (Energy bound). *Under Assumptions 2.1, 2.2, 2.7, there exists a constant $C > 0$ such that, for any $t \geq 0$ and $n = 1, 2, \dots$,*

$$\mathbb{E}_{\mu_n}[\mathcal{H}_n(n^2t)] \leq Cn(t+1). \quad (6.1)$$

Proof. Taking the time derivative of the relative entropy \mathbf{H}_n defined by (2.28) we have

$$\begin{aligned} \frac{d}{dt}\mathbf{H}_n(n^2t) &= n^2 \int_{\Omega_n} \mathcal{F}_n(n^2t) \partial_{p_n} f_n(n^2t) \, d\nu_{T_-} - 2\gamma n^2 T_- \int_{\Omega_n} \frac{(\partial_{p_0} f_n(n^2t))^2}{f_n(n^2t)} \, d\nu_{T_-} \\ &\quad + n^2 \gamma \int_{\Omega_n} f_n(t) S_{\text{flip}} \log f_n(t) \, d\nu_{T_-}. \end{aligned} \quad (6.2)$$

Since the last term on the right hand side of (6.2) involving S_{flip} is negative, we have

$$\begin{aligned} \mathbf{H}_n(n^2t) + T_- n^2 \int_0^t ds \int_{\Omega_n} \frac{(\partial_{p_0} f_n(n^2s))^2}{f_n(n^2s)} \, d\nu_{T_-} \\ \leq \mathbf{H}_n(0) + n^2 \int_0^t \mathcal{F}_n(n^2s) ds \int_{\Omega_n} \partial_{p_n} f_n(n^2s) \, d\nu_{T_-} \\ = \mathbf{H}_n(0) + \frac{1}{T_-} \int_0^t \mathcal{F}_n(n^2s) \bar{p}_n(n^2s) \, ds = \mathbf{H}_n(0) + \frac{nW_n(t)}{T_-}. \end{aligned} \quad (6.3)$$

By the assumption (2.29) on the initial entropy and Theorem 2.4 which gives the limiting work, we therefore obtain the entropy bound

$$\mathbf{H}_n(n^2t) \leq Cn(t+1), \quad n \geq 1, t \geq 0. \quad (6.4)$$

Then, as we did already for the initial time in (2.31), we use the entropy inequality and from (6.4) we conclude that:

$$\begin{aligned} \mathbb{E}_{\mu_n}[\mathcal{H}_n(n^2t)] &= \int_{\Omega_n} \left(\sum_{x=0}^n \mathcal{E}_x \right) f_n(n^2t) \, d\nu_{T_-} \\ &\leq \frac{1}{\alpha} \left\{ \log \left(\int_{\Omega_n} \exp \left\{ \frac{\alpha}{2} \sum_{x=0}^n (p_x^2 + r_x^2) \right\} \, d\nu_{T_-} \right) + \mathbf{H}_n(n^2t) \right\} \end{aligned} \quad (6.5)$$

for any $t \geq 0$ and $\alpha > 0$. Hence for any $\alpha \in (0, T_-^{-1})$ we can find $C_\alpha > 0$ such that

$$\mathbb{E}_{\mu_n}[\mathcal{H}_n(n^2t)] \leq \frac{1}{\alpha} (C_\alpha n + \mathbf{H}_n(n^2t)), \quad t \geq 0, \quad (6.6)$$

and (6.1) follows. \square

From the proof of Theorem 6.1 (see (6.3)) we also obtain the following:

Corollary 6.2. *There exists $C > 0$ such that, for any n and $t \geq 0$:*

$$T_- \int_0^t ds \int_{\Omega_n} \frac{[\partial_{p_0} f_n(n^2s)]^2}{f_n(n^2s)} \, d\nu_{T_-} \leq \frac{C(t+1)}{n}. \quad (6.7)$$

Finally, we give an important corollary about the boundary behavior:

Corollary 6.3 (Current estimate and boundary temperature). *For any $t_* > 0$, there exists $C = C(t_*) > 0$ such that, for any n and $t \in [0, t_*]$,*

$$\left| \int_0^t (T_- - \mathbb{E}_{\mu_n}[p_0^2(n^2s)]) \, ds \right| \leq \frac{C}{n} \quad (6.8)$$

and

$$\sup_{x=0, \dots, n} \left| \int_0^t \mathbb{E}_{\mu_n}[j_{x-1, x}(n^2s)] \, ds \right| \leq \frac{C}{n}. \quad (6.9)$$

Proof. Recall that the time derivative of the microscopic energy is a local gradient, see (2.13). We conclude that the total energy evolves in time according to the equation

$$\mathbb{E}_{\mu_n}[\mathcal{H}_n(n^2t)] = \mathbb{E}_{\mu_n}[\mathcal{H}_n(0)] + 2\gamma n^2 \int_0^t (T_- - \mathbb{E}_{\mu_n}[p_0^2(n^2s)]) ds + nW_n(t). \quad (6.10)$$

Thanks to Theorem 6.1, see the energy bound (6.1), and Theorem 2.4 about the limiting work, we conclude (6.8).

Let us now prove (6.9). Using once again the local gradient (2.13) we conclude that

$$\left| \int_0^t j_{x-1,x}(n^2s) ds - \int_0^t j_{x,x+1}(n^2s) ds \right| = \frac{1}{n^2} |\mathcal{E}_x(n^2t) - \mathcal{E}_x(0)|$$

Hence, by the boundary estimate (6.8) and the energy bound (6.1) we have

$$\left| \int_0^t \mathbb{E}_{\mu_n}[j_{x,x+1}(n^2s)] ds \right| \leq \frac{1}{n^2} \sum_{y=0}^x \mathbb{E}_{\mu_n}[\mathcal{E}_y(n^2t) + \mathcal{E}_y(0)] + \frac{C}{n} \leq \frac{C'}{n}$$

and (6.9) follows. \square

7. THE MACROSCOPIC ENERGY EQUATIONS

The main goal of this section is to prove both convergences (2.36) and (2.40) asserted in Theorems 2.11 and 2.13. Using the current identity (2.13), one can easily see that an important quantity which needs to be controlled is $\langle\langle j_{x,x+1} \rangle\rangle_t$, where $\langle\langle \cdot \rangle\rangle_t$ has been defined in (2.17). This is why we start with some preliminary computation, which gives an adequate *fluctuation-dissipation relation*. Then, we sketch the proof at the end of this section, and postpone the proofs of intermediate technical results to the forthcoming sections.

7.1. Covariance matrix and fluctuation-dissipation relation. Recall the following notations, for the fluctuating parts of the configurations:

$$\begin{aligned} r'_x(t) &:= r_x(t) - \bar{r}_x(t), & x = 1, \dots, n, \\ p'_x(t) &:= p_x(t) - \bar{p}_x(t), & x = 0, \dots, n. \end{aligned} \quad (7.1)$$

We introduce the block *covariance matrix*:

$$S(t) := \begin{bmatrix} S^{(r)}(t) & S^{(r,p)}(t) \\ S^{(p,r)}(t) & S^{(p)}(t) \end{bmatrix}, \quad (7.2)$$

where

$$\begin{aligned} S^{(r)}(t) &= \left(\mathbb{E}_{\mu_n}[r'_x(t)r'_y(t)] \right)_{x,y=1,\dots,n}, & S^{(r,p)}(t) &= \left(\mathbb{E}_{\mu_n}[r'_x(t)p'_y(t)] \right)_{x=1,\dots,n,y=0,\dots,n}, \\ S^{(p)}(t) &= \left(\mathbb{E}_{\mu_n}[p'_x(t)p'_y(t)] \right)_{x,y=0,\dots,n} & \text{and} & S^{(p,r)}(t) = \left(S^{(r,p)}(t) \right)^T. \end{aligned} \quad (7.3)$$

Similarly to (2.17), we use the notation

$$\langle\langle S \rangle\rangle_t := \frac{1}{t} \int_0^t S(n^2s) ds.$$

We rewrite the time average of expression (2.14) in terms of the second moment matrix $S^{(p,r)}$ and of the work functional, as follows:

$$\langle\langle j_{x,x+1} \rangle\rangle_t = -\langle\langle S_{x,x+1}^{(p,r)} \rangle\rangle_t - \langle\langle \bar{p}_x \bar{r}_{x+1} \rangle\rangle_t, \quad x = 0, \dots, n-1, \quad (7.4)$$

and at the boundaries

$$\langle\langle j_{n,n+1} \rangle\rangle_t = -\frac{1}{nt} W_n(t), \quad \langle\langle j_{-1,0} \rangle\rangle_t = 2\gamma (T - \langle\langle p_0^2 \rangle\rangle_t). \quad (7.5)$$

Finally, the time average of the microscopic density of thermal energy $\mathcal{E}'_x(t)$, see (2.39), is given by

$$\langle\langle \mathcal{E}'_x \rangle\rangle_t = \frac{1}{2} \left(\langle\langle S_{x,x}^{(p)} \rangle\rangle_t + \langle\langle S_{x,x}^{(r)} \rangle\rangle_t \right). \quad (7.6)$$

The main result of this section is the following:

Lemma 7.1 (Fluctuation-dissipation relation). *We have*

$$\langle\langle S_{x,x+1}^{(p,r)} \rangle\rangle_t = -\frac{1}{4\gamma} \nabla \mathcal{U}_x(t) + \mathcal{V}_x(t), \quad \text{for any } x = 0, \dots, n-1, \quad (7.7)$$

where

$$\mathcal{U}_x(t) := \langle\langle \mathcal{E}'_x \rangle\rangle_t + \frac{1}{2} \langle\langle S_{x,x+1}^{(r)} \rangle\rangle_t + \frac{1}{2} \langle\langle S_{x-1,x}^{(p)} \rangle\rangle_t + \gamma \langle\langle S_{x,x}^{(p,r)} \rangle\rangle_t, \quad x = 0, \dots, n$$

$$\mathcal{V}_x(t) := \frac{1}{n^2 t} \mathbb{E}_{\mu_n} [V_x^e(0) - V_x^e(n^2 t)], \quad x = 0, \dots, n-1,$$

with

$$V_x^e := \frac{1}{8\gamma} (2r'_{x+1} p'_x + p'_{x+1} r'_{x+1} + p'_x r'_x) + \frac{1}{4} (r'_{x+1})^2. \quad (7.8)$$

By convention $r_0 = 0$, $\langle\langle S_{0,1}^{(r)} \rangle\rangle_t = 0$ and $\langle\langle S_{n,n+1}^{(r)} \rangle\rangle_t = 0$. Moreover, for any $t > 0$ there exists $C = C(t) > 0$ such that

$$\sum_{x=0}^{n-1} |\mathcal{V}_x(t)| \leq \frac{C}{n}, \quad n = 1, 2, \dots \quad (7.9)$$

Proof. We use the following straightforward fluctuation-dissipation decompositions of the energy current: for $x = 0, \dots, n-1$ we have

$$j_{x,x+1} = -\frac{1}{4\gamma} \nabla U_x^e + \mathcal{G}_t V_x^e, \quad (7.10)$$

with $U_x^e := \mathcal{E}_x + \frac{1}{2} (r_x r_{x+1} + p_{x-1} p_x) + \gamma p_x r_x$. Here the convention at the right extremity reads $r_{n+1} \equiv \mathcal{F}_n$. Relation (7.10) also holds with the centered quantities, namely

$$-\bar{p}_x \bar{r}_{x+1} = -\frac{1}{4\gamma} \nabla \bar{U}_x^e + \bar{\mathcal{G}}_t \bar{V}_x^e, \quad (7.11)$$

with \bar{U}^e and \bar{V}^e be defined as U^e and V^e , but with every r and p replaced with \bar{r} and \bar{p} . From (7.10), its centered version, and (7.4), we conclude (7.7).

Besides, (7.9) is a direct consequence of the energy bound (6.1) given in Theorem 6.1 together with the control of ℓ^2 norms of averages, given in (3.1) and (3.2) from Proposition 3.1. \square

7.2. Limit of the energy functionals. Consider now the evolution of the energy distribution functional

$$\xi_n^e(\varphi, t) := \frac{1}{n} \sum_{x=0}^n \varphi_x \mathbb{E}_{\mu_n} [\mathcal{E}_x(n^2 t)], \quad (7.12)$$

where $\varphi : [0, 1] \rightarrow \mathbb{R}$ is continuous and $\varphi_x = \varphi(\frac{x}{n})$. By a standard approximation argument it is enough to consider test functions $\varphi \in \mathcal{C}^\infty[0, 1]$ such that

$$\text{supp } \varphi'' \subset (0, 1), \quad \varphi(0) = 0 \quad \text{and} \quad \varphi'(1) = 0. \quad (7.13)$$

The above implies that there exists $u_* > 0$ such that

$$\varphi(u) = \begin{cases} \varphi'(0)u, & u \in [0, u_*] \\ \varphi(1), & u \in (1 - u_*, 1]. \end{cases} \quad (7.14)$$

From equation (2.13) we get

$$\xi_n^e(\varphi, t) - \xi_n^e(\varphi, 0) = -\frac{n^2 t}{n} \sum_{x=0}^n \varphi_x \nabla^* \langle \langle j_{x,x+1} \rangle \rangle_t.$$

Hence, summing by parts, and then using (7.4) and (7.5), plus the fact that $\varphi_0 = 0$, we obtain

$$\begin{aligned} \xi_n^e(\varphi, t) - \xi_n^e(\varphi, 0) &= t \sum_{x=0}^{n-1} n \nabla \varphi_x \langle \langle j_{x,x+1} \rangle \rangle_t - nt \varphi_n \langle \langle j_{n,n+1} \rangle \rangle_t \\ &= -t \sum_{x=0}^{n-1} (n \nabla \varphi_x) \langle \langle S_{x,x+1}^{(p,r)} \rangle \rangle_t - t \sum_{x=0}^{n-1} (n \nabla \varphi_x) \langle \langle \bar{p}_x \bar{r}_{x+1} \rangle \rangle_t + \varphi(1) W_n(t). \end{aligned} \quad (7.15)$$

Recall that $W_n(t)$ converges to $W(t)$, as $n \rightarrow +\infty$ (from Theorem 2.4). This takes care of the last term.

Let us focus on the second term on the right hand side. Using the second equation of (3.5) and then performing the integration by parts (in time), we can rewrite

$$\begin{aligned} \langle \langle \bar{p}_x \bar{r}_{x+1} \rangle \rangle_t &= \frac{1}{2\gamma} \langle \langle \nabla \bar{r}_x \bar{r}_{x+1} \rangle \rangle_t - \frac{1}{2\gamma n^2} \langle \langle \frac{d\bar{p}_x}{ds} \bar{r}_{x+1} \rangle \rangle_t \\ &= \frac{1}{2\gamma} \langle \langle \nabla \bar{r}_x \bar{r}_{x+1} \rangle \rangle_t + \frac{1}{2\gamma n^2} \langle \langle \bar{p}_x \frac{d\bar{r}_{x+1}}{ds} \rangle \rangle_t + \frac{1}{2\gamma n^2 t} [\bar{p}_x(0) \bar{r}_{x+1}(0) - \bar{p}_x(n^2 t) \bar{r}_{x+1}(n^2 t)] \\ &= \frac{1}{2\gamma} \langle \langle \nabla \bar{r}_x \bar{r}_{x+1} \rangle \rangle_t + \frac{1}{2\gamma} \langle \langle \bar{p}_x \nabla \bar{p}_x \rangle \rangle_t + \frac{1}{2\gamma n^2 t} [\bar{p}_x(0) \bar{r}_{x+1}(0) - \bar{p}_x(n^2 t) \bar{r}_{x+1}(n^2 t)]. \end{aligned}$$

We use this decomposition to express the second term of (7.15). Then

$$-t \sum_{x=0}^{n-1} (n \nabla \varphi_x) \langle \langle \bar{p}_x \bar{r}_{x+1} \rangle \rangle_t = B_n(t) + C_n(t) + D_n(t) \quad (7.16)$$

where

$$\begin{aligned} B_n(t) &:= -\frac{t}{2\gamma} \sum_{x=0}^{n-1} (n \nabla \varphi_x) \langle \langle \nabla \bar{r}_x \bar{r}_{x+1} \rangle \rangle_t \\ C_n(t) &:= -\frac{t}{2\gamma} \sum_{x=0}^{n-1} (n \nabla \varphi_x) \langle \langle \bar{p}_x \nabla \bar{p}_x \rangle \rangle_t \\ D_n(t) &:= -\frac{1}{2\gamma n^2} \sum_{x=0}^{n-1} (n \nabla \varphi_x) [\bar{p}_x(0) \bar{r}_{x+1}(0) - \bar{p}_x(n^2 t) \bar{r}_{x+1}(n^2 t)]. \end{aligned}$$

Let us consider each of these contributions separately. We start with $B_n(t)$. Using the identity $(\bar{r}_{x+1} - \bar{r}_x) \bar{r}_{x+1} = \frac{1}{2}(\bar{r}_{x+1}^2 - \bar{r}_x^2) + \frac{1}{2}(\bar{r}_{x+1} - \bar{r}_x)^2$ we rewrite it as

$$-\frac{t}{4\gamma} \sum_{x=0}^{n-1} (n \nabla \varphi_x) \langle \langle \nabla(\bar{r}_x^2) \rangle \rangle_t - \frac{t}{4\gamma} \sum_{x=0}^{n-1} (n \nabla \varphi_x) \langle \langle (\nabla \bar{r}_x)^2 \rangle \rangle_t.$$

Using (4.10) we conclude

$$t \sum_{x=0}^{n-1} (n \nabla \varphi_x) \langle \langle (\nabla \bar{r}_x)^2 \rangle \rangle_t = n \sum_{x=0}^{n-1} \nabla \varphi_x \int_0^t (\nabla \bar{r}_x)^2(n^2 s) ds \xrightarrow[n \rightarrow \infty]{} 0. \quad (7.17)$$

Performing the summation by parts we obtain

$$\begin{aligned}
& -\frac{1}{4\gamma} \sum_{x=0}^{n-1} (n\nabla\varphi_x) \int_0^t \nabla \bar{r}_x^2(n^2s) \, ds \\
& = \frac{1}{4\gamma n} \sum_{x=1}^{n-1} (n^2\Delta\varphi_x) \int_0^t \bar{r}_x^2(n^2s) \, ds - \frac{n\nabla\varphi_{n-1}}{4\gamma} \int_0^t \bar{r}_n^2(n^2s) \, ds \\
& \xrightarrow{n \rightarrow \infty} \frac{1}{4\gamma} \int_0^t \, ds \int_0^1 \varphi''(u) r^2(s, u) \, du
\end{aligned} \tag{7.18}$$

from (4.9) together with (7.14) for the boundary term (note that we have $n\nabla\varphi_{n-1} \equiv 0$ for n large). Above, $r(t, u)$ is the solution to (2.23). Therefore the limit of $B_n(t)$ is given by (7.18). From the Cauchy-Schwarz inequality (applied twice) and (3.1), we bound

$$\begin{aligned}
|C_n(t)| & = \left| \frac{1}{2\gamma} \sum_{x=0}^{n-1} (n\nabla\varphi_x) \int_0^t \bar{p}_x(n^2s) \nabla \bar{p}_x(n^2s) \, ds \right| \\
& \leq C \|\varphi'\|_\infty \left| \sum_{x=0}^{n-1} \int_0^t \bar{p}_x^2(n^2s) \, ds \right|^{1/2} \left| \sum_{x=0}^{n-1} \int_0^t (\bar{p}_{x+1} - \bar{p}_x)^2(n^2s) \, ds \right|^{1/2} \leq \frac{C}{n}
\end{aligned} \tag{7.19}$$

Therefore $C_n(t)$ vanishes as $n \rightarrow +\infty$. Finally, by Proposition 3.1, we also have

$$D_n(t) = -\frac{1}{2\gamma n^2} \sum_{x=0}^{n-1} n\nabla\varphi_x (\bar{r}_{x+1}(0)\bar{p}_x(0) - \bar{r}_{x+1}(n^2t)\bar{p}_x(n^2t)) \xrightarrow{n \rightarrow \infty} 0.$$

It remains to deal with the first term on the right hand side of (7.15). We use the fluctuation-dissipation relation given in (7.7). Then we perform the summation by parts, and use the estimate from (7.9). Doing this, we obtain that this term is equal to

$$\begin{aligned}
-t \sum_{x=0}^{n-1} (n\nabla\varphi_x) \langle\langle S_{x,x+1}^{(p,r)} \rangle\rangle_t & = -\frac{t}{4\gamma} \sum_{x=1}^{n-1} (n\nabla\varphi_x) \nabla (\mathcal{U}_x(t)) + o_n(1) \\
& = \frac{t}{4\gamma n} \sum_{x=2}^{n-1} (n^2\Delta\varphi_x) \mathcal{U}_x(t) + \frac{t}{4\gamma} (n\nabla\varphi_1) \mathcal{U}_1(t) - \frac{t}{4\gamma} (n\nabla\varphi_{n-1}) \mathcal{U}_n(t).
\end{aligned} \tag{7.20}$$

As before we know that $n\nabla\varphi_{n-1} \equiv 0$ for a sufficiently large n .

In order to treat the first two terms on the right hand side, we state the following important result, which will be proven in Section 9, after preliminary work done in Section 8.2.

Theorem 7.2 (Local equilibrium and boundary moments). *For any $\varphi \in \mathcal{C}[0, 1]$ compactly supported in $(0, 1)$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle S_{x,x+h}^{(p,r)} \rangle\rangle_t = 0, \quad \text{for any } h \in \{-1, 0, 1, 2\}, \tag{7.21}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=0}^{n-1} \varphi_x \langle\langle S_{x,x+1}^{(p)} \rangle\rangle_t = 0 \tag{7.22}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=1}^{n-1} \varphi_x \langle\langle S_{x,x+1}^{(r)} \rangle\rangle_t = 0, \tag{7.23}$$

Moreover, at the boundaries:

$$\lim_{n \rightarrow \infty} \langle\langle S_{1,1}^{(p)} \rangle\rangle_t = T_-, \quad \lim_{n \rightarrow \infty} \langle\langle S_{0,1}^{(p)} \rangle\rangle_t = 0, \quad (7.24)$$

$$\lim_{n \rightarrow \infty} \langle\langle S_{1,1}^{(p,r)} \rangle\rangle_t = 0, \quad (7.25)$$

$$\lim_{n \rightarrow \infty} \langle\langle S_{1,1}^{(r)} \rangle\rangle_t = T_-, \quad \lim_{n \rightarrow \infty} \langle\langle S_{1,2}^{(r)} \rangle\rangle_t = 0. \quad (7.26)$$

Let us show how to use Theorem 7.2. By its definition (recall Lemma 7.1), $\mathcal{U}_1(t)$ can be fully expressed in terms of the covariance matrix as

$$\mathcal{U}_1(t) = \frac{1}{2} (\langle\langle S_{1,1}^{(p)} \rangle\rangle_t + \langle\langle S_{1,1}^{(r)} \rangle\rangle_t + \langle\langle S_{1,2}^{(r)} \rangle\rangle_t + \langle\langle S_{0,1}^{(p)} \rangle\rangle_t) + \gamma \langle\langle S_{1,1}^{(p,r)} \rangle\rangle_t.$$

As a consequence of (7.24)–(7.26), we are able to conclude that

$$\frac{t}{4\gamma} (n \nabla \varphi_1) \mathcal{U}_1(t) \xrightarrow{n \rightarrow \infty} \frac{1}{4\gamma} \varphi'(0) T_-.$$

From the definition of $\mathcal{U}_x(t)$ and the fact that

$$\mathcal{E}'_x = \mathcal{E}_x - \frac{1}{2} \bar{p}_x^2 - \frac{1}{2} \bar{r}_x^2, \quad (7.27)$$

we write

$$\frac{t}{4\gamma n} \sum_{x=2}^{n-1} (n^2 \Delta \varphi_x) \mathcal{U}_x(t) = \text{I}_n(t) + \text{II}_n(t)$$

where

$$\begin{aligned} \text{I}_n(t) &:= \frac{t}{4\gamma n} \sum_{x=2}^n (n^2 \Delta \varphi_x) \left(\langle\langle \mathcal{E}_x \rangle\rangle_t - \frac{1}{2} \langle\langle \bar{p}_x^2 \rangle\rangle_t - \frac{1}{2} \langle\langle \bar{r}_x^2 \rangle\rangle_t \right) \\ \text{II}_n(t) &:= \frac{t}{4\gamma n} \sum_{x=2}^n (n^2 \Delta \varphi_x) \left(\frac{1}{2} \langle\langle S_{x,x+1}^{(r)} \rangle\rangle_t + \frac{1}{2} \langle\langle S_{x-1,x}^{(p)} \rangle\rangle_t + \gamma \langle\langle S_{x,x}^{(p,r)} \rangle\rangle_t \right). \end{aligned}$$

As a direct consequence of Theorem 7.2, we conclude that $\text{II}_n(t)$ vanishes as $n \rightarrow +\infty$. The limit of $\text{I}_n(t)$ is easier: the first contribution involving $\langle\langle \mathcal{E}_x \rangle\rangle_t$ will allow us to close the limit equation satisfied by $\xi_n^e(\varphi, t)$, while the remaining part can be handled thanks to the previous sections: from (4.8) and (4.9) we easily get

$$\text{I}_n(t) - \frac{1}{4\gamma} \int_0^t \xi_n^e(\varphi'', s) ds \xrightarrow{n \rightarrow \infty} -\frac{1}{8\gamma} \int_0^t \varphi''(u) r^2(s, u) ds.$$

Summarizing the entire argument we have proven that:

$$\begin{aligned} &\lim_{n \rightarrow \infty} \left[\xi_n^e(\varphi, t) - \xi_n^e(\varphi, 0) - \frac{1}{4\gamma} \int_0^t \xi_n^e(\varphi'', s) ds \right] \\ &= \frac{1}{8\gamma} \int_0^t ds \int_0^1 du \varphi''(u) r^2(s, u) + \varphi'(0) \frac{T_-}{4\gamma} + \varphi(1) W(t), \quad (7.28) \end{aligned}$$

which identifies the limit as the solution of the initial-boundary value problem (2.37). The compactness of the sequence $\{\xi_n^e\}$ in $\mathcal{C}([0, t_*], \mathcal{M}_K([0, 1]))$ follows by the same argument as for $\{\xi_n^r\}$, and this concludes the proof of Theorem 2.11. \square

Let us conclude this section with the outline of the proof of Theorem 2.13. First of all, the convergence (2.40) is obvious thanks to Theorem 2.11, the decomposition (7.27) and the fact that we already know the limit of the fields (4.8) and (4.9). The second convergence result (2.41) follows from an *equipartition result* stated below:

Proposition 7.3 (Equipartition). *For any $\varphi \in \mathcal{C}[0, 1]$ compactly supported in $(0, 1)$ we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=1}^n \varphi_x \left[\langle\langle (p'_x)^2 \rangle\rangle_t - \langle\langle (r'_x)^2 \rangle\rangle_t \right] = 0. \quad (7.29)$$

Proposition 7.3 will also be proved in Section 9.

8. EVOLUTION OF THE COVARIANCE MATRIX AND CONSEQUENCES

8.1. Resolution. In this section we state and solve the system of equations satisfied by the covariance matrix defined in (7.2).

From (2.4)–(2.5) and (3.7) we get the following system of equations

$$dr'_x(t) = \nabla^* p'_x(t) dt, \quad (8.1)$$

$$dp'_x(t) = \nabla r'_x(t) dt - 2\gamma p'_x(t-) dt + 2(1 - \delta_{x,0}) p_x(t-) d\tilde{N}_x(\gamma t) + \delta_{x,0} \sqrt{4\gamma T_-} d\tilde{w}_-(t)$$

for $x = 0, \dots, n$. Here $\{\tilde{N}_x(t) := N_x(t) - t; x = 1, \dots, n\}$ are independent martingales. Furthermore, we define

$$\Sigma(\mathbf{p}) = \begin{bmatrix} 0_n & 0_{n+1,n} \\ 0_{n,n+1} & D(\mathbf{p}) \end{bmatrix}, \quad \text{with } D(\mathbf{p}) = \begin{pmatrix} \sqrt{4\gamma T_-} & 0 & 0 & \dots & 0 \\ 0 & 2p_1 & 0 & \dots & 0 \\ 0 & 0 & 2p_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2p_n \end{pmatrix}. \quad (8.2)$$

The column vector solution $\mathbf{X}'(t) = [\mathbf{r}'(t), \mathbf{p}'(t)]^T$ of (8.1) satisfies

$$\mathbf{X}'(t) = e^{-At} \mathbf{X}'(0) + \int_0^t e^{-A(t-s)} \Sigma(\mathbf{p}(s)) dM(s), \quad t \geq 0. \quad (8.3)$$

Here A is defined by (3.8) and $(M(t))_{t \geq 0}$ is $(2n+1)$ -dimensional column vector martingale

$$dM(s) = [0, \dots, 0, d\tilde{w}_-(s), d\tilde{N}_1(\gamma s), \dots, d\tilde{N}_n(\gamma s)]^T.$$

Recall definition (7.2) for the covariance matrix $S(t)$. From (8.3) we easily get

$$\frac{1}{n^2} \frac{d}{dt} S(t) = -AS(t) - S(t)A^T + 4\gamma \Sigma_2(\mathbf{p}^2(t)), \quad (8.4)$$

where

$$\Sigma_2(\mathbf{p}^2) = \begin{bmatrix} 0_n & 0_{n+1,n} \\ 0_{n,n+1} & D_2(\mathbf{p}^2) \end{bmatrix}, \quad \text{with } D_2(\mathbf{p}^2) = \begin{pmatrix} T_- & 0 & 0 & \dots & 0 \\ 0 & \mathbb{E}_{\mu_n}[p_1^2] & 0 & \dots & 0 \\ 0 & 0 & \mathbb{E}_{\mu_n}[p_2^2] & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathbb{E}_{\mu_n}[p_n^2] \end{pmatrix}.$$

After time averaging as in (2.17), we obtain the following central equation:

$$A \langle\langle S \rangle\rangle_t + \langle\langle S \rangle\rangle_t A^T = 4\gamma \Sigma_2(\langle\langle \mathbf{p}^2 \rangle\rangle_t) + R(n^2 t). \quad (8.5)$$

where

$$R(t) = \frac{1}{t} \mathbb{E}_{\mu_n} [S(0) - S(t)] =: \begin{pmatrix} R^{(r)}(t) & R^{(r,p)}(t) \\ R^{(p,r)}(t) & R^{(p)}(t) \end{pmatrix}. \quad (8.6)$$

Remark 8.1. *Note that the error term $R(n^2 t)$ can be roughly estimated from the energy bound (6.1) given in Theorem 6.1. One can easily get that: for any $n \geq 1$, any $t > 0$,*

$$\sup_{\ell} \sum_x |R_{x+\ell, x}^{(\alpha)}(n^2 t)| \leq \frac{C}{n} \left(1 + \frac{1}{t}\right), \quad \alpha \in \{p, r, pr, rp\}, \quad (8.7)$$

where the summation in the first variable is understood in the modulo sense.

The matrix equation (8.5) which is of the form $AX + XA^T = \Gamma$, with A given by (3.8) and Γ being some fixed data, can be explicitly solved using the Fourier basis (A.1) and (A.2). The computations are quite involved but straightforward, and in order to make the presentation clearer we postpone some details to Appendix B.

Since $t > 0$ is fixed, in order to simplify the notation we write in the following $R^{(\alpha)} \equiv R^{(\alpha)}(n^2t)$, $S^{(\alpha)} \equiv S^{(\alpha)}(n^2t)$ and $\langle\langle S^{(\alpha)} \rangle\rangle \equiv \langle\langle S^{(\alpha)} \rangle\rangle_t$, and similarly for other quantities. First, let us define, for $j, j' = 0, \dots, n$ and $\ell, \ell' = 1, \dots, n$,

$$\begin{aligned} \tilde{S}_{\ell,j}^{(r,p)} &:= \langle \phi_\ell, \langle\langle S^{(r,p)} \rangle\rangle \psi_j \rangle, & \tilde{S}_{j,\ell}^{(p,r)} &:= \langle \psi_j, \langle\langle S^{(p,r)} \rangle\rangle \phi_\ell \rangle, \\ \tilde{S}_{\ell,\ell'}^{(r)} &:= \langle \phi_\ell, \langle\langle S^{(r)} \rangle\rangle \phi_{\ell'} \rangle, & \tilde{S}_{j,j'}^{(p)} &:= \langle \psi_j, \langle\langle S^{(p)} \rangle\rangle \psi_{j'} \rangle. \end{aligned} \quad (8.8)$$

Analogous definitions of $\tilde{R}_{j,j'}^{(\alpha)}$ can be made for the entries of R . We also let

$$\begin{aligned} \tilde{S}_{\ell,j}^{(r,p)}(n^2t) &:= \langle \phi_\ell, S^{(r,p)}(n^2t) \psi_j \rangle, & \tilde{S}_{j,\ell}^{(p,r)}(n^2t) &:= \langle \psi_j, S^{(p,r)}(n^2t) \phi_\ell \rangle, \\ \tilde{S}_{\ell,\ell'}^{(r)}(n^2t) &:= \langle \phi_\ell, S^{(r)}(n^2t) \phi_{\ell'} \rangle, & \tilde{S}_{j,j'}^{(p)}(n^2t) &:= \langle \psi_j, S^{(p)}(n^2t) \psi_{j'} \rangle. \end{aligned} \quad (8.9)$$

Finally, let us define (recall the right hand side of (8.5)):

$$\tilde{F}_{j,j'}(t) := \sum_{y=1}^n \psi_j(y) \psi_{j'}(y) p_y^2(t) + \psi_j(0) \psi_{j'}(0) T_-, \quad j, j' = 0, \dots, n. \quad (8.10)$$

The main technical result of this section gives the explicit formular of any quantity of the form $\tilde{S}_{k,k'}^{(\alpha)}$ as a linear combination of $\langle\langle \tilde{F}_{k,k'} \rangle\rangle$ and of all the $\tilde{R}_{k,k'}^{(\beta)}$ for $\beta \in \{p, r, rp, pr\}$. Namely:

Lemma 8.2 (Resolution of the covariance matrix). *For any $j, j' = 1, \dots, n$ and any $\alpha \in \{p, r, rp, pr\}$ we have*

$$\tilde{S}_{j,j'}^{(\alpha)} = \Theta_\alpha(\lambda_j, \lambda_{j'}) \langle\langle \tilde{F}_{j,j'} \rangle\rangle + \sum_{\beta \in \{p, r, rp, pr\}} \Xi_\beta^{(\alpha)}(\lambda_j, \lambda_{j'}) \tilde{R}_{j,j'}^{(\beta)}, \quad (8.11)$$

where the functions Θ_α and $\Xi_\beta^{(\alpha)}$ are defined as follows: let us introduce the symmetric function $\theta(c, c') := (c - c')^2 + 8\gamma^2(c + c') = \theta(c', c)$, and define

$$\begin{aligned} \Theta_p(c, c') &= \frac{8\gamma^2(c + c')}{\theta(c, c')}, & \Theta_r(c, c') &= \frac{16\gamma^2\sqrt{cc'}}{\theta(c, c')}, \\ \Theta_{p,r}(c, c') &= \frac{4\gamma\sqrt{c'}(c - c')}{\theta(c, c')}, & \Theta_{r,p}(c, c') &= \Theta_{p,r}(c', c), \end{aligned} \quad (8.12)$$

and also, when $\alpha = \beta$,

$$\begin{aligned} \Xi_p^{(p)}(c, c') &= \frac{2\gamma(c + c')}{\theta(c, c')}, & \Xi_r^{(r)}(c, c') &= \frac{2\gamma(8\gamma^2 + c + c')}{\theta(c, c')}, \\ \Xi_{p,r}^{(p,r)}(c, c') &= \frac{4\gamma c'}{\theta(c, c')}, & \Xi_{r,p}^{(r,p)}(c, c') &= \Xi_{p,r}^{(p,r)}(c', c), \end{aligned} \quad (8.13)$$

and when $\alpha \neq \beta$,

$$\begin{aligned}
\Xi_r^{(p)}(c, c') &= \Xi_p^{(r)}(c, c') = \frac{4\gamma\sqrt{cc'}}{\theta(c, c')}, \\
\Xi_{p,r}^{(r,p)}(c, c') &= \Xi_{r,p}^{(p,r)}(c, c') = -\frac{4\gamma\sqrt{cc'}}{\theta(c, c')}, \\
\Xi_{r,p}^{(p)}(c, c') &= -\Xi_p^{(r,p)}(c, c') = \frac{\sqrt{c}(c-c')}{\theta(c, c')}, \\
\Xi_{p,r}^{(p)}(c, c') &= -\Xi_p^{(p,r)}(c, c') = \Xi_{r,p}^{(p)}(c', c), \\
\Xi_r^{(p,r)}(c, c') &= -\Xi_{p,r}^{(r)}(c, c') = \frac{\sqrt{c}(c-c'+8\gamma^2)}{\theta(c, c')}, \\
\Xi_r^{(r,p)}(c, c') &= -\Xi_{r,p}^{(r)}(c, c') = \Xi_r^{(p,r)}(c', c).
\end{aligned} \tag{8.14}$$

When either $j = 0$, or $j' = 0$, formula (8.11) is still valid. When both $j = j' = 0$, then by convention, we let $\Theta_p(0, 0) = 1$, $\Xi_p^{(p)}(0, 0) = \frac{1}{4\gamma}$ and $\Theta_\alpha(0, 0) = \Xi_\alpha^{(\beta)}(0, 0) = 0$ for all $\alpha \neq p$, and all β .

8.2. Consequences of the resolution.

8.2.1. Estimates on the kinetic energy.

Proposition 8.3. *For any $t > 0$ there exists $C = C(t) > 0$ such that, for any $n \geq 1$,*

$$\sum_{x=0}^{n-1} (\langle\langle p_x^2 \rangle\rangle_t - \langle\langle p_{x+1}^2 \rangle\rangle_t)^2 \leq \frac{C}{n} \quad \text{and} \quad \sup_{x=0, \dots, n} \langle\langle p_x^2 \rangle\rangle_t \leq C. \tag{8.15}$$

Proof. The proof is divided into four steps: first, we express $\langle\langle p_x^2 \rangle\rangle_t$ thanks to the resolution (8.11). Then, we obtain an intermediate result which is formulated in Lemma 8.4 below. Finally we show how to prove (8.15).

STEP 1 – From (8.11), and inverse Fourier transforms, we have

$$\langle\langle S_{x,x}^{(p)} \rangle\rangle_t = \sum_{y=0}^n M_{x,y} \langle\langle p_y^2 \rangle\rangle_t + (T_- - \langle\langle p_0^2 \rangle\rangle_t) M_{x,0} + \mathfrak{r}_{x,x}^{(p)}(t), \tag{8.16}$$

where

$$M_{x,y} := \sum_{j,j'=0}^n \Theta_p(\lambda_j, \lambda_{j'}) \psi_j(x) \psi_{j'}(x) \psi_j(y) \psi_{j'}(y) \tag{8.17}$$

and

$$\mathfrak{r}_{x,x}(t) := \sum_{j,j'=0}^n \sum_{\beta \in \{r,p,rp,pr\}} \Xi_\beta^{(p)}(\lambda_j, \lambda_{j'}) \tilde{R}_{j,j'}^{(\beta)} \psi_j(x) \psi_{j'}(x). \tag{8.18}$$

It has been shown in [KLO23a, Appendix A] that

$$\sum_{y'=0}^n M_{x,y'} = \sum_{y'=0}^n M_{y',x} \equiv 1 \quad \text{and} \quad M_{x,y} > 0 \quad \text{for all } x, y = 0, \dots, n. \tag{8.19}$$

Recall that $\langle\langle p_x^2 \rangle\rangle_t = \langle\langle S_{x,x}^{(p)} \rangle\rangle_t + \langle\langle \bar{p}_x^2 \rangle\rangle_t$. By Proposition 3.1 and Corollary 6.3, see the boundary estimate (6.8), we conclude that for each $t > 0$ there exists $C > 0$ such that, for any $n \geq 1$,

$$\sum_{x=0}^n \langle\langle \bar{p}_x^2 \rangle\rangle_t \leq \frac{C}{n} \quad \text{and} \quad |T_- - \langle\langle p_0^2 \rangle\rangle_t| \leq \frac{C}{n}. \tag{8.20}$$

From (8.20) we infer therefore that

$$\langle\langle p_x^2 \rangle\rangle_t = \sum_{y=0}^n M_{x,y} \langle\langle p_y^2 \rangle\rangle_t + \rho_x(t) + \mathfrak{r}_{x,x}^{(p)}(t), \quad (8.21)$$

where $\rho_x(t)$ satisfies: for any $t > 0$ there exists $C > 0$ such that, for any $n \geq 1$,

$$\sup_{s \in [0,t]} \sum_{x=0}^n |\rho_x(s)| \leq \frac{C}{n}. \quad (8.22)$$

STEP 2 – We now prove an intermediate result:

Lemma 8.4. *For any $t > 0$ there exists $C = C(t) > 0$ such that, for any $n \geq 1$,*

$$\sum_{x=0}^{n-1} (\langle\langle p_x^2 \rangle\rangle_t - \langle\langle p_{x+1}^2 \rangle\rangle_t)^2 \leq \frac{C}{n} \sup_{x=0,\dots,n} \langle\langle p_x^2 \rangle\rangle_t. \quad (8.23)$$

Proof of Lemma 8.4. The following lower bound on the matrix $[M_{x,y}]$ comes from [KLO23a, Proposition 7.1] (see also [BLL04]): there exists $c_* > 0$ such that, for any $n \geq 1$, and any $(f_x) \in \mathbb{R}^{n+1}$,

$$\sum_{x,y=0}^n (\delta_{x,y} - M_{x,y}) f_y f_x \geq c_* \sum_{x=0}^{n-1} (\nabla f_x)^2. \quad (8.24)$$

Multiplying both sides of (8.21) by $\langle\langle p_x^2 \rangle\rangle_t$, summing over x and using (8.24) together with estimate (8.22) we immediately obtain: for any $t > 0$ there is $C > 0$ such that

$$\sum_{x=0}^{n-1} (\langle\langle p_x^2 \rangle\rangle_t - \langle\langle p_{x+1}^2 \rangle\rangle_t)^2 \leq C \sum_{x=0}^n |\rho_x(t)| \langle\langle p_x^2 \rangle\rangle_t + C \mathfrak{F}_n(t)$$

where

$$\mathfrak{F}_n(t) := \sum_{x=0}^n \mathfrak{r}_{x,x}^{(p)}(t) \langle\langle p_x^2 \rangle\rangle_t.$$

We now claim that there is $C = C(t) > 0$ such that

$$\mathfrak{F}_n(t) \leq \frac{C}{n} \sup_{x=0,\dots,n} \langle\langle p_x^2 \rangle\rangle_t. \quad (8.25)$$

From last estimate and (8.22), we immediately conclude the proof of Proposition 8.3. Let us prove the claim. Recall (8.6). We can rewrite $\mathfrak{F}_n(t)$ as

$$\mathfrak{F}_n(t) = \mathfrak{F}_n(0, t) - \mathfrak{F}_n(n^2 t, t), \quad (8.26)$$

where (see (8.9))

$$\begin{aligned} \mathfrak{F}_n(s, t) &:= \frac{1}{n^2 t} \sum_{j,j'=0}^n \sum_{\beta \in \{r,p,rp,pr\}} \Xi_{\beta}^{(p)}(\lambda_j, \lambda_{j'}) \widetilde{S}_{j,j'}^{(\beta)}(s) a_{j,j'}(\langle\langle p^2 \rangle\rangle_t), \quad s \geq 0 \\ a_{j,j'}(\langle\langle p^2 \rangle\rangle_t) &:= \sum_{x=0}^n \psi_j(x) \psi_{j'}(x) \langle\langle p_x^2 \rangle\rangle_t. \end{aligned}$$

We show in Appendix that the matrix $[\sum_{\beta} \Xi_{\beta}^{(p)}(\lambda_j, \lambda_{j'}) \widetilde{S}_{j,j'}^{(\beta)}(s)]_{j,j'}$ is non-negative definite (see Lemma B.1, (B.10)), since it is the Fourier image of a covariance matrix. Therefore we can use the following estimate which can be found *e.g.* in [MOA10, H1g, p. 340].

Lemma 8.5. *Suppose that A and B are two symmetric $m \times m$ matrices and A is assumed to be non-negative definite. Then*

$$|\mathrm{Tr}(AB)| \leq \mathrm{Tr}(A) \sup_{\|\xi\|=1} |B\xi \cdot \xi|. \quad (8.27)$$

By (8.27) we have

$$\begin{aligned}\mathfrak{F}_n(s, t) &= \frac{1}{n^2 t} \operatorname{Tr} \left(\sum_{\beta} [\Xi_{\beta}^{(p)}(\lambda_j, \lambda_{j'}) \tilde{S}_{j, j'}^{(\beta)}(s)] [a_{\ell, \ell'}(\langle\langle p^2 \rangle\rangle_t)] \right) \\ &\leq \frac{1}{n^2 t} \operatorname{Tr} \left(\sum_{\beta} [\Xi_{\beta}^{(p)}(\lambda_j, \lambda_{j'}) \tilde{S}_{j, j'}^{(\beta)}(s)] \right) \sup_{\|\xi\|=1} |[a_{j, j'}(\langle\langle p^2 \rangle\rangle_t)] \xi \cdot \xi|. \end{aligned} \quad (8.28)$$

First, note that

$$\sup_{\|\xi\|=1} [a_{j, j'}(\langle\langle p^2 \rangle\rangle_t)] \xi \cdot \xi = \sup_{\|\xi\|=1} \sum_{x=0}^n \sum_{j, j'=0}^n \xi_j \xi_{j'} \psi_j(x) \psi_{j'}(x) \langle\langle p_x^2 \rangle\rangle_t = \sup_{\|\widehat{\xi}\|=1} \sum_{x=0}^n |\widehat{\xi}_x|^2 \langle\langle p_x^2 \rangle\rangle_t,$$

with $\widehat{\xi}_x := \sum_{j=0}^n \psi_j(x) \xi_j$. Therefore we have

$$\sup_{\|\xi\|=1} [a_{j, j'}(\langle\langle p^2 \rangle\rangle_t)] \xi \cdot \xi \leq \sup_{x=0, \dots, n} \langle\langle p_x^2 \rangle\rangle_t. \quad (8.29)$$

Since $\Xi_{\beta}^{(p)} = \frac{1}{4\gamma} \Theta_{\beta}$, $\beta = p, r$ and $\Xi_{\beta}^{(p)} = -\frac{1}{4\gamma} \Theta_{\beta}$, $\beta = pr, rp$, by Lemma B.1 we get

$$\operatorname{Tr} \left(\sum_{\beta} [\Xi_{\beta}^{(p)}(\lambda_j, \lambda_{j'}) \tilde{S}_{j, j'}^{(\beta)}(s)] \right) \leq 2 \left(\sum_{j=0}^n \tilde{S}_{j, j}^{(p)}(s) + \sum_{j=1}^n \tilde{S}_{j, j}^{(r)}(s) \right) \leq 4 \mathbb{E}_{\mu_n} [\mathcal{H}_n(s)]. \quad (8.30)$$

Combining (8.28) with (8.29) and (8.30) we get

$$\mathfrak{F}_n(s, t) \leq \frac{4}{n^2 t} \mathbb{E}_{\mu_n} [\mathcal{H}_n(s)] \sup_{x=0, \dots, n} \langle\langle p_x^2 \rangle\rangle_t.$$

From (8.26) and using the energy bound (6.1), the claim (8.25) follows.

STEP 3 – Using the triangle and then Cauchy-Schwarz inequalities we conclude

$$\begin{aligned} \sup_{x=0, \dots, n} \langle\langle p_x^2 \rangle\rangle_t &\leq \langle\langle p_0^2 \rangle\rangle_t + \sum_{x=0}^{n-1} \left| \langle\langle p_x^2 \rangle\rangle_t - \langle\langle p_{x+1}^2 \rangle\rangle_t \right| \\ &\leq \langle\langle p_0^2 \rangle\rangle_t + \sqrt{n} \left\{ \sum_{x=0}^{n-1} \left(\langle\langle p_x^2 \rangle\rangle_t - \langle\langle p_{x+1}^2 \rangle\rangle_t \right)^2 \right\}^{1/2}. \end{aligned} \quad (8.31)$$

Denote $D_n := \sum_{x=0}^{n-1} \left(\langle\langle p_x^2 \rangle\rangle_t - \langle\langle p_{x+1}^2 \rangle\rangle_t \right)^2$. We can summarize (8.23) and (8.31) as follows: for any $t > 0$ there exists $C > 0$ such that

$$\begin{aligned} D_n &\leq \frac{C}{n} \sup_{x=0, \dots, n} \langle\langle p_x^2 \rangle\rangle_t, \\ \sup_{x=0, \dots, n} \langle\langle p_x^2 \rangle\rangle_t &\leq \langle\langle p_0^2 \rangle\rangle_t + \sqrt{n} D_n^{1/2} \leq \langle\langle p_0^2 \rangle\rangle_t + C + C \left(\sup_{x=0, \dots, n} \langle\langle p_x^2 \rangle\rangle_t \right)^{1/2}, \end{aligned} \quad (8.32)$$

for all $n \geq 1$. Thus the second estimate of (8.15) follows, which in turn implies the first estimate of (8.15) as well. \square

We end this section with two important corollaries.

Corollary 8.6. *For any $t > 0$ there exists $C = C(t) > 0$ such that, for any $n \geq 1$,*

$$\sup_{x=0} \left| \langle\langle p_x p_0 \rangle\rangle_t \right| \leq \frac{C}{n^{1/2}}, \quad \sum_{x=1}^n \left| \langle\langle r_x p_0 \rangle\rangle_t \right| \leq C n^{1/2}. \quad (8.33)$$

Proof. We only prove the estimate for the momenta average, as the argument for the mixed momenta-stretch average is similar. Note that for $x \neq 0$, integrating by parts in p_0 we get

$$\begin{aligned} \int_0^t \mathbb{E}_{\mu_n}[(p_x p_0)(n^2 s)] ds &= -T_- \int_0^t ds \int_{\Omega_n} p_x f_n(n^2 s; \mathbf{r}, \mathbf{p}) \partial_{p_0} g_{T_-}(\mathbf{r}, \mathbf{p}) d\mathbf{r} d\mathbf{p} \quad (8.34) \\ &= T_- \int_0^t ds \int_{\Omega_n} p_x \partial_{p_0} f_n(n^2 s) d\nu_{T_-} \\ &= T_- \int_0^t ds \int_{\Omega_n} p_x f_n^{1/2}(n^2 s) \frac{\partial_{p_0} f_n(n^2 s)}{f_n^{1/2}(n^2 s)} d\nu_{T_-}. \end{aligned}$$

The absolute value of the utmost right hand side can be estimated using the Cauchy-Schwarz inequality by

$$T_- \left\{ \int_0^t \mathbb{E}_{\mu_n}[p_x^2(n^2 s)] ds \right\}^{1/2} \left\{ \int_0^t ds \int_{\Omega_n} \frac{[\partial_{p_0} f_n(n^2 s)]^2}{f_n(n^2 s)} d\nu_{T_-} \right\}^{1/2} \leq \frac{C}{n^{1/2}}, \quad (8.35)$$

by virtue of (6.7) and (8.15). Then the first estimate of (8.33) follows. The argument for the second estimate of (8.33) is similar, except that in the first part of (8.35) we need to keep the summation over x and then use the energy bound (6.1). \square

Finally, we give a last consequence of the kinetic energy estimate which will be used in the forthcoming section.

Using the definition of $\widetilde{F}_{j,j'}(t)$ (see (8.10)) and the trigonometric identity

$$\begin{aligned} \psi_j(y) \psi_{j'}(y) &= \frac{1}{n+1} \left(1 - \frac{\delta_{j',0}}{2}\right)^{1/2} \left(1 - \frac{\delta_{j,0}}{2}\right)^{1/2} \left[\cos\left(\frac{\pi(j+j')(2y+1)}{2(n+1)}\right) \right. \\ &\quad \left. + \cos\left(\frac{\pi(j-j')(2y+1)}{2(n+1)}\right) \right] \end{aligned}$$

we can write,

$$\langle\langle \widetilde{F}_{j,j'} \rangle\rangle_t = \left(1 - \frac{\delta_{j',0}}{2}\right)^{1/2} \left(1 - \frac{\delta_{j,0}}{2}\right)^{1/2} \left[\widehat{\mathfrak{F}}(j-j') + \widehat{\mathfrak{F}}(j+j') \right]. \quad (8.36)$$

where, for $j = -2(n+1), \dots, 2n+1$,

$$\widehat{\mathfrak{F}}(j) := \frac{1}{n+1} \sum_{y=1}^n \cos\left(\frac{\pi j(2y+1)}{2(n+1)}\right) \langle\langle p_y^2 \rangle\rangle_t + \frac{T_-}{n+1} \cos\left(\frac{\pi j}{2(n+1)}\right). \quad (8.37)$$

By another elementary trigonometric identity we have also

$$\begin{aligned} \widehat{\mathfrak{F}}(j) &= \frac{1}{2(n+1) \sin\left(\frac{\pi j}{2(n+1)}\right)} \sum_{y=1}^n \sin\left(\frac{\pi j y}{n+1}\right) \nabla^* \langle\langle p_y^2 \rangle\rangle_t + \frac{T_-}{n+1} \cos\left(\frac{\pi j}{2(n+1)}\right) \\ &= \frac{1}{\sqrt{2^3(n+1)} \sin\left(\frac{\pi j}{2(n+1)}\right)} \sum_{y=1}^n \phi_j(y) \nabla^* \langle\langle p_y^2 \rangle\rangle_t + \frac{T_-}{n+1} \cos\left(\frac{\pi j}{2(n+1)}\right). \end{aligned} \quad (8.38)$$

Corollary 8.7. *For any $t_* > 0$ there exists $C > 0$ such that*

$$\sum_{j=-2n-2}^{2n+1} |\widehat{\mathfrak{F}}(j)| \leq C, \quad t \in [0, t_*]. \quad (8.39)$$

Proof. Since $\widehat{\mathfrak{F}}(2n+2-j) = -\widehat{\mathfrak{F}}(-j)$ and $\widehat{\mathfrak{F}}(j-2n-2) = -\widehat{\mathfrak{F}}(j)$, we have

$$\sum_{j=-2n-2}^{2n+1} |\widehat{\mathfrak{F}}(j)| = 2 \sum_{j=-n-1}^n |\widehat{\mathfrak{F}}(j)|.$$

Since

$$\left| \frac{j}{2(n+1) \sin\left(\frac{\pi j}{2(n+1)}\right)} \right| \leq \frac{1}{2}, \quad j = -n, \dots, n+1,$$

and defining

$$P_{n,j} := \frac{1}{\sqrt{2^3(n+1) \sin\left(\frac{\pi j}{2(n+1)}\right)}} \sum_{y=1}^n \phi_j(y) \nabla^* \langle p_y^2 \rangle_t, \quad (8.40)$$

we have

$$|j P_{n,j}| \leq \frac{(n+1)^{1/2}}{2^{3/2}} \left| \sum_{y=1}^n \phi_j(y) (\nabla^* \langle p_y^2 \rangle_t) \right|.$$

In consequence, using the Plancherel identity

$$\sum_{j=-n}^{n+1} j^2 P_{n,j}^2 \leq \frac{n+1}{8} \sum_{y=1}^n [\nabla^* \langle p_y^2 \rangle_t]^2 \leq C,$$

by virtue of (8.15). Hence, by Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{j=-n-1}^n |\widehat{\mathfrak{F}}(j)| &\leq 2T_- + \sum_{j=-n-1}^n |P_{n,j}| \\ &\leq 2T_- + 2^{1/2} \left\{ \sum_{j=-n-1}^n j^2 P_{n,j}^2 \right\}^{1/2} \left\{ \sum_{j=0}^n \frac{1}{(j+1)^2} \right\}^{1/2} \leq C'. \end{aligned}$$

□

8.2.2. *Consequences of Assumption 2.8.* Recall that the Fourier transforms of the fluctuation fields read as

$$\tilde{p}_j(t) = \sum_{x=0}^n \psi_j(x) p'_x(t) \quad \text{and} \quad \tilde{r}_j(t) = \sum_{x=1}^n \phi_j(x) r'_x(t).$$

Recall also the position functional which has been defined in (2.32). We have the relation

$$\tilde{q}_j(n^2 s) = \lambda_j^{-1/2} \tilde{r}_j(n^2 s). \quad (8.41)$$

Therefore, by (8.41) there exist $c, C > 0$, absolute constants, such that

$$cn^2 \sum_{j=1}^n \frac{\langle (\tilde{r}'_j)^2 \rangle_t}{j^2} \leq \sum_{x=0}^n \langle (q'_x)^2 \rangle_t = \sum_{j=1}^n \frac{\langle (\tilde{r}'_j)^2 \rangle_t}{\lambda_j} \leq Cn^2 \sum_{j=1}^n \frac{\langle (\tilde{r}'_j)^2 \rangle_t}{j^2}. \quad (8.42)$$

We now prove the following technical bound, which will imply the extension of Assumption 2.8 at positive time:

Proposition 8.8. *For any $t_* > 0$ there exists a constant $C = C(t_*) > 0$ such that for any $n \geq 1$, for any $j = 1, \dots, n$,*

$$\frac{\mathbb{E}_{\mu_n} [(\tilde{r}'_j(n^2 t))^2]}{j^2} \leq C \left(1 + \frac{\mathbb{E}_{\mu_n} [(\tilde{r}'_j(0))^2]}{j^2} \right), \quad t \in [0, t_*]. \quad (8.43)$$

Proof. Recall the definition of $\tilde{F}_{j,j'}$ in (8.10). From (8.11), and several easy computations when the functions in (8.12)–(8.14) are evaluated at $c = c'$: for instance we have $\Theta_p(\lambda_j, \lambda_j) = 1$ and

$$\Xi_p^{(p)}(\lambda_j, \lambda_j) = \Xi_r^{(p)}(\lambda_j, \lambda_j) = \frac{1}{4\gamma}, \quad \text{and} \quad \Xi_{r,p}^{(p)}(\lambda_j, \lambda_j) = \Xi_{p,r}^{(p)}(\lambda_j, \lambda_j) = 0,$$

we get the following system (writing for short \mathbb{E} for \mathbb{E}_{μ_n}): for any $j = 1, \dots, n$,

$$\begin{aligned}
\mathbb{E}[\tilde{p}_j(n^2t)^2] &= \tilde{F}_j(t) - \frac{1}{4\gamma n^2} \frac{d}{dt} \mathbb{E}[\tilde{p}_j(n^2t)^2] - \frac{1}{4\gamma n^2} \frac{d}{dt} \mathbb{E}[\tilde{r}_j(n^2t)^2], \\
\mathbb{E}[\tilde{r}_j(n^2t)^2] &= \tilde{F}_j(t) - \frac{1}{4\gamma n^2} \frac{d}{dt} \mathbb{E}[\tilde{p}_j(n^2t)^2] - \frac{1}{n^2} \left(\frac{1}{4\gamma} + \frac{\gamma}{\lambda_j} \right) \frac{d}{dt} \mathbb{E}[\tilde{r}_j(n^2t)^2] \\
&\quad + \frac{1}{n^2 \sqrt{\lambda_j}} \frac{d}{dt} \mathbb{E}[\tilde{p}_j(n^2t) \tilde{r}_j(n^2t)], \\
\mathbb{E}[\tilde{p}_j(n^2t) \tilde{r}_j(n^2t)] &= -\frac{1}{2n^2 \sqrt{\lambda_j}} \frac{d}{dt} \mathbb{E}[\tilde{r}_j(n^2t)^2].
\end{aligned} \tag{8.44}$$

Let us shorten notation and use the energy bound (6.1), and also the fact that $\psi_j^2(y) \leq C/n$, in order to write the following: for any $t \geq 0$,

$$\tilde{F}_j(t) := \mathbb{E}[\tilde{F}_{j,j}(n^2t)] = \sum_{y=1}^n \psi_j^2(y) \mathbb{E}[p_y^2(n^2t)] + \psi_j^2(0) T_- \leq C(t+1). \tag{8.45}$$

Summing the two equations of (8.44) sideways we get: for $j = 1, \dots, n$,

$$\begin{aligned}
\frac{1}{4\gamma} \frac{d}{dt} \left\{ \mathbb{E}[(\tilde{p}_j(n^2t))^2] + 2\mathbb{E}[(\tilde{r}_j(n^2t))^2] \right\} + \frac{d}{dt} \mathbb{E} \left[\left(\frac{1}{2\sqrt{\gamma}} \tilde{p}_j(n^2t) + \left(\frac{\gamma}{\lambda_j} \right)^{1/2} \tilde{r}_j(n^2t) \right)^2 \right] \\
= -n^2 \left(\mathbb{E}[(\tilde{p}_j(n^2t))^2] + \mathbb{E}[(\tilde{r}_j(n^2t))^2] \right) + 2n^2 \tilde{F}_j(t).
\end{aligned} \tag{8.46}$$

Let us turn to the proof of (8.43). Starting from (8.46), we first integrate in time and then drop the first term on the left hand side (bounded from below by 0). This yields:

$$\begin{aligned}
\mathbb{E} \left[\left(\frac{1}{2\sqrt{\gamma}} \tilde{p}_j(n^2t) + \left(\frac{\gamma}{\lambda_j} \right)^{1/2} \tilde{r}_j(n^2t) \right)^2 \right] \leq -n^2 \int_0^t \mathbb{E}[(\tilde{p}_j(n^2s))^2] + \mathbb{E}[(\tilde{r}_j(n^2s))^2] ds \\
+ 2n^2 \int_0^t \tilde{F}_j(s) ds + G_j,
\end{aligned} \tag{8.47}$$

where

$$G_j := \mathbb{E} \left[\left(\frac{1}{2\sqrt{\gamma}} \tilde{p}_j(0) + \left(\frac{\gamma}{\lambda_j} \right)^{1/2} \tilde{r}_j(0) \right)^2 \right] + \frac{1}{4\gamma} \mathbb{E}[(\tilde{p}_j(0))^2] + \frac{1}{2\gamma} \mathbb{E}[(\tilde{r}_j(0))^2].$$

Below we sum and subtract $\frac{1}{2\sqrt{\gamma}} \tilde{p}_j(n^2t)$ and use the standard inequality $(a+b)^2 \leq 2a^2 + 2b^2$, and we get

$$\begin{aligned}
\frac{\gamma}{\lambda_j} \mathbb{E}[(\tilde{r}_j(n^2t))^2] &\leq \frac{1}{2\gamma} \mathbb{E}[(\tilde{p}_j(n^2t))^2] + 2\mathbb{E} \left[\left(\frac{1}{2\sqrt{\gamma}} \tilde{p}_j(n^2t) + \left(\frac{\gamma}{\lambda_j} \right)^{1/2} \tilde{r}_j(n^2t) \right)^2 \right] \\
&\leq \frac{1}{2\gamma} \mathbb{E}[(\tilde{p}_j(n^2t))^2] + n^2 \int_0^t \tilde{F}_j(s) ds + G_j,
\end{aligned}$$

from (8.47). Using the Cauchy-Schwarz inequality and estimate (8.45) to bound the first and second term on the right hand side respectively and the fact that

$C\frac{j^2}{n} \leq \lambda_j \leq c\frac{j^2}{n}$ for universal constants, we conclude that

$$\begin{aligned} cn^2 \frac{\mathbb{E}[(\tilde{r}'_j(n^2t))^2]}{j^2} &\leq C \sum_{x=0}^n \mathbb{E}[(p'_x(n^2t))^2] + Cn^2 \\ &\quad + Cn^2 \frac{\mathbb{E}[(\tilde{r}'_j(0))^2]}{j^2} + C \sum_{x=0}^n \mathbb{E}[(p'_x(0))^2 + (r'_x(0))^2]. \end{aligned}$$

Dividing both sides by n^2 and using the energy bound (6.1) we conclude (8.43). \square

We now give some asymptotics of the functionals involving positions.

Corollary 8.9. *Under Assumption 2.8 for any $t > 0$ we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n^3} \sum_{x=0}^n \mathbb{E}_{\mu_n} [(q'_x)^2(n^2t)] = 0. \quad (8.48)$$

Proof. From (8.42) and (8.43), for any $\kappa \in (0, 1)$ we can write

$$\begin{aligned} \frac{1}{n^3} \sum_{x=0}^n \langle\langle (q'_x)^2 \rangle\rangle_t &\leq C \frac{1}{n} \sum_{j=1}^n \frac{\langle\langle (\tilde{r}'_j)^2 \rangle\rangle_t}{j^2} \leq C \frac{1}{n} \sum_{j=1}^{n^\kappa} \frac{\langle\langle (\tilde{r}'_j)^2 \rangle\rangle_t}{j^2} + C \frac{1}{n^{1+2\kappa}} \sum_{j=n^\kappa+1}^n \langle\langle (\tilde{r}'_j)^2 \rangle\rangle_t \\ &\leq Cn^{\kappa-1} + C \frac{1}{n} \sum_{j=1}^{n^\kappa} \frac{\mathbb{E}_{\mu_n}[(\tilde{r}'_j(0))^2]}{j^2} + \frac{C}{n^{2\kappa}}. \end{aligned} \quad (8.49)$$

By Assumption 2.8 and (8.42), we conclude (8.48). \square

Proposition 8.10. *For any $\varphi \in \mathcal{C}^2[0, 1]$ such that $\text{supp } \varphi \subset (0, 1)$ we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle q'_x p'_0 \rangle\rangle_t = 0. \quad (8.50)$$

Proof. Define

$$\widehat{\varphi^{(o)}}(j) := \frac{1}{n} \sum_{x=0}^n \varphi_x \sin\left(\frac{\pi j x}{n+1}\right), \quad \widehat{\varphi^{(e)}}(j) := \frac{1}{n} \sum_{x=0}^n \varphi_x \cos\left(\frac{\pi j x}{n+1}\right). \quad (8.51)$$

Since $\varphi \in \mathcal{C}^2[0, 1]$ and $\text{supp } \varphi \subset (0, 1)$, by [KLO23a, Lemma B.1] there exists $C > 0$ such that

$$|\widehat{\varphi^{(\iota)}}(j)| \leq \frac{C}{\chi_n^2(j)}, \quad j \in \mathbb{Z}, \iota = \{e, o\}, \quad (8.52)$$

where χ_n is a $2(n+1)$ -periodic extension of the function $\chi_n(j) = (1+j) \wedge (2n+3-j)$, for $j = 0, \dots, 2n+1$. We can write

$$\frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle q'_x p'_0 \rangle\rangle_t = \frac{2}{n} \sum_{j=1}^n \sum_{j'=0}^n \left(1 - \frac{\delta_{j',0}}{2}\right)^{1/2} \cos\left(\frac{\pi j'}{2(n+1)}\right) \lambda_j^{-1/2} \widetilde{S}_{j,j'}^{(r,p)}(\lambda_j, \lambda_{j'}) \widehat{\varphi^{(o)}}(j). \quad (8.53)$$

Using the decomposition (8.11) we can rewrite

$$\frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle q'_x p'_0 \rangle\rangle_t = \mathbb{I}_n + \sum_{\beta \in \{r,p,rp,pr\}} \mathbb{I}_{n,\beta},$$

where

$$\begin{aligned} \mathbb{I}_n &= \frac{2}{n} \sum_{j=1}^n \sum_{j'=0}^n \left(1 - \frac{\delta_{j',0}}{2}\right)^{1/2} \cos\left(\frac{\pi j'}{2(n+1)}\right) \lambda_j^{-1/2} \Theta_{r,p}(\lambda_j, \lambda_{j'}) \langle\langle \widetilde{F}_{j,j'} \rangle\rangle_t \widehat{\varphi^{(o)}}(j), \\ \mathbb{I}_{n,\beta} &= \frac{2}{n} \sum_{j=1}^n \sum_{j'=0}^n \left(1 - \frac{\delta_{j',0}}{2}\right)^{1/2} \cos\left(\frac{\pi j'}{2(n+1)}\right) \lambda_j^{-1/2} \Xi_\beta^{(r,p)}(\lambda_j, \lambda_{j'}) \widetilde{R}_{j,j'}^{(\beta)} \widehat{\varphi^{(o)}}(j). \end{aligned}$$

We can write, recalling (8.36) and the definition of $\Theta_{r,p}$ (see (8.12))

$$\begin{aligned} I_n &= \frac{4}{n} \sum_{j=1}^n \widehat{\varphi^{(o)}}(j) \sum_{j'=0}^n \left(1 - \frac{\delta_{j',0}}{2}\right) \left(1 - \frac{\delta_{j,0}}{2}\right)^{1/2} \frac{4\gamma(\lambda_{j'} - \lambda_j)}{(\lambda_j - \lambda_{j'})^2 + 8\gamma^2(\lambda_j + \lambda_{j'})} \\ &\quad \times \cos\left(\frac{\pi j'}{2(n+1)}\right) \left[\widehat{\mathfrak{F}}(j - j') + \widehat{\mathfrak{F}}(j + j')\right]. \end{aligned}$$

Hence

$$|I_n| \leq \frac{C}{n} \sum_{j=1}^n |\widehat{\varphi^{(o)}}(j)| \sum_{j'=-n-1}^n \left[|\widehat{\mathfrak{F}}(j - j')| + |\widehat{\mathfrak{F}}(j + j')|\right] \leq \frac{C'}{n},$$

by virtue of (8.39) and (8.52).

Since the estimates for terms $I_{n,\beta}$ are similar to each other, we demonstrate them for $I_{n,r}$. Then,

$$\begin{aligned} I_{n,r} &= \frac{2}{n^3 t} \sum_{j=1}^n \widehat{\varphi^{(o)}}(j) \sum_{j'=0}^n \left(1 - \frac{\delta_{j',0}}{2}\right)^{1/2} \frac{\lambda_{j'}^{1/2}(\lambda_{j'} - \lambda_j + 8\gamma^2)}{\lambda_j^{1/2}[(\lambda_j - \lambda_{j'})^2 + 8\gamma^2(\lambda_j + \lambda_{j'})]} \\ &\quad \times \cos\left(\frac{\pi j'}{2(n+1)}\right) \left(\mathbb{E}[(\tilde{r}'_j \tilde{r}'_{j'})(0)] - \mathbb{E}[(\tilde{r}'_j \tilde{r}'_{j'})(n^2 t)]\right). \end{aligned}$$

Since $(\pi j/n)^2 \geq \lambda_j \geq 2(j/n)^2$ for $j = 1, \dots, n$, there exists $C > 0$ such that

$$\frac{\lambda_{j'}^{1/2} |\lambda_{j'} - \lambda_j + 8\gamma^2|}{\lambda_j^{1/2} [(\lambda_j - \lambda_{j'})^2 + 8\gamma^2(\lambda_j + \lambda_{j'})]} \leq \frac{C n^2 j'}{j[j^2 + (j')^2]}$$

for $j, j' = 1, \dots, n$. This in turn leads to an estimate

$$\begin{aligned} |I_{n,r}| &\leq \frac{C}{nt} \sum_{j=1}^n \frac{|\widehat{\varphi^{(o)}}(j)|}{jj'} \sum_{j'=1}^n \left[\left\{ \mathbb{E}[(\tilde{r}'_j(0))^2] \right\}^{1/2} \left\{ \mathbb{E}[(\tilde{r}'_{j'}(0))^2] \right\}^{1/2} \right. \\ &\quad \left. + \left\{ \mathbb{E}[(\tilde{r}'_j(n^2 t))^2] \right\}^{1/2} \left\{ \mathbb{E}[(\tilde{r}'_{j'}(n^2 t))^2] \right\}^{1/2} \right]. \end{aligned}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned} &\frac{1}{n} \sum_{j,j'=1}^n \frac{|\widehat{\varphi^{(o)}}(j)|}{jj'} \left\{ \mathbb{E}[(\tilde{r}'_j(n^2 s))^2] \right\}^{1/2} \left\{ \mathbb{E}[(\tilde{r}'_{j'}(n^2 s))^2] \right\}^{1/2} \\ &\leq \frac{1}{n} \left\{ \sum_{j,j'=1}^n \frac{\mathbb{E}[(\tilde{r}'_j(n^2 s))^2]}{j^2(j')^2} \right\}^{1/2} \left\{ \sum_{j,j'=1}^n (\widehat{\varphi^{(o)}})^2(j) \mathbb{E}[(\tilde{r}'_{j'}(n^2 s))^2] \right\}^{1/2} \quad (8.54) \end{aligned}$$

for any $s \in [0, t]$. Using (8.52) we can estimate the right hand side by

$$C \left\{ \frac{1}{n} \sum_{j=1}^n \frac{\mathbb{E}[(\tilde{r}'_j(n^2 s))^2]}{j^2} \right\}^{1/2} \left\{ \frac{1}{n} \sum_{j'=1}^n \mathbb{E}[(\tilde{r}'_{j'}(n^2 s))^2] \right\}^{1/2} \rightarrow 0,$$

as $n \rightarrow +\infty$, by (8.49). This ends the proof of (8.50). \square

9. PROOF OF EQUIPARTITION AND LOCAL EQUILIBRIUM

In this section we prove both Theorem 7.2 and Proposition 7.3, which were used in Section 7 in order to identify the limit. Let $\varphi \in \mathcal{C}^2[0, 1]$ be compactly supported in $(0, 1)$. Recall that we have introduced the notation $\varphi_x := \varphi(\frac{x}{n})$.

9.1. Proof of Proposition 7.3. We prove the equipartition result. Similarly to (2.32) the position functional (not recentered) is given by

$$q_x = \sum_{y=1}^x r_y, \quad x = 1, \dots, n, \quad \text{and} \quad q_0 = 0. \quad (9.1)$$

We have

$$\mathcal{G}_t \left(\frac{1}{n} \sum_{x=0}^n \varphi_x (p_x q_x + \gamma q_x^2) \right) = \frac{1}{n} \sum_{x=0}^n \varphi_x \left(p_x^2 + q_x \nabla r_x - p_x p_0 - 2\gamma q_x p_0 \right),$$

As in the proof of Lemma 7.1, see (7.11), a similar identity can be written for the time evolution of the averages, namely the same above sums but with \bar{p}_x , \bar{q}_x , and \bar{r}_x instead. Therefore, we can write the following relation for the recentered quantities (namely q'_x, r'_x, p'_x , recall (7.1)), which is obtained after integrating in time:

$$\begin{aligned} & \frac{1}{n} \sum_{x=0}^n \varphi_x \left[\langle\langle (p'_x)^2 \rangle\rangle_t + \langle\langle q'_x \nabla r'_x \rangle\rangle_t - \langle\langle p'_x p'_0 \rangle\rangle_t - 2\gamma \langle\langle q'_x p'_0 \rangle\rangle_t \right] \\ &= \frac{1}{n^3 t} \sum_{x=0}^n \varphi_x \mathbb{E}_{\mu_n} \left[(p'_x q'_x + \gamma (q'_x)^2)(n^2 t) - (p'_x q'_x + \gamma (q'_x)^2)(0) \right], \end{aligned} \quad (9.2)$$

We now use the identity $\nabla(q'_x r'_x) = (r'_{x+1})^2 + q'_x \nabla r'_x$, valid for $x = 1, \dots, n-1$, and summing by parts we obtain

$$\begin{aligned} \frac{1}{n} \sum_{x=0}^n \varphi_x \left[\langle\langle (p'_x)^2 \rangle\rangle_t - \langle\langle (r'_{x+1})^2 \rangle\rangle_t \right] &= \frac{1}{n} \sum_{x=0}^n (\nabla^* \varphi_x) \langle\langle q'_x r'_x \rangle\rangle_t + \frac{1}{n} \sum_{x=0}^n \varphi_x \left(\langle\langle p'_x p'_0 \rangle\rangle_t - 2\gamma \langle\langle q'_x p'_0 \rangle\rangle_t \right) \\ &+ \frac{1}{n^3 t} \sum_{x=0}^n \varphi_x \mathbb{E}_{\mu_n} \left[(p'_x q'_x + \gamma (q'_x)^2)(n^2 t) - (p'_x q'_x + \gamma (q'_x)^2)(0) \right]. \end{aligned}$$

Thanks to Corollary 8.6, proved below, which estimates the contribution from $\langle\langle p_x p_0 \rangle\rangle_t$, together with a Cauchy-Schwarz inequality similar to (7.19) for the contribution coming from the $\langle\langle \bar{p}_x \bar{p}_0 \rangle\rangle_t$, we can prove directly

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle p'_x p'_0 \rangle\rangle_t = 0.$$

By Proposition 8.9,

$$\lim_{n \rightarrow +\infty} \frac{1}{n^3} \sum_{x=0}^n \mathbb{E}_{\mu_n} \left[(q'_x)^2 (n^2 t) \right] = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle q'_x p'_0 \rangle\rangle_t = 0.$$

This also gives the bound

$$\left| \frac{1}{n^3} \sum_{x=0}^n \varphi_x \mathbb{E}_{\mu_n} \left[(p'_x q'_x)(n^2 t) \right] \right| \leq \frac{\|\varphi\|_\infty}{n} \left(\frac{1}{n} \sum_{x=0}^n \mathbb{E}_{\mu_n} \left[(p'_x)^2 (n^2 t) \right] \right)^{\frac{1}{2}} \left(\frac{1}{n^3} \sum_{x=0}^n \mathbb{E}_{\mu_n} \left[(q'_x)^2 (n^2 t) \right] \right)^{\frac{1}{2}},$$

which therefore vanishes as $n \rightarrow +\infty$ (recall the energy bound (6.1) and also Proposition 3.1 in order to bound the first term on the right hand side). Similarly,

$$\left| \frac{1}{n} \sum_{x=0}^n (\nabla^* \varphi_x) \langle\langle q'_x r'_x \rangle\rangle_t \right| \leq \|\varphi'\|_\infty \left(\frac{1}{n} \sum_{x=0}^n \langle\langle (r'_x)^2 \rangle\rangle_t \right)^{\frac{1}{2}} \left(\frac{1}{n^3} \sum_{x=0}^n \langle\langle (q'_x)^2 \rangle\rangle_t \right)^{\frac{1}{2}}, \quad (9.3)$$

which, from the same reasons, vanishes as $n \rightarrow +\infty$. This concludes the proof of equipartition as stated in Proposition 7.3. \square

9.2. Proof of Theorem 7.2. We start with the case $h = 0$, which is the easiest one. We have $r_x p_x = -j_{x-1,x} + \frac{1}{2} \mathcal{G}_t r_x^2$ and similarly $\bar{r}_x \bar{p}_x = -\bar{r}_x \bar{p}_{x-1} + \frac{1}{2} \bar{\mathcal{G}}_t \bar{r}_x^2$ for $x = 1, \dots, n$. In consequence

$$\begin{aligned} \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle r'_x p'_x \rangle\rangle_t &= \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle \bar{r}_x \bar{p}_{x-1} \rangle\rangle_t \\ &\quad - \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle j_{x-1,x} \rangle\rangle_t + \frac{1}{2n^3 t} \sum_{x=0}^n \varphi_x \mathbb{E}[(r'_x)^2(n^2 t) - (r'_x)^2(0)]. \end{aligned} \quad (9.4)$$

The second and third terms on the right vanish, as $n \rightarrow +\infty$, by virtue of the current estimate (6.9) and the energy bound (6.1). Using the Cauchy-Schwarz inequality together with estimates (3.1) and (3.2) we conclude that also the first term vanishes.

Let $h \in \{-1, 1, 2\}$, and by convention in the following, the boundary quantities out of $\{1, \dots, n\}$ vanish. One can check directly that:

$$r_x p_{x+h} = -q_{x+h} \nabla^* p_x + r_x p_0 + \mathcal{G}_t (r_x q_{x+h}),$$

and similarly for the average quantities. Hence,

$$\begin{aligned} \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle r_x p_{x+h} \rangle\rangle_t &= -\frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle \bar{r}_x \bar{p}_{x+h} \rangle\rangle_t - \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle q'_{x+h} \nabla^* p'_x \rangle\rangle_t \\ &\quad + \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle r'_x p'_0 \rangle\rangle_t + \frac{1}{2n^3 t} \sum_{x=0}^n \varphi_x \mathbb{E}[(r'_x q'_{x+h})(n^2 t) - (r'_x q'_{x+h})(0)]. \end{aligned} \quad (9.5)$$

The first term on the right hand side vanishes with $n \rightarrow +\infty$ as in (9.4).

Estimate (8.33) combined with estimates (3.1) and (3.2) imply that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle r'_x p'_0 \rangle\rangle_t = 0.$$

Concerning the second term we have

$$\begin{aligned} \frac{1}{n} \sum_{x=0}^n \varphi_x \langle\langle q'_{x+h} \nabla^* p'_x \rangle\rangle_t &= \frac{1}{n} \sum_{x=0}^n \sum_{j,j'=1}^n \phi_j(x+h) \nabla^* \psi_{j'}(x) \varphi_x \langle\langle \tilde{q}_j \tilde{p}'_{j'} \rangle\rangle_t \\ &= \frac{1}{n} \sum_{j,j'=1}^n \left(\frac{\lambda_{j'}}{\lambda_j} \right)^{1/2} \tilde{S}_{j,j'}^{(r,p)}(\lambda_j, \lambda_{j'}) \left\{ \left[\widehat{\varphi^{(e)}}(j-j') - \widehat{\varphi^{(e)}}(j+j') \right] \cos\left(\frac{\pi j h}{n+1}\right) \right. \\ &\quad \left. + \left[\widehat{\varphi^{(o)}}(j-j') + \widehat{\varphi^{(o)}}(j+j') \right] \sin\left(\frac{\pi j h}{n+1}\right) \right\}. \end{aligned} \quad (9.6)$$

The fact that it vanishes, as $n \rightarrow +\infty$, can be argued as in (8.53).

Concerning (7.22), note that

$$\mathcal{G}_t(p_x p_{x+1}) = -4\gamma p_x p_{x+1} + (r_{x+1} - r_x) p_{x+1} + (r_{x+2} - r_{x+1}) p_x,$$

and, taking into account (7.21), (7.22) follows;

Concerning (7.23), it follows from the relation

$$\mathcal{G}_t(p_x r_x) = p_x^2 - r_x^2 - p_x p_{x-1} + r_{x+1} r_x - 2\gamma p_x r_x,$$

recalling the previous results (7.21) and (7.22), and the equipartition result (7.29).

For the boundary identities, the first one in (7.24) is a consequence of the decomposition

$$\langle\langle p_1^2 \rangle\rangle_t - \langle\langle \bar{p}_1^2 \rangle\rangle_t = (\langle\langle p_1^2 \rangle\rangle_t - \langle\langle p_0^2 \rangle\rangle_t) + \langle\langle p_0^2 \rangle\rangle_t - \langle\langle \bar{p}_1^2 \rangle\rangle_t$$

plus the *kinetic energy bound* obtained in Proposition 8.3 for the first term; the boundary estimate (6.8) for the second term; and the control of the average (3.1)

for the last term. The second limit follows from (4.2), (6.7) and a simple integration by parts.

About (7.25), note that

$$\begin{aligned}\langle\langle S_{1,1}^{(p,r)} \rangle\rangle_t &= -\langle\langle j_{0,1} \rangle\rangle_t - \langle\langle \bar{p}_1 \bar{r}_1 \rangle\rangle_t + \langle\langle r_1(p_1 - p_0) \rangle\rangle_t = -\langle\langle j_{0,1} \rangle\rangle_t - \langle\langle \bar{p}_0 \bar{r}_1 \rangle\rangle_t + \frac{1}{2} \langle\langle \mathcal{G}_t(r_1^2) \rangle\rangle_t \\ &= -\langle\langle j_{0,1} \rangle\rangle_t - \langle\langle \bar{p}_0 \bar{r}_1 \rangle\rangle_t + \frac{1}{2n^2} \mathbb{E}_{\mu_n} [r_1^2(n^2t) - r_1^2(0)]\end{aligned}$$

that tend to 0 from Proposition 3.1 and Corollary 6.3.

To prove the first of (7.26), notice that $\mathcal{G}_t(r_1 p_0) = r_1^2 + p_1 p_0 - p_0^2 - 2\gamma r_1 p_0$. Hence,

$$\frac{1}{n^2} \mathbb{E}_{\mu_n} [(p_0 r_1)(n^2t) - (p_0 r_1)(0)] = \langle\langle r_1^2 \rangle\rangle_t - \langle\langle p_0^2 \rangle\rangle_t + \langle\langle p_0 p_1 \rangle\rangle_t - 2\gamma \langle\langle p_0 r_1 \rangle\rangle_t, \quad (9.7)$$

and the first of (7.26) follows from (7.24) and (7.25).

We finally prove the second of (7.26). We have $\mathcal{G}_t(r_1 p_1) = r_1 r_2 - r_1^2 + p_1^2 - p_1 p_0 - 2\gamma p_1 r_1$. Hence,

$$\frac{1}{n^2} \mathbb{E}_{\mu_n} [(p_1 r_1)(n^2t) - (p_1 r_1)(0)] = \langle\langle r_2 r_1 \rangle\rangle_t + \langle\langle p_1^2 \rangle\rangle_t - \langle\langle r_1^2 \rangle\rangle_t - \langle\langle p_0 p_1 \rangle\rangle_t - 2\gamma \langle\langle p_1 r_1 \rangle\rangle_t. \quad (9.8)$$

The second of (7.26) then follows from (7.24), (7.25) and the first of (7.26). \square

APPENDIX A. FOURIER COEFFICIENTS. GRADIENT AND DIVERGENCE OPERATORS

A.1. Orthonormal basis. We introduce two orthonormal basis in the respective spaces \mathbb{R}^{n+1} and \mathbb{R}^n , which are used in our assumptions, and in several proofs.

Let us define

$$\psi_j(x) := \left(\frac{2 - \delta_{0,j}}{n+1} \right)^{1/2} \cos \left(\frac{\pi j(2x+1)}{2(n+1)} \right), \quad x, j = 0, \dots, n. \quad (A.1)$$

and

$$\phi_j(x) := \left(\frac{2}{n+1} \right)^{1/2} \sin \left(\frac{j\pi x}{n+1} \right), \quad x, j = 1, \dots, n. \quad (A.2)$$

It turns out that $\{\psi_j\}_{j=0,\dots,n}$ and $\{\phi_j\}_{j=1,\dots,n}$ constitute orthonormal basis of \mathbb{R}^{n+1} and \mathbb{R}^n , respectively. Given two sequences $g = (g_0, \dots, g_n) \in \mathbb{R}^{n+1}$ and $f = (f_1, \dots, f_n) \in \mathbb{R}^n$ we define their respective Fourier transform as

$$\tilde{g}_j := \sum_{x=0}^n \psi_j(x) g_x, \quad j = 0, \dots, n \quad (A.3)$$

$$\tilde{f}_j := \sum_{x=1}^n \phi_j(x) f_x, \quad j = 1, \dots, n. \quad (A.4)$$

Then, the inverse transforms are given by

$$g_x = \sum_{j=0}^n \tilde{g}_j \psi_j(x), \quad x = 0, \dots, n \quad (A.5)$$

$$f_x = \sum_{j=1}^n \tilde{f}_j \phi_j(x), \quad x = 1, \dots, n.$$

In the proofs, we regularly use the fact that:

$$\psi_j^2(x) \leq \frac{C}{n}, \quad \text{with } C > 0 \text{ a universal constant.} \quad (A.6)$$

A.2. Gradient and divergence operators. Let $\mathbb{I}_n := \{0, \dots, n\}$ and take a function $f : \mathbb{I}_n \rightarrow \mathbb{R}$. Then, it can be represented as a vector in finite dimensional space, and its *divergence* $\nabla^* : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is defined as follows:

$$f = \begin{pmatrix} f_0 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \nabla^* f := \begin{pmatrix} f_1 - f_0 \\ \vdots \\ f_n - f_{n-1} \end{pmatrix}$$

so that $\nabla^* f_x = f_x - f_{x-1}$, for any $x = 1, \dots, n$.

The *gradient* operator $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ assigns to each vector

$$g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} \quad \text{a vector} \quad \nabla g = \begin{pmatrix} g_1 \\ g_2 - g_1 \\ \vdots \\ g_n - g_{n-1} \\ -g_n \end{pmatrix}$$

so that $\nabla g_x = g_{x+1} - g_x$ for any $x \in \{0, \dots, n\}$, with the convention $g_0 = g_{n+1} = 0$.

The matrices corresponding to the divergence and gradient are given by (3.9). Note that $\nabla^T = -\nabla^*$, i.e. for any $f : \{0, \dots, n\} \rightarrow \mathbb{R}$ and $g : \{1, \dots, n\} \rightarrow \mathbb{R}$,

$$\sum_{x=0}^n \nabla g_x f_x = - \sum_{x=1}^n g_x \nabla^* f_x. \quad (\text{A.7})$$

Recall the vectors ψ_j , $j = 0, \dots, n$ and ϕ_j , $j = 1, \dots, n$ defined in (A.1) and (A.2). We have the following orthogonality relations: for $x, x', j, j' = 0, \dots, n$,

$$\begin{aligned} \sum_{k=0}^n \psi_k(x) \psi_k(x') &= \delta_{x,x'}, & \sum_{y=0}^n \psi_j(y) \psi_{j'}(y) &= \delta_{j,j'}, \\ \sum_{k=1}^n \phi_k(x) \phi_k(x') &= \delta_{x,x'}, & \sum_{y=1}^n \phi_j(y) \phi_{j'}(y) &= \delta_{j,j'} \end{aligned} \quad (\text{A.8})$$

and the following identities:

$$(\nabla^* \psi_j)_x = -\lambda_j^{1/2} \phi_j(x), \quad x, j = 1, \dots, n \quad (\text{A.9})$$

$$(\nabla \phi_j)_x = \lambda_j^{1/2} \psi_j(x), \quad x = 0, \dots, n, j = 1, \dots, n, \quad (\text{A.10})$$

where

$$\lambda_j := 4 \sin^2 \left(\frac{j\pi}{2(n+1)} \right). \quad (\text{A.11})$$

Define also

$$\begin{aligned} \Delta_{\mathcal{N}} f_x &:= \nabla \nabla^* f_x, & x &= 0, \dots, n, \\ \Delta_{\mathcal{D}} g_x &:= \nabla^* \nabla g_x, & x &= 1, \dots, n. \end{aligned} \quad (\text{A.12})$$

It follows from (A.9) and (A.12) that

$$\Delta_{\mathcal{D}} \phi_j = \lambda_j^{1/2} \nabla^* \psi_j = -\lambda_j \phi_j, \quad \Delta_{\mathcal{N}} \psi_j = -\lambda_j^{1/2} \nabla \phi_j = -\lambda_j \psi_j \quad (\text{A.13})$$

i.e. ϕ_j , λ_j , $j = 1, \dots, n$ and ψ_j , λ_j , $j = 0, \dots, n$ are the eigenvectors and corresponding eigenvalues of $\Delta_{\mathcal{D}}$ and $\Delta_{\mathcal{N}}$, respectively.

APPENDIX B. RESOLUTION OF THE COVARIANCE MATRIX

B.1. Explicit resolution. Here we prove Lemma 8.2, which gives the full resolution of $\tilde{S}_{j,j'}^{(\alpha)}$. Equation (8.5) leads to the following equations on the blocks defined in (7.2):

$$\left[\langle\langle S^{(p,r)} \rangle\rangle \right]^T = \langle\langle S^{(r,p)} \rangle\rangle, \quad \langle\langle S^{(r,p)} \rangle\rangle \nabla = \nabla^* \langle\langle S^{(p,r)} \rangle\rangle + R^{(r)}, \quad (\text{B.1})$$

and moreover

$$\begin{aligned} -\nabla \langle\langle S^{(r)} \rangle\rangle + 2\gamma \langle\langle S^{(p,r)} \rangle\rangle + \langle\langle S^{(p)} \rangle\rangle \nabla &= R^{(p,r)}, \\ \langle\langle S^{(r)} \rangle\rangle \nabla^* + 2\gamma \langle\langle S^{(r,p)} \rangle\rangle - \nabla^* \langle\langle S^{(p)} \rangle\rangle &= R^{(r,p)}, \\ -\nabla \langle\langle S^{(r,p)} \rangle\rangle + \langle\langle S^{(p,r)} \rangle\rangle \nabla^* &= 4\gamma D_2(\langle\langle \mathbf{p}^2 \rangle\rangle) - 4\gamma \langle\langle S^{(p)} \rangle\rangle + R^{(p)}. \end{aligned}$$

With the notation (8.8) for the Fourier coefficients, we get for $j, j' = 1, \dots, n$:

$$\begin{aligned} \lambda_{j'}^{1/2} \tilde{S}_{j,j'}^{(r,p)} &= -\lambda_j^{1/2} \tilde{S}_{j,j'}^{(p,r)} + \tilde{R}_{j,j'}^{(r)} \\ -\lambda_j^{1/2} \tilde{S}_{j,j'}^{(r)} + 2\gamma \tilde{S}_{j,j'}^{(p,r)} + \lambda_{j'}^{1/2} \tilde{S}_{j,j'}^{(p)} &= \tilde{R}_{j,j'}^{(p,r)} \\ -\lambda_{j'}^{1/2} \tilde{S}_{j,j'}^{(r)} + 2\gamma \tilde{S}_{j,j'}^{(r,p)} + \lambda_j^{1/2} \tilde{S}_{j,j'}^{(p)} &= \tilde{R}_{j,j'}^{(r,p)} \\ -\lambda_j^{1/2} \tilde{S}_{j,j'}^{(r,p)} - \lambda_{j'}^{1/2} \tilde{S}_{j,j'}^{(p,r)} &= 4\gamma \langle\langle \tilde{F}_{j,j'} \rangle\rangle - 4\gamma \tilde{S}_{j,j'}^{(p)} + \tilde{R}_{j,j'}^{(p)}. \end{aligned} \quad (\text{B.2})$$

The above equalities also hold when $j = 0$, or $j' = 0$, provided we let $\tilde{r}_0(n^2t) \equiv 0$. This system of equations takes the form

$$\begin{pmatrix} -\lambda_{j'}^{1/2} & \lambda_j^{1/2} & 0 & 2\gamma \\ -\lambda_j^{1/2} & \lambda_{j'}^{1/2} & 2\gamma & 0 \\ 0 & 4\gamma & -\lambda_{j'}^{1/2} & -\lambda_j^{1/2} \\ 0 & 0 & \lambda_j^{1/2} & \lambda_{j'}^{1/2} \end{pmatrix} \begin{pmatrix} \tilde{S}_{j,j'}^{(r)} \\ \tilde{S}_{j,j'}^{(p)} \\ \tilde{S}_{j,j'}^{(p,r)} \\ \tilde{S}_{j,j'}^{(r,p)} \end{pmatrix} = \begin{pmatrix} \tilde{R}_{j,j'}^{(r,p)} \\ \tilde{R}_{j,j'}^{(p,r)} \\ 4\gamma \langle\langle \tilde{F}_{j,j'} \rangle\rangle + \tilde{R}_{j,j'}^{(p)} \\ \tilde{R}_{j,j'}^{(r)} \end{pmatrix}.$$

A long but straightforward linear algebra resolution leads to (8.11).

B.2. Matrix $\mathbf{\Pi}(\Gamma)$ and its properties. Consider a non-negative definite symmetric block matrix

$$\Gamma := \begin{pmatrix} [\Gamma_{x,x'}^{(r)}]_{x,x'=0,\dots,n} & [\Gamma_{x,x'}^{(r,p)}]_{x,x'=0,\dots,n} \\ [\Gamma_{x,x'}^{(p,r)}]_{x,x'=0,\dots,n} & [\Gamma_{x,x'}^{(p)}]_{x,x'=0,\dots,n} \end{pmatrix}.$$

Let $\widehat{\Gamma}$ be its Fourier transform obtained as in (8.8). Define the matrix $\mathbf{\Pi}(\Gamma)$

$$\mathbf{\Pi}(\Gamma)_{j,j'} := \sum_{\iota=pr, r} \Theta_\iota(\lambda_j, \lambda_{j'}) \widehat{\Gamma}_{j,j'}^{(\iota)} - \sum_{\iota=pr, rp} \Theta_\iota(\lambda_j, \lambda_{j'}) \widehat{\Gamma}_{j,j'}^{(\iota)}, \quad j, j' = 0, \dots, n$$

with $\Theta_\iota(\lambda_j, \lambda_{j'})$ given by (8.12).

Lemma B.1. *The matrix $\mathbf{\Pi}(\Gamma)$ is non-negative definite. There exists $C > 0$ such that*

$$\text{Tr} [\mathbf{\Pi}(\Gamma)] \leq 2\text{Tr} [\Gamma]. \quad (\text{B.3})$$

Proof. Consider the stationary solution of the system of equations

$$\begin{aligned} du_x(t) &= \nabla^* v_x(t) dt + dw_x^{(r)}(t), \quad x = 0, \dots, n, \\ dv_x(t) &= \nabla u_x(t) dt - 2\gamma v_x(t) dt + dw_x^{(p)}(t), \quad x = 0, \dots, n, \end{aligned} \quad (\text{B.4})$$

where $\mathbf{W}(t) = [w_0^{(r)}(t), \dots, w_n^{(r)}(t), w_0^{(p)}(t), \dots, w_n^{(p)}(t)]^T$ is a $2n+2$ -dimensional Brownian motion with the covariance matrix Γ .

Let $\mathbf{u}(t) = [u_0(t), \dots, u_n(t)]^T$ and $\mathbf{v}(t) = [v_0(t), \dots, v_n(t)]^T$. The process $\mathbf{Y}(t) = [\mathbf{u}(t), \mathbf{v}(t)]^T$ is Gaussian of zero mean, given by

$$\mathbf{Y}(t) = \int_{-\infty}^t e^{-A(t-s)} d\mathbf{W}(s), \quad t \geq 0, \quad (\text{B.5})$$

with the matrix A given by (3.8). Its covariance matrix C

$$C = \begin{bmatrix} C^{(u)} & C^{(u,v)} \\ C^{(v,u)} & C^{(v)} \end{bmatrix}, \quad (\text{B.6})$$

where

$$\begin{aligned} C^{(u)} &= \left(\mathbb{E}[u_x(t)u_y(t)] \right)_{x,y=0,\dots,n}, & C^{(u,v)} &= \left(\mathbb{E}[u_x(t)v_y(t)] \right)_{x=0,\dots,n,y=0,\dots,n}, \\ C^{(v)} &= \left(\mathbb{E}[v_x(t)v_y(t)] \right)_{x,y=0,\dots,n} & \text{and } C^{(v,u)} &= \left(C^{(u,v)} \right)^T, \end{aligned} \quad (\text{B.7})$$

satisfies equation

$$AC + CA^T = \Gamma, \quad (\text{B.8})$$

Let us define the matrix \tilde{C} as a block matrix, whose respective entries are given by

$$\begin{aligned} \tilde{C}_{j,j'}^{(r,p)} &:= \langle \phi_j, C^{(r,p)} \psi_{j'} \rangle, & \tilde{C}_{j',j}^{(p,r)} &:= \langle \psi_{j'}, C^{(p,r)} \phi_j \rangle \\ \tilde{C}_{j,j'}^{(r)} &:= \langle \phi_j, C^{(r)} \phi_{j'} \rangle, & \tilde{C}_{\ell,\ell'}^{(p)} &:= \langle \psi_\ell, C^{(p)} \psi_{\ell'} \rangle. \end{aligned}$$

Repeating the argument made between (B.1) and (B.2) we have for $j, j' = 0, \dots, n$:

$$\begin{aligned} \lambda_{j'}^{1/2} \tilde{C}_{j,j'}^{(r,p)} &= -\lambda_j^{1/2} \tilde{C}_{j,j'}^{(p,r)} + \widehat{\Gamma}_{j,j'}^{(r)}, \\ -\lambda_j^{1/2} \tilde{C}_{j,j'}^{(r)} + 2\gamma \tilde{C}_{j,j'}^{(p,r)} + \lambda_{j'}^{1/2} \tilde{C}_{j,j'}^{(p)} &= \widehat{\Gamma}_{j,j'}^{(r,p)} \\ -\lambda_{j'}^{1/2} \tilde{C}_{j,j'}^{(r)} + 2\gamma \tilde{C}_{j,j'}^{(r,p)} + \lambda_j^{1/2} \tilde{C}_{j,j'}^{(p)} &= \widehat{\Gamma}_{j,j'}^{(p,r)}, \\ -\lambda_j^{1/2} \tilde{C}_{j,j'}^{(r,p)} - \lambda_{j'}^{1/2} \tilde{C}_{j,j'}^{(p,r)} &= -4\gamma \tilde{C}_{j,j'}^{(p)} + \widehat{\Gamma}_{j,j'}^{(p)}, \end{aligned} \quad (\text{B.9})$$

which leads to

$$\tilde{C}_{j,j'}^{(\iota)} = \sum_{\iota' \in I} \Xi_{\iota'}^{(\iota)} (\lambda_j, \lambda_{j'}) \widehat{\Gamma}_{j,j'}^{(\iota')} \quad j, j' = 0, \dots, n, \quad (\text{B.10})$$

where $\iota \in \{p, r, rp, pr\}$. Since

$$\Xi_{\iota}^{(p)} = \begin{cases} \frac{1}{4\gamma} \Theta_{\iota}, & \text{for } \iota = p, r, \\ -\frac{1}{4\gamma} \Theta_{\iota}, & \text{for } \iota = pr, rp \end{cases}$$

in the particular case $\iota = p$ we get

$$\tilde{C}_{j,j'}^{(p)} = \frac{1}{4\gamma} \sum_{\iota'=p,r} \Theta_{\iota'} (\lambda_j, \lambda_{j'}) \widehat{\Gamma}_{j,j'}^{(\iota')} - \frac{1}{4\gamma} \sum_{\iota'=pr,rp} \Theta_{\iota'} (\lambda_j, \lambda_{j'}) \widehat{\Gamma}_{j,j'}^{(\iota')} \quad (\text{B.11})$$

for $j = 0, \dots, n$, $j' = 0, \dots, n$. Therefore $[\tilde{C}_{j,j'}^{(p)}] = \frac{1}{4\gamma} \mathbf{\Pi}(\Gamma)$ is non-negative definite as a Fourier image of a covariance matrix. Note that

$$\text{Tr}(\mathbf{\Pi}(\Gamma)) = \sum_{j=0}^n \sum_{\iota'=p,r} \Theta_{\iota'} (\lambda_j, \lambda_j) \widehat{\Gamma}_{j,j}^{(\iota')} - \sum_{j=0}^n \sum_{\iota'=pr,rp} \Theta_{\iota'} (\lambda_j, \lambda_j) \widehat{\Gamma}_{j,j}^{(\iota')}.$$

We have

$$\Theta_p(\lambda_j, \lambda_j) = 1, \quad 0 \leq \Theta_r(\lambda_j, \lambda_j) \leq 1 \quad \text{and} \quad \Theta_{p,r}(\lambda_j, \lambda_j) = \Theta_{r,p}(\lambda_j, \lambda_j) = 0.$$

The estimate (B.3) then follows. \square

REFERENCES

- [BBJ⁺16] G. Basile, C. Bernardin, M. Jara, T. Komorowski, and S. Olla, *Thermal conductivity in harmonic lattices with random collisions*, Lecture Notes in Physics, vol. 921, ch. 5, Springer, 2016, <https://doi.org/10.1007/978-3-319-29261-8-5>.
- [Ber07] C. Bernardin, *Hydrodynamics for a system of harmonic oscillators perturbed by a conservative noise*, Stochastic Processes and their Applications **117** (2007), no. 4, 487–513.
- [BLL04] F. Bonetto, Joel L. Lebowitz, and J. Lukkarinen, *Fourier’s Law for a Harmonic Crystal with Self-Consistent Stochastic Reservoirs*, J. Stat. Phys. **116** (2004), 783–813.
- [BO05] C. Bernardin and S. Olla, *Fourier’s Law for a Microscopic Model of Heat Conduction*, J. Stat. Phys. **121** (2005), 271–289.
- [GKLO24] P. L. Garrido, T. Komorowski, Joel L. Lebowitz, and S. Olla, *On the Behaviour of a Periodically Forced and Thermostatted Harmonic Chain*, J. Stat. Phys. **191** (2024), 30.
- [Kel75] J.L. Kelley, *General topology*, Graduate Texts in Mathematics, Springer New York, 1975.
- [KL13] C. Kipnis and C. Landim, *Scaling limits of interacting particle systems*, Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, 2013.
- [KLO23a] T. Komorowski, Joel L. Lebowitz, and S. Olla, *Heat flow in a periodically forced, thermostatted chain*, Communications in Mathematical Physics **400** (2023), 2181–2225.
- [KLO23b] ———, *Heat Flow in a Periodically Forced, Thermostatted Chain II*, J. Stat. Phys. **190** (2023), 87.
- [KLOS23] T. Komorowski, Joel L. Lebowitz, S. Olla, and M. Simon, *On the conversion of work into heat: Microscopic models and macroscopic equations*, Ensaios Matemáticos **38** (2023), 325–341.
- [KOS18] T. Komorowski, S. Olla, and M. Simon, *Macroscopic evolution of mechanical and thermal energy in a harmonic chain with random flip of velocities*, Kinet. Relat. Models **11** (2018), no. 3, 615–645.
- [KOS20] ———, *An open microscopic model of heat conduction: evolution and non-equilibrium stationary states*, Communications in Mathematical Sciences **18** (2020), 751–780.
- [KOS21] ———, *Hydrodynamic limit for a chain with thermal and mechanical boundary forces*, Electronic Journal of Probability **26** (2021), 1–49.
- [MOA10] A.W. Marshall, I. Olkin, and B.C. Arnold, *Inequalities: Theory of majorization and its applications*, Springer Series in Statistics, Springer New York, 2010.
- [Oll19] S. Olla, *Role of conserved quantities in Fourier’s law for diffusive mechanical systems*, Comptes Rendus Physique **20** (2019), no. 5, 429–441.
- [RLL04] Z. Rieder, J. L. Lebowitz, and E. Lieb, *Properties of a harmonic crystal in a stationary nonequilibrium state*, pp. 373–378, Springer Berlin Heidelberg, Berlin, Heidelberg, 2004.

TOMASZ KOMOROWSKI, INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, WARSAW, POLAND.

Email address: komorow@hektor.umcs.lublin.pl

STEFANO OLLA, CEREMADE, UNIVERSITÉ PARIS DAUPHINE - PSL RESEARCH UNIVERSITY, and INSTITUT UNIVERSITAIRE DE FRANCE, and GSSI, L’AQUILA

Email address: olla@ceremade.dauphine.fr

UNIVERSITE CLAUDE BERNARD LYON 1, CNRS, ECOLE CENTRALE DE LYON, INSA LYON, UNIVERSITÉ JEAN MONNET, ICJ UMR5208, 69622 VILLEURBANNE, FRANCE, and GSSI, L’AQUILA

Email address: msimon@math.univ-lyon1.fr