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BLOCH FUNCTIONS IN THE BALL WITH UNIVERSAL NON-TANGENTIAL LIMITS

STÉPHANE CHARPENTIER, NICOLAS ESPOULLIER, RACHID ZAROUF

ABSTRACT. We prove the existence of functions f in the little Bloch space of the unit ball \mathbb{B}_n of \mathbb{C}^n with the property that, given any measurable function φ on the unit sphere \mathbb{S}_n , there exist a sequence $(r_n)_n$, $r_n \in (0, 1)$, converging to 1, such that for every $z \in \mathbb{B}_n$,

$$f(r_n(\zeta - z) + z) \rightarrow \varphi(\zeta) \text{ as } n \rightarrow \infty, \text{ for almost every } \zeta \in \mathbb{S}_n.$$

The set of such functions is residual in the little Bloch space.

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

Let $\mathbb{B}_n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 = |z_1|^2 + \dots + |z_n|^2 < 1\}$ denote the open unit ball of \mathbb{C}^n and $\mathbb{D} = \mathbb{B}_1$ the open unit disc. Without possible confusions, we shall indifferently denote by $|\cdot|$ the modulus of a complex number and the euclidean norm in \mathbb{C}^n . For f holomorphic on \mathbb{B}_n , the radial derivative of f is defined by

$$\mathcal{R}(f) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}.$$

The Bloch space \mathcal{B} consists in all functions f holomorphic on \mathbb{B}_n such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2) |\mathcal{R}(f)(z)| < \infty.$$

Endowed with the norm $\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z \in \mathbb{B}_n} (1 - |z|^2) |\mathcal{R}(f)(z)|$, it is a Banach space. The little Bloch space is the closed subspace of \mathcal{B} , denoted by \mathcal{B}_0 , consisting of all functions f in \mathcal{B} such that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) |\mathcal{R}(f)(z)| = 0.$$

Contrary to the Bloch space, the little Bloch space is a separable Banach space, in which the set of all polynomials is dense. A function in \mathcal{B} (resp. \mathcal{B}_0) will be called a Bloch function (resp. a little Bloch function). For equivalent definitions of the Bloch space and the little Bloch space, and for the standard results that will not be used later, we refer to [25, Chapter 3].

In this note, we are interested in the boundary behaviour of little Bloch functions for $n \geq 1$. It easily follows from a classical tauberian theorem due to Hardy and Littlewood [15] that the lacunary series $\sum_k z^{2^k}$ is a Bloch function with finite radial limits at no points of the unit circle $\mathbb{T} = \partial\mathbb{D}$. By Plessner's theorem, this implies that the image by such functions of almost all radii in \mathbb{D} is dense in \mathbb{C} . In the unit ball of \mathbb{C}^n , proving the existence of Bloch functions with radial limits at no point of \mathbb{S}_n is far more involved, and was done by Ullrich [24]. His proof is constructive and partly relies on ideas of [1]. It makes use of homogeneous polynomials of Ryll-Wojtaszczyk type, that appears in the construction of inner functions in the ball. Note that a result in [3] implies that for any $f \in \mathcal{B}$, \mathbb{S}_n contains a dense set of points at which any f admits radial limit in $\mathbb{C} \cup \{\infty\}$. Thus, for any function f in \mathcal{B}

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with finite radial limit at no point of \mathbb{S}_n , there exists a dense set of points ζ in \mathbb{S}_n such that $|f(r\zeta)| \rightarrow \infty$ as $r \rightarrow 1^-$.

The main result of this note shows that quasi-all functions in \mathcal{B}_0 (in the sense of Baire category theorem) carry an even more erratic non-tangential boundary behaviour, near a.e. point of \mathbb{S}_n . We recall that a subset of separable Fréchet space is residual if it contains a dense countable intersection of open sets. Let us denote by m the normalized Lebesgue measure on \mathbb{S}_n .

Main theorem. *Let $(r_n)_n$ be a sequence of real numbers in $(0, 1)$, converging to 1 as $n \rightarrow \infty$. There exists a function f in \mathcal{B}_0 with the following property \mathcal{P} : given any m -measurable function φ on \mathbb{S}_n , there exists a sequence $(n_k)_k$ of integers such that for any $z \in \mathbb{B}_n$ and m -a.e. $\zeta \in \mathbb{S}_n$,*

$$f(r_{n_k}(\zeta - z) + z\zeta) \rightarrow \varphi(\zeta) \text{ as } k \rightarrow \infty.$$

The set of such functions is residual in \mathcal{B}_0 .

In [4], Bayart proved the same statement, where \mathcal{B}_0 is replaced by the Fréchet space $H(\mathbb{B}_n)$ of all holomorphic functions on \mathbb{B}_n , or by any little growth space $H_{w,0}^\infty$ of the ball. Given a (continuous) weight function $w(t) : [0, 1) \rightarrow (0, +\infty)$ with $w(0^+) = 0$, the growth space H_w^∞ consists of all functions f holomorphic in \mathbb{B}_n such that

$$\sup_{z \in \mathbb{B}_n} w(1 - |z|)|f(z)| < \infty.$$

The space $H_{w,0}^\infty$ is defined as the closed subspace of H_w^∞ consisting of those f for which $w(1 - |z|)|f(z)| \rightarrow 0$ as $|z| \rightarrow 1$. Since there exist weights w for which H_w^∞ is contained in all the Bergman spaces A^p , $p \geq 1$, the space $\bigcap_{p \geq 1} A^p$ contains a (lot of) function(s) satisfying the property \mathcal{P} . In [9], an extension of this result to $H(D)$, where D is a pseudoconvex domain of \mathbb{C}^n , is obtained. We observe that in the space $H(\mathbb{D})$, it is known [8] that quasi-all functions satisfy a much stronger property. In passing, the above theorem answers a question posed in [10] for the Bloch space of the unit disc.

The above results typically fall within the theory of universality [5, 13], and it is now well-understood that most of results of this kind follow from a Baire category argument combined with a diagonal argument, and eventually reduce to a suitable *simultaneous approximation*. The general idea is that, if we are given a (reasonable) separable Fréchet space X continuously embedded in $H(\mathbb{B}_n)$, in order to build a function f in X with some prescribed boundary behaviour m -a.e. on \mathbb{S}_n , it is enough, roughly speaking, to find a function f in X that simultaneously approximates 0 in X and any given continuous function on some *large* subset of \mathbb{S}_n . We point out that simultaneous polynomial approximation in spaces of holomorphic functions is an independent active topic of research. We refer to the seminal thesis work of Khrushchev [17] and to the recent paper [21] (see also the references therein).

Following this strategy, in order to get our main result, we will prove a slightly more precise version of the following simultaneous approximation lemma (see Lemma 2.1).

Lemma (Simultaneous approximation in the Bloch space of the ball). *For any $\varepsilon > 0$, any $g \in \mathcal{B}_0$, and any function φ continuous on \mathbb{S}_n , there exist a measurable set $E \subset \mathbb{S}_n$, with $m(E) \geq 1 - \varepsilon$, and a function $f \in \mathcal{B}_0$ such that*

$$\|f - g\|_{\mathcal{B}} \leq \varepsilon \quad \text{and} \quad \sup_{z \in E} |f(z) - \varphi(z)| \leq \varepsilon.$$

For any holomorphic function space X on \mathbb{B}_n , that contains polynomials, if the set of all polynomials is dense in X and contained in the pointwise multipliers of X , then proving the above lemma reduces to the case where $g = 0$ and $\varphi = 1$. Now, a general approach to simultaneously approximate 0 in X and 1 on a *large* subset of \mathbb{S}_n consists in selecting

the approximant f as equal to $P \circ I$, where P is a polynomial (usually given by Oka-Weyl theorem - or Runge's theorem in \mathbb{C}) that simultaneously approximate uniformly 1 an arbitrarily *large* proper compact subset of \mathbb{S}_n and 0 in a compact subset of \mathbb{B}_n , and where I is an inner mapping chosen so that $P \circ I$ is small in X norm, and so that it *transports* the approximation property of P on (another) *large* subset of \mathbb{S}_n . For example, in [4], the author made use of a similar simultaneous approximation result for $H_{w,0}^\infty$, that was due to Jordan [16]. Therein, the polynomial P is chosen using a Mergelyan's type result for the ball [14], and the inner mapping is $z\Phi^k(z)$, $z \in \mathbb{B}_n$, where ϕ is any inner function vanishing at 0, and the power k large enough (depending on the weight w).

A similar approach is also followed by Limani [21, Section 4.2], who basically proves the above lemma for $n = 1$. The main difficulty consists in showing the existence of an inner function I such that $\|P \circ I\|_{\mathcal{B}}$ is arbitrarily small, which essentially reduces to the existence of an inner function I with arbitrarily small Bloch norm. It turns out that this can be achieved using a deep result of Aleksandrov, Anderson and Nicolau [2]. We mention that this allows Limani to prove the existence of so-called Menshov universal functions in the Bloch space of \mathbb{D} , answering a question posed in [6]. We also refer to [18, 19] for results of a similar nature in other Banach spaces of holomorphic functions on \mathbb{D} .

In order to prove our simultaneous approximation lemma for the Bloch space of the ball, we follow the same strategy as Limani. When $n > 1$, the first difficulty is that one does not have a proper Mergelyan's theorem, and the second one is that inner functions are usually much more difficult to construct. Nevertheless, a combination of simple geometric ideas, that served already in [9], and a partial generalisation of Aleksandrov-Anderson-Nicolau's result, due to Doubtsov [11], will allow us to obtain the main theorem.

The paper is organised as follows: the second section is devoted to the proof of Lemma 2.1. In the third one, we outline the proof of the Main Result. Some extensions are also discussed.

2. SIMULTANEOUS APPROXIMATION IN THE BLOCH SPACE OF THE BALL

Let us start with some notations. By $\mathcal{C}(\mathbb{S}_n)$, we denote the space of all continuous functions on \mathbb{S}_n , endowed with the supremum norm. Given a subset $E \subset \mathbb{C}^n$, the notation $\|\cdot\|_{E,\infty}$ will stand for the essential supremum norm of $L^\infty(E)$. If $E \subset \mathbb{S}_n$ and a compact subset L of \mathbb{B}_n are given, we will denote by E_L^* the set

$$E_L^* := \{r(z - \zeta) + z : z \in L, \zeta \in E, 0 \leq r \leq 1\}.$$

Lemma 2.1. *Let $\varepsilon > 0$, let L be a compact subset of \mathbb{B}_n , let $g \in \mathcal{B}_0$, and let $\varphi \in \mathcal{C}(\mathbb{S}_n)$. There exist a set $E \subset \mathbb{S}_n$ with $\sigma(E) \geq 1 - \varepsilon$ and a function f continuous on E_L^* , such that*

- (i) $\|f - g\|_{\mathcal{B}} \leq \varepsilon$;
- (ii) $\|f - \varphi\|_{E,\infty} \leq \varepsilon$.

The density of the polynomials in \mathcal{B}_0 show that in order to prove Lemma 2.1, it is enough to prove it for $g = 0$, upon replacing φ by $\varphi - Q$, with Q close to g in \mathcal{B}_0 .

The remaining of the section is devoted to the proof of this lemma for $g = 0$. It is based on two results. The first one is geometric and follows immediately from [9, Lemma 3.6], for e.g.

Lemma 2.2. *Let $\varepsilon > 0$ and let U be a domain in \mathbb{C}^n such that $U \cap \mathbb{S}_n \neq \emptyset$. There exists a measurable subset E of $U \cap \mathbb{S}_n$ such that:*

- (i) $\sigma(U \cap \mathbb{S}_n \setminus E) < \varepsilon$;
- (ii) for any compact set $L \subset \mathbb{B}_n$, the set $E \cup L$ is polynomially convex.

The second one deals with the existence of inner function in the ball with a specific behaviour at the boundary. It is a combination of two results by Aleksandrov-Anderson-Nicolau [2] and Doubtsov [11].

Theorem 2.3. *Let $\eta > 0$ be fixed. There exists a non-constant inner function $I : \mathbb{B}_n \rightarrow \mathbb{C}$ such that*

$$\frac{(1 - |z|^2)|\mathcal{R}I(z)|}{1 - |I(z)|^2} \leq \eta, \quad z \in \mathbb{B}_n.$$

In particular, $\|I\|_{\mathcal{B}} \leq \eta$.

Proof. Let us fix $\eta > 0$. Let μ be a pluriharmonic probability measure such that the slices μ_ζ are uniformly symmetric and singular for all $\zeta \in \mathbb{S}_n$. Such measures exist, by Corollary 2.3 in [11]. Then define the non-constant singular inner function

$$\Theta(z) = \exp \left(- \int_{\mathbb{S}_n} \left(\frac{2}{(1 - \langle z, \zeta \rangle)^n} - 1 \right) d\mu(\zeta) \right).$$

Then, again by [11], one has

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)|\mathcal{R}\Theta(z)|}{1 - |\Theta(z)|^2} = 0.$$

In particular, there exists a positive constant $M < \infty$ such that

$$\frac{(1 - |z|^2)|\mathcal{R}\Theta(z)|}{1 - |\Theta(z)|^2} \leq M, \quad z \in \mathbb{B}_n.$$

By Theorem 1 in [2] applied with $\phi(t) = \eta t/M$, there exists an inner function $g : \mathbb{D} \rightarrow \mathbb{D}$ such that

$$\frac{(1 - |w|^2)|g'(w)|}{1 - |g(w)|^2} \leq \frac{\eta}{M}, \quad w \in \mathbb{D}.$$

Let us set $I = g \circ \Theta$. Then I is a non-constant inner function on \mathbb{B}_n such that, for any $z \in \mathbb{B}_n$,

$$\frac{(1 - |z|^2)|\mathcal{R}I(z)|}{1 - |I(z)|^2} = \frac{(1 - |z|^2)|g'(\Theta(z))||\mathcal{R}\Theta(z)|}{1 - |g(\Theta(z))|^2} \leq \frac{\eta}{M} \frac{(1 - |z|^2)|\mathcal{R}\Theta(z)|}{1 - |\Theta(z)|^2} \leq \eta.$$

□

We are now ready to prove Lemma 2.1. The notation $A \lesssim_P B$ shall be used when there exists a positive constant C , depending on the parameter P , such that $A \leq CB$.

Proof of Lemma 2.1 with $g = 0$. Let $\varepsilon > 0$, let L be a compact subset of \mathbb{B}_n , and let $\varphi \in \mathcal{C}(\mathbb{S}_n)$.

First, by uniform continuity of φ , there exist open domains U_1, \dots, U_l in \mathbb{C}^n , pairwise disjoint, with $U_j \cap \mathbb{S}_n \neq \emptyset$ for any j , and complex numbers c_1, \dots, c_l , such that

- (a) $\sigma(\mathbb{S}_n \setminus (\overline{U_1} \cup \dots \cup \overline{U_l})) \leq \varepsilon/5$;
- (b) $|\varphi(z) - c_k| \leq \varepsilon/2$ for any $z \in \overline{U_k} \cap \mathbb{S}_n$ and any $k \in \{1, \dots, l\}$.

Now, applying Lemma 2.2 to each U_k , we choose a compact set $F \subset \mathbb{S}_n \cap (U_1 \cup \dots \cup U_l)$ with $\sigma(F) \geq 1 - \varepsilon/4$, such that $F \cup \{0\}$ is polynomially convex. Since the continuous function equal to 0 at 0 and to c_k on U_k , $k \in \{1, \dots, l\}$, extends holomorphically to an open neighborhood of $\{0\} \cup \overline{U_1} \cup \dots \cup \overline{U_l}$, the Oka-Weyl theorem (see [23], e.g.) ensures the existence of a polynomial Q such that

$$|Q(z) - \varphi(z)| \leq \varepsilon/2, \quad z \in F.$$

Let also P be a polynomial such that

$$(1) \quad |P(0)| \leq \frac{\varepsilon}{2\|Q\|_{\infty, \mathbb{B}_n}} \quad \text{and} \quad |P(z) - 1| \leq \frac{\varepsilon}{2\|Q\|_{\infty, \mathbb{B}_n}}, \quad z \in F.$$

Let now I be given by Theorem 2.3 with the same $\eta > 0$ as above. We define the mapping $J : z = (z_1, \dots, z_n) \mapsto I(z)z = (z_1 I(z), \dots, z_n I(z))$, $z \in \mathbb{B}_n$. It is clear that J is an inner mapping from \mathbb{B}_n to \mathbb{B}_n that vanishes at 0. Now, a simple calculation shows that

$$\mathcal{R}(P \circ J)(z) = (\mathcal{R}(I)(z) + I(z)) \sum_{i=1}^n z_i \frac{\partial P}{\partial z_i}(J(z)),$$

Hence, since P is a polynomial, we get for any $z \in \mathbb{B}_n$,

$$\begin{aligned} |\mathcal{R}(P \circ J)(z)| &\leq \sum_{i=1}^n \left| \frac{\partial P}{\partial z_i}(J(z)) \right| (|I(z)| + |\mathcal{R}(I)(z)|) \\ &\lesssim_P |I(z)| + |\mathcal{R}(I)(z)|. \end{aligned}$$

Moreover, as it is well-known,

$$|I(z)| \lesssim_P \|I\|_{\mathcal{B}} \log\left(\frac{1}{1 - |z|^2}\right), \quad z \in \mathbb{B}_n.$$

Then for any $z \in \mathbb{B}_n$,

$$(1 - |z|^2) |\mathcal{R}(P \circ J)(z)| \lesssim_P (1 - |z|^2) |\mathcal{R}(I)(z)| + (1 - |z|^2) \log\left(\frac{1}{1 - |z|^2}\right) \|I\|_{\mathcal{B}}.$$

By Theorem 2.3, the function $P \circ J$ belongs to \mathcal{B}_0 , and one has $\|I\|_{\mathcal{B}} \leq \eta$. Therefore

$$(1 - |z|^2) |\mathcal{R}(P \circ J)(z)| \lesssim_P \eta \left(1 + (1 - |z|^2) \log\left(\frac{1}{1 - |z|^2}\right) \right), \quad z \in \mathbb{B}_n,$$

hence $(1 - |z|^2) |\mathcal{R}(P \circ J)(z)| \lesssim_P \eta$, $z \in \mathbb{B}_n$.

We set $f = Q.(P \circ J)$. Using the fact that the multiplication by Q is a bounded linear operator on \mathcal{B}_0 [25, Theorem 3.21], and upon choosing η small enough, we deduce by the previous that

$$(1 - |z|^2) |\mathcal{R}(f)(z)| \lesssim_P \eta, \quad z \in \mathbb{B}_n.$$

Moreover, since J vanishes at 0, by (1), we have $|P \circ J(0)| \leq \varepsilon/2$. All in all we have proven the inequality $\|f\|_{\mathcal{B}} \leq \varepsilon$.

To finish the proof, observe that, by (1) and the definition of Q , for any $z \in J^{-1}(F) \cap F$, we have

$$|f(z) - \varphi(z)| \leq |Q(z)| |(P(J(z)) - 1)| + |Q(z) - \varphi(z)| \leq \varepsilon.$$

Since $J = Iz$, with I inner, Loewner's lemma (see [22, Example 14.2]) implies

$$\sigma(J^{-1}(F)) = \sigma(F) \geq 1 - \varepsilon/4,$$

hence $\sigma(J^{-1}(F) \cap F) \geq 1 - \varepsilon/2$. We set $\tilde{E} = J^{-1}(F) \cap F$. The function f being bounded on \mathbb{B}_n , we may apply Egoroff's theorem (see [4, Lemma 1] for details) to choose a compact set E so that f and E satisfy all the desired properties. \square

Remark 2.4. For $f \in H(\mathbb{B}_n)$, let f_r denote the dilate of f defined by $f_r(z) = f(rz)$, $z \in \mathbb{B}_n$. We recall that H^∞ is continuously embedded into \mathcal{B}_0 and that $\|f_r - f\|_{\mathcal{B}} \rightarrow 0$ as $r \rightarrow 1$ for any $f \in \mathcal{B}_0$ [25]. Therefore, upon choosing $r \in (0, 1)$ close enough to 1 and approximating f_r by a polynomial uniformly on $\overline{\mathbb{B}_n}$, the approximating function f in Lemma 2.1 can be chosen as a polynomial.

3. BLOCH FUNCTIONS WITH UNIVERSAL RADIAL LIMITS

3.1. Outline of the proof of the Main Theorem. Let us restate the Main Theorem below.

Theorem 3.1. *Let $(r_n)_n$ be a sequence of real numbers in $(0, 1)$, converging to 1 as $n \rightarrow \infty$. There exists a function f in \mathcal{B}_0 with the following property \mathcal{P} : given any m -measurable function φ on \mathbb{S}_n , there exists a sequence $(n_k)_k$ of integers such that for any $z \in \mathbb{B}_n$ and m -a.e. $\zeta \in \mathbb{S}_n$,*

$$f(r_{n_k}(\zeta - z) + z\zeta) \rightarrow \varphi(\zeta) \text{ as } k \rightarrow \infty.$$

The set of such functions is residual in \mathcal{B}_0 .

Since the proof of Theorem 3.1 is identical to that of Theorem 1 in [4], upon replacing [4, Lemma 2] by Lemma 2.1, we shall only sketch it.

Let $(r_n)_n$ be a sequence in $(0, 1)$, convergent to 1^- . We shall first identify a countable intersection of dense open sets in \mathcal{B}_0 , consisting of functions that all satisfy the property \mathcal{P} , and then apply the Baire category theorem. To do so, we fix an exhaustion $(L_s)_s$ of \mathbb{B}_n by compact sets, and a dense sequence $(\varphi_l)_l$ in $\mathcal{C}(\mathbb{S}_n)$. For $p \in \mathbb{N}$, let us denote by \mathcal{E}_p the set of all measurable subsets E of \mathbb{S}_n with $m(E) \geq 1 - 2^{-p}$. Now, for $l, n, p, s \in \mathbb{N}$, we define the set

$$U(l, n, p, s) = \bigcup_{m \geq n} \bigcup_{E \in \mathcal{E}_p} \{f \in \mathcal{B}_0 : \sup_{z \in L_s} \sup_{\zeta \in E} |f(r_m(\zeta - z) + z) - \varphi_l(\zeta)| < 2^{-s}\}.$$

Then we claim that every function f in $\bigcap_{l, n, p, s \in \mathbb{N}} U(l, n, p, s)$ satisfies the property \mathcal{P} . Indeed, using Lusin's theorem, one first checks that given any measurable function φ on \mathbb{S}_n , there exist sequences $(m_k)_k$, $m_k \in \mathbb{N}$, and $(E(k))_k$, $E(k) \in \mathcal{E}_k$, such that

$$\sup_{z \in L_k} \sup_{\zeta \in E(k)} |f(r_{m_k}(\zeta - z) + z) - \varphi_l(\zeta)| < 2^{-k}.$$

Note that, for the choice of r_{m_k} , it is used that f is absolutely continuous on $E(k)_{L_k}^*$ (see the beginning of Section 2 for the notation). Now, setting $E = \bigcup_{M \in \mathbb{N}} \bigcap_{k \geq M} E(k)$, it is easily checked that f has the desired property for any $\zeta \in E$.

To finish the proof, it remains to check that each set $U(l, n, p, s)$ is open and dense in \mathcal{B}_0 . The fact that \mathcal{B}_0 is continuously embedded into $H(\mathbb{B}_n)$ immediately implies that $U(l, n, p, s)$ is open. The density is a straightforward application of Lemma 2.1.

3.2. Further extensions and remarks.

3.2.1. Weighted Bloch spaces. For $n = 1$, the main theorem can be extended to a scale of generalized Bloch spaces. Let $w : (0, 1) \rightarrow (0, +\infty)$ be a non-decreasing weight (that is $w(t) \rightarrow 0$ as $t \rightarrow 0^+$). The weighted Bloch space $\mathcal{B}(w)$ associated with w is defined as the space of all functions f holomorphic in \mathbb{D} , such that

$$\|f\|_{\mathcal{B}(w)} := |f(0)| + \sup_{|z| < 1} \frac{1 - |z|}{w(1 - |z|)} |f'(z)| < \infty,$$

and the little weighted Bloch space $\mathcal{B}_0(w)$ as the subspace of $\mathcal{B}(w)$ consisting of all f such that

$$\frac{1 - |z|}{w(1 - |z|)} |f'(z)| \rightarrow 0 \text{ as } |z| \rightarrow 1^-.$$

Applying Mergelyan's theorem together with [2, Theorem 5.2], one can follow the same steps as to prove Theorem 3.1 (see also [21, Section 4.2]), in order to get the following improvement in the case $n = 1$.

Theorem 3.2. *Let w be a non-decreasing weight such that, for some $\varepsilon > 0$, the function $w(t)/t^{1-\varepsilon}$ is decreasing. If*

$$\int_x^1 \frac{w(t)^2}{t} dt = +\infty,$$

then there exists a residual set of functions in $\mathcal{B}_0(w)$ satisfying the property \mathcal{P} .

Interestingly, the integral condition in the previous result is sharp. Indeed, Proposition 1.1 in [12] shows that if $\int_x^1 \frac{w(t)^2}{t} dt < +\infty$, then every function in $\mathcal{B}(w)$ has radial limits almost everywhere. Since the latter result holds true for the unit ball \mathbb{B}_n as well, it is natural to wonder whether Theorem 3.2 also holds for $n \geq 2$.

3.2.2. Cluster sets along curves of Bloch functions. Using the notation of the proof of Theorem 3.1, we have seen that all functions of the G_δ -dense subset $\bigcap_{l,n,p,s \in \mathbb{N}} U(l, n, p, s)$ of \mathcal{B}_0 satisfy the property \mathcal{P} . We shall see that, in fact, they also satisfy another property along certain paths in \mathbb{B}_n that end at a point of \mathbb{S}_n .

For $\zeta \in \mathbb{S}_n$, let us say that a (continuous) path $\gamma : [0, 1) \rightarrow \mathbb{B}_n$ *terminates non-tangentially* at ζ if $\gamma(t) \rightarrow \zeta$ as $t \rightarrow 1$, and there exists a compact set $L \subset \mathbb{B}_n$ such that for any $t \in [0, 1)$, $\gamma(t) \in \{\zeta\}_L^*$ where, following the notation introduced at the beginning of Section 2,

$$\{\zeta\}_L^* := \{r(z - \zeta) + z : z \in L, 0 \leq r \leq 1\}.$$

Now, it is clear that for any path γ terminating non-tangentially at some $\zeta \in \mathbb{S}_n$, one has $\gamma([0, 1)) \cap \{r(z - \zeta) + z : z \in L\} \neq \emptyset$, for any $r \in [0, 1)$ large enough. Therefore, for any function f in $\bigcap_{l,n,p,s \in \mathbb{N}} U(l, n, p, s)$, there exists a set $E \in \mathbb{S}_n$, with $m(E) = 1$, such that any path in \mathbb{B}_n terminating non-tangentially at a point $\zeta \in E$ is mapped by f to a dense subset of \mathbb{C} . In particular, we have:

Corollary 3.3. *There exists a residual set in \mathcal{B}_0 , every element f of which satisfies the following property: there exists a set $E \subset \mathbb{S}_n$, with $m(E) = 1$, such that for any path γ terminating non-tangentially at a point of E , the cluster set of f along γ is maximal - that is the set $f(\gamma([0, 1)))$ is dense in \mathbb{C} .*

This result complements the fact that, for every function f in the Bloch space, there exists a dense set of points in \mathbb{S}_n at which f admits (finite or infinite) radial limit (this can be seen as a consequence of a result in [3]). For $n = 1$, since every Bloch function is normal, it turns out that every radial limit of a non-constant function in \mathcal{B} is also a non-tangential limit [20]. In contrast, we recall that there exist a residual set in $H(\mathbb{B}_n)$ of functions having a maximal cluster set along any path terminating at a point of \mathbb{S}_n , with finite length [9]. For the case of the disc, the assumption *with finite length* can be dropped (see [7] or [8]).

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