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Advances in a Quantum Information-based Color Perception Theory

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Abstract

In this contribution it is shown how a recent quantum information-based theory of color perception permits to account in a natural way for several well-known properties and also to predict new ones. The quantum model is based on a completely different paradigm with respect to the one followed in classical colorimetry and it relies on the hypothesis that color sensations are the result of (perceptual) quantum measurements performed by human observers.

1 Introduction

As it is well-known, the classical CIE (Commission Internationale de l'Éclairage) theory is a color reproduction (and *not* color perception) model based on the integration of spectral radiance of light stimuli over the visual spectrum, weighted by the sensitivity functions of the LMS cones. The role of retinal neurons and that of the brain is completely disregarded. As remarked by several vision scientists, noticeably de Valois in [16], while the cones ignite the visual process, color perception is mainly build up by ganglion cells through chromatic opposition, which is not taken into account at all in the original CIE theory.

To remedy this and other problems, the CIE tried to create appearance-friendly color spaces, self-declared 'perceptually-uniform', the most famous of which is the CIELab one. Unfortunately, the construction of such spaces is performed firstly through non-linear and ad hoc manipulations of the non-perceptual XYZ space and, secondly, by imposing chromatic opposition 'at hand' to fit the data of an experimental setting with very strict radiometric constraints.

For these and other reasons, this kind of spaces are theoretically deceiving and they have a questionable practical usefulness. However, in spite of these problems, they are ubiquitously employed in colorimetry, image processing and computer vision, also in situations where their use is implausible.

In this contribution, it is shown that a mathematically rigorous theory of color perception can be built if, instead of concentrating on light spectra and trying to turn them into color sensations, one focuses on the mathematical properties of the last ones, which are strong and robust enough to permit the identification of an intrinsically hyperbolic chromatic space. The fact that this geometric feature emerges naturally is of major importance because *all* the perceptual experiments performed throughout the last century regarding color vision exhibited an hyperbolic behavior.

It turns out that, if the hyperbolic chromatic space is interpreted as the state space of a quantum system, then the rigorous machinery of quantum information theory permits to explain several well-known color perception features, such as, for instance, chromatic opponency, and also to predict new phenomena and equations that can be efficiently used in image processing.

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The plan of the paper is the following: in section 2 the origins of the quantum color perception model will be recalled, in section 3 more recent advances will be discussed and, finally, in section 4 perspectives to further extend the theory in order to encompass and explain appearance phenomena will conclude the paper.

2 The quantum color perception model

Disclaimer: the construction of the quantum color perception model is the result of the collaboration between Michel Berthier, professor at Université de La Rochelle, France, the author of this extended abstract, and their PhD students.

The results that have led to this theory have been published in the following papers [6, 4, 11, 7, 5, 8, 12, 9]. Due to space limitation, here it will be provided only an essential recap of the model that has been developed, the interested reader can find further information in the papers just quoted.

The roots of this work may be founded in the golden era of color study, i.e. the second part of the nineteenth century, when it became more and more clear to renowned mathematicians and physicists such as Riemann, Maxwell, Grassmann and von Helmholtz, that the ensemble of perceptual colors \mathcal{C} is not merely a set of sensations, but a space with a rich mathematical structure.

In [27], Schrödinger condensed the fragmented knowledge about \mathcal{C} into a coherent set of axioms which imply that \mathcal{C} is a *3-dimensional regular convex cone*.

In [26], Resnikoff completed Schrödinger's axiomatic system by showing the existence of a transitive group action on \mathcal{C} , which implies that \mathcal{C} is also a *homogenous space*. This was a piece of major importance to understand the puzzling nature of \mathcal{C} because it implied that \mathcal{C} can only take two forms: either $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, or $\mathbb{R}^+ \times \mathbf{H}$, where \mathbf{H} is a 2-dimensional hyperbolic space, see [25] for more details. For reasons that will be clearer soon, it is important to stress that \mathcal{C} represents an *ideal space of perceptual colors in isolation*, meaning that \mathcal{C} contains all the possible color sensations reported by a trichromatic human being who observes a single light source in isolation and without considering the real lower and upper bounds of human vision as light intensity varies.

The flat space $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ is the geometric prototype of any classical CIE color spaces, whereas $\mathbb{R}^+ \times \mathbf{H}$ is a much more interesting space from an algebraic, geometric and also perceptual point of view. $\mathbb{R}^+ \times \mathbf{H}$ is isomorphic to $\overline{\mathcal{H}^+(2, \mathbb{R})}$, the space of 2×2 positive-semidefinite real symmetric matrices, and to $\overline{\mathcal{L}^+}$, the closure of the future lightcone in the 3-dimensional Minkowski space. From now on, \mathcal{C} will be identified with either $\overline{\mathcal{H}^+(2, \mathbb{R})}$ or $\overline{\mathcal{L}^+}$.

In the last part of his paper, Resnikoff remarked that these two spaces coincide with *the domain of positivity of the only two non-associative 3-dimensional formally real Jordan algebras*: $\mathcal{H}(2, \mathbb{R})$, the vector space of real symmetric 2×2 matrices endowed with the Jordan matrix product, i.e. $A \circ B = (AB + BA)/2$, $A, B \in \mathcal{H}(2, \mathbb{R})$, and $\mathbb{R} \oplus \mathbb{R}^2$, which is called *spin factor* when equipped with the Jordan product $(\alpha, \mathbf{v}) \circ (\beta, \mathbf{w}) = (\alpha\beta + \mathbf{v} \cdot \mathbf{w}, \alpha\mathbf{w} + \beta\mathbf{v})$, $\alpha, \beta \in \mathbb{R}$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and $\mathbf{v} \cdot \mathbf{w}$ denotes the Euclidean inner product of \mathbb{R}^2 , see [3] for more information about Jordan algebras. The only fact that is essential to recap in this document is that $\mathcal{H}(2, \mathbb{R})$ and $\mathbb{R} \oplus \mathbb{R}^2$ are naturally isomorphic as Jordan algebras via the transformation

$$\chi : \begin{array}{ccc} \mathcal{H}(2, \mathbb{R}) & \xrightarrow{\sim} & \mathbb{R} \oplus \mathbb{R}^2 \\ A = \begin{pmatrix} \alpha + v_1 & v_2 \\ v_2 & \alpha - v_1 \end{pmatrix} & \longmapsto & \chi(A) = (\alpha, (v_1, v_2))^t. \end{array} \quad (1)$$

Here is where the quantum assumption comes into play: on one side, Jordan algebras were built by the German mathematical physicist Pascual Jordan to provide a more suitable framework for quantum observables than that of Hermitian operators on Hilbert spaces, on the other side, there are several reasons why interpreting perceptual colors as quantum observables makes more sense than treating them as classical one, see in particular [5, 12] for a detailed discussion about this topic.

The quantum states associated to these observables are more easily computed by considering $\mathcal{H}(2, \mathbb{R})$ instead of the spin factor: in this case, states are represented by *density matrices*, i.e. unit trace matrices belonging to $\overline{\mathcal{H}^+}(2, \mathbb{R})$, explicitly,

$$\mathcal{S} = \left\{ \rho(s_1, s_2) \equiv \frac{1}{2} \begin{pmatrix} 1 + s_1 & s_2 \\ s_2 & 1 - s_1 \end{pmatrix}, s_1, s_2 \in \mathbb{R}, s_1^2 + s_2^2 \leq 1 \right\} \cong \mathcal{D}, \quad (2)$$

where \mathcal{D} is the unit disk in \mathbb{R}^2 . The normalization of the trace is a *linear constraint* responsible for the fact that \mathcal{S} is embedded in a 2-dimensional space. Notice that \mathcal{S} is the analog, in this quantum framework, of the chromatic diagram build by the CIE through a projective transformation, which is a much more difficult operation to handle than the trace normalization.

\mathcal{S} is the state space of the easiest quantum system, called *rebit*, the real analog of a qubit, i.e. a system which can only be measured in two states. Physical examples of qubits are given by an electron with spin up or down and a photon polarized horizontally or vertically. However, regarding rebits, no concrete examples were available before the development of the quantum color perception model, in which the two states that can be measured have a precise, and very important, meaning. To understand their interpretation, it is enough to write a generic density matrix in the so-called *Bloch representation*, i.e. to decompose ρ on the basis of $\mathcal{H}(2, \mathbb{R})$ given by $(\sigma_j)_{j=0}^2$, where $\sigma_0 = I_2$ and

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3)$$

are the two real Pauli matrices. $(\sigma_j)_{j=0}^2$ is an orthogonal basis of $\mathcal{H}(2, \mathbb{R})$ with respect to the Hilbert-Schmidt (HS) inner product, i.e. $\langle A, B \rangle_{\text{HS}} = \text{Tr}(AB)$, for all $A, B \in \mathcal{H}(2, \mathbb{R})$, and $\|\sigma_j\|_{\text{HS}} = 2$ for all $j = 0, 1, 2$, so

$$\rho(s_1, s_2) = \rho_0 + \frac{s_1}{2}\sigma_1 + \frac{s_2}{2}\sigma_2 \equiv \rho_0 + \frac{1}{2}\mathbf{v}_\rho \cdot \vec{\sigma}, \quad (4)$$

where $\rho_0 := I_2/2$ and $\mathbf{v}_\rho = (s_1, s_2)^t = (\text{Tr}(\rho\sigma_1), \text{Tr}(\rho\sigma_2))^t \in \mathcal{D}$ is called *Bloch vector* and \mathcal{D} is referred to as the Bloch disk. Notice that the components of the Bloch vector are the expectation values of the two real Pauli matrices on the state defined by ρ , so $\mathbf{v}_\rho = (\langle \sigma_1 \rangle_\rho, \langle \sigma_2 \rangle_\rho)^t$.

Using polar coordinates, i.e. $(s_1, s_2) = (r \cos \vartheta, r \sin \vartheta)$, $r \in [0, 1]$, $\vartheta \in [0, 2\pi)$, one easily finds that the generic expression of a density matrix can be written as follows

$$\rho(r, \vartheta) = \rho_0 + \frac{\langle \sigma_1 \rangle_\rho}{2} [\rho(1, 0) - \rho(1, \pi)] + \frac{\langle \sigma_2 \rangle_\rho}{2} [\rho(1, \pi/2) - \rho(1, 3\pi/2)]. \quad (5)$$

It turns out that, for all ϑ , $\rho(1, \vartheta)$ is a rank-1 projector, i.e. a *pure state*, and $\rho(1, \vartheta_1)$, $\rho(1, \vartheta_2)$ project on orthogonal directions precisely when ϑ_1 and ϑ_2 correspond to antipodal points on the unit circle. Since orthogonality in quantum theories represents incompatible states, eq. (5) codifies a *generic chromatic state* as the *superposition of two chromatic opponencies between incompatible states*, red-green and yellow-blue in Hering's theory, see [20], weighted by the expectation values of the real density matrices, *plus an offset state* represented by ρ_0 , which is parameterized by the center of the Bloch disk \mathcal{D} .

While the points of the $\partial\mathcal{D}$, the unit circle, parameterize pure states, it is well-known, see e.g. [19], that it exists only one maximally mixed state in the case of a rebit and it is parameterized by the center of \mathcal{D} . This is precisely the state ρ_0 which is also characterized by the fact of having maximal the von Neumann entropy, defined as $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$, and not carrying any chromatic information. For this reason, ρ_0 is called the *achromatic state*.

Thanks to this final identification, it follows that formula (5) is precisely the quantum description of the chromatic information that can be gathered from an isolated color stimulus in Hering's theory, as admirably resumed by the physiologist D. Hubel in [21].

3 The growing importance of quantum information for the advancement of the color perception model

As previously remarked, the observables belonging to the infinite cone $\overline{\mathcal{L}^+}$ and the chromatic states contained in the Bloch disk \mathcal{D} represent an ideal situation in which the real limitations of human ability to perceive colors are not taken into account. However, the visible threshold and glare limits imply that the space of perceived colors perceived by a *real* normal trichromatic observer is actually a finite-volume convex subset, usually called *color solid*, of the infinite cone $\overline{\mathcal{L}^+}$.

The modern quantum information theory offers a surprisingly reach and well-suited set of tools to handle this and other issues properly. The main observation from which all the results that will be discussed in this section descend is that *a perceived color is the result of a (quantum) measurement* that a human observer performs by looking at a color stimulus. Since the quantum model is based on data gathered from observers adapted to the illuminant conditions of a visual scene, the hypothesis of *adaptation* must also be assumed to hold true, where, in the quantum color perception theory, an observer is considered adapted when the state chosen to match the perception of a non-selective patch, or a broadband light spectrum, is ρ_0 , i.e. the achromatic state.

As it is well-known, in quantum theories, even of isolated systems, the measurement process is intrinsically probabilistic and the concept that encodes this feature is called *effect*, see e.g. [23, 14, 19] for an overview on this fundamental object. In the quantum color perception framework, an effect is defined to be an element $\eta_{\mathbf{e}}$ of $\overline{\mathcal{H}^+(2, \mathbb{R})}$ bounded between the null and the identity 2×2 matrix (with respect to the Loewner ordering of positive semi-definite matrices, i.e. $B \leq A \iff A - B \in \overline{\mathcal{H}^+(2, \mathbb{R})}$). The matrix $\eta_{\mathbf{e}}$ can be written explicitly as follows

$$\eta_{\mathbf{e}} = \begin{pmatrix} e_0 + e_1 & e_2 \\ e_2 & e_0 - e_1 \end{pmatrix}, \quad (6)$$

with $e_0, e_1, e_2 \in \mathbb{R}$ belonging to the following *effect space*:

$$\mathcal{E} = \left\{ (e_0, e_1, e_2) \in \mathbb{R}^3, e_0 \in [0, 1], e_1^2 + e_2^2 \leq \min_{e_0 \in [0, 1]} \{ (1 - e_0)^2, e_0^2 \} \right\}. \quad (7)$$

As it can be seen in Figure 1, \mathcal{E} is a *closed convex double cone* with a circular basis of radius $1/2$ located at height $e_0 = 1/2$ and vertices in $(0, 0, 0)$ and $(1, 0, 0)$, associated to *the null and the unit effect*, respectively [12]. The geometry of \mathcal{E} happens to be in perfect agreement with that of the perceived color spaces advocated by Ostwald and de Valois, see e.g. [15]. Thus, if we assume perceived colors to be the result of measurements represented by effects, then it is natural to bound the infinite cone $\overline{\mathcal{L}^+}$ of *virtually perceivable colors* to the color solid \mathcal{E} of *actually perceived colors*.

The *effect vector* in the Bloch disk associated to $\eta_{\mathbf{e}}$ is given by

$$\mathbf{v}_{\mathbf{e}} := \begin{pmatrix} e_1 & e_2 \\ e_0 & e_0 \end{pmatrix}^t. \quad (8)$$

Effects parameterize a fundamental class of state transformations called *Lüders operations*, which are *convex-linear positive functions* $\psi_{\mathbf{e}}$ defined on the state space \mathcal{S} and satisfying the constraint:

$$0 \leq \text{Tr}(\psi_{\mathbf{e}}(\rho)) \leq 1, \quad \text{for all } \rho \in \mathcal{S}. \quad (9)$$

This implies that ρ will lose the property of having unit trace after a Lüders operation, becoming a so-called *generalized density matrix* representing a *post-measurement generalized state* which does not belong to the Bloch disk anymore, but to the color solid \mathcal{E} . The value of $\text{Tr}(\psi_{\mathbf{e}}(\rho))$ has actually a very important meaning: in fact it has been proven in [9] that $\text{Tr}(\psi_{\mathbf{e}}(\rho))$ can be associated to the *brightness* $\mathcal{B}_{\mathbf{e}}(\rho)$ of the perceived color $\psi_{\mathbf{e}}(\rho) \in \mathcal{E}$.

This interpretation permits to recover the three-dimensionality of color perception in the quantum model: the chromatic information is embedded in the state $\rho \in \mathcal{S}$, which defines the object

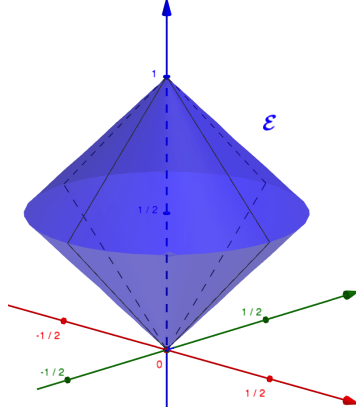


Figure 1: The double cone representing the effect space can be interpreted as the color solid of actually perceived colors inside the infinite cone $\overline{\mathcal{L}^+}$.

of a perceptual measurement associated to an effect \mathbf{e} and, once the observation is done, the real value $\mathcal{B}_{\mathbf{e}}(\rho)$ contains the achromatic information of brightness.

The analytical expression of the post-measurement generalized state $\psi_{\mathbf{e}}(\rho)$, see [14], is:

$$\psi_{\mathbf{e}}(\rho) = \eta_{\mathbf{e}}^{1/2} \rho \eta_{\mathbf{e}}^{1/2}, \quad (10)$$

$\eta_{\mathbf{e}}^{1/2}$ is the so-called *Kraus operator* associated to \mathbf{e} and it is the square root of $\eta_{\mathbf{e}}$, i.e. the only symmetric and positive semi-definite matrix such that $\eta_{\mathbf{e}}^{1/2} \eta_{\mathbf{e}}^{1/2} = \eta_{\mathbf{e}}$. Straightforward computations lead to the following explicit formula for brightness

$$\mathcal{B}_{\mathbf{e}}(\rho) = \text{Tr}(\psi_{\mathbf{e}}(\rho)) = e_0(1 + \mathbf{v}_{\mathbf{e}} \cdot \mathbf{v}_{\rho}), \quad (11)$$

which permits to define the chromatic state belonging to \mathcal{S} uniquely associated to $\psi_{\mathbf{e}}$ as follows:

$$\varphi_{\mathbf{e}}(\rho) = \frac{\psi_{\mathbf{e}}(\rho)}{e_0(1 + \mathbf{v}_{\mathbf{e}} \cdot \mathbf{v}_{\rho})}. \quad (12)$$

Remarkably, in [12], it has been shown that *the state change* $\rho \mapsto \psi_{\mathbf{e}}(\rho)$ *induced by the act of observing a color* is implemented through a *3-dimensional normalized Lorentz boost* in the direction of $\mathbf{v}_{\mathbf{e}}$ and that the post-measurement chromatic state vector is the Einstein-Poincaré relativistic sum of $\mathbf{v}_{\mathbf{e}}$ and \mathbf{v}_{ρ} , i.e.

$$\mathbf{v}_{\varphi_{\mathbf{e}}(\rho)} = \mathbf{v}_{\mathbf{e}} \oplus \mathbf{v}_{\rho}, \quad (13)$$

defined as follows: if $\|\mathbf{v}_{\mathbf{e}}\| < 1$, then

$$\mathbf{v}_{\mathbf{e}} \oplus \mathbf{v}_{\rho} = \frac{1}{1 + \mathbf{v}_{\mathbf{e}} \cdot \mathbf{v}_{\rho}} \left\{ \mathbf{v}_{\mathbf{e}} + \frac{1}{\gamma_{\mathbf{v}_{\mathbf{e}}}} \mathbf{v}_{\rho} + \frac{\gamma_{\mathbf{v}_{\mathbf{e}}}}{1 + \gamma_{\mathbf{v}_{\mathbf{e}}}} (\mathbf{v}_{\mathbf{e}} \cdot \mathbf{v}_{\rho}) \mathbf{v}_{\mathbf{e}} \right\}, \quad (14)$$

where $\gamma_{\mathbf{v}_{\mathbf{e}}}$ is the \mathbf{e} -Lorentz factor defined by

$$\gamma_{\mathbf{v}_{\mathbf{e}}} = \frac{1}{\sqrt{1 - \|\mathbf{v}_{\mathbf{e}}\|^2}}, \quad (15)$$

and, if $\|\mathbf{v}_{\mathbf{e}}\| = 1$,

$$\mathbf{v}_{\mathbf{e}} \oplus \mathbf{v}_{\rho} = \mathbf{v}_{\mathbf{e}}. \quad (16)$$

This formalizes the link between color perception and relativistic theories guessed in [28] on a heuristic basis.

As recalled at the beginning of this section, an observer is supposed to be adapted to the visual scene when the perceptual measurement is performed, thus the post-measurement state $\varphi_{\mathbf{e}}(\rho)$

should be free from possible spurious chromatic features induced by a non-neutral illumination. This has been confirmed in the papers [18] and [10]. In particular, in the last one, an efficient white balance algorithm was built through the use of split-quaternions, which permit to implement the Lüders operation via a two-sided multiplication, or ‘sandwich’. Figure 2 show a result of this algorithm in comparison with the classical von Kries white balance.

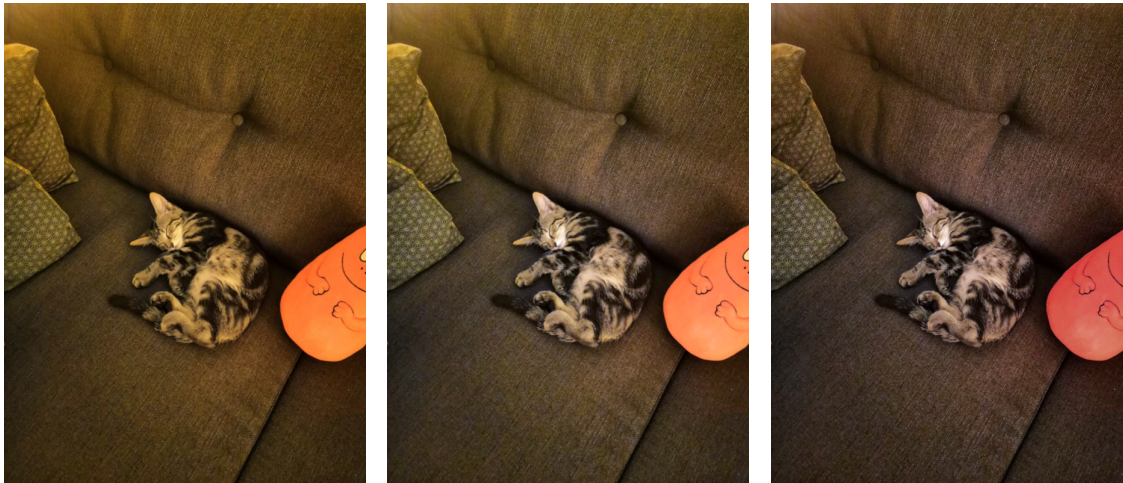


Figure 2: *Left*: input image of ‘Panko’. *Center*: output image after the von Kries white balance. *Right*: output image after the Lüders white balance implemented with split quaternions. The white balanced images have been obtained using the same illuminant estimation. It can be seen that the Lüders white balance overperforms the von Kries one in reducing the presence of the orange cast.

Finally, also the chromatic attributes of hue and saturation have been rigorously defined in the quantum framework. Given the perceived color $\psi_{\mathbf{e}}(\rho) \in \mathcal{E}$, its saturation is

$$\begin{aligned} \text{Sat}(\psi_{\mathbf{e}}(\rho)) &= R(\rho_{\varphi_{\mathbf{e}}(\rho)} || \rho_{\mathbf{0}}) \\ &= \frac{1}{2} \log_2(1 - r_{\varphi_{\mathbf{e}}(\rho)}^2) + \frac{r_{\varphi_{\mathbf{e}}(\rho)}}{2} \log_2 \left(\frac{1 + r_{\varphi_{\mathbf{e}}(\rho)}}{1 - r_{\varphi_{\mathbf{e}}(\rho)}} \right), \end{aligned} \quad (17)$$

where R is the *relative entropy* between the states appearing as its arguments and $r_{\varphi_{\mathbf{e}}(\rho)} = ||\mathbf{v}_{\varphi_{\mathbf{e}}(\rho)}||$, while its hue is the pure chromatic state $\varphi_{\mathbf{e}}^*(\rho)$ defined by

$$\varphi_{\mathbf{e}}^*(\rho) := \arg \min_{\rho \in \mathcal{PS}} R(\rho || \rho_{\varphi_{\mathbf{e}}(\rho)}), \quad (18)$$

where \mathcal{PS} are the pure chromatic states, parameterized by the points of the border $\partial\mathcal{D}$ of the Bloch disk. The explicit expression of the density matrix associated to the pure chromatic state $\varphi_{\mathbf{e}}^*(\rho)$ is

$$\rho_{\varphi_{\mathbf{e}}^*(\rho)} = \frac{1}{2} \begin{pmatrix} 1 + \cos \vartheta_{\varphi_{\mathbf{e}}(\rho)} & \sin \vartheta_{\varphi_{\mathbf{e}}(\rho)} \\ \sin \vartheta_{\varphi_{\mathbf{e}}(\rho)} & 1 - \cos \vartheta_{\varphi_{\mathbf{e}}(\rho)} \end{pmatrix}, \quad (19)$$

where

$$\cos \vartheta_{\rho, \varphi_{\mathbf{e}}(\rho)} = \frac{\mathbf{v}_{\rho} \cdot \mathbf{v}_{\varphi_{\mathbf{e}}(\rho)}}{r_{\varphi_{\mathbf{e}}(\rho)}}. \quad (20)$$

4 Conclusion and future perspectives

In this contribution it has been recalled that a radical change of paradigm with respect to the classical CIE theory permits to build a mathematically rigorous color perception model.

This change is grounded on the fact that the color space is not built starting from the extremely complicated, and still misunderstood, mix of physics and psychophysics that lead to the concept of metameric classes of color spectra, but solely from the perceptual features exhibited by color sensations. These last ones determine uniquely the only two possible forms that the color space can have and, once the flat option is discarded because not coherent with experimental data, only a single intrinsically hyperbolic color space \mathcal{C} remains.

\mathcal{C} is the domain of positivity of two isomorphic Jordan algebras of dimension 3 over \mathbb{R} , interpreted as the state cone of a quantum system, which happens to be a rebit, the real counterpart of a qubit. Surprisingly, this fact has a tight link with Hering’s opponency theory of color perception: in fact, the outcomes of perceptual measurements that can be performed on color stimuli can only express the weights of chromatic opponency between two mutually incompatible couples of pure states, which, in Hering’s theory, are identified with red-green and yellow-blue.

It is important to stress that this result is obtained by considering the simplest, most natural, representation of density matrices associated to chromatic states, i.e. the Bloch representation, and not by performing *a posteriori* statistical analysis, see e.g. [13, 24], or by introducing opponency ‘by hand’, as done by the CIE in the construction of the so-called ‘uniform color spaces’.

Starting from the quantum interpretation of \mathcal{C} , the powerful tools and results of quantum theories permit to proceed in the analysis of color perception without imposing any superstructure at will, but solely by identifying judiciously the color perceptual features that correspond to either well-known results of quantum information or by proving new ones in the context of rebit systems.

Following this line of work, a whole new vocabulary for color perception has been created in [9], with concepts as hue, saturation, brightness, lightness, chroma, colorfulness appearing in a mathematically well-defined system of definitions free from the typical circularity of standard color dictionaries.

Indirect proofs of the soundness of this new proposal are provided by the possibility to explain quite easily the phenomenon of lightness constancy and also to create new, non-linear, white balance algorithms able to reduce color cast better than classical methods, see [9, 10]. In doing so, the concept of state transformation via Lüders operations plays a fundamental role and it is truly remarkable that this kind of functions are tightly related to relativistic theories, as proven in [12], which, in turn, is completely coherent with the hyperbolic nature of the color space \mathcal{C} .

The strategy adopted to build the quantum color perception model is clearly much more in line with the habits of mathematical physics than with those of classical colorimetry, however, despite its high level of abstraction, this approach has the clear advantage to avoid the creation of a plethora of different proposals for color spaces and attributes, something that has been plaguing this research field for almost a century.

Up to this point, no incoherence with known results has been found but, of course, the construction of the quantum color perception model is still ongoing and there are several research paths that must be explored in order to consider this novel proposal a benchmark that can compete also empirically, and not only theoretically, with the classical CIE theory.

First of all, it is imperative to associate color stimuli, in the form of light spectra, to chromatic states. In its ‘operational’ definition, a quantum state is taken to be an equivalence class of *preparations* of a system in order to perform measurements of its observables, colors in this case, where the equivalence consists in the fact that the measurement outcomes must agree independently of the preparation. Concretely, this means that if two color stimuli produce the same set of (perceptual) measurement outcomes for color attributes, then they must be considered as two representatives of the equivalence class that defines the same chromatic state. Notice that this is very different than considering them as metameric stimuli in the canonical colorimetric sense, in fact, metameric stimuli are supposed to give rise to the same tristimulus values, while in the quantum model no reference to how the cones transform light spectra is made. This is of crucial importance, because, as stressed in the introduction, the role of opponency mechanisms and of higher brain functions taking place in the visual zones may lead to same chromatic sensation even for stimuli having equal tristimulus values, see e.g. [17].

Consequently, the experiments apt to determine the correspondence between perceptual chromatic states and color stimuli must switch from the canonical integration of these last ones over

the LMS cones sensitivities, to the Hurvich-Jameson hue cancellation and hue scaling experiments for the determination of the degree of opponency, see e.g. [22], which, as exhibited by formula (5), would provide the expectation values of the real Pauli matrices that appear in the density matrices.

Another extremely important perspective is the rigorous mathematical comprehension of the phenomenon of adaptation: being a dynamical event, it is likely that it should be expressed either in the form of the result of a differential equation, as e.g. the Lindblad master equation [2], or of a so-called quantum channel [19]. The last hypothesis is what led to the complete classification of rebit channels in [1].

Preliminary results have shown that, as a result of adaptation, a chromatic state changes in a way that involves, at once, the perception of hue, saturation and brightness, which is a further evidence of the importance of having precise definitions of such attributes. More precise and detailed studies are needed to obtain quantitative formulae which can be confronted to empirical data.

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