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Combinatorial properties of Circulant Hadamard matrices

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Abstract We describe combinatorial properties of the defining row of a circulant Hadamard matrix by exploiting its orthogonality to subsequent rows, and show how to exclude several particular forms of these matrices.

1 Introduction and basic definitions

Throughout this paper, the i -row of a matrix H of size $n \times n$ is denoted by H_i , $1 \leq i \leq n$. A (± 1) -matrix H of size $n \times n$ is called *Hadamard matrix* if its rows are mutually orthogonal. It is a *circulant matrix* if for $H_1 = (h_1, \dots, h_n)$, its i -row is given by $H_i = (h_{1-i+1}, \dots, h_{n-i+1})$, the subscripts being taken modulo n . In this case, the first row H_1 will be called the *defining row* of H .

An example of a 4×4 circulant Hadamard matrix is the following:

$$H = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

A conjecture of Ryser (cf. [2, pp.97]) states that there is no circulant Hadamard matrix unless $n = 1$ or $n = 4$. This conjecture is still open although many partial results have been obtained. For recent results and further details, we refer to [1], [3] and [4].

The objective of this paper is to exhibit some combinatorial properties of the defining row H_1 of such a matrix H that we obtained by exploiting the orthogonality of H_1 to some of its subsequent rows. It turns out that these properties allow to exclude several particular forms for circulant Hadamard matrices. We think that a further development of this “combinatorial approach” may contribute to exclude other cases of such matrices and help to determine the maximum number k for which the first k rows of a circulant (± 1) -matrix are mutually orthogonal.

The order n (for $n > 1$) of a circulant Hadamard matrix H is well known to be of the form $n = 4m = 4h^2$, where h is an odd integer. Also, by orthogonality, the entries of any two different rows have the same sign in exactly $2m$ columns, and a different sign in the other $2m$ ones.

Finally, the property of being circulant allows any row of H to be represented as a circular sequence of “+” and “-”. Figure 1 illustrates the first two rows of such a matrix H for $n = 12$.

Some further definitions are required. Let us consider the first row H_1 of a circulant Hadamard matrix H . A maximal sequence of consecutive +1’s (respectively -1’s) will be called a *positive block* (respectively

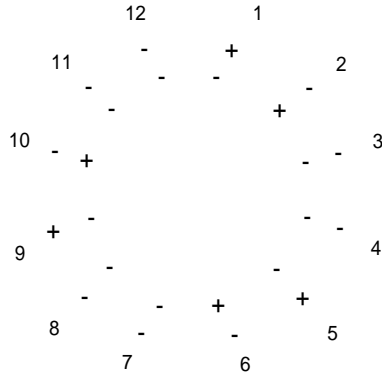


Figure 1: The first two rows of a circulant Hadamard matrix of order 12

negative block), denoted by B . If the size of such a block equals k , for some positive integer k , we will speak of a *positive* (respectively *negative*) k -block. The family of all blocks is denoted by \mathcal{B} , that of all k -blocks by \mathcal{B}_k and that of all k -blocks with $k \geq i$, for a given positive integer i , by $\mathcal{B}_{\geq i}$. Thus, H_1 decomposes into a sequence of alternating, positive or negative blocks. For convenience, we suppose the first and last block of H_1 to be of opposite sign. Finally, a maximal alternating sequence of blocks in \mathcal{B}_k (in $\mathcal{B}_{\geq k}$) will be called a k -alternating sequence ($\geq k$ -alternating sequence, respectively), the *length* of which is given by the number of blocks forming it.

2 Our results

Lemma 2.1 *The total number of blocks in the first row H_1 of a circulant Hadamard matrix H equals $2m$.*

Proof: We consider H_1 and H_2 of H and observe that the number of columns with entries of different sign equals the total number of blocks. Hence, $|\mathcal{B}| = 2m$. \square

Alternatively, if we count the number of columns whose entries have equal sign, we obtain $\sum_{B \in \mathcal{B}} (|B| - 1)$, which equals $4m - |\mathcal{B}|$ and which by orthogonality, gives $2m$. Again, $|\mathcal{B}| = 2m$. \square

Lemma 2.2 *The total number of 1-blocks in H_1 equals m .*

Proof: We consider H_1 and H_3 of H and count the number of columns with entries of equal sign. We observe that:

- in a 1-alternating sequence, this number equals the length of the sequence minus 1, plus 1 arising from the ≥ 2 -block preceding the sequence,
- a block $B \in \mathcal{B}_{\geq 3}$ contributes with $|B| - 2$,
- no block $B \in \mathcal{B}_2$ contributes to this number (except those preceding a 1-alternating path, whose contribution is already taken into account).

By orthogonality of H_1 and H_3 , we obtain:

$$\begin{aligned}
2m &= |\mathcal{B}_1| + \sum_{B \in \mathcal{B}_{\geq 3}} (|B| - 2) \\
&= |\mathcal{B}_1| + \sum_{B \in \mathcal{B}_{\geq 2}} (|B| - 2) \\
&= |\mathcal{B}_1| + \left(\sum_{B \in \mathcal{B}_{\geq 2}} |B| \right) - 2|\mathcal{B}_{\geq 2}| \\
&= |\mathcal{B}_1| + (4m - |\mathcal{B}_1|) - 2(|\mathcal{B}| - |\mathcal{B}_1|) \\
&= 4m - 4m + 2|\mathcal{B}_1|.
\end{aligned}$$

Hence, $|\mathcal{B}_1| = m$.

Alternatively, we may count the number of columns of H_1 and H_3 , with entries of different signs. We denote by α_1 the number of 1-alternating sequences, by $\alpha_{\geq 2}$ the number of ≥ 2 -alternating sequences, by Λ the set of all ≥ 2 -alternating sequences, and by $|\ell|$ the length of an element ℓ of Λ . We observe that:

- any 1-alternating sequence contributes with 2,
- any ≥ 2 -alternating sequence ℓ contributes with $2(|\ell| - 1)$.

Thus, we obtain:

$$2m = 2\alpha_1 + 2\left(\sum_{\ell \in \Lambda} (|\ell| - 1)\right). \quad (1)$$

Since 1-alternating sequences and ≥ 2 -alternating sequences are themselves alternating within H_1 , $\alpha_1 = \alpha_{\geq 2}$, and we get from (1):

$$m = \alpha_1 + |\mathcal{B}_{\geq 2}| - \alpha_{\geq 2} = |\mathcal{B}_{\geq 2}| = |\mathcal{B}| - |\mathcal{B}_1| = 2m - |\mathcal{B}_1| \text{ by Lemma 2.1.}$$

$$\text{Hence, } |\mathcal{B}_1| = m. \quad \square$$

Lemma 2.3 *The number of blocks of size 2 and the number of 1-alternating sequences add up to m :*

$$|\mathcal{B}_2| + \alpha_1 = m.$$

Proof: We consider H_1 and H_4 of H and count the number of columns with entries of equal sign. We observe that:

- any 1- alternating sequence is preceded and followed by a block of size ≥ 2 ,
- a block $B \in \mathcal{B}_{\geq 3}$ contributes with $|B| - 3$,
- any 2-block $B \in \mathcal{B}_2$ is preceded and followed by an element of opposite sign.

By orthogonality of H_1 and H_4 and by Lemmata 2.1 and 2.2, we obtain:

$$\begin{aligned} 2m &= \sum_{B \in \mathcal{B}_{\geq 3}} (|B| - 3) + |\mathcal{B}_2| + 2\alpha_1 \\ &= 4m - |\mathcal{B}_1| - 2|\mathcal{B}_2| - 3|\mathcal{B}_{\geq 3}| + |\mathcal{B}_2| + 2\alpha_1 \\ &= 3m - |\mathcal{B}_2| - 3(|\mathcal{B}| - m - |\mathcal{B}_2|) + 2\alpha_1 \\ &= 2|\mathcal{B}_2| + 2\alpha_1. \end{aligned}$$

$$\text{Hence, } |\mathcal{B}_2| + \alpha_1 = m. \quad \square$$

Remark 2.4 *It would be interesting to know whether counting the number of columns with entries of different sign provides the same condition on $|\mathcal{B}_2|$, similar to that obtained for $|\mathcal{B}_1|$. For the sake of completeness and to make the reader more familiar with our approach, we have included such a condition at this place.*

For given n , let α_1, α_2 and $\alpha_{\geq 3}$ be the number of 1-, 2- and ≥ 3 -alternating sequences, and let $\alpha_{2, \geq 3}$ denote the number of pairs of blocks B, B' with $B \in \mathcal{B}_2, B' \in \mathcal{B}_{\geq 3}$ or vice versa. We obtain the following:

Lemma 2.5

$$\alpha_1 + \alpha_{2,\geq 3} = \alpha_2 + \alpha_{\geq 3}.$$

Proof: Any 1-, 2- or ≥ 3 -alternating sequence is preceded (followed) by a 2- or 3-, 1- or ≥ 3 -, 1- or 2-alternating sequence, respectively.

Let us examine the first case, i.e. that of a 1-alternating sequence. Counting the number of columns with entries of different signs gives us a contribution of $(\ell - 1)$ for such a sequence, if ℓ denotes the length of it (see Figure 2).

$$+ + - - \mid + - \boxed{\begin{array}{cccccc} - & + & \cdots & - & + & - & + \\ + & - & \cdots & + & - & + & - \end{array}} - - -$$

Figure 2: A 1-alternating sequence preceded (followed) by a 2- (≥ 3 -) one, and the contribution of $\ell - 1$.

For a 2-alternating sequence preceded and followed by a 1-alternating one, the total contribution will be $(\ell - 1)$ again. If, however, such a sequence is preceded or followed by a ≥ 3 -alternating sequence (see Figure 3), one or two consecutive pairs of blocks B, B' with $B \in \mathcal{B}_2, B' \in \mathcal{B}_{\geq 3}$ or vice versa, will appear and contribute by an additional value of 1.

$$- + - \mid + + - \boxed{\begin{array}{c} + \\ - \end{array}} + \boxed{\begin{array}{c} - \\ + \end{array}} \cdots \boxed{\begin{array}{c} + \\ - \end{array}} + \boxed{\begin{array}{c} - \\ + \end{array}} - \boxed{\begin{array}{c} + \\ - \end{array}} \mid + \boxed{\begin{array}{c} - \\ + \end{array}} \boxed{\begin{array}{c} - \\ + \end{array}} +$$

Figure 3: A 2-alternating sequence preceded (followed) by a 1- (≥ 3 -) one.

In that case, there is another contribution of 1 (encircled by a dashed line), that will be included in the contribution of the ≥ 3 -alternating sequence.

Finally, for ≥ 3 -alternating sequences, we obtain the situation depicted in Figure 4. The overall contribution of such a ≥ 3 -alternating sequence is $3\ell - 1$. Also, observe that the pair of blocks B, B' with $B \in \mathcal{B}_{\geq 3}$, the last block of this sequence, and $B' \in \mathcal{B}_2$, the first of the following 2-alternating one, again contribute by 1 to the value $\alpha_{2,\geq 3}$.

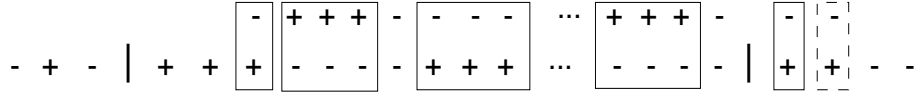


Figure 4: A ≥ 3 -alternating sequence preceded (followed) by a 1 – (2–) one.

Altogether, counting the number of columns with entries of different signs in H_1 and H_4 gives the condition:

$$\sum_{i \in I} (\ell_i - 1) + \sum_{j \in J} (\ell_j - 1) + \sum_{k \in K} (3\ell_k - 1) + \alpha_{2, \geq 3} = 2m,$$

where I, J, K , denote the sets of 1-, 2- and ≥ 3 -alternating sequences, respectively, and ℓ their length.

We obtain:

$$|\mathcal{B}_1| - \alpha_1 + |\mathcal{B}_2| - \alpha_2 + 3|\mathcal{B}_{\geq 3}| - \alpha_{\geq 3} + \alpha_{2, \geq 3} = 2m.$$

Since, by Lemma 2.1,

$$|\mathcal{B}_1| + |\mathcal{B}_2| + 3|\mathcal{B}_{\geq 3}| = 4m - 2|\mathcal{B}_1| - 2|\mathcal{B}_2|,$$

$$4m - \alpha_1 - \alpha_2 - \alpha_{\geq 3} + \alpha_{2, \geq 3} = 2|\mathcal{B}_1| + 2|\mathcal{B}_2|.$$

By Lemma 2.2, we obtain:

$$2m - \alpha_1 - \alpha_2 - \alpha_{\geq 3} + \alpha_{2, \geq 3} = 2|\mathcal{B}_2|.$$

Since, by Lemma 2.3,

$$2m - 2|\mathcal{B}_2| = 2\alpha_1,$$

we obtain as stated

$$\alpha_1 + \alpha_{2, \geq 3} = \alpha_2 + \alpha_{\geq 3}.$$

□

We are now ready to show the nonexistence of a circulant Hadamard matrix for the following four particular situations:

- i) $\alpha_1 = 1$, ii) $\alpha_1 = m$, iii) $\alpha_1 = m - 1$, iv) $\alpha_1 = 2$

Theorem 2.6 *If in H_1 , there is only one 1-alternating sequence (of length m), or if there are m such sequences (all of length 1), or if the number of 1-alternating sequences is $m-1$, then there is no circulant Hadamard matrix H with H_1 as first row.*

Proof: We consider the three cases.

Case $\alpha_1 = 1$:

We know already, from Lemmata 2.1, 2.2 and 2.3, that:

- the total number of blocks is $2m$,
- that of 1-blocks is m ,
- $|\mathcal{B}_2| + \alpha_1 = m$.

Therefore, since $n = 4m$, there is exactly one more block of size $m+2$, and if we suppose that this particular block precedes the 1-alternating sequence (clockwise), we obtain the configuration depicted in Figure 5:

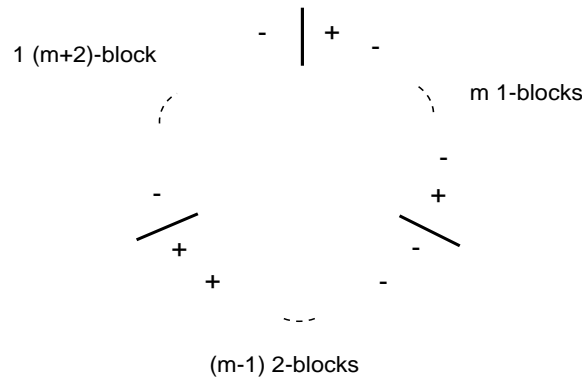


Figure 5: A first representation of H_1 for $\alpha_1 = 1$

Clockwise evaluation of the scalar product of H_1 and H_5 gives:

$$(m-4) + 0 + 2(m-3) + 0 + (m-2) + 0 = 4m - 12,$$

which allows for orthogonality only if $m = 3$, contradicting the condition on m to be an odd square integer.

By symmetry, we obtain the same result if the particular $(m+2)$ -block follows the 1-alternating sequence.

If, now, the $(m + 2)$ -block is situated “in between” the sequence of 2-blocks, as depicted in Figure 6,

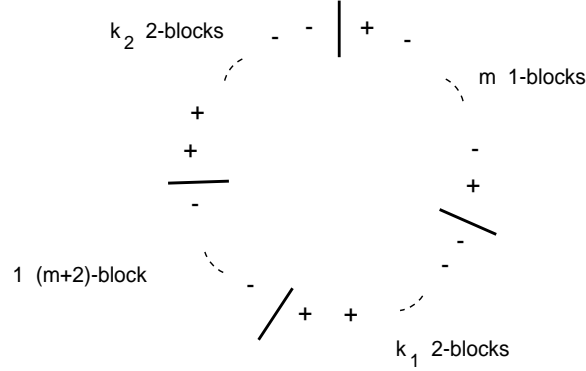


Figure 6: A second representation of H_1 for $\alpha_1 = 1$ and $(m - 1) = k_1 + k_2$

we may suppose that

$$m - 1 = k_1 + k_2, \text{ with } k_1, k_2 \text{ both even or both odd.}$$

Evaluating the scalar product of H_1 and H_5 gives:

$$(m - 4) + 0 + 2(k_1 - 2) + 0 + (m - 2) + 2(k_2 - 2) + 0 = 4m - 16.$$

Orthogonality implies $m = 4$, contradicting the condition on m to be odd. We obtain the same result for the case $k_1 = 1$ or $k_2 = 1$, so that altogether, H_1 cannot be the defining row of a circulant Hadamard matrix.

Case $\alpha_1 = m$:

By Lemma 2.3, there is no 2-block in H_1 and m blocks remain to cover $3m$ elements, i.e. besides the m 1-blocks, there are exactly m 3-blocks, and both types alternate in H_1 . But then the scalar product of H_1 and H_5 gives $4m > 0$ and H_1 cannot define a circulant Hadamard matrix either.

Case $\alpha_1 = m - 1$:

In this case there is exactly one consecutive pair of 1-blocks, and by Lemma 2.3, there is exactly one 2-block, so that $3m - 2$ elements have to

be covered by $m - 1$ blocks of size ≥ 3 . This is only possible with $(m - 2)$ 3-blocks and one 4-block. Moreover, the unique 2-block, say B , cannot be preceded *and* followed by a ≥ 3 -block. Therefore, we come up with two possibilities:

- a) B is preceded and followed by a 1-alternating sequence.
- b) B is preceded or followed (but not both) by a ≥ 3 -block.

Situation a) is depicted in Figure 7:

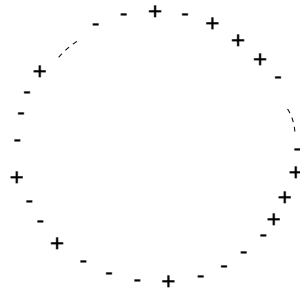


Figure 7: Illustration of possibility a)

Just observe, that in this situation exactly two ≥ 3 -blocks follow each other. We count the number of columns with different signs in H_1 and H_5 and observe that a contribution (of 2) to this number is given only by

- the two consecutive 1-blocks;
- the unique 2-block;
- the unique 4-block;
- exactly one of the 3-blocks.

By orthogonality,

$$2m = 8,$$

a contradiction to the condition on m to be odd.

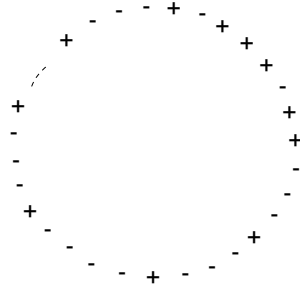


Figure 8: Illustration of possibility b)

Situation b) is depicted in Figure 8:

In this situation, no 3-block contributes any more, and orthogonality of H_1 and H_5 gives

$$2m = 6,$$

a contradiction to the condition that m has to be an odd square integer. \square

In the following, we will show how the orthogonality of H_1 and H_5 can be exploited to obtain a further condition involving $|\mathcal{B}_2|$ and $|\mathcal{B}_3|$, which we will use afterwards to exclude the situation where $\alpha_1 = 2$.

For this, we need to describe the possible types of 1-alternating and 2-alternating sequences which (up to symmetry) is done in Figures 9 and 10:

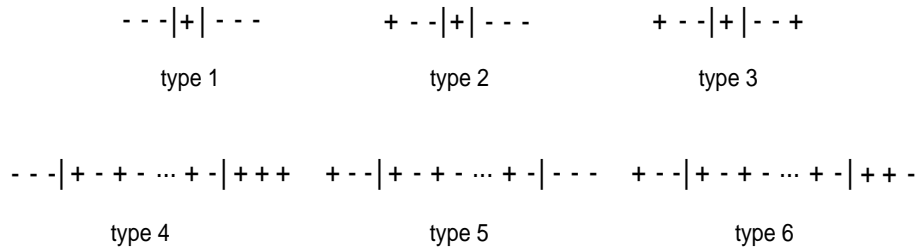


Figure 9: 1-alternating sequences of length 1 and ℓ , respectively

It is not difficult to verify that if we count the columns of H_1 and H_5 with equal sign, the contribution of these 6 types will be 3, 2, 1 and ℓ , $\ell - 1$, $\ell - 2$,

respectively.

For 2-alternating sequences, we obtain 3 similar types, depicted in Figure 10, whose contributions will be 2ℓ , $2\ell - 1$, $2\ell - 2$, respectively:

$$\begin{array}{ccc}
 \dots|++ \dots ++| \dots & +-|++ \dots ++| \dots & +-|++ \dots ++| \dots \\
 \text{type 1} & \text{type 2} & \text{type 3}
 \end{array}$$

Figure 10: 2-alternating sequences of length ℓ

with the properties that

- type 1 is preceded and followed by a ≥ 3 -block;
- type 2 is preceded by a 1-alternating sequence and followed by a ≥ 3 -block (or vice versa);
- type 3 is preceded and followed by a 1-alternating sequence.

If now α_1^i denotes the number of 1-alternating sequences of type i , for $i = 1, \dots, 6$, and α_2^j the number of 2-alternating sequences of type j , for $j = 1, \dots, 3$, we get

Lemma 2.7 *If H is a circulant Hadamard matrix, then one has*

$$2|\mathcal{B}_2| + 2|\mathcal{B}_3| = (\alpha_1^2 + 2\alpha_1^3) + (2\alpha_1^4 + 3\alpha_1^5 + 4\alpha_1^6) + (\alpha_2^2 + 2\alpha_2^3).$$

Proof:

We can describe the total number of columns of H_1 and H_5 , whose entries have equal sign, by the following expression

$$L := 3\alpha_1^1 + 2\alpha_1^2 + \alpha_1^3 + C + D,$$

where

$$C := \sum_{i \in I} \ell_i + \sum_{j \in J} (\ell_j - 1) + \sum_{k \in K} (\ell_k - 2),$$

and

$$D := \sum_{p \in P} 2\ell_p + \sum_{q \in Q} (2\ell_q - 1) + \sum_{r \in R} (2\ell_r - 2) + \sum_{B \in \mathcal{B}_{\geq 4}} (|B| - 4) + |\mathcal{B}_3|,$$

expression in which $\sum_{j \in J} (\ell_j - 1)$, for instance, relates to all 1-alternating sequences S_j , $j \in J$, whose contribution equals $\ell_j - 1$, i.e., which are of type 5.

We know the following:

$$i) \alpha_1^1 + \alpha_1^2 + \alpha_1^3 + \sum_{i \in I} \ell_i + \sum_{j \in J} \ell_j + \sum_{k \in K} \ell_k = |\mathcal{B}_1| = m, \text{ by Lemma 2.2,}$$

$$ii) \sum_{p \in P} \ell_p + \sum_{q \in Q} \ell_q + \sum_{r \in R} \ell_r = |\mathcal{B}_2|, \quad \text{and}$$

$$iii) \alpha_1^1 + \alpha_1^2 + \alpha_1^3 + \alpha_1^4 + \alpha_1^5 + \alpha_1^6 = \alpha_1.$$

Therefore,

$$\begin{aligned} 3\alpha_1^1 + 2\alpha_1^2 + \alpha_1^3 + C &= 2\alpha_1^1 + \alpha_1^2 + |\mathcal{B}_1| - \alpha_1^5 - 2\alpha_1^6 \\ &= \alpha_1 + m + \alpha_1^1 - \alpha_1^3 - \alpha_1^4 - 2\alpha_1^5 - 3\alpha_1^6. \end{aligned}$$

We also have

$$\sum_{p \in P} 2\ell_p + \sum_{q \in Q} (2\ell_q - 1) + \sum_{r \in R} (2\ell_r - 2) = 2|\mathcal{B}_2| - \alpha_2^2 - 2\alpha_2^3.$$

Thus, if we put

$$\delta := \alpha_1 + m + \alpha_1^1 - \alpha_1^3 - \alpha_1^4 - 2\alpha_1^5 - 3\alpha_1^6 + 2|\mathcal{B}_2| - \alpha_2^2 - 2\alpha_2^3,$$

then

$$\begin{aligned} L &= \delta + \sum_{B \in \mathcal{B}_{\geq 4}} |B| - 4|\mathcal{B}_{\geq 4}| + |\mathcal{B}_3| \\ &= \delta + 4m - |\mathcal{B}_1| - 2|\mathcal{B}_2| - 3|\mathcal{B}_3| - 4(|\mathcal{B}| - |\mathcal{B}_1| - |\mathcal{B}_2| - |\mathcal{B}_3|) + |\mathcal{B}_3| \\ &= \delta - m + 2|\mathcal{B}_2| + 2|\mathcal{B}_3|, \text{ by Lemmata 2.1 and 2.2} \\ &= \delta - \alpha_1 + |\mathcal{B}_2| + 2|\mathcal{B}_3|, \text{ by Lemma 2.3} \\ &= m + (\alpha_1^1 - \alpha_1^3 - \alpha_1^4 - 2\alpha_1^5 - 3\alpha_1^6) + 3|\mathcal{B}_2| + 2|\mathcal{B}_3| - \alpha_2^2 - 2\alpha_2^3. \end{aligned}$$

This last expression, by orthogonality, has to be equal to $2m$.

So, by Lemma 2.3 and by iii) above, we get

$$(\alpha_1^1 - \alpha_1^3 - \alpha_1^4 - 2\alpha_1^5 - 3\alpha_1^6) = (\alpha_1 - \alpha_1^2 - 2\alpha_1^3 - 2\alpha_1^4 - 3\alpha_1^5 - 4\alpha_1^6)$$

and

$$|\mathcal{B}_2| + \alpha_1 = m = \alpha_1 - \alpha_1^2 - 2\alpha_1^3 - 2\alpha_1^4 - 3\alpha_1^5 - 4\alpha_1^6 + 3|\mathcal{B}_2| + 2|\mathcal{B}_3| - \alpha_2^2 - 2\alpha_2^3,$$

which gives our result:

$$2|\mathcal{B}_2| + 2|\mathcal{B}_3| = (\alpha_1^2 + 2\alpha_1^3) + (2\alpha_1^4 + 3\alpha_1^5 + 4\alpha_1^6) + (\alpha_2^2 + 2\alpha_2^3).$$

□

Lemma 2.7 can now be used to exclude another situation:

Theorem 2.8 *If H_1 contains exactly two 1-alternating sequences, i.e. if $\alpha_1 = 2$, then H is not a circulant Hadamard matrix.*

Proof:

Lemma 2.3 implies that $|\mathcal{B}_2| = m - \alpha_1 = m - 2$. There are two blocks left, by Lemma 2.1, to cover the remaining $m + 4$ elements. Hence $|\mathcal{B}_3| \in \{0, 1\}$ for $m \geq 3$. Also, $\alpha_2^1 + \alpha_2^2 + \alpha_2^3 \leq 4$, with equality if the two 1-alternating sequences and the two remaining blocks from $\mathcal{B}_{\geq 3}$ are all alternating with a 2-alternating sequence.

Since

$$(\alpha_1^2 + 2\alpha_1^3) + (2\alpha_1^4 + 3\alpha_1^5 + 4\alpha_1^6) + (\alpha_2^2 + 2\alpha_2^3) \leq 4 + 4 + 8 = 16,$$

we get by Lemma 2.7

$$m - 2 + |\mathcal{B}_3| = |\mathcal{B}_2| + |\mathcal{B}_3| \leq 8, \text{ with } |\mathcal{B}_3| \in \{0, 1\}.$$

Hence $m \leq 10$ or $m \leq 9$, both cases for which no circulant Hadamard matrix exists. □

3 Conclusion and further work

Future work should include an analysis of further rows of H along the same line. This is what we are currently doing.

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