



HAL
open science

Combinatorial properties of Circulant Hadamard matrices

Reinhardt Euler, Luis H Gallardo, Olivier Rahavandrainy

► **To cite this version:**

Reinhardt Euler, Luis H Gallardo, Olivier Rahavandrainy. Combinatorial properties of Circulant Hadamard matrices. Contemporary mathematics, 2016, 658. hal-04612986

HAL Id: hal-04612986

<https://hal.science/hal-04612986v1>

Submitted on 15 Jun 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Combinatorial properties of Circulant Hadamard matrices

Reinhardt Euler

Lab-STICC UMR CNRS 6285, University of Brest,
20, Av. Le Gorgeu, C.S. 93837, 29238 Brest Cedex 3, France,

Luis H. Gallardo, Olivier Rahavandrainy

Laboratoire de Mathématiques UMR CNRS 6205, University of Brest,
6, Avenue Le Gorgeu, C.S. 93837, 29238 Brest Cedex 3, France.

e-mail: {Reinhardt.Euler,Luis.Gallardo,Olivier.Rahavandrainy}
@univ-brest.fr

Keywords: Hadamard matrix, Circulant, Combinatorial Properties

AMS Subject Classification: 15B34

Abstract We describe combinatorial properties of the defining row of a circulant Hadamard matrix by exploiting its orthogonality to subsequent rows, and show how to exclude several particular forms of these matrices.

1 Introduction and basic definitions

Throughout this paper, the i -row of a matrix H of size $n \times n$ is denoted by H_i , $1 \leq i \leq n$. A (± 1) -matrix H of size $n \times n$ is called *Hadamard matrix* if its rows are mutually orthogonal. It is a *circulant matrix* if for $H_1 = (h_1, \dots, h_n)$, its i -row is given by $H_i = (h_{1-i+1}, \dots, h_{n-i+1})$, the subscripts being taken modulo n . In this case, the first row H_1 will be called the *defining row* of H .

An example of a 4×4 circulant Hadamard matrix is the following:

$$H = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

A conjecture of Ryser (cf. [2, pp.97]) states that there is no circulant Hadamard matrix unless $n = 1$ or $n = 4$. This conjecture is still open although many partial results have been obtained. For recent results and further details, we refer to [1], [3] and [4].

The objective of this paper is to exhibit some combinatorial properties of the defining row H_1 of such a matrix H that we obtained by exploiting the orthogonality of H_1 to some of its subsequent rows. It turns out that these properties allow to exclude several particular forms for circulant Hadamard matrices. We think that a further development of this “combinatorial approach” may contribute to exclude other cases of such matrices and help to determine the maximum number k for which the first k rows of a circulant (± 1) -matrix are mutually orthogonal.

The order n (for $n > 1$) of a circulant Hadamard matrix H is well known to be of the form $n = 4m = 4h^2$, where h is an odd integer. Also, by orthogonality, the entries of any two different rows have the same sign in exactly $2m$ columns, and a different sign in the other $2m$ ones.

Finally, the property of being circulant allows any row of H to be represented as a circular sequence of “+” and “-”. Figure 1 illustrates the first two rows of such a matrix H for $n = 12$.

Some further definitions are required. Let us consider the first row H_1 of a circulant Hadamard matrix H . A maximal sequence of consecutive +1’s (respectively -1’s) will be called a *positive block* (respectively

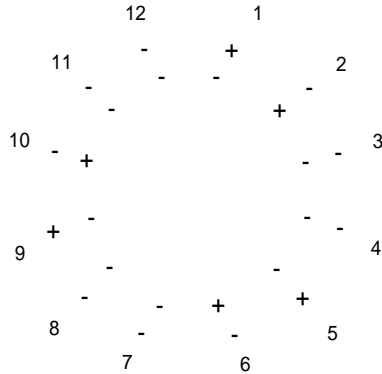


Figure 1: The first two rows of a circulant Hadamard matrix of order 12

negative block), denoted by B . If the size of such a block equals k , for some positive integer k , we will speak of a *positive* (respectively *negative*) k -block. The family of all blocks is denoted by \mathcal{B} , that of all k -blocks by \mathcal{B}_k and that of all k -blocks with $k \geq i$, for a given positive integer i , by $\mathcal{B}_{\geq i}$. Thus, H_1 decomposes into a sequence of alternating, positive or negative blocks. For convenience, we suppose the first and last block of H_1 to be of opposite sign. Finally, a maximal alternating sequence of blocks in \mathcal{B}_k (in $\mathcal{B}_{\geq k}$) will be called a k -alternating sequence ($\geq k$ -alternating sequence, respectively), the *length* of which is given by the number of blocks forming it.

2 Our results

Lemma 2.1 *The total number of blocks in the first row H_1 of a circulant Hadamard matrix H equals $2m$.*

Proof: We consider H_1 and H_2 of H and observe that the number of columns with entries of different sign equals the total number of blocks. Hence, $|\mathcal{B}| = 2m$. \square

Alternatively, if we count the number of columns whose entries have equal sign, we obtain $\sum_{B \in \mathcal{B}} (|B| - 1)$, which equals $4m - |\mathcal{B}|$ and which by orthogonality, gives $2m$. Again, $|\mathcal{B}| = 2m$. \square

Lemma 2.2 *The total number of 1-blocks in H_1 equals m .*

Proof: We consider H_1 and H_3 of H and count the number of columns with entries of equal sign. We observe that:

- in a 1-alternating sequence, this number equals the length of the sequence minus 1, plus 1 arising from the ≥ 2 -block preceding the sequence,
- a block $B \in \mathcal{B}_{\geq 3}$ contributes with $|B| - 2$,
- no block $B \in \mathcal{B}_2$ contributes to this number (except those preceding a 1-alternating path, whose contribution is already taken into account).

By orthogonality of H_1 and H_3 , we obtain:

$$\begin{aligned}
2m &= |\mathcal{B}_1| + \sum_{B \in \mathcal{B}_{\geq 3}} (|B| - 2) \\
&= |\mathcal{B}_1| + \sum_{B \in \mathcal{B}_{\geq 2}} (|B| - 2) \\
&= |\mathcal{B}_1| + \left(\sum_{B \in \mathcal{B}_{\geq 2}} |B| \right) - 2|\mathcal{B}_{\geq 2}| \\
&= |\mathcal{B}_1| + (4m - |\mathcal{B}_1|) - 2(|\mathcal{B}| - |\mathcal{B}_1|) \\
&= 4m - 4m + 2|\mathcal{B}_1|.
\end{aligned}$$

Hence, $|\mathcal{B}_1| = m$.

Alternatively, we may count the number of columns of H_1 and H_3 , with entries of different signs. We denote by α_1 the number of 1-alternating sequences, by $\alpha_{\geq 2}$ the number of ≥ 2 -alternating sequences, by Λ the set of all ≥ 2 -alternating sequences, and by $|\ell|$ the length of an element ℓ of Λ . We observe that:

- any 1-alternating sequence contributes with 2,
- any ≥ 2 -alternating sequence ℓ contributes with $2(|\ell| - 1)$.

Thus, we obtain:

$$2m = 2\alpha_1 + 2\left(\sum_{\ell \in \Lambda} (|\ell| - 1)\right). \quad (1)$$

Since 1-alternating sequences and ≥ 2 -alternating sequences are themselves alternating within H_1 , $\alpha_1 = \alpha_{\geq 2}$, and we get from (1):

$$m = \alpha_1 + |\mathcal{B}_{\geq 2}| - \alpha_{\geq 2} = |\mathcal{B}_{\geq 2}| = |\mathcal{B}| - |\mathcal{B}_1| = 2m - |\mathcal{B}_1| \text{ by Lemma 2.1.}$$

$$\text{Hence, } |\mathcal{B}_1| = m. \quad \square$$

Lemma 2.3 *The number of blocks of size 2 and the number of 1-alternating sequences add up to m :*

$$|\mathcal{B}_2| + \alpha_1 = m.$$

Proof: We consider H_1 and H_4 of H and count the number of columns with entries of equal sign. We observe that:

- any 1- alternating sequence is preceded and followed by a block of size ≥ 2 ,
- a block $B \in \mathcal{B}_{\geq 3}$ contributes with $|B| - 3$,
- any 2-block $B \in \mathcal{B}_2$ is preceded and followed by an element of opposite sign.

By orthogonality of H_1 and H_4 and by Lemmata 2.1 and 2.2, we obtain:

$$\begin{aligned} 2m &= \sum_{B \in \mathcal{B}_{\geq 3}} (|B| - 3) + |\mathcal{B}_2| + 2\alpha_1 \\ &= 4m - |\mathcal{B}_1| - 2|\mathcal{B}_2| - 3|\mathcal{B}_{\geq 3}| + |\mathcal{B}_2| + 2\alpha_1 \\ &= 3m - |\mathcal{B}_2| - 3(|\mathcal{B}| - m - |\mathcal{B}_2|) + 2\alpha_1 \\ &= 2|\mathcal{B}_2| + 2\alpha_1. \end{aligned}$$

$$\text{Hence, } |\mathcal{B}_2| + \alpha_1 = m. \quad \square$$

Remark 2.4 *It would be interesting to know whether counting the number of columns with entries of different sign provides the same condition on $|\mathcal{B}_2|$, similar to that obtained for $|\mathcal{B}_1|$. For the sake of completeness and to make the reader more familiar with our approach, we have included such a condition at this place.*

For given n , let α_1, α_2 and $\alpha_{\geq 3}$ be the number of 1-, 2- and ≥ 3 -alternating sequences, and let $\alpha_{2, \geq 3}$ denote the number of pairs of blocks B, B' with $B \in \mathcal{B}_2, B' \in \mathcal{B}_{\geq 3}$ or vice versa. We obtain the following:

Lemma 2.5

$$\alpha_1 + \alpha_{2,\geq 3} = \alpha_2 + \alpha_{\geq 3}.$$

Proof: Any 1-, 2- or ≥ 3 -alternating sequence is preceded (followed) by a 2- or 3-, 1- or ≥ 3 -, 1- or 2-alternating sequence, respectively. Let us examine the first case, i.e. that of a 1-alternating sequence. Counting the number of columns with entries of different signs gives us a contribution of $(\ell - 1)$ for such a sequence, if ℓ denotes the length of it (see Figure 2).

$$+ + - - \mid + - \boxed{\begin{array}{cccccc} - & + & \cdots & - & + & - & + \\ + & - & \cdots & + & - & + & - \end{array}} - - -$$

Figure 2: A 1-alternating sequence preceded (followed) by a 2 – (≥ 3 –) one, and the contribution of $\ell - 1$.

For a 2-alternating sequence preceded and followed by a 1-alternating one, the total contribution will be $(\ell - 1)$ again. If, however, such a sequence is preceded or followed by a ≥ 3 -alternating sequence (see Figure 3), one or two consecutive pairs of blocks B, B' with $B \in \mathcal{B}_2, B' \in \mathcal{B}_{\geq 3}$ or vice versa, will appear and contribute by an additional value of 1.

$$- + - \mid + + - \boxed{\begin{array}{c} + \\ - \end{array}} + \boxed{\begin{array}{c} - \\ + \end{array}} \cdots \boxed{\begin{array}{c} + \\ - \end{array}} + \boxed{\begin{array}{c} - \\ + \end{array}} - \boxed{\begin{array}{c} + \\ - \end{array}} \mid + \boxed{\begin{array}{c} - \\ + \end{array}} \boxed{\begin{array}{c} - \\ + \end{array}} +$$

Figure 3: A 2-alternating sequence preceded (followed) by a 1 – (≥ 3 –) one.

In that case, there is another contribution of 1 (encircled by a dashed line), that will be included in the contribution of the ≥ 3 -alternating sequence.

Finally, for ≥ 3 -alternating sequences, we obtain the situation depicted in Figure 4. The overall contribution of such a ≥ 3 -alternating sequence is $3\ell - 1$. Also, observe that the pair of blocks B, B' with $B \in \mathcal{B}_{\geq 3}$, the last block of this sequence, and $B' \in \mathcal{B}_2$, the first of the following 2-alternating one, again contribute by 1 to the value $\alpha_{2,\geq 3}$.

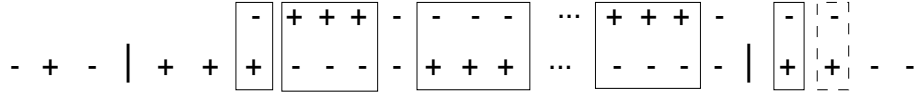


Figure 4: A ≥ 3 -alternating sequence preceded (followed) by a 1 – (2–) one.

Altogether, counting the number of columns with entries of different signs in H_1 and H_4 gives the condition:

$$\sum_{i \in I} (\ell_i - 1) + \sum_{j \in J} (\ell_j - 1) + \sum_{k \in K} (3\ell_k - 1) + \alpha_{2, \geq 3} = 2m,$$

where I, J, K , denote the sets of 1-, 2- and ≥ 3 -alternating sequences, respectively, and ℓ their length.

We obtain:

$$|\mathcal{B}_1| - \alpha_1 + |\mathcal{B}_2| - \alpha_2 + 3|\mathcal{B}_{\geq 3}| - \alpha_{\geq 3} + \alpha_{2, \geq 3} = 2m.$$

Since, by Lemma 2.1,

$$|\mathcal{B}_1| + |\mathcal{B}_2| + 3|\mathcal{B}_{\geq 3}| = 4m - 2|\mathcal{B}_1| - 2|\mathcal{B}_2|,$$

$$4m - \alpha_1 - \alpha_2 - \alpha_{\geq 3} + \alpha_{2, \geq 3} = 2|\mathcal{B}_1| + 2|\mathcal{B}_2|.$$

By Lemma 2.2, we obtain:

$$2m - \alpha_1 - \alpha_2 - \alpha_{\geq 3} + \alpha_{2, \geq 3} = 2|\mathcal{B}_2|.$$

Since, by Lemma 2.3,

$$2m - 2|\mathcal{B}_2| = 2\alpha_1,$$

we obtain as stated

$$\alpha_1 + \alpha_{2, \geq 3} = \alpha_2 + \alpha_{\geq 3}.$$

□

We are now ready to show the nonexistence of a circulant Hadamard matrix for the following four particular situations:

- i) $\alpha_1 = 1$, ii) $\alpha_1 = m$, iii) $\alpha_1 = m - 1$, iv) $\alpha_1 = 2$

Theorem 2.6 *If in H_1 , there is only one 1-alternating sequence (of length m), or if there are m such sequences (all of length 1), or if the number of 1-alternating sequences is $m-1$, then there is no circulant Hadamard matrix H with H_1 as first row.*

Proof: We consider the three cases.

Case $\alpha_1 = 1$:

We know already, from Lemmata 2.1, 2.2 and 2.3, that:

- the total number of blocks is $2m$,
- that of 1-blocks is m ,
- $|\mathcal{B}_2| + \alpha_1 = m$.

Therefore, since $n = 4m$, there is exactly one more block of size $m+2$, and if we suppose that this particular block precedes the 1-alternating sequence (clockwise), we obtain the configuration depicted in Figure 5:

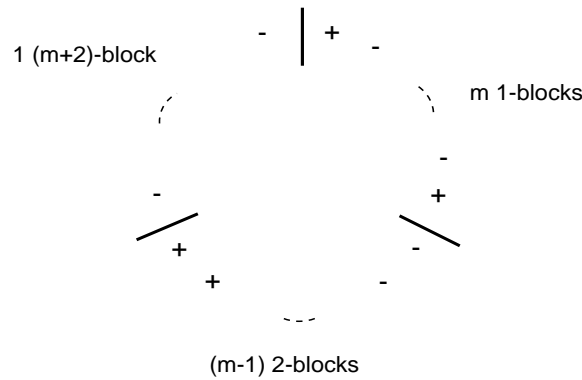


Figure 5: A first representation of H_1 for $\alpha_1 = 1$

Clockwise evaluation of the scalar product of H_1 and H_5 gives:

$$(m-4) + 0 + 2(m-3) + 0 + (m-2) + 0 = 4m - 12,$$

which allows for orthogonality only if $m = 3$, contradicting the condition on m to be an odd square integer.

By symmetry, we obtain the same result if the particular $(m+2)$ -block follows the 1-alternating sequence.

If, now, the $(m + 2)$ -block is situated “in between” the sequence of 2-blocks, as depicted in Figure 6,

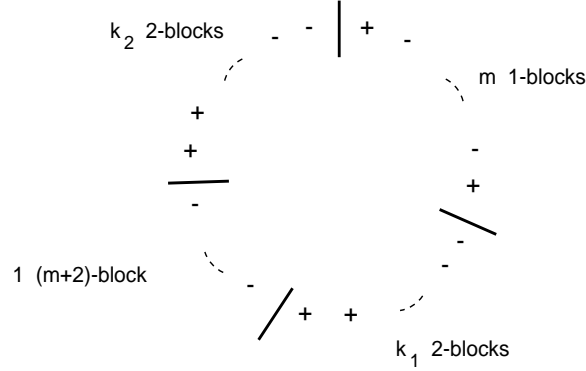


Figure 6: A second representation of H_1 for $\alpha_1 = 1$ and $(m - 1) = k_1 + k_2$

we may suppose that

$$m - 1 = k_1 + k_2, \text{ with } k_1, k_2 \text{ both even or both odd.}$$

Evaluating the scalar product of H_1 and H_5 gives:

$$(m - 4) + 0 + 2(k_1 - 2) + 0 + (m - 2) + 2(k_2 - 2) + 0 = 4m - 16.$$

Orthogonality implies $m = 4$, contradicting the condition on m to be odd. We obtain the same result for the case $k_1 = 1$ or $k_2 = 1$, so that altogether, H_1 cannot be the defining row of a circulant Hadamard matrix.

Case $\alpha_1 = m$:

By Lemma 2.3, there is no 2-block in H_1 and m blocks remain to cover $3m$ elements, i.e. besides the m 1-blocks, there are exactly m 3-blocks, and both types alternate in H_1 . But then the scalar product of H_1 and H_5 gives $4m > 0$ and H_1 cannot define a circulant Hadamard matrix either.

Case $\alpha_1 = m - 1$:

In this case there is exactly one consecutive pair of 1-blocks, and by Lemma 2.3, there is exactly one 2-block, so that $3m - 2$ elements have to

be covered by $m - 1$ blocks of size ≥ 3 . This is only possible with $(m - 2)$ 3-blocks and one 4-block. Moreover, the unique 2-block, say B , cannot be preceded *and* followed by a ≥ 3 -block. Therefore, we come up with two possibilities:

- a) B is preceded and followed by a 1-alternating sequence.
- b) B is preceded or followed (but not both) by a ≥ 3 -block.

Situation a) is depicted in Figure 7:

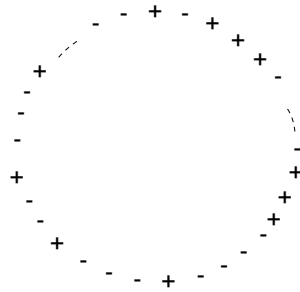


Figure 7: Illustration of possibility a)

Just observe, that in this situation exactly two ≥ 3 -blocks follow each other. We count the number of columns with different signs in H_1 and H_5 and observe that a contribution (of 2) to this number is given only by

- the two consecutive 1-blocks;
- the unique 2-block;
- the unique 4-block;
- exactly one of the 3-blocks.

By orthogonality,

$$2m = 8,$$

a contradiction to the condition on m to be odd.

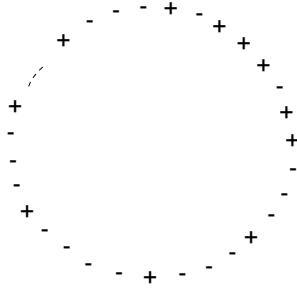


Figure 8: Illustration of possibility b)

Situation b) is depicted in Figure 8:

In this situation, no 3-block contributes any more, and orthogonality of H_1 and H_5 gives

$$2m = 6,$$

a contradiction to the condition that m has to be an odd square integer. \square

In the following, we will show how the orthogonality of H_1 and H_5 can be exploited to obtain a further condition involving $|\mathcal{B}_2|$ and $|\mathcal{B}_3|$, which we will use afterwards to exclude the situation where $\alpha_1 = 2$.

For this, we need to describe the possible types of 1-alternating and 2-alternating sequences which (up to symmetry) is done in Figures 9 and 10:

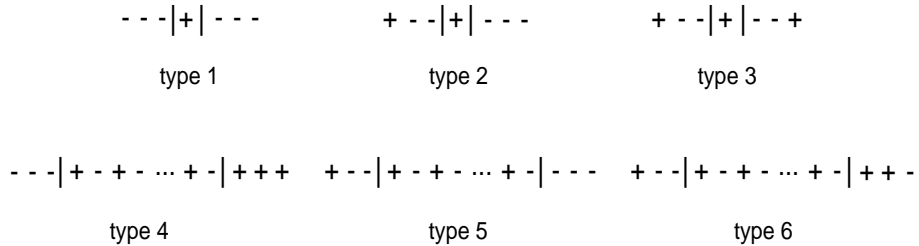


Figure 9: 1-alternating sequences of length 1 and ℓ , respectively

It is not difficult to verify that if we count the columns of H_1 and H_5 with equal sign, the contribution of these 6 types will be 3, 2, 1 and $\ell, \ell - 1, \ell - 2$,

respectively.

For 2-alternating sequences, we obtain 3 similar types, depicted in Figure 10, whose contributions will be 2ℓ , $2\ell - 1$, $2\ell - 2$, respectively:

$$\begin{array}{ccc}
 \dots|++ \dots ++| \dots & +-|++ \dots ++| \dots & +-|++ \dots ++| \dots \\
 \text{type 1} & \text{type 2} & \text{type 3}
 \end{array}$$

Figure 10: 2-alternating sequences of length ℓ

with the properties that

- type 1 is preceded and followed by a ≥ 3 -block;
- type 2 is preceded by a 1-alternating sequence and followed by a ≥ 3 -block (or vice versa);
- type 3 is preceded and followed by a 1-alternating sequence.

If now α_1^i denotes the number of 1-alternating sequences of type i , for $i = 1, \dots, 6$, and α_2^j the number of 2-alternating sequences of type j , for $j = 1, \dots, 3$, we get

Lemma 2.7 *If H is a circulant Hadamard matrix, then one has*

$$2|\mathcal{B}_2| + 2|\mathcal{B}_3| = (\alpha_1^2 + 2\alpha_1^3) + (2\alpha_1^4 + 3\alpha_1^5 + 4\alpha_1^6) + (\alpha_2^2 + 2\alpha_2^3).$$

Proof:

We can describe the total number of columns of H_1 and H_5 , whose entries have equal sign, by the following expression

$$L := 3\alpha_1^1 + 2\alpha_1^2 + \alpha_1^3 + C + D,$$

where

$$C := \sum_{i \in I} \ell_i + \sum_{j \in J} (\ell_j - 1) + \sum_{k \in K} (\ell_k - 2),$$

and

$$D := \sum_{p \in P} 2\ell_p + \sum_{q \in Q} (2\ell_q - 1) + \sum_{r \in R} (2\ell_r - 2) + \sum_{B \in \mathcal{B}_{\geq 4}} (|B| - 4) + |\mathcal{B}_3|,$$

expression in which $\sum_{j \in J} (\ell_j - 1)$, for instance, relates to all 1-alternating sequences S_j , $j \in J$, whose contribution equals $\ell_j - 1$, i.e., which are of type 5.

We know the following:

$$i) \alpha_1^1 + \alpha_1^2 + \alpha_1^3 + \sum_{i \in I} \ell_i + \sum_{j \in J} \ell_j + \sum_{k \in K} \ell_k = |\mathcal{B}_1| = m, \text{ by Lemma 2.2,}$$

$$ii) \sum_{p \in P} \ell_p + \sum_{q \in Q} \ell_q + \sum_{r \in R} \ell_r = |\mathcal{B}_2|, \quad \text{and}$$

$$iii) \alpha_1^1 + \alpha_1^2 + \alpha_1^3 + \alpha_1^4 + \alpha_1^5 + \alpha_1^6 = \alpha_1.$$

Therefore,

$$\begin{aligned} 3\alpha_1^1 + 2\alpha_1^2 + \alpha_1^3 + C &= 2\alpha_1^1 + \alpha_1^2 + |\mathcal{B}_1| - \alpha_1^5 - 2\alpha_1^6 \\ &= \alpha_1 + m + \alpha_1^1 - \alpha_1^3 - \alpha_1^4 - 2\alpha_1^5 - 3\alpha_1^6. \end{aligned}$$

We also have

$$\sum_{p \in P} 2\ell_p + \sum_{q \in Q} (2\ell_q - 1) + \sum_{r \in R} (2\ell_r - 2) = 2|\mathcal{B}_2| - \alpha_2^2 - 2\alpha_2^3.$$

Thus, if we put

$$\delta := \alpha_1 + m + \alpha_1^1 - \alpha_1^3 - \alpha_1^4 - 2\alpha_1^5 - 3\alpha_1^6 + 2|\mathcal{B}_2| - \alpha_2^2 - 2\alpha_2^3,$$

then

$$\begin{aligned} L &= \delta + \sum_{B \in \mathcal{B}_{\geq 4}} |B| - 4|\mathcal{B}_{\geq 4}| + |\mathcal{B}_3| \\ &= \delta + 4m - |\mathcal{B}_1| - 2|\mathcal{B}_2| - 3|\mathcal{B}_3| - 4(|\mathcal{B}| - |\mathcal{B}_1| - |\mathcal{B}_2| - |\mathcal{B}_3|) + |\mathcal{B}_3| \\ &= \delta - m + 2|\mathcal{B}_2| + 2|\mathcal{B}_3|, \text{ by Lemmata 2.1 and 2.2} \\ &= \delta - \alpha_1 + |\mathcal{B}_2| + 2|\mathcal{B}_3|, \text{ by Lemma 2.3} \\ &= m + (\alpha_1^1 - \alpha_1^3 - \alpha_1^4 - 2\alpha_1^5 - 3\alpha_1^6) + 3|\mathcal{B}_2| + 2|\mathcal{B}_3| - \alpha_2^2 - 2\alpha_2^3. \end{aligned}$$

This last expression, by orthogonality, has to be equal to $2m$.

So, by Lemma 2.3 and by iii) above, we get

$$(\alpha_1^1 - \alpha_1^3 - \alpha_1^4 - 2\alpha_1^5 - 3\alpha_1^6) = (\alpha_1 - \alpha_1^2 - 2\alpha_1^3 - 2\alpha_1^4 - 3\alpha_1^5 - 4\alpha_1^6)$$

and

$$|\mathcal{B}_2| + \alpha_1 = m = \alpha_1 - \alpha_1^2 - 2\alpha_1^3 - 2\alpha_1^4 - 3\alpha_1^5 - 4\alpha_1^6 + 3|\mathcal{B}_2| + 2|\mathcal{B}_3| - \alpha_2^2 - 2\alpha_2^3,$$

which gives our result:

$$2|\mathcal{B}_2| + 2|\mathcal{B}_3| = (\alpha_1^2 + 2\alpha_1^3) + (2\alpha_1^4 + 3\alpha_1^5 + 4\alpha_1^6) + (\alpha_2^2 + 2\alpha_2^3).$$

□

Lemma 2.7 can now be used to exclude another situation:

Theorem 2.8 *If H_1 contains exactly two 1-alternating sequences, i.e. if $\alpha_1 = 2$, then H is not a circulant Hadamard matrix.*

Proof:

Lemma 2.3 implies that $|\mathcal{B}_2| = m - \alpha_1 = m - 2$. There are two blocks left, by Lemma 2.1, to cover the remaining $m + 4$ elements. Hence $|\mathcal{B}_3| \in \{0, 1\}$ for $m \geq 3$. Also, $\alpha_2^1 + \alpha_2^2 + \alpha_2^3 \leq 4$, with equality if the two 1-alternating sequences and the two remaining blocks from $\mathcal{B}_{\geq 3}$ are all alternating with a 2-alternating sequence.

Since

$$(\alpha_1^2 + 2\alpha_1^3) + (2\alpha_1^4 + 3\alpha_1^5 + 4\alpha_1^6) + (\alpha_2^2 + 2\alpha_2^3) \leq 4 + 4 + 8 = 16,$$

we get by Lemma 2.7

$$m - 2 + |\mathcal{B}_3| = |\mathcal{B}_2| + |\mathcal{B}_3| \leq 8, \text{ with } |\mathcal{B}_3| \in \{0, 1\}.$$

Hence $m \leq 10$ or $m \leq 9$, both cases for which no circulant Hadamard matrix exists. □

3 Conclusion and further work

Future work should include an analysis of further rows of H along the same line. This is what we are currently doing.

References

- [1] R. Craigen, G. Faucher, R. Low, T. Wares, *Circulant partial Hadamard matrices*, Lin. Alg. Appl. **439**, 3307-3317, 2013.
- [2] Philip J. Davis, *Circulant Matrices*, Chelsea Publishing New York, N.Y., 1979, (Reprinted 1994).
- [3] R. Euler, L. Gallardo, O. Rahavandrainy, *Sufficient conditions for a conjecture of Ryser about Hadamard Circulant matrices*, Lin. Alg. Appl. **437**, 2877-2886, 2012.
- [4] K. H. Leung, B. Schmidt, *New restrictions on possible orders of circulant Hadamard matrices*, Designs, Codes and Cryptography **64**, 143-151, 2012.