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LOCAL CLASSIFICATION OF PAIRS OF SINGULAR FOLIATIONS IN DIMENSION 2.

ADJARATOU ARAME DIAW

ABSTRACT. We study the analytic classification of pairs of singular holomorphic foliations in $(\mathbb{C}^2, 0)$. We show that a pair of reduced, non-degenerate foliations with common separatrices and a simple tangency is entirely determined by a complete set of invariants composed of: the pair of holonomy representations, the couple of Camacho-Sad indices and the common separatrices.

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INTRODUCTION

Pairs of foliations appear in many domains of mathematics namely in partial differential equations and differential geometry. Their classification has application to the quadratic differential equations (i.e the theory of the holomorphic 2-web on manifolds) and to the study of Hilbert modular surfaces using modular foliations (see [2]). By flow-box theorem, every pair of tranverse regular foliations on complex surface can be locally straightened as a pair of foliations by vertical lines and horizontal lines. Therefore, all the important features of pairs of foliations occur on a the tangency set. In fact, the classification of the pair of foliations depends heavily on the nature of the tangency set. However, in [14], O.Thom shows that the classification of pairs of foliation is quite complicated even if the foliations are regular and the tangency set is normal crossings. Which makes it possible to affirm that such a classification proves even more difficult in the case where the foliations

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are singular. In [4], we state that almost all the local models of a configuration (i.e. pair of foliations + the tangency set) after resolution of the singularities on a compact complex surface are known except one. Therefore, in this article, we are going to focus on this last case. In other terms, we want to give an analytic/formal classification of a pair of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ on a complex surface S satisfying the following conditions:

- (1) The tangency set $Tang(\mathcal{F}_1, \mathcal{F}_2) = T$ consists of invariant algebraic curves by foliations \mathcal{F}_1 and \mathcal{F}_2 ;
- (2) the divisor T is normal crossings such that $Sing(\mathcal{F}_1) = Sing(\mathcal{F}_2) = Sing(T)$;
- (3) The singularities of \mathcal{F}_1 and \mathcal{F}_2 are reduced and non-degenerated.

To do this, we will first focus on the local classification of such pairs in the neighborhood of singular points. More precisely, in this article, our main goal is to give a classification of germs of pairs of reduced singular foliations on $(\mathbb{C}^2, 0)$ with simple tangency. Our main result is the following:

Theorem A (Main result). Let $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ be germs of pairs of foliations defined by:

$$\mathcal{F}_i = \langle xdy - \lambda_i y(1 + f_i(x, y))dx \rangle, \quad \mathcal{G}_i = \langle xdy - \lambda_i y(1 + g_i(x, y))dx \rangle,$$

such that $\lambda_1 \neq \lambda_2$, both non-zero. If there exists, for $i = 1, 2$, a germ of analytic diffeomorphism $\Phi_i \in \text{Diff}(\mathbb{C}^2, 0)$ preserving each component of the curve of the separatrices $\{xy = 0\}$ and conjugating \mathcal{F}_i to \mathcal{G}_i , i.e.

$$\Phi_i^* \mathcal{G}_i = \mathcal{F}_i,$$

then there exists a germ of analytic diffeomorphism $\Phi \in \text{Diff}(\mathbb{C}^2, 0)$ conjugating the pairs of foliations,

$$\Phi^* \mathcal{G}_i = \mathcal{F}_i, \quad i = 1, 2.$$

This main result can be reformulated as:

Theorem A(bis). Let $(\mathcal{F}_1, \mathcal{F}_2)$ be a germ of a pair of foliations defined as:

$$\mathcal{F}_i = \langle xdy - \lambda_i y(1 + f_i(x, y))dx \rangle \quad i = 1, 2,$$

such that $\lambda_1 \neq \lambda_2$, both non-zero. Any germ of analytic diffeomorphism $\Phi \in \text{Diff}(\mathbb{C}^2, 0)$ preserving each component of the curve of the separatrices $\{xy = 0\}$ decomposes in the form

$$\Phi = \Phi_1 \circ \Phi_2$$

where Φ_i is a germ of analytic diffeomorphism preserving the foliation \mathcal{F}_i , i.e

$$\Phi_i^* \mathcal{F}_i = \mathcal{F}_i, \quad i = 1, 2.$$

The proof of the main result will be done in two different ways. At first, we are going to give a proof based on the path method introduced by Jürgen Moser (see [11]). This method consists essentially in replacing the problem of conjugation of pairs of foliations to that of conjugation of path of pairs by making an unfolding of the pair of foliations in $(\mathbb{C}^3, 0)$. Furthermore, using this technique, we notice that, we can even generalize our main result for pairs of foliation with any order of tangency $k \geq 1$ along separatrices.

On the other hand, we give another proof of this result by introducing a new method which is sharper than the path method in the sense that it highlights all the complete invariants of a pair of foliations. Indeed, in the method of the path, only the order of tangency of the pair of foliations is taken into account.

Inspired by the work of F. Loray in (see [5]) and that of recently published by F. Loray, F. Touzet and O. Thom (see [7]), we define an invariant, a meromorphic one-form along a separatrix which measures the divergence of a pair of foliations along this separatrix.

Thus, by considering pairs of foliations with this same invariant, we can construct a local conjugation of the pairs of foliations in the neighborhood of a point in the punctured horizontal separatrix, denoted by $H := \{y = 0\} \setminus \{0\}$. After this, it suffices to find a way to extend this conjugation around the punctured disc H . In this work, we show that this extension is possible only if the foliation \mathcal{F}_1 (resp. \mathcal{F}_2) has the same monodromy as the foliation \mathcal{G}_1 (resp. \mathcal{G}_2). This is summarized by the following theorem:

Theorem B. Let $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ be two germs of pairs of foliations. Let us denote by $h_{\mathcal{F}_i}$ (resp. $h_{\mathcal{G}_i}$) the germ of holonomy of the foliation \mathcal{F}_i (resp. \mathcal{G}_i) with respect to the horizontal separatrix.

If there are two germs of analytic diffeomorphisms φ_1 and φ_2 in $(\mathbb{C}, 0)$ which conjugate the holonomies along the horizontal separatrix, i.e.

$$h_{\mathcal{G}_1} = \varphi_1 \circ h_{\mathcal{F}_1} \circ \varphi_1^{-1} \quad \text{and} \quad h_{\mathcal{G}_2} = \varphi_2 \circ h_{\mathcal{F}_2} \circ \varphi_2^{-1}.$$

then there exists a neighborhood U of a corona on H and a germ of analytic diffeomorphism Φ on this neighborhood conjugating the pairs of foliations:

$$\Phi^* \mathcal{F}_i = \mathcal{G}_i \quad \text{on } U.$$

Finally, the additional work that remains to be done is to find an analytic extension of this germ diffeomorphism Φ in the neighborhood of the origin $0 \in \mathbb{C}^2$. To obtain this kind of extension, we will use the fact that there is a transversely formal germ of diffeomorphism $\widehat{\Phi}$ defined on the disk and which conjugates the pairs of foliations. Furthermore, it is important to emphasize that the existence of this transversely formal conjugation relies on the formal version of the theorem A(bis).

In this paper, we also pointed out some special results concerning pairs of foliations no longer having simple tangency divisor. However, it is important to mention that, the study of such pairs deserves more attention and we would hope to investigate it in another work.

The structure of this paper is composed of four sections. In the first section, we define all of the basic notions required for the understanding of the classification of pairs of singular foliations. The second section is entirely devoted to the analytic classification of pairs of foliations by passing through the path method. In the third section, we introduce the notion of transversely formal classification of pairs of foliations with a simple tangency and finally, in the fourth section, we give a complete list of analytic invariant to classify the germs of pairs of singular foliations.

Notations. Denote the ring of formal series on $(\mathbb{C}^2, 0)$ by $\widehat{\mathcal{O}}_2$, its maximal ideal by $\widehat{\mathfrak{m}}_2$ and the group of formal diffeomorphisms of $(\mathbb{C}^2, 0)$ by $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$. The convergent settings of the previous sets are the germs of holomorphic functions on $(\mathbb{C}^2, 0)$ denoted by \mathcal{O}_2 , its maximal idea denoted by \mathfrak{m}_2 , and the group of diffeomorphisms of $(\mathbb{C}^2, 0)$ by $\text{Diff}(\mathbb{C}^2, 0)$. Let us denote by $\mathcal{X}(\mathbb{C}^2, 0)$ (resp. $\widehat{\mathcal{X}}(\mathbb{C}^2, 0)$) the set of germ of holomorphic (resp. formal) vector field at $0 \in \mathbb{C}^2$, with an isolated singularity at the origin. In addition, note that, in the sequel, any germ of vector fields will be identified by one of its representative in a sufficiently small neighborhood of $0 \in \mathbb{C}^2$.

1. BASIC ISSUES OF LOCAL NATURE

1.1. Family of objects one wants to study. Let S be a complex surface. A local one-form on S is a pair (U, ω) of an open set $U \subseteq S$ and a holomorphic one-form ω defined in U .

Definition 1.1. A singular foliation on S is a collection $\mathcal{F} = \{(U_i, \omega_i)_{i \in I}\}$ of local one-forms with isolated zeros such that:

- (1) $\{U_i\}$ is an open covering of S .
- (2) For each pair $i, j \in I$, if $U_i \cap U_j \neq \emptyset$, then

$$\omega_i = \varphi_{ij} \omega_j$$

for some non-vanishing holomorphic function φ_{ij} defined on $U_i \cap U_j$.

Remark 1.2. The singular locus of the foliation \mathcal{F} is given by the union of the singular locus of each one-form ω_i . We have:

$$\text{Sing}(\mathcal{F}) = \bigcup_{i \in I} \text{Sing}(\omega_i).$$

Note that, by duality we can define the foliation \mathcal{F} by replacing the one-forms ω_i by their dual vector fields X_i (i.e $\forall i \in I, \omega_i(X_i) = 0$). This dual is uniquely determined up a multiplicative invertible function.

We say that a local one-form (U, ω) on S is a local generator of the foliation \mathcal{F} if the augmented collection

$$\{(U_i, \omega_i)_{i \in I}\} \cup \{(U, \omega)\}$$

satisfies the conditions (1) and (2) of the above definition. From now on, we will suppose that the collection \mathcal{F} is saturated, meaning that it contains all such generators.

In this work, we are interested in a germ of holomorphic singular foliations. Roughly speaking, a germ of foliation at $0 \in \mathbb{C}^2$ is given by a germ of local generator (U, ω) at 0 . Throughout this article, unless otherwise stated, any germ of a foliation is represented by a holomorphic one-form ω defined on an open neighborhood of $0 \in \mathbb{C}^2$ and having a unique singularity at the origin.

Definition 1.3 (Formal vector fields).

A formal vector field $\widehat{X} \in \widehat{\mathcal{X}}(\mathbb{C}^2, 0)$ is a derivation of the ring of formal series. In the local coordinate (x, y) , it can be written as

$$\widehat{X} = \hat{a}_1(x, y)\partial_x + \hat{a}_2(x, y)\partial_y$$

where $\hat{a}_i \in \widehat{\mathfrak{m}}_2$, for any $i = 1, 2$.

Remark 1.4. It is well-known that if the linear part of a formal vector field \widehat{X} is nilpotent, then its formal flow given by

$$\exp(t\widehat{X})(x, y) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \widehat{X}^j(x, y)$$

converges in the formal setting and defines a formal diffeomorphism of $(\mathbb{C}^2, 0)$.

Next, for any $k \geq 1$, let us denote by $\widehat{\mathcal{X}}_k(\mathbb{C}^2, 0)$ the set of formal vector fields \widehat{X} defined as $\widehat{X} = \hat{a}_1(x, y)\partial_x + \hat{a}_2(x, y)\partial_y$ where $\hat{a}_i \in \widehat{\mathfrak{m}}_2^k$.

Thanks to Cerveau-Moussu we have the following result. For more details the reader can see [3].

Proposition 1.5. *The flow of a nilpotent formal vector field $\widehat{X} \in \widehat{\mathcal{X}}_{k>1}(\mathbb{C}^2, 0)$ is polynomial in variable t , i.e. $\exp(t\widehat{X})(x, y) \in (\mathbb{C}[t][[x, y]])^2$.*

In particular, the time one-map of the formal flow verifies the following formula which is called the Baker-Campbell-Hausdorff formula.

Lemma 1.6 (Baker-Campbell-Hausdorff formula). *Let \widehat{X} and \widehat{Y} be two nilpotent formal vector fields in $\widehat{\mathcal{X}}(\mathbb{C}^2, 0)$. Then, there exists a nilpotent formal vector field $\widehat{Z} \in \widehat{\mathcal{X}}(\mathbb{C}^2, 0)$ such that:*

$$\exp(\widehat{X}) \circ \exp(\widehat{Y}) = \exp(\widehat{Z}),$$

Moreover, we have:

$$\widehat{Z} = \widehat{X} + \widehat{Y} + \frac{1}{2}[\widehat{X}, \widehat{Y}] + \frac{1}{12}[\widehat{X}, [\widehat{X}, \widehat{Y}]] + \dots$$

In this work, we also consider special formal vector fields which are called a logarithmic vector fields introduced by K. Saito [for more details, see [12]]. Let us consider T a reduced analytic curve of $(\mathbb{C}^2, 0)$ and for any point $p \in T$, let \mathcal{O}_p be is the germ of holomorphic functions at p .

Definition 1.7. A germ of holomorphic vector field X at p is logarithmic with respect the divisor T , if for any reduced equation h_p of T in a neighborhood of p , we have the following:

$$X.h_p = f_p h_p, \quad \text{where } f_p \in \mathcal{O}_p.$$

We denote by $Der(-\log T)_p$, the set of germs at p of logarithmic vector fields with respect T . Furthermore, if a vector field is logarithmic at any point of T , we shall say that the vector field is logarithmic along the divisor T .

Thanks to K. Saito, we have:

Proposition 1.8. *Let (v_1, v_2) be two germs of vector fields in $Der(-\log T)_p$. If there exists an invertible function $u \in \mathcal{O}_p$ such that the function determinant $\det(v_1, v_2)$ verifies*

$$\det(v_1, v_2) = u h_p,$$

then the vector fields v_1 and v_2 form a basis of the \mathcal{O}_p -module $Der(-\log T)_p$.

Remark 1.9. Let us recall that, in [12], K. Saito stated that the above Proposition is also true in the formal setting.

1.2. The notion of equivalence of germs of foliations.

Definition 1.10. Let \mathcal{F} and \mathcal{G} be two germs of foliations on $(\mathbb{C}^2, 0)$. Assume that the foliation \mathcal{F} (resp. \mathcal{G}) is represented by the one-form ω (resp. ω').

(1) The one-forms ω and ω' define the same foliation if and only if we have:

$$\omega \wedge \omega' = 0.$$

(2) \mathcal{F} is analytically conjugated to \mathcal{G} (or $\mathcal{F} \sim \mathcal{G}$) if and only if there exists a germ of analytic diffeomorphism $\Phi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that

$$\Phi^* \omega' \wedge \omega = 0.$$

In formal sense, we have:

Definition 1.11. The germs of foliations \mathcal{F} and \mathcal{G} are formally equivalent if there exists a germ of formal diffeomorphism $\widehat{\Phi}: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that $\widehat{\Phi}^* \mathcal{G} = \mathcal{F}$, i.e

$$\widehat{\Phi}^* \omega' \wedge \omega = 0$$

in formal one-forms.

In general, it is not accurate to obtain an analytic conjugation of foliations from a formal one, but sometimes, we can perform the formal equivalence of the foliations by using the notion of transversely formal conjugation.

Definition 1.12 (Transversely formal series.) Let $V \subset \mathbb{C}$ be an open set (it can contain 0 or not) and $C := V \times \{0\} \subset \mathbb{C}^2$ its embedding into the horizontal axis. A formal power series $\sum_{n \geq 1} a_n(x)y^n$ is transversely formal along C , if all the functions $x \mapsto a_n(x)$ are analytic on C . Note that the power series can be divergent along the y -direction. We denote by $\mathcal{O}(C)[[y]]$ the group of formal power series which are transversely formal along C .

Remark 1.13. We shall say that a vector field is a transversely formal vector field on (\mathbb{C}^2, C) if it can be written as

$$\widehat{X} = \widehat{A}(x, y)\partial_x + \widehat{B}(x, y)\partial_y,$$

where the functions \widehat{A} and \widehat{B} are transversely formal along C . Likewise, a formal diffeomorphism $\widehat{\Phi}$ is transversely formal along C , if it can be expressed in the form

$$\widehat{\Phi}(x, y) = (x + y\widehat{f}(x, y), y\widehat{g}(x, y))$$

where $\widehat{f}, \widehat{g} \in \mathcal{O}(C)[[y]]$ and $\widehat{g}(x, 0)$ does not vanish along C .

We denote by $\text{Diff}(C \times (\mathbb{C}, 0))$ the group of transversely formal diffeomorphisms.

1.3. The invariant to understand if foliations are equivalent. If two singular foliations are equivalent, the most important invariants are the separatrices, the Camacho-Sad indices and the holonomies.

Let \mathcal{F} be a germ foliation defined on $(\mathbb{C}^2, 0)$. Let us consider (U, ω) a representative of \mathcal{F} where U is an open neighborhood of $0 \in \mathbb{C}^2$ and ω is a holomorphic one-form defined by

$$\omega = A(x, y)dx + B(x, y)dy$$

such that $A(0, 0) = B(0, 0) = 0$ and $A(x, 0) = 0$. Thus, it is clear that the curve $\{y = 0\}$ is an invariant curve and we shall call it the horizontal separatrix of \mathcal{F} . In general, we have the following definition:

Definition 1.14. A separatrix of \mathcal{F} is a germ of an analytic curve at 0, invariant by the foliation.

Remark 1.15. Note that the number of separatrices of a foliation is invariant under the action of the analytic diffeomorphism on $(\mathbb{C}^2, 0)$.

Now, if we consider $X = B(x, y)\partial_x - A(x, y)\partial_y$, the dual of the one-form ω , then the eigenvalues μ_1, μ_2 of the linear part $DX(0)$ of X at $0 \in \mathbb{C}^2$ are well defined. If the eigenvalue μ_2 is not zero, then the quotient $\lambda = \frac{\mu_1}{\mu_2}$ is called the Camacho-Sad index of \mathcal{F} with respect to the horizontal separatrix. This quotient λ is unchanged by multiplication of X by a non-vanishing holomorphic function. Moreover, one can see that it depends only on the germ of the foliation.

More precisely, if a foliation admits a smooth separatrix, then the Camacho-Sad index is defined as follows:

Definition 1.16. Let C be a smooth separatrix of \mathcal{F} . Assume that local coordinates are chosen such that $C := \{y = 0\}$. The Camacho-Sad index of \mathcal{F} with respect to C is defined as:

$$CS(\mathcal{F}, C, 0) = -\text{Res}_{x=0} \frac{\partial}{\partial y} \left(\frac{A}{B} \right) (x, 0),$$

where $\text{Res}_{x=0}$ is the residue at $0 \in \mathbb{C}$.

Remark 1.17. It is important to remark that the Camacho-Sad index is invariant by any change of coordinates preserving the horizontal separatrix: we shall say that it is an analytic invariant of the foliation.

Let us now discuss of the concept of holonomy which is one of the most important analytic invariant of the foliation.

Let $f: (\mathbb{C}^2, p_0) \rightarrow (\mathbb{C}, 0)$ be a local submersion defined on an open set U_{p_0} at p_0 , such that $\omega \wedge df \equiv 0$. Then, f is a minimal first integral of \mathcal{F} at p_0 : any other first integral of \mathcal{F} on U_{p_0} is written in the form $\varphi \circ f$, where φ is a function in one variable such that $\varphi(0) = 0$. Furthermore, f can be extended analytically along any path in the horizontal separatrix C . Hence, we have:

Definition 1.18. The holonomy of the foliation along the loop γ , ($\gamma(0) = \gamma(1) = p_0$) defined inside $C \setminus \{0\}$, is the unique germ of diffeomorphism $h_\gamma^0: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ such that the germ of analytic continuation of f along the loop γ is given by

$$f^\gamma = h_\gamma^0 \circ f.$$

Remark 1.19. Note that the holonomy depends only on the homotopy class of the loop. In addition, if $g: (\mathbb{C}^2, p_0) \rightarrow (\mathbb{C}, 0)$ is an another minimal first integral at the point p_0 , then there exists a diffeomorphism $\varphi \in \text{Diff}(\mathbb{C}, 0)$ such that $g = \varphi \circ f$ and such that the germ of analytic continuation of g along the loop γ is given by

$$g^\gamma = (\varphi \circ h_\gamma^0 \circ \varphi^{-1}) \circ g.$$

Hence, the germ of holonomy $h_\gamma^0: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ of the foliation \mathcal{F} along the loop γ

is well-defined modulo conjugation by an element of $\text{Diff}(\mathbb{C}, 0)$. Finally, if we denote by $\text{Diff}(\mathbb{C}, 0)/\sim$ the group of germs of diffeomorphisms modulo conjugation, we have the following representation associated to the foliation.

$$\begin{cases} \pi_1(C, p_0) & \rightarrow & \text{Diff}(\mathbb{C}, 0)/\sim \\ \gamma & \rightarrow & h_\gamma^0 \end{cases}$$

The image of this representation is called the holonomy group of the horizontal separatrix C . We denote it by $h_{\mathcal{F}}$.

By definition, the holonomy group of a germ of foliation is well-defined and it is an analytic invariant in the sense that if two germs of foliations at 0 are conjugated by a germ of diffeomorphism which preserves the curve C , then their holonomies groups related to C are conjugated.

These invariants are not always complete, i.e. do not classify the foliation, in fact, it depends on the complexity of the singularity. However, there exists a kind of singularities for which the classification by means these invariants of foliations is well-understood.

Definition 1.20. Let \mathcal{F} be a foliation on $(\mathbb{C}^2, 0)$. The singularity 0 is said to be reduced in Seidenberg sense if for every one-form ω with an isolated singularity at 0 defining \mathcal{F} , one of the following case necessarily occurs:

- **Non-degenerate singularity:** the linear part of the dual vector field of ω has two eigenvalues different from zero and the ratio between these eigenvalues is not a positive rational number. Thus, thanks to Briot-Bouquet, up to analytic change of coordinates, the foliation is defined by

$$\omega = x[1 + f(x, y)]dy - y[\lambda + g(x, y)]dx$$

with ($\lambda \neq 0$), $\lambda \notin \mathbb{Q}^+$ and the functions f, g vanish at the origin $0 \in \mathbb{C}^2$. The separatrices are the axes of coordinates $\{x = 0\}$ and $\{y = 0\}$ and moreover, in this case, λ is the camacho-Sad index at 0 along the horizontal separatrix $\{y = 0\}$.

When $\lambda \in \mathbb{C}^* \setminus \mathbb{R}_-$, then we shall say that the singularity is of Poincaré type, if $\lambda \in \mathbb{R}_-^*$, then the singularity is of Siegel type.

- **Saddle-Node singularity:** the linear part of the dual vector field of ω has exactly one eigenvalue different from zero. Up to holomorphic change of coordinates, the foliation is defined by

$$\omega = [x(1 + \mu y^k) + yR(x, y)]dy - y^{k+1}dx$$

with $\mu \in \mathbb{C}$, $k \in \mathbb{N}$ and $R(0, 0) = 0$ and such that the function $R(0, y)$ has order up to k at 0. The curve $\{y = 0\}$ is invariant by the foliation and it is called the strong separatrix. In addition, there also exists a formal separatrix which may or may not correspond to an analytic separatrix, i.e. it can be divergent.

These two kind of singularities are considered as the most simple singularities of a foliation: they are said reduced and the class of reduced singular points is stable under blow-up. In dimension 2, Seidenberg states that every isolated singularity of a foliation becomes reduced after a finite number of blow-ups. Details and proofs can be found in [13].

In the sequel, we only consider the germs of reduced foliations on $(\mathbb{C}^2, 0)$. Let us recall the formal normal form of reduced foliations due to Briot-Bouquet, Poincaré-Dulac. Mainly, the normal forms are determined by the Camacho-Sad index. If there is no resonance between the eigenvalues, then the normal form is linear. However, the linearizing diffeomorphism can be formal. Siegel and Brjuno showed that if the eigenvalues verify a certain diophantian condition, then the linearizing diffeomorphism is analytic. For more details about the analytic and formal normalization, we refer the reader to [1].

The following proposition is the most important result about normal forms, furthermore, in [4], we showed that each normalizing diffeomorphism can be constructed as a transversely formal diffeomorphism along the horizontal separatrix. We have:

Proposition 1.21. *Let $\mathcal{F} = \langle xdy - \lambda ydx + h.o.t \rangle$ be of reduced foliation on $(\mathbb{C}^2, 0)$.*

- (1) *If $\lambda \in \mathbb{R}_- \setminus \mathbb{Q}_-$, then there exists $\widehat{\Phi} \in \text{Diff}(C \times \widehat{(\mathbb{C}, 0)})$ such that:*

$$\widehat{\Phi}^*\mathcal{F} = \langle xdy - \lambda ydx \rangle .$$

- (2) *$\lambda = -\frac{p}{q} \in \mathbb{Q}_-^*$, then there exists $\widehat{\Phi} \in \text{Diff}(C \times \widehat{(\mathbb{C}, 0)})$ such that either:*

$$\widehat{\Phi}^*\mathcal{F} = \langle qxdy + pydx \rangle$$

or

$$\widehat{\Phi}^*\mathcal{F} = \langle q(1 + \alpha u^k)xdy + p(1 + (\alpha - 1)u^k)ydx \rangle ,$$

where $u = x^p y^q$. The integer $k > 0$ and the complex number α are formal invariants.

- (3) *If $\lambda \in \mathbb{C} \setminus \{\mathbb{R}_- \cup \mathbb{Q}_+\}$, then there exists an analytic diffeomorphism $\Phi \in \text{Diff}(\mathbb{C}^2, 0)$ such that*

$$\Phi^*\mathcal{F} = \langle xdy - \lambda ydx \rangle .$$

- (4) *If $\lambda = 0$, then there exists $\widehat{\Phi} \in \text{Diff}(C \times \widehat{(\mathbb{C}, 0)})$ such that*

$$\widehat{\Phi}^*\mathcal{F} = \langle y^{k+1}dx - y(1 + \mu x^k)dx \rangle ,$$

where $k \in \mathbb{N}^*$, and the function f is holomorphic. Moreover, k and μ are formal invariants.

Thanks to the above Proposition, reduced foliations are transversely formally classified by formal invariants which are λ , α , μ and k .

1.4. Analytic classification of reduced foliations. Now, we are going to recall the analytic classification of the reduced foliations around the neighborhood of the origin $0 \in \mathbb{C}^2$. Thanks to J.F. Mattei, R. Moussu and Martinet-Ramis (see [10] and [9] for more details), a foliation in the Siegel domain (resp. a foliation in the saddle-node domain) is entirely determined by their holonomies. More precisely, we have:

Theorem 1.22. *Let*

$$\mathcal{F}_i = \langle x[1 + f_i(x, y)]dy - y[\lambda + g_i(x, y)]dx \rangle,$$

($i = 1, 2$) be two germs of non-degenerate foliations at $0 \in \mathbb{C}^2$, such that $\lambda \in \mathbb{R}^-$. Denote by h_1 (resp. h_2) the holonomy map of the foliation \mathcal{F}_1 (resp. \mathcal{F}_2) with respect to the horizontal separatrix $\{y = 0\}$. Then, the holonomy maps are analytically conjugated by a germ in $\text{Diff}(\mathbb{C}, 0)$ if and only if the foliations \mathcal{F}_1 and \mathcal{F}_2 are conjugated by a germ of an analytic diffeomorphism $\Phi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ preserving the horizontal separatrix.

Remark 1.23. Likewise, if we consider

$$\mathcal{F}_i = \langle [x(1 + \mu_i y^{k_i}) + yR_i(x, y)]dy - y^{k_i+1}dx \rangle,$$

($i = 1, 2$) two germs of saddles nodes at $0 \in \mathbb{C}^2$. Then the same result follows.

In short, these previous result state that a convergent separatrix and the holonomy related to it are complete invariants for reduced singularities.

The case of a foliation of Poincaré type is more accurate. Indeed, the only complete invariant is the Camacho-Sad index, due to the fact that such foliations are always analytically linearizable.

From now on, we want to generalize the classification of foliations to that of pairs of foliations on $(\mathbb{C}^2, 0)$. Using the same spirit, we want to classify the pairs of foliations by developing a complete set of invariants.

Definition 1.24. Let $(\mathcal{F}_1, \mathcal{F}_2)$ be a germ of a pair of singular foliations on $(\mathbb{C}^2, 0)$. Let us consider the one-forms ω_i ($i = 1, 2$) which are representatives of the germs \mathcal{F}_i on $(\mathbb{C}^2, 0)$. We shall denote it by $\mathcal{F}_i := \langle \omega_i \rangle$. If we assume that $\omega_1 \wedge \omega_2 = f(x, y)dx \wedge dy$, then the tangency divisor of the pair $(\mathcal{F}_1, \mathcal{F}_2)$ is a germ at the origin of a divisor defined as

$$\text{Tang}(\mathcal{F}_1, \mathcal{F}_2) := \{\text{div}(f)\}.$$

Remark 1.25. Note that up to a multiplicative non-zero coefficient, the tangency divisor does not depend on the chosen coordinates. In the sequel, we denote the tangency divisor by T .

Motivated by the result in [4], we are interested in the case where the foliations \mathcal{F}_i , ($i = 1, 2$) are reduced with common separatrices $\{xy = 0\}$ and such that $T = \text{div}(x^{k_1}y^{k_2})$. In this case, k_1 (resp. k_2) is the order of contact of the pair $(\mathcal{F}_1, \mathcal{F}_2)$ along the separatrix $\{x = 0\}$ (resp. $\{y = 0\}$). In particular, if $k_1 = k_2 = 1$, then we shall say that the tangency is simple.

Definition 1.26. Let $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ be two germs of pairs of foliations on $(\mathbb{C}^2, 0)$ and having the same tangency divisor T . We shall say that the pair $(\mathcal{F}_1, \mathcal{F}_2)$ is analytically conjugated to the $(\mathcal{G}_1, \mathcal{G}_2)$ if there exists a germ of an analytic diffeomorphism $\Phi \in \text{Diff}(\mathbb{C}^2, 0)$ such that

$$\Phi^* \mathcal{G}_i = \mathcal{F}_i, \quad \forall i = 1, 2 \quad \text{and} \quad \Phi|_T = \text{Id}.$$

More precisely, if $\mathcal{F}_i := \langle \omega_i \rangle$ and $\mathcal{G}_i := \langle \alpha_i \rangle$, $i = 1, 2$, then there exists a germ of analytic diffeomorphism defined as

$$\Phi(x, y) = (x + xy\Phi_1(x, y), y + xy\Phi_2(x, y))$$

and such that

$$\Phi^* \alpha_i \wedge \omega_i = 0, \quad \forall i = 1, 2.$$

Remark 1.27. In particular, if we have $\Phi^* \mathcal{F}_i = \mathcal{F}_i$ and $\Phi|_T = Id$, we shall say that the Φ preserves (or Φ is a symmetry of) the pair of foliations $(\mathcal{F}_1, \mathcal{F}_2)$. We denote by $\text{Diff}(\mathbb{C}^2, T, 0)$ (resp. $\widehat{\text{Diff}}(\mathbb{C}^2, T, 0)$) the germs of analytic (resp. formal) diffeomorphisms which are tangent to identity along T .

2. LOCAL ANALYTIC CLASSIFICATION OF PAIRS OF FOLIATIONS: PATH METHOD

Let us first consider the problem of the conjugation of the pairs of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ when \mathcal{F}_1 and \mathcal{G}_1 as well as \mathcal{F}_2 and \mathcal{G}_2 are independently conjugated.

Theorem 2.1. *Let $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ be the germs of pairs of foliations defined on $(\mathbb{C}^2, 0)$ by:*

$$\mathcal{F}_i = \langle xdy - \lambda_i y(1 + f_i(x, y))dx \rangle, \quad \mathcal{G}_i = \langle xdy - \lambda_i y(1 + g_i(x, y))dx \rangle,$$

$i = 1, 2$, with $\lambda_1 \neq \lambda_2$. Note that, $\lambda_1 \neq \lambda_2 \iff T := \{xy = 0\}$.

Assume that \mathcal{F}_1 and \mathcal{G}_1 (resp. \mathcal{F}_2 and \mathcal{G}_2) are conjugated by a germ of an analytic diffeomorphism $\Phi_1: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ (resp. $\Phi_2: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$) tangent to identity along the tangency divisor $T := \{xy = 0\}$. Then, there exists a germ of an analytic diffeomorphism $\Phi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \in \text{Diff}(\mathbb{C}^2, T, 0)$ which conjugates $(\mathcal{F}_1, \mathcal{F}_2)$ to $(\mathcal{G}_1, \mathcal{G}_2)$, i.e. $\Phi^* \mathcal{G}_1 = \mathcal{F}_1$ and $\Phi^* \mathcal{G}_2 = \mathcal{F}_2$.

Proof. Let us assume that the germs of foliations \mathcal{F}_i and \mathcal{G}_i , $i = 1, 2$, are defined by the germs of one-forms $\omega_i = xdy - \lambda_i y(1 + f_i(x, y))dx$ and $\alpha_i = xdy - \lambda_i y(1 + g_i(x, y))dx$ respectively.

By assumption, there exist germs of invertible functions F, G such that $\Phi_1^* \alpha_1 = F\omega_1$ and $\Phi_2^* \alpha_2 = G\omega_2$. Thus, without loss of generality, by replacing the generators ω_1 and ω_2 by $F\omega_1$ and $G\omega_2$ respectively, we can assume that $\Phi_1^* \alpha_1 = \omega_1$ and $\Phi_2^* \alpha_2 = \omega_2$.

In the same way, even if we conjugate the germs of one-forms α_1 and α_2 by Φ_2^{-1} , we can suppose that $\phi_2 = Id$ and so $\alpha_2 = \omega_2 = \alpha$. Therefore, to conjugate the pairs $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ simultaneously is amount to conjugate the pairs $(\mathcal{F}_1, \mathcal{G})$ and $(\mathcal{G}_1, \mathcal{G})$ simultaneously, where $\mathcal{G} = \langle \alpha \rangle$. In addition, we can assume that

$$\Phi_1(x, y) = (x, y(1 + u(x, y))), \quad u(0, 0) = 0.$$

Now, let us consider the one parameter family of the germs of analytic diffeomorphisms $(\Phi_t)_{t \in [0, 1]}$ defined by

$$\Phi_t(x, y) = (x, y(1 + tu(x, y))),$$

the projection map

$$\pi: \mathbb{C}^2 \times [0, 1] \rightarrow \mathbb{C}^2, \quad \pi(x, y, t) = (x, y),$$

and the map $\Phi: (\mathbb{C}^2, 0) \times [0, 1] \rightarrow (\mathbb{C}^2, 0)$ defined as

$$\Phi(x, y, t) = \Phi_t(x, y) = (x, y(1 + tu(x, y))).$$

If we can pull back the foliation \mathcal{G}_1 defined by α_1 on $(\mathbb{C}^2, 0)$ to a foliation of codimension 1 on $(\mathbb{C}^2, 0) \times [0, 1]$ by Φ , then this previous is defined as:

$$\Omega := \Phi^* \alpha_1 = A(x, y, t)dx + B(x, y, t)dy + C(x, y, t)dt$$

where:

$$\begin{aligned} A(x, y, t) &= y(xt\partial_x u(x, y) - \lambda_1(1 + tu(x, y))(1 + g_1 \circ \Phi_t(x, y, t))) \\ B(x, y, t) &= x(1 + t(u(x, y) + y\partial_y u(x, y))) \\ C(x, y, t) &= xyu(x, y) \end{aligned}$$

Furthermore, we have:

$$\Omega_{|\mathbb{C}^2 \times \{0\}} = \alpha_1 \quad \text{and} \quad \Omega_{|\mathbb{C}^2 \times \{1\}} = \omega_1.$$

Note that the notation $\Omega_{|\mathbb{C}^2 \times \{0\}} = \alpha_1$ (resp. $\Omega_{|\mathbb{C}^2 \times \{1\}} = \omega_1$) means the intersections of the leaves of the codimension one foliation defined by Ω with the plane $\mathbb{C}^2 \times \{0\}$ (resp. $\mathbb{C}^2 \times \{1\}$) is given by the leaves of the one dimensional foliation defined by the one-form α_1 (resp. ω_1).

Next, we want to find out an analytic diffeomorphism on $(\mathbb{C}^2, 0) \times [0, 1]$, of the form $h(x, y, t) = (h_1(x, y, t), h_2(x, y, t), t)$ such that:

- the foliation defined by Ω is conjugated by h to the foliation defined by the one-form $\pi^*\alpha_1$;
- the foliation defined by $\pi^*\alpha$ is invariant by h .

It amounts to construct a vector field

$$Z = U(x, y, t)\partial_x + V(x, y, t)\partial_y + \partial_t \quad \text{such that} \quad \alpha(Z) = 0 \quad \text{and} \quad \Omega(Z) = 0.$$

Therefore, we have to solve the following system of equations:

$$\begin{cases} AU + BV &= -xyu(x, y) \\ -\lambda_2 y(1 + g_2)U + xV &= 0 \end{cases}$$

As $\lambda_1 \neq \lambda_2$, the discriminant $\Delta(x, y) = xA(x, y) + \lambda_2 yB(x, y)(1 + g_2(x, y))$ of the previous system can be seen as

$$\Delta(x, y) = xy\Delta'(x, y), \quad \text{with} \quad \Delta'(0, 0) \neq 0.$$

Hence, the solutions of the system are

$$U(x, y, t) = -\frac{xu(x, y)}{\Delta'(x, y)} \quad \text{and} \quad V(x, y, t) = -\frac{\lambda_2 yu(x, y)(1 + g_2(x, y))}{\Delta'(x, y)}$$

which are holomorphic on $(\mathbb{C}^2, 0) \times [0, 1]$.

Finally, by integrating the holomorphic vector field Z , we obtain a germ of an analytic diffeomorphism $h = h(x, y, 1)$ which verifies:

$$h^*\omega_1 = \alpha_1 \quad \text{and} \quad h^*\alpha = \alpha.$$

□

One direct consequence of the previous theorem is the following:

Corollary 2.2. *Any pair of foliations of Poincaré type with simple tangency is simultaneously linearizable.*

Proof. Let

$$\mathcal{F}_1 = \langle xdy - \lambda_1 y(1 + f_1(x, y))dx \rangle, \quad \text{and} \quad \mathcal{F}_2 = \langle xdy - \lambda_2 y(1 + f_2(x, y))dx \rangle,$$

with $\lambda_1 \neq \lambda_2 \in \mathbb{C} \setminus \{\mathbb{R}_- \cup \mathbb{Q}_+\}$. Thanks to the Proposition 1.21 and the Theorem 2.1, there exists $\Phi \in \text{Diff}(\mathbb{C}^2, T, 0)$ such that:

$$\Phi^*\mathcal{F}_1 = \langle xdy - \lambda_1 ydx \rangle \quad \text{and} \quad \Phi^*\mathcal{F}_2 = \langle xdy - \lambda_2 ydx \rangle.$$

□

Another result that follows to the previous is the following generalization of the theorem of Mattei-Moussu for the pairs of foliations.

Theorem 2.3. *Let*

$$\mathcal{F}_i = \langle xdy - \lambda_i y[1 + f_i(x, y)]dx \rangle,$$

and

$$\mathcal{G}_i = \langle xdy - \lambda_i y[1 + g_i(x, y)]dx \rangle,$$

($i = 1, 2$), be two germs of pairs of foliations such that $\lambda_i \in \mathbb{R}_*^*$ and $\lambda_1 \neq \lambda_2$. Denote by $h_{\mathcal{F}_i}$ (resp. $h_{\mathcal{G}_i}$) the holonomy map of the foliation \mathcal{F}_i (resp. \mathcal{G}_i) with respect to the horizontal separatrix $\{y = 0\}$. Then, the pair of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ is conjugated to the pair of foliations $(\mathcal{G}_1, \mathcal{G}_2)$ by an element of $\text{Diff}(\mathbb{C}^2, T, 0)$ (with $T = \{xy = 0\}$) if and only if there exists analytic diffeomorphisms $\varphi_i \in \text{Diff}(\mathbb{C}, 0)$ such that for any $i = 1, 2$ we have:

$$h_{\mathcal{G}_1} = \varphi_1 \circ h_{\mathcal{F}_1} \circ \varphi_1^{-1} \quad \text{and} \quad h_{\mathcal{G}_2} = \varphi_2 \circ h_{\mathcal{F}_2} \circ \varphi_2^{-1}.$$

The proof of this previous theorem follows from Theorem 2.1 combined with Mattei-Moussu theorem. Finally, pairs of non-degenerate foliations with simple tangency have only the Camacho-Sad indices and the corresponding holonomies with respect to the horizontal separatrix as complete invariants.

To find the complete invariants of the pairs of foliations in the case where the tangency divisor is no longer reduced (for example if $\lambda_1 = \lambda_2$), is more complicated. However, as in the simple tangency case, if the conjugating diffeomorphisms Φ_i are sufficiently tangent to the identity along T , we have the same result:

Theorem 2.4. *Let $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ be two germs of pairs of foliations having the tangency divisor $T = T_1 \cup T_2$. Assume that the order of contact of the pairs along the common separatrix T_i is equal to k_i , ($i = 1, 2$). If there exists germs of analytic diffeomorphisms tangent to the identity up to order k_1 (resp. up to order k_2) along T_1 (resp. T_2), $\Phi_i: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$, $i = 1, 2$ and such that $\Phi_i^* \mathcal{G}_i = \mathcal{F}_i$, then the pairs $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ are conjugated by a germ of an analytic diffeomorphism which is tangent to the identity along T .*

Proof. The proof is the same as in the case of simple tangency (i.e. when Camacho-Sad indices of the foliations were different.)

Without loss of generality, we can set that $\mathcal{F}_2 = \mathcal{G}_2$ and so $\Phi_2 = id$. Let us consider $\omega_1 = A_1(x, y)dx + B_1(x, y)dy$, $\alpha_1 = a_1(x, y)dx + b_1(x, y)dy$ and $\alpha = M(x, y)dx + N(x, y)dy$, the germs of one-forms at $0 \in \mathbb{C}^2$ which define respectively the foliations \mathcal{F}_1 , \mathcal{G}_1 and $\mathcal{F}_2 = \mathcal{G}_2 = \mathcal{G}$. Assume that

$$\Phi_1(x, y) = (x + x^{k_1+1}y^{k_2+1}\phi(x, y), y + x^{k_1+1}y^{k_2+1}\psi(x, y)), \quad \psi(0, 0) \neq 0, \quad \phi(0, 0) \neq 0.$$

Now, we can pull back the foliation $\mathcal{G}_1 = \langle \alpha_1 \rangle$ to a foliation of codimension 1 on $(\mathbb{C}^2, 0) \times [0, 1]$ by the germ of map

$$\tilde{\Phi}(x, y, t) = (x + t(x^{k_1+1}y^{k_2+1}\phi(x, y)), y + t(x^{k_1+1}y^{k_2+1}\psi(x, y))).$$

Thus, we have:

$$\tilde{\Omega} = \tilde{\Phi}^* \alpha_1 = \tilde{a}_1(x, y, t)dx + \tilde{b}_1(x, y, t)dy + \tilde{c}_1(x, y, t)dt$$

where

$$\tilde{a}_1(x, y, t) = a_1(\tilde{\Phi}) + t \left[a_1(\tilde{\Phi}) \partial_x (x^{k_1+1}y^{k_2+1}\phi(x, y)) + b_1(\tilde{\Phi}) \partial_x (x^{k_1+1}y^{k_2+1}\psi(x, y)) \right]$$

$$\tilde{b}_1(x, y, t) = b_1(\tilde{\Phi}) + t \left[a_1(\tilde{\Phi}) \partial_y (x^{k_1+1}y^{k_2+1}\phi(x, y)) + b_1(\tilde{\Phi}) \partial_y (x^{k_1+1}y^{k_2+1}\psi(x, y)) \right]$$

and

$$\tilde{c}_1(x, y, t) = a_1(\tilde{\Phi})x^{k_1+1}y^{k_2+1}\phi(x, y) + b_1(\tilde{\Phi})x^{k_1+1}y^{k_2+1}\psi(x, y).$$

Moreover, $\tilde{\Omega}_{|\mathbb{C}^2 \times \{0\}} = \alpha_1$ and $\tilde{\Omega}_{|\mathbb{C}^2 \times \{1\}} = \omega_1$.

Using the path method, the theorem follows if there exists a holomorphic vector field

$$Z = U(x, y, t)\partial_x + V(x, y, t)\partial_y + \partial_t \quad \text{such that} \quad \alpha(Z) = 0 \quad \text{and} \quad \tilde{\Omega}(Z) = 0.$$

Therefore, we have to solve the following system of equations:

$$\begin{cases} \tilde{a}_1U + \tilde{b}_1V + \tilde{c}_1 & = 0 \\ MU + NV & = 0 \end{cases}$$

As the order $ord_{T_i}(\alpha_1 \wedge \alpha) = k_i$, $i = 1, 2$, the function $a_1N - b_1M = x^{k_1}y^{k_2}R(x, y)$ where $R(0, 0) \neq 0$. Thus, we obtain that the determinant can be defined as

$$\tilde{\Delta} = \tilde{a}_1N - \tilde{b}_1M = x^{k_1}y^{k_2}\tilde{\Delta}'(x, y), \quad \text{with} \quad \tilde{\Delta}'(0, 0) \neq 0.$$

Hence, if we set that $\tilde{c}_1(x, y) = x^{k_1+1}y^{k_2+1}\tilde{c}'_1(x, y)$, we obtain that:

$$U = -xy\frac{N\tilde{c}'_1}{\tilde{\Delta}'} \quad \text{and} \quad V = xy\frac{M\tilde{c}'_1}{\tilde{\Delta}'}$$

which are the holomorphic solutions of the system. And the theorem follows. \square

A direct inspection yields this following result:

Corollary 2.5. *Let*

$$\mathcal{F}_i = \langle xdy + y(\lambda + f_i(x, y))dx \rangle, \quad i = 1, 2,$$

and

$$\mathcal{G} = \langle xdy + \lambda ydx \rangle, \quad \lambda = \frac{p}{q}, \quad (p, q) \in \mathbb{N}^* \times \mathbb{N}^*,$$

be germs of foliations on $(\mathbb{C}^2, 0)$ such that for any $i = 1, 2$, $Tang(\mathcal{F}_i, \mathcal{G}) = div(x^p y^q)$. Assume that there exist germs of analytic diffeomorphisms $\Phi_i(x, y) = (x, y + x^{p+1}y^{q+1}u_i(x, y))$, $i = 1, 2$, and $u_i(0, 0) \neq 0$, such that $\Phi_1^*\mathcal{F}_2 = \mathcal{F}_1$ and $\Phi_2^*\mathcal{G} = \mathcal{G}$. Then, there exists a germ of an analytic diffeomorphism

$$\Phi(x, y) = (x, y(1 + x^p y^q u(x, y))) \in \text{Diff}(\mathbb{C}^2, 0) \quad \text{such that} \quad \Phi^*\mathcal{F}_2 = \mathcal{F}_1 \quad \text{and} \quad \Phi^*\mathcal{G} = \mathcal{G}.$$

Proof. The proof of this corollary is based on the previous Theorem 2.4. Indeed, if $\lambda = \frac{p}{q} \in \mathbb{Q}_+^*$, then, any analytic diffeomorphism which preserves the foliation generated by $xdy + \lambda ydx$ and which preserves each separatrix is of the form

$$\Phi(x, y) = (x, y(1 + x^p y^q U(x, y)))$$

where $U(x, y) = a_{pq} + \sum_{k>1} a_k x^{kp} y^{kq}$, with $a_{pq} \in \mathbb{C}$. \square

3. TRANSVERSELY FORMAL CLASSIFICATION OF PAIRS OF FOLIATIONS

From now on, we will establish new techniques that make it possible to classify the pairs of foliations according to their associated invariants. This classification is rather in the sense of generalizing the Mattei-Moussu theorem for the pairs of foliations. Furthermore, this method has been used in [8].

First of all, we want to find the formal normal form of pair of foliations at the neighborhood of $0 \in \mathbb{C}^2$. Unless otherwise mentioned, we rule out the saddle-node case.

Lemma 3.1. *(Decomposition of a formal logarithmic vector field.)*

Let $\widehat{X} = x^{k_1}\widehat{f}\partial_x + y^{k_2}\widehat{g}\partial_y$, with $k_i > 0, i = 1, 2$, be a formal logarithmic vector field along $T := \{xy = 0\}$. Assume that \widehat{X} is transversely formal along the divisor $\{y = 0\}$, then there exists \widehat{F} and \widehat{G} , transversely formal power series along $\{y = 0\}$ such that:

$$\exp(\widehat{X}) = \exp(\widehat{F}v_1) \circ \exp(\widehat{G}v_2),$$

where $v_1 = x\partial_x + \lambda_1 y(1 + f_1(x, y))\partial_y$, $v_2 = x\partial_x + \lambda_2 y(1 + f_2(x, y))\partial_y$, and $\lambda_1 \neq \lambda_2$.

Proof. As $\lambda_1 \neq \lambda_2$, both non-zero, the vector fields v_1 and v_2 form a basis of the formal logarithmic vector fields along T (see Proposition 1.8). Thus, there exists transversely formal power series F and G along the divisor $\{y = 0\}$, such that

$$\widehat{X} = Fv_1 + Gv_2.$$

Now, let F_i and G_i be the homogeneous components of degree i of the functions F and G respectively. Consider the sequence of transversely formal power series defined as:

$$U_n = \exp(-F_n v_1) \circ \dots \circ \exp(-F_1 v_1).$$

We claim that there exists a transversely formal series \widehat{F} such that the sequence (U_n) converges to $\exp(-\widehat{F}v_1)$ in the sense of Krull topology.

First, using Baker-Campbell-Hausdorff formula, we can deduce that for any integer $n \geq 1$, there exists transversely formal power series $\widehat{F(n)}$ along $\{y = 0\}$, such that:

$$\exp(-F_n v_1) \circ \dots \circ \exp(-F_1 v_1) = \exp\left(-\widehat{F(n)}v_1\right).$$

Next, thanks to Baker-Campbell-Hausdorff formula, we can see that for any $k > 1$, there exists $n_0 = k$ such that for all $n \geq n_0$, we have

$$J^k(\widehat{F(n)}) = J^k(\widehat{F(n+1)}),$$

(where J^k is the jet of order k).

As a consequence, we can construct a transversely formal series \widehat{F} such that

$$\text{for any } k > 1, \quad J^k(\widehat{F}) = J^k(\widehat{F(k)}).$$

Lastly, it is well-known that modulus the terms of orders greater than $k + 1$, two homogeneous vector fields of degree greater than $k + 1$, necessarily commute, thus, we can find an integer $N = N(k) \geq k$ such that for any $n > N$, we have:

$$\begin{aligned} J^k(U_n) &= J^k(\exp(-F_k v_1) \circ \dots \circ \exp(-F_1 v_1)) \\ &= J^k\left(\exp\left(-\widehat{F(k)}v_1\right)\right) \\ &= J^k\left(\exp\left(-\widehat{F}v_1\right)\right) \end{aligned}$$

Hence, in the sense of Krull topology, we conclude that the sequence (U_n) converge to $\exp(-\widehat{F}v_1)$.

Likewise, using the same reasoning, we obtain that the sequence

$$V_n = \exp(-G_1 v_2) \circ \dots \circ \exp(-G_n v_2)$$

converges to $\exp(-\widehat{G}v_2)$, where \widehat{G} is a transversely formal power series along $\{y = 0\}$.

From another side, due to the Baker-Campbell-Hausdorff formula, we can see by recurrence the following property $P_{n \geq 1}$:

$${}^n U_n = \exp(-F_n v_1) \circ \dots \circ \exp(-F_1 v_1) \quad \text{and} \quad V_n = \exp(-G_1 v_2) \circ \dots \circ \exp(-G_n v_2)$$

verify

$$J^{n+1} \left(\exp(-F_n v_1) \circ \dots \circ \exp(-F_1 v_1) \circ \exp(\widehat{X}) \circ \exp(-G_1 v_2) \circ \dots \circ \exp(-G_n v_2) \right) = Id^n$$

Finally, if $n \rightarrow +\infty$, we obtain:

$$\exp(\widehat{X}) = \exp(\widehat{F}v_1) \circ \exp(\widehat{G}v_2),$$

and the lemma follows. \square

This previous Lemma can be reformulated in the sense of transversely formal diffeomorphisms along $\{y = 0\}$. We have:

Lemma 3.2. (*Decomposition of a transversely formal diffeomorphism.*)

Consider $v_1 = x\partial_x + \lambda_1 y(1 + f_1(x, y))\partial_y$ and $v_2 = x\partial_x + \lambda_2 y(1 + f_2(x, y))\partial_y$ two vector fields such that $\lambda_1 \neq \lambda_2$.

Any transversely formal diffeomorphism $\widehat{\phi}(x, y) = (x + yx\widehat{\phi}_1(x, y), y + xy\widehat{\phi}_2(x, y))$ along the divisor $\{y = 0\}$ can be decomposed in the form

$$\widehat{\phi} = \widehat{\phi}_1 \circ \widehat{\phi}_2,$$

such that:

- for any $i = 1, 2$, $\widehat{\phi}_i$ is a transversely formal diffeomorphism along the divisor $\{y = 0\}$;
- if \mathcal{F}_1 and \mathcal{F}_2 are the foliations defined by the vector fields v_1, v_2 , respectively, then

$$\widehat{\phi}_1^* \mathcal{F}_1 = \mathcal{F}_1 \quad \text{and} \quad \widehat{\phi}_2^* \mathcal{F}_2 = \mathcal{F}_2.$$

Proof. The proof of this Lemma relies heavily on the fact that the Camacho-Sad of the vector fields v_i are different (i.e. the tangency divisor of the pair of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ is simple). As, $J_{(0,0)}^1 \widehat{\phi} = Id$, where $J_{(0,0)}^1$ is the jet of order 1 at the origin, there exists a transversely formal vector field \widehat{X} along $\{y = 0\}$ and logarithmic along the axes of coordinates such that:

$$\widehat{\phi} = \exp(\widehat{X}).$$

Moreover, by Lemma 3.1, we have:

$$\widehat{\phi} = \left(\exp(\widehat{F}_1 v_1) \right) \circ \left(\exp(\widehat{F}_2 v_2) \right),$$

where $\widehat{F}_1 v_1$ and $\widehat{F}_2 v_2$ are transversely formal power series. Finally, it suffices to set

$$\widehat{\phi}_1 = \exp(\widehat{F}_1 v_1) \quad \text{and} \quad \widehat{\phi}_2 = \exp(\widehat{F}_2 v_2).$$

\square

From now on, we can establish the transversely formal conjugation of the pairs of foliations with simple tangency.

Theorem 3.3 (Transversely formal conjugacy of pairs of foliations).

Let $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ be two germs of pairs of foliations defined on $(\mathbb{C}^2, 0)$ by

$$\mathcal{F}_i := \langle x\partial_x + \lambda_i y(1 + f_i(x, y))\partial_y \rangle,$$

$$\mathcal{G}_i := \langle x\partial_x + \lambda_i y(1 + g_i(x, y))\partial_y \rangle,$$

$i = 1, 2$, and $\lambda_1 \neq \lambda_2$.

Then there exists germs of transversely formal diffeomorphisms $\widehat{\zeta}_i \in \text{Diff}(\mathbb{C}^2, \widehat{(T, 0)})$ (with $T := \{xy = 0\}$) such that

$$\begin{cases} \widehat{\zeta}_1^* \mathcal{G}_1 = \mathcal{F}_1 \\ \widehat{\zeta}_2^* \mathcal{G}_2 = \mathcal{F}_2 \end{cases}$$

if and only if there exists a transversely formal diffeomorphism $\widehat{\zeta} \in \text{Diff}(\mathbb{C}^2, \widehat{(T, 0)})$ which conjugates simultaneously the pairs of foliations:

$$\begin{cases} \widehat{\zeta}^* \mathcal{G}_1 = \mathcal{F}_1 \\ \widehat{\zeta}^* \mathcal{G}_2 = \mathcal{F}_2 \end{cases}$$

Proof. As the transversely formal diffeomorphism $\widehat{\zeta}_1 \circ \widehat{\zeta}_2^{-1}$ is tangent to the identity along the tangency divisor $T := \{xy = 0\}$, by Lemma 3.2, we can find two transversely formal diffeomorphisms \widehat{g}_1 and \widehat{g}_2 along the divisor $\{y = 0\}$ which preserve the foliations \mathcal{G}_1 and \mathcal{G}_2 and such that:

$$\widehat{\zeta}_1 \circ \widehat{\zeta}_2^{-1} = \widehat{g}_1 \circ \widehat{g}_2.$$

Therefore, if we set that $\widehat{\zeta} = \widehat{g}_1^{-1} \circ \widehat{\zeta}_1$, we obtain the transversely formal diffeomorphism along the divisor $\{y = 0\}$ which conjugate the pairs. \square

Remark 3.4. In the same way, it can be seen that the convergent version of the Lemma 3.2 implies the main result (see Theorem A).

Thanks to the transversely formal classification of reduced foliations (see Proposition 1.21) and Theorem 3.3, we obtain the following Corollary which gives the transversely formal normal forms of pairs of foliations without a saddle-node singularity.

Corollary 3.5. (*Transversely formal normal form*) Let

$$\mathcal{F}_i = \langle \omega_i = xdy - y\lambda_i(1 + f_i(x, y))dx \rangle,$$

($i = 1, 2$), $\lambda_1 \neq \lambda_2$, both non-zero, be a germ of pairs of foliations. Let us denote by \mathcal{F}_i^0 , $i = 1, 2$, the foliation defined by the formal normal form of ω_i . Then, the pair $(\mathcal{F}_1, \mathcal{F}_2)$ is transversely formal conjugated to the pair of its normal forms $(\mathcal{F}_1^0, \mathcal{F}_2^0)$.

Remark 3.6.

These previous results work in saddle-node case, when the central manifold, i.e. the separatrix tangent to the 0-eigendirection, is convergent (see [4]). It is also important to mention that if the tangency of the pairs of foliations is not simple, then we cannot use these previous techniques in order to find a normal form of the pair. This is really another work that deserves to be studied in another paper.

4. ANALYTIC CLASSIFICATION OF PAIRS OF SINGULAR FOLIATIONS ON $(\mathbb{C}^2, 0)$

Before giving an analytic classification of the germs of pairs of singular foliations, we need to better understand the classification of pairs of regular foliations along a common leaf. More precisely, the invariants that occur in this classification.

4.1. Local classification of the pair of regular foliations with common leaf.

Let us consider $(\mathcal{F}, \mathcal{G})$ a pair of regular foliations in the neighborhood of a common leaf $H := \{y = 0\} \setminus \{0\}$. Assume that the order of tangency of the pair along H is equal to $k \geq 1$. Using the flow-box theorem, for any point $(x_0, 0) \in H$, we can find a local analytic diffeomorphism which preserves the common leaf H point-wise and conjugates the pair $(\mathcal{F}, \mathcal{G})$ to the pair $(\langle dy \rangle, \langle dy + y^k B(x, 0)dx \rangle)$, where $B(x, 0)$ is not identically zero. Moreover, we can prove that:

- if $k = 1$, then the one-form $\omega = B(x, 0)dx$ can be extended to a global one-form on the common leaf H ;
- if $k > 1$, then the one-form ω is well-defined up to a constant term, however, the local coordinates $\varphi(x, 0) = \int_0^x \omega$, define a global affine structure on H .

Thus, the pair of foliation is related to a one-form ω which measures the discrepancy at order k between the foliations \mathcal{F} and \mathcal{G} where the foliation \mathcal{F} is straightened as the foliation defined by the one-form dy . More precisely, if we deal with a pair of foliation $(\mathcal{F}, \mathcal{G})$ with simple tangency, we have the following key lemma due by F.Loray, F.Touzert and O.Thom (see [[7],page 18] or the reader can also see [5] for more details).

Lemma 4.1. *Let us consider $(\mathcal{F}, \mathcal{G})$ a pair of regular foliations in the neighborhood of a common leaf with $H := \{y = 0\} \setminus \{0\}$. Assume that the order of tangency of the pair along H is equal to 1 and let ω be the discrepancy one-form of the pair.*

Consider (F_0, G_0) be a pair of local minimal first integrals of the pair $(\mathcal{F}, \mathcal{G})$, defined in the neighborhood of $(x_0, 0) \in H$ such that on which $H := \{F_0 = 0\} = \{G_0 = 0\}$. Then, there exists a unique system of local coordinates (x', y') in the neighborhood of $(x_0, 0)$ which fixes H point-wise such that:

$$F_0 \sim y' \quad \text{et} \quad G_0 \sim \exp\left(\int_{x_0}^{x'} \omega\right) y'.$$

In addition, using this above key lemma, we can prove that this discrepancy one-form ω , is a complete invariant to classify locally pairs of regular foliations with simple tangency. In fact, we have:

Proposition 4.2 (Local simultaneous conjugacy). *Let $(\mathcal{F}, \mathcal{G})$ and $(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ be two pairs of regular foliations defined in the neighborhood of a common leaf H with simple tangency divisor. Let us consider (F_0, G_0) and $(\tilde{F}_0, \tilde{G}_0)$ be two pairs of local minimal first integrals of the pair $(\mathcal{F}, \mathcal{G})$ and $(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ respectively, defined in the neighborhood of the point $(x_0, 0) \in H$. If we assume that the discrepancy one-forms ω and $\tilde{\omega}$ associated to the pairs $(\mathcal{F}, \mathcal{G})$ and $(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ respectively are equal, then there exists a neighborhood of a point $(x_0, 0) \in H$, such that the pair of local first integral (F_0, G_0) and $(\tilde{F}_0, \tilde{G}_0)$ are simultaneously conjugated by a unique analytic diffeomorphism which fixes point-wise the common leaf H . Hence, the pairs of foliations $(\mathcal{F}, \mathcal{G})$ and $(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ are locally simultaneously conjugated.*

Proof. Let (F_0, G_0) and $(\tilde{F}_0, \tilde{G}_0)$ be two pairs of local minimal first integrals of pairs $(\mathcal{F}, \mathcal{G})$ and $(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ respectively, defined in the neighborhood of the point $(x_0, 0) \in H$. Then, thanks to Lemma 4.1, there exist unique local coordinates $\phi(x', y')$ and $\tilde{\phi}(x', y')$, in the neighborhood of $(x_0, 0)$, both fixing point-wise the germ of common leaf H and such that:

$$\left\{ \begin{array}{l} F_0 \circ \phi(x', y') = y' \\ G_0 \circ \phi(x', y') = \exp\left(\int_{x_0}^{x'} \omega\right) y' \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \tilde{F}_0 \circ \tilde{\phi}(x', y') = y' \\ \tilde{G}_0 \circ \tilde{\phi}(x', y') = \exp\left(\int_{x_0}^{x'} \tilde{\omega}\right) y' \end{array} \right.$$

As the pairs of foliations have the same discrepancy one-form, $\omega = \tilde{\omega}$, then the analytic diffeomorphism $\phi_0 = \tilde{\phi} \circ \phi^{-1}$, conjugates the pair of first integrals $(\tilde{F}_0, \tilde{G}_0)$ to (F_0, G_0) . Finally, this conjugation induces a local conjugation of pairs of foliations $(\mathcal{F}, \mathcal{G})$ and $(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ \square

4.2. Global classification of pairs of regular foliations along a common leaf.

Now, the natural question is to ask how to construct an analytical extension of this local conjugation of the pairs of regular foliations along the common leaf. Thus, we will see in the following, that to have the existence of this analytical extension, it suffices that the two pairs have the same monodromy. In this case, we can extend local conjugation of the pairs of foliations in the neighborhood of H .

Proposition 4.3. *Let $(\mathcal{F}, \mathcal{G})$ and $(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ be two pairs of regular foliations defined in the neighborhood of a common leaf H with simple tangency. Let us consider $h_{\mathcal{F}}$ (resp. $h_{\tilde{\mathcal{F}}}$) and $h_{\mathcal{G}}$ (resp. $h_{\tilde{\mathcal{G}}}$) the holonomies of the foliation \mathcal{F} (resp. $\tilde{\mathcal{F}}$) and of the foliation \mathcal{G} (resp. $\tilde{\mathcal{G}}$) with respect to the common leaf H . If there exist analytic diffeomorphisms $\varphi_i \in \text{Diff}(\mathbb{C}, 0)$, $i = 1, 2$, such that*

$$h_{\tilde{\mathcal{F}}} = \varphi_1 \circ h_{\mathcal{F}} \circ \varphi_1^{-1} \quad \text{and} \quad h_{\tilde{\mathcal{G}}} = \varphi_2 \circ h_{\mathcal{G}} \circ \varphi_2^{-1},$$

Then, for any corona $C_{r,R} = \{(x, 0) / r < |x| < R\}$ inside H , there exist neighborhoods $U, V \supset C_{r,R}$ and a local analytic diffeomorphism $\Phi: (U, C_{r,R}) \rightarrow (V, C_{r,R})$ fixing the corona $C_{r,R}$ point-wise and such that:

$$\begin{cases} \Phi^* \tilde{\mathcal{F}} = \mathcal{F} \\ \Phi^* \tilde{\mathcal{G}} = \mathcal{G} \end{cases}$$

Proof. Let $C_{r,R} = \{(x, 0) / r < |x| < R\}$ be a corona inside H and $(x_0, 0) \in C_{r,R}$. Consider, $\gamma(t) = (x_0 \exp(2\pi it), 0)$ a loop at the point $(x_0, 0)$ contained in $C_{r,R}$. Assume that there exists analytic diffeomorphisms φ_i , $i = 1, 2$ such that the holonomies of the foliations along the loop γ verify:

$$h_{\tilde{\mathcal{F}}} = \varphi_1 \circ h_{\mathcal{F}} \circ \varphi_1^{-1} \quad \text{and} \quad h_{\tilde{\mathcal{G}}} = \varphi_2 \circ h_{\mathcal{G}} \circ \varphi_2^{-1}.$$

Let us fix (F_0, G_0) and $(\tilde{F}_0, \tilde{G}_0)$ be two pairs of local minimal first integrals of $(\mathcal{F}, \mathcal{G})$ and $(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ respectively defined in the neighborhood of $(x_0, 0)$.

Without loss of generality, we can suppose that the monodromy of the local first integral F_0 (resp. G_0) is the same as that of \tilde{F}_0 (resp. \tilde{G}_0) along the loop γ . It means that if $h_{\mathcal{F}} \circ F_0$ and $h_{\mathcal{G}} \circ G_0$ are analytic continuations of the local first integrals F_0 and G_0 respectively along the loop γ , then $h_{\mathcal{F}} \circ \tilde{F}_0$ and $h_{\mathcal{G}} \circ \tilde{G}_0$ are respectively those of \tilde{F}_0 and \tilde{G}_0 . Indeed, to obtain this, it suffices to replace the local first integrals \tilde{F}_0, \tilde{G}_0 to $\varphi_1 \circ \tilde{F}_0, \varphi_2 \circ \tilde{G}_0$ respectively. Next, thanks to Proposition 4.2, there exists a unique local of coordinates $\phi_0(x, y)$ defined in the neighborhood $(x_0, 0)$ which conjugates the pairs of first integrals (F_0, G_0) and $(\tilde{F}_0, \tilde{G}_0)$. Thus, using the uniqueness of this local conjugation between pairs of first integrals, we can deduce that the local coordinates $\phi_0(x, y)$ admits an analytic continuation ϕ_γ along the loop γ which verifies that

$$h_{\mathcal{F}} \circ \tilde{F}_0 \circ \phi_\gamma = h_{\mathcal{F}} \circ F_0 \quad \text{and} \quad h_{\mathcal{G}} \circ \tilde{G}_0 \circ \phi_\gamma = h_{\mathcal{G}} \circ G_0.$$

Once more, by the uniqueness of $\phi_0(x, y)$, we have $\phi_0 = \phi_\gamma$ in the neighborhood of $(x_0, 0)$ and the Proposition follows. \square

From now on, we are going to extend these previous results to a germ of pairs of singular foliations.

4.3. Analytic classification of pairs of foliation in the neighborhood of a corona.

For germs of pairs of singular foliations with simple tangency, the discrepancy one-form ω , is defined as the associated one to the germ of pairs of induced regular foliations.

Furthermore, we can extend it to a meromorphic one form with simple pole along the germ of the horizontal separatrix $\{y = 0\}$. In fact, if we consider

$$\mathcal{F}_1 := \langle v_1 = x\partial_x + \lambda_1 y(1 + f_1(x, y))\partial_y \rangle \quad \text{and} \quad \mathcal{F}_2 := \langle v_2 = x\partial_x + \lambda_2 y(1 + f_2(x, y))\partial_y \rangle,$$

be a pair of foliations on $(\mathbb{C}^2, 0)$, such that $\lambda_1 \neq \lambda_2$, then the meromorphic discrepancy one-form along the separatrix $\{y = 0\}$ can be defined as

$$\omega_{\mathcal{F}} = [\lambda_1(1 + f_1(x, 0)) - \lambda_2(1 + f_2(x, 0))] \frac{dx}{x}.$$

By a direct inspection, we notice that the residue of this previous meromorphic one-form $\omega_{\mathcal{F}} = (\lambda_1(1 + f_1(x, 0)) - \lambda_2(1 + f_2(x, 0))) \frac{dx}{x}$ at 0 is equal to $\lambda_1 - \lambda_2$. Consequently, there exists a local change of coordinates in the neighborhood of $0 \in \mathbb{C}$ (see [page 20, [6]]) such that the one-form can be written as:

$$\omega_{\mathcal{F}} = \left((\lambda_1 - \lambda_2) \frac{dx}{x} \right).$$

Therefore, without loss of generality, we can always assume that two germs of pairs of foliations defined as

$$\mathcal{F}_i := \langle x\partial_x + \lambda_i y(1 + f_i(x, y))\partial_y \rangle,$$

$$\mathcal{G}_i := \langle x\partial_x + \lambda_i y(1 + g_i(x, y))\partial_y \rangle,$$

$i = 1, 2$, and $\lambda_1 \neq \lambda_2$, have the same discrepancy one-forms.

We have the following:

Theorem 4.4 (Analytic classification in the neighborhood of a corona). *Let $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ be two pairs of foliations on $(\mathbb{C}^2, 0)$ defined by*

$$\mathcal{F}_i := \langle x\partial_x + \lambda_i y(1 + f_i(x, y))\partial_y \rangle,$$

$$\mathcal{G}_i := \langle x\partial_x + \lambda_i y(1 + g_i(x, y))\partial_y \rangle,$$

$i = 1, 2$, and $\lambda_1 \neq \lambda_2$. If there exist germs of analytic diffeomorphisms $\varphi_i \in \text{Diff}(\mathbb{C}, 0)$ such that for any $i = 1, 2$ we have:

$$h_{\mathcal{G}_1} = \varphi_1 \circ h_{\mathcal{F}_1} \circ \varphi_1^{-1} \quad \text{and} \quad h_{\mathcal{G}_2} = \varphi_2 \circ h_{\mathcal{F}_2} \circ \varphi_2^{-1},$$

then for any corona $C_{r,R} = \{(x, 0) / r < |x| < R\}$ inside $C := \{y = 0\}$, there exist neighborhoods $U, V \supset C_{r,R}$ and a germ of analytic diffeomorphism $\Phi: (U, C_{r,R}) \rightarrow (V, C_{r,R})$ fixing the corona $C_{r,R}$ point-wise and such that:

$$\begin{cases} \Phi^* \mathcal{G}_1 = \mathcal{F}_1 \\ \Phi^* \mathcal{G}_2 = \mathcal{F}_2 \end{cases}$$

Proof. According to the previous remark, modulo conjugation, two pairs of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ with the same pair of Camacho-Sad indices always have the same discrepancy one-form. Then, we can assume that $\omega_{\mathcal{F}} = \omega_{\mathcal{G}}$. Therefore, thanks to Proposition 4.2, if we fix two pairs of minimal local first integrals of the pairs of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$, we obtain a local conjugation of the pairs of foliations in the neighborhood of any point $(x_0, 0), x_0 \neq 0$. Moreover, as the pairs of holonomies of pairs of foliations are conjugated, the theorem follows thanks to Proposition 4.3. \square

Method to construct an analytic conjugation of pairs of foliations on the disc

According to what has just been seen, in order to construct a conjugation between the pairs of foliations on a disk, it suffices to show that any analytic conjugation of the pairs in the neighborhood of a corona can be extended into an analytic conjugation on the disk. The main idea to obtain this is the following: Let $\Phi: (U, C_{r,R}) \rightarrow (V, C_{r,R})$ be an analytic diffeomorphism in the neighborhood of the corona which conjugates the pair of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ to the pair of foliations $(\mathcal{G}_1, \mathcal{G}_2)$. Assume that there exists a transversely formal diffeomorphism $\widehat{\zeta}: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ along the horizontal separatrix $\{y = 0\}$ which conjugates the pair $(\mathcal{F}_1, \mathcal{F}_2)$ to the pair $(\mathcal{G}_1, \mathcal{G}_2)$. Then, the diffeomorphism $\delta = \Phi^{-1} \circ \widehat{\zeta}: (U, C_{r,R}) \rightarrow (U, C_{r,R})$ is a transversely formal symmetry of the pair $(\mathcal{F}_1, \mathcal{F}_2)$ in the neighborhood of the corona, $(U, C_{r,R})$ fixing point-wise the corona.

- If $\delta = Id$, i.e. $\Phi = \widehat{\zeta}$, then as the diffeomorphism $\widehat{\zeta}$ is transversely formal along the disk $\{|x| < R, y = 0\}$, there are no terms in x with a negative index in its development in Laurent series on $(U, C_{r,R})$. Thus, $\Phi = \widehat{\zeta}$ extends analytically on a disk containing the origin.
- If $\delta \neq Id$, as this symmetry is transversely formal on the neighborhood $(U, C_{r,R})$, then its development in Laurent series in the neighborhood $(U, C_{r,R})$ is of the form:

$$\delta(x, y) = \left(x + \sum_{n \geq 1, m \in \mathbb{Z}} a_{mn} x^m y^n, \sum_{n \geq 1, m \in \mathbb{Z}} b_{mn} x^m y^n \right).$$

Therefore, if $\delta = \Phi^{-1} \circ \widehat{\zeta}$ extends in a transversely formal symmetry along the disk $\{|x| < R, y = 0\}$, then its development in Laurent series does not have terms in x of negative index, so we have $a_{mn} = b_{mn} = 0, \forall m < 0$. This property obviously implies that $\Phi = \widehat{\zeta} \circ \delta^{-1}$ is without a negative term in the Laurent expansion (δ and ζ formally extend on the disk), and so converges in the neighborhood of the origin.

In addition, as any pair of foliations with simple tangency is transversely formally conjugated to its pair of normal forms (see the Corollary 3.5), then the transversely formal symmetries on $(U, C_{r,R})$ of the pair $(\mathcal{F}_1, \mathcal{F}_2)$ extend as a transversely formal symmetry along the disk $\{|x| < R, y = 0\}$ if and only if the transversely formal symmetries on $(U, C_{r,R})$ of the pair of normal forms $(\mathcal{F}_1^0, \mathcal{F}_2^0)$ extend along the disk. Therefore, the problem of classifying analytically the pairs of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$ on $(\mathbb{C}^2, 0)$ can be seen as to that of classifying the symmetries of the pairs of normal forms which are transversely formal on $(U, C_{r,R})$ and which, moreover, fix the corona point-wise.

4.4. On the transversely formal symmetries of pairs of singular foliations. Let

$$\mathcal{F}_i = \langle \omega_i = xdy - y\lambda_i(1 + f_i(x, y))dx \rangle,$$

($i = 1, 2$), $\lambda_1 \neq \lambda_2$, both non-zero, be a pair of foliations on $(\mathbb{C}^2, 0)$. Let us denote by \mathcal{F}_i^0 , $i = 1, 2$, the foliation defined by the formal normal form of ω_i . We have the following:

Theorem 4.5. *Let*

$$\delta(x, y) = \left(x + \sum_{n \geq 1} a_n(x)y^n, \sum_{n \geq 1} b_n(x)y^n \right),$$

be a transversely formal symmetry on $(U, C_{r,R})$ of the pair of formal normal forms $(\mathcal{F}_1^0, \mathcal{F}_2^0)$. Suppose that neither \mathcal{F}_1^0 nor \mathcal{F}_2^0 have no holomorphic first integrals. We have:

- If $\mathcal{F}_1^0 := \langle \omega_1 = xdy - \lambda_1 y dx \rangle$ and $\mathcal{F}_2^0 := \langle \omega_2 = xdy - \lambda_2 y dx \rangle$, then :

$$\delta(x, y) = (x, by) \quad \text{where } b \in \mathbb{C}^*.$$

- If

$$\mathcal{F}_1^0 := \langle \omega_1 = xdy - \lambda_1 ydx \rangle$$

and

$$\mathcal{F}_2^0 := \langle \omega_2 = p(1 + (\alpha - 1)u_1^k)ydx + q(1 + \alpha u_1^k)xdy \rangle$$

where $(p, q) = 1$ and $u_1 = x^p y^q$ then :

$$\delta(x, y) = (x, by) \circ \exp(t_2 V_2) \quad \text{where } b^{kq} = 1, \quad t_2 \in \mathbb{C},$$

and such that the vector field V_2 is holomorphic on $(\mathbb{C}^2, 0)$.

- If

$$\mathcal{F}_1^0 := \langle \omega_1 = p(1 + (\alpha - 1)u_1^k)ydx + q(1 + \alpha u_1^k)xdy \rangle$$

and

$$\mathcal{F}_2^0 := \langle \omega_2 = p'(1 + (\beta - 1)u_2^{k'})ydx + q'(1 + \beta u_2^{k'})xdy \rangle$$

where $(p, q) = 1$, $(p', q') = 1$, $\frac{p}{q} \neq \frac{p'}{q'}$, $u_1 = x^p y^q$, and $u_2 = x^{p'} y^{q'}$, then

$$\delta(x, y) = (x, by) \circ \exp(t_1 V_1 + t_2 V_2) \quad \text{where } b^{kq} = 1, \quad b^{k'q'} = 1, \quad t_i \in \mathbb{C},$$

and such that the vector fields V_i , $(i = 1, 2)$ are holomorphic on $(\mathbb{C}^2, 0)$.

Before proving this theorem, we need to establish some key lemmas.

Lemma 4.6. *Let Ω_1 and Ω_2 be two closed meromorphic formal one-forms defined on the neighborhood of a corona $(U, C_{R,r})$, and such that their associated foliations are transverse outside the corona. Let us consider the set*

$$\widehat{\mathcal{X}}(\Omega_1, \Omega_2) = \{\text{formal vector fields on } (U, C_{R,r}) \text{ symmetric both to } \Omega_1 \text{ and } \Omega_2\}.$$

Then, there exist two commuting formal vector fields V_1 and V_2 such that for any $\widehat{X} \in \widehat{\mathcal{X}}(\Omega_1, \Omega_2)$, there exist constants t_1 and t_2 verifying:

$$\widehat{X} = t_1 V_1 + t_2 V_2.$$

Proof. The vector fields V_i , $(i = 1, 2)$, are defined as:

$$\begin{cases} \Omega_1(V_1) = 1 \\ \Omega_2(V_1) = 0 \end{cases} \quad \text{et} \quad \begin{cases} \Omega_1(V_2) = 0 \\ \Omega_2(V_2) = 1 \end{cases}$$

Therefore, the vector fields V_i are common Lie symmetries of the one-forms Ω_i on $(U, C_{R,r})$. Moreover, due to the fact that the one-forms Ω_i are closed, a direct inspection shows that $[V_1, V_2] = 0$. Let $\widehat{X} \in \widehat{\mathcal{X}}(\Omega_1, \Omega_2)$. Assume that \widehat{X} preserves the divisor of tangency $Tang(\Omega_1, \Omega_2)$. As the vector fields V_i are transverse outside the tangency divisor, there exists formal meromorphic functions \widehat{h}_i , $i = 1, 2$, such that

$$\widehat{X} = \widehat{h}_1 V_1 + \widehat{h}_2 V_2.$$

Using the commutativity and the transversality of vector fields V_1 and V_2 , the functions $\widehat{h}_i = t_i$ become constants. The lemma follows. \square

Remark 4.7. According to the normal forms of germs of non-degenerate foliations, for every $i = 1, 2$, the germ of normal forms \mathcal{F}_i^0 is generated by a germ of 1-holomorphic form ω_i which has an integrating factor denoted by F_i . Thus, the one-form

$$\Omega_i := \frac{\omega_i}{F_i}$$

is a germ of closed meromorphic 1-form. We shall say that Ω_i is the germ of one-closed meromorphic form associated with the germ of foliations \mathcal{F}_i^0 . For instance, in the Siegel domain, it is defined as follows :

- if $\omega := xdy - \lambda_1 y dx$, then its associated closed 1-meromorphic form is

$$\Omega := \frac{dy}{y} - \lambda_1 \frac{dx}{x};$$

- if $\omega := p(1 + (\alpha - 1)u_1^k)ydx + q(1 + \alpha u_1^k)x dy$, then $pqxyu_1^k$ is an integrating factor of ω and therefore its associated closed meromorphic 1-form is:

$$\Omega := \frac{1 + (\alpha - 1)u_1^k}{qu_1^k} \frac{dx}{x} + \frac{1 + \alpha u_1^k}{pu_1^k} \frac{dy}{y}.$$

In both cases, any other closed meromorphic 1-form defining the foliation is of the form $c\Omega$ for a constant $c \in \mathbb{C}^*$. Furthermore, if the Camacho-Sad indices of the foliations generated by the closed one-forms Ω_i , $i = 1, 2$, are different, then the vector fields V_i defined in the Lemma 4.6, extend holomorphically on $(\mathbb{C}^2, 0)$.

We also have the following lemma which makes it possible to characterize the symmetries of the closed forms associated with the normal forms in the Siegel domain.

Lemma 4.8. *Let $\mathcal{F}_\omega = \langle \omega = xdy - \lambda y(1 + f(x, y))dx \rangle$ be a germ of formal normal forms in the Siegel domain without a non-constant first integral at the origin.*

If $\Upsilon: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ is a transversely formal symmetry of \mathcal{F}_ω along a corona $C_{R,r}$ in $y = 0$, then Υ is a (transversely formal) symmetry of the associated closed meromorphic 1-form Ω which additionally verifies :

$$\Upsilon^*\Omega = \Omega.$$

Proof. Since $\Upsilon: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ is a symmetry of the foliation defined by $\omega = 0$, then it verifies the equation

$$\Upsilon^*\omega \wedge \omega = 0 \quad \Longleftrightarrow \quad \Upsilon^*\Omega \wedge \Omega = 0.$$

Therefore, there is a transversely formal function F along $C_{R,r}$ such that

$$\Upsilon^*\Omega = F\Omega.$$

Hence,

$$d\Omega = 0 \quad \Longrightarrow \quad dF \wedge \Omega = 0.$$

Thus, the function F is a first integral of \mathcal{F}_ω on the corona. Thanks to Mattei-Moussu (see [10]), F extends at the origin and must therefore be constant by hypotheses. We then have

$$\Upsilon^*\Omega = cste\Omega.$$

As the symmetry Υ fixes the horizontal separatrix, the residue of the 1-meromorphic form Ω along the horizontal separatrix is invariant by the symmetry Υ hence the constant $cste = 1$. Note that, the case where the residue of Ω is zero, is called a case resonant (see Proposition 1.21 ,with $\alpha = 0$). In this case, we can verify that Ω must be invariant by the centralizer of the holonomy of \mathcal{F}_ω (in restriction to a transversal) [see [6, Prop 1.3.2 and §5.2.2)]; since the Υ commute with the holonomy, it is in the centralizer and must therefore preserve Ω . \square

Now we can give a proof of the Theorem 4.5.

Proof. Let

$$\delta(x, y) = \left(x + \sum_{n \geq 1} a_n(x)y^n, \sum_{n \geq 1} b_n(x)y^n \right),$$

be a transversely formal symmetry on $(U, C_{r,R})$ of the pair of formal normal forms $(\mathcal{F}_1^0, \mathcal{F}_2^0)$, and assume that the foliations do not have non-constant holomorphic first integrals.

Denote by Ω_1, Ω_2 , the closed meromorphic one-forms associated to \mathcal{F}_1^0 and \mathcal{F}_2^0 respectively. Then, by Lemma 4.8, the symmetry δ can be decomposed a:

$$\delta(x, y) = (x, by) \circ \exp(\widehat{X})(x, y),$$

($b \neq 0$), such that the diffeomorphism $(x, y) \mapsto (x, by)$ preserves the pair $(\mathcal{F}_1^0, \mathcal{F}_2^0)$, and the formal vector fields $\widehat{X} \in \widehat{\mathcal{X}}(\Omega_1, \Omega_2)$. Thus, thanks to the Lemma 4.6 and the Remark 4.7, the vector fields \widehat{X} can be written as $\widehat{X} = t_1V_1 + t_2V_2$, where $t_i, i = 1, 2$, are constants and the vector fields $V_i, i = 1, 2$, are holomorphic on $(\mathbb{C}^2, 0)$. \square

Remark 4.9. The case where one of the normals forms admits a holomorphic first integral (i.e. $\omega = px dy + qy dx$) is more complicated due to the existence of infinitely many symmetries. We need to use other techniques for this particular case. We are currently working on it and we will publish it very soon in another article.

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