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► **To cite this version:**

Henri Bourlès. Structure at infinity and defect of transfer matrices with time-varying coefficients, with application to exact model-matching. Linear Algebra and its Applications, 2024, 10.1016/j.laa.2024.06.009 . hal-04612555

**HAL Id: hal-04612555**

**<https://hal.science/hal-04612555>**

Submitted on 14 Jun 2024

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# Structure at infinity and defect of transfer matrices with time-varying coefficients, with application to exact model-matching

Henri Bourlès\*

June 12, 2024

## Abstract

We study the structure at infinity of transfer matrices with time-varying coefficients. Such transfer matrices have their entries in a skew field  $\mathbf{F}$  of rational fractions, i.e. of quotients of skew polynomials. Any skew rational fraction is the quotient of two proper ones, the latter forming a ring  $\mathbf{F}_{pr}$  (a subring of  $\mathbf{F}$ ) on which a “valuation at infinity” is defined. A transfer matrix  $G$  has both a “generalized degree” and a valuation at infinity, the sum of which is the opposite of the “defect” of  $G$ . The latter was first defined by Forney in the time-invariant case to be the difference between the total number of poles and the total number of zeros of  $G$  (poles and zeros at infinity included and multiplicities accounted for). In our framework, which covers both continuous- and discrete-time systems, the classic relation between the defect and Forney’s left- and right-minimal indices is extended to the time-varying case. The exact model-matching problem is also completely solved. These results are illustrated through an example belonging to the area of power systems.

**Keywords:** defect, model matching, time-varying, transfer matrix

**AMS subject classification** 93B25, 93N10, 93C05

## 1 Introduction

Linear time-varying (LTV) systems, i.e. systems with time-varying coefficients, are encountered in many application fields ([8], Chapter 9). The transfer matrix of an LTV continuous-time system was defined by Fliess [16]. The coefficients of such a transfer matrix belong to a skew field  $\mathbf{F}$  of skew rational fractions since the system coefficients do not commute with the derivation. This formalism was extended in [4] to LTV continuous- and discrete-time systems in a unified approach that will be followed in the sequel. This extension brings into play a “generalized derivation” which reduces to the usual derivation operator  $a \mapsto$

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$da/dt$  in the continuous-time case and to the “finite difference operator”  $a(t) \mapsto a(t+1) - a(t)$  in the discrete-time case (where  $a$  is any system coefficient). The non-commutativity between the system coefficients and the above generalized derivation involves a “commutation rule” (see (2) below) which was introduced by Ore [34]. There result computations on skew polynomials which, although they are generally more complicated than in the linear time-invariant (LTI) case (where only usual polynomials are involved), can be implemented using existing softwares, e.g. MAPLE’s ORETOOLS or ORE\_ALGEBRA packages [29].

In the LTI case, it is well-known that, taking into account the poles and zeros at infinity, the total number of poles of a rational transfer function is equal to the total number of its zeros. This is no longer true for a rational transfer matrix  $G$  (with several rows and/or columns): the difference between the total number of poles and the total number of zeros of  $G$  is a non-negative integer, called the *defect* of  $G$  and denoted by  $\text{def}(G)$ . This notion was introduced by Forney [17] and slightly generalized by Kung and Kailath [28] (see also ([23], Section 6.5.4)). As was shown in [28], by Verghese [44], and by Verghese, van Dooren and Kailath [45] (see also ([23], Theorem 6.5-11)),  $\text{def}(G)$  is the sum of the left- and right-minimal indices of  $G$  (those indices were introduced in [17], generalizing the classic Kronecker indices of matrix pencils).

In control theory, proper transfer functions play a prominent role since a controller is practically realizable only if its transfer function is proper. Proper transfer functions  $g = n/d$ , i.e. those such that  $\deg(g) := \deg(n) - \deg(d) \leq 0$ , where  $n, d$  are polynomials and  $d \neq 0$ , form a subring  $\mathbf{F}_{pr}$  of the field  $\mathbf{F}$  of all transfer functions. A transfer matrix is called proper if its entries are all proper. The “exact model-matching problem” consists, given two transfer matrices  $A, B$ , to determine whether there exists a proper transfer matrix  $H$  such that  $HA = B$  or  $AH = B$  ( $H$  is the transfer matrix of the post-compensator in the first case, and of the pre-compensator in the second case) [47] (see also [17], [28] and [23]) and, when the answer is positive, to parametrize explicitly all these proper transfer matrices  $H$ .

For the reader’s convenience, notations, conventions and background are recalled in Section 2. The notion of proper transfer matrix is extended to the LTV case in an obvious way (an LTV system with a proper transfer matrix is called a proper system, and an intrinsic characterization of such systems was given in [14]; see also [35]). The structure of the ring  $\mathbf{F}_{pr}$  and the properties of proper transfer matrices are studied in Section 4. Using the notions of “generalized degree” and of “valuation at infinity” of a matrix with entries in  $\mathbf{F}$ , introduced in Sections 3 and 4 respectively, the defect of a transfer matrix and its relation to Forney’s left- and right-minimal indices are extended to the LTV case in Section 5. The exact model-matching problem is completely solved in Section 6. The solution  $H$  and its parametrization are similar to those given by Marinescu [30] but with the important difference that in [30],  $H$  and the parameter on which it depends are assumed to be matrices with entries in a ring of formal skew power series whereas our solution  $H$  has its entries in  $\mathbf{F}_{pr}$ , which is much more natural. In addition, we do not assume that  $A$  and  $B$  are full column- or row-rank as in [30], thus our solution  $H$  depends on two parameters

(instead of one in [30]) which are matrices with entries in  $\mathbf{F}_{pr}$ . All results are constructive. The calculation of the defect  $\text{def}(G)$  and that of the solution  $H$  to the model matching problem in the LTV case are illustrated in Section 7 through an example belonging to the area of electric power systems.

## 2 Notation, conventions and background

### 2.1 Notation and conventions

The list of notations below is followed by some explanations when necessary.

$:=$	Equal by definition
$\rightarrow$	Surjection
$\mathbb{Z}$	Set of all integers,
$\mathbb{N}, \mathbb{N}^\times$	Set of all non-negative integers, of all natural integers
$\mathbf{R}^\times$	Set of nonzero elements in a ring $\mathbf{R}$
$(a)$	Two-sided ideal generated by an element $a$ in a ring $\mathbf{R}$
$\mathbf{U}(\mathbf{R})$	Group of units in a ring $\mathbf{R}$
$\mathfrak{M}_n(\mathbf{R})$	Ring of $n \times n$ matrices with entries in a ring $\mathbf{R}$
$\text{GL}_n(\mathbf{R})$	Group of unimodular $n \times n$ matrices with entries in $\mathbf{R}$
$M \cong_{\mathbf{R}} N$	Isomorphic $\mathbf{R}$ -modules $M, N$
$\text{tor}(M)$	Torsion submodule of an $\mathbf{R}$ -module $M$
$\mathbf{Q}(\mathbf{R})$	Quotient skew field of an Ore domain $\mathbf{R}$
$\text{col}\{A_i\}_{i \leq 1 \leq n}$	Matrix composed of the submatrices $A_i$ ( $i = 1, \dots, n$ ) arranged in column (the submatrices $A_1, \dots, A_n$ all having the same number of rows)
$\mathbf{K}, 1_{\mathbf{K}}$	Commutative field, identity map of $\mathbf{K}$
$\alpha, \delta$	Automorphism, $\alpha$ -derivation of $\mathbf{K}$ ((1))
$\mathbf{A} = \mathbf{K}[s; \alpha, \delta]$	Ring of skew polynomials with coefficients in $\mathbf{K}$ and indeterminate $s$ equipped with the commutation rule (2)
$\mathbf{F} = \mathbf{K}(s; \alpha, \delta)$	Quotient skew field of $\mathbf{A} = \mathbf{K}[s; \alpha, \delta]$
$\text{diag}\{n_i\}_{1 \leq i \leq k}$	Matrix (not necessarily square) with $n_1, \dots, n_k$ on its main diagonal and zeros elsewhere
$\dim_{\mathbf{K}}\{M\}$	Dimension over $\mathbf{K}$ of an $\mathbf{A}$ -module $M$
$\text{deg}(g)$	Degree of a skew rational fraction $g \in \mathbf{F}$ ((6))
$\mathbf{F}_{pr}$	Subring of $\mathbf{F}$ consisting of proper skew rational fractions (Definition 2)
$\text{deg}_F(G)$	$F$ -degree of a matrix $G \in \mathbf{F}^{p \times m}$ (Definition 3)
$\sim_{\mathbf{R}}$	Equivalence of matrices (Subsection 2.2)
$a \parallel b, a b$	Total divisor, divisor (Subsections 2.2, 4.1)
$\approx_{\mathbf{R}}$	Similarity (Subsection 2.2)
$\text{deg det}$	Determinantal degree (Subsection 2.3)
$\text{coker}_{\mathbf{R}}(\bullet A)$	Cokernel of the right multiplication by a matrix $A$ with entries in $\mathbf{R}$
$\text{deg}_M(G)$	Generalized degree of a matrix $G \in \mathbf{F}^{p \times m}$ (Definition 9)

$\deg_M^z(G)$	Generalized degree of $G$ relative to zeros
$\deg_M^p(G)$	Generalized degree of $G$ relative to poles
$\nu_\infty$	Valuation at infinity ((28), (35))
$\sigma$	Normed uniformizer of $\nu_\infty$ (i.e. $\sigma = s^{-1}$ )
$\triangleleft$	Normal subgroup (Subsection 4.3)
$d_\infty^z(G)$	Degree of the zero at infinity of $G$ (Definition 21)
$d_\infty^p(G)$	Degree of the pole at infinity of $G$
def	Defect (Definition 26)

Recall that a matrix  $V \in \mathfrak{M}_n(\mathbf{R})$  is said to be unimodular if it has an inverse  $V^{-1} \in \mathfrak{M}_n(\mathbf{R})$ . In the sequel, all  $\mathbf{R}$ -modules (where  $\mathbf{R}$  is a ring) are left modules. “Principal ideal domain” means “left and right principal ideal domain”, “Ore domain” means “left and right Ore domain”, “Euclidean domain” means “left and right Euclidean domain”, etc., and all these rings are noncommutative except when explicitly stated.

In what follows,  $\mathbf{K}$  is the field of system coefficients. An  $\alpha$ -derivation  $\delta$  of  $\mathbf{K}$  is an additive map  $\delta : \mathbf{K} \rightarrow \mathbf{K}$  such that for any  $a, b \in \mathbf{K}$ ,

$$(ab)^\delta = a^\alpha b^\delta + a^\delta b \quad (1)$$

where  $\alpha$  is an endomorphism of  $\mathbf{K}$ ,  $a^\alpha := \alpha(a)$  and  $a^\delta := \delta(a)$  is the “generalized derivative” of  $a$  with respect to time (see (3), (5)). In the two cases considered in the sequel (i.e. the continuous-time case and the discrete-time case),  $\alpha$  is an automorphism commuting with  $\delta$ . Besides,  $\mathbf{A} = \mathbf{K}[s; \alpha, \delta]$  is the ring of skew polynomials with coefficients in  $\mathbf{K}$  and indeterminate  $s$ , such that the following commutation rule holds ([12], Section 8.3):

$$sa = a^\alpha s + a^\delta, \quad (2)$$

as explained below. The ring  $\mathbf{A}$  is then an Euclidean domain, thus a principal ideal domain, and so an Ore domain ([12], Sections 0.8, 0.10). Therefore, it admits a left quotient skew field  $\mathbf{F} := \mathbf{Q}(\mathbf{A}) = \mathbf{K}(s; \alpha, \delta)$  (of left fractions  $d^{-1}n$ ) which coincides with its right quotient skew field (of right fractions  $n'd'^{-1}$ ) where  $n, n', d, d' \in \mathbf{A}, d, d' \neq 0$ . This formalism is classic in mathematics since Ore’s seminal paper [34] and was used in [4], [5] (see also [50]) to unify the continuous- and discrete-time cases as we will now briefly explain.

1. *Continuous-time systems*: For such a system, the time  $t$  belongs the real line and  $(\mathbf{K}, \delta)$  is a differential field ([26], Section 1.1). The field  $\mathbb{C}(t)$  of rational fractions with complex coefficients and the field  $\mathcal{M}$  of meromorphic functions, both equipped with the usual derivation  $d/dt$ , are typical differential fields [49], [20], [21]. So are also the field  $\mathcal{P}_{loc,m}$  ( $m \in \mathbb{N}^\times$ ) of locally convergent Laurent series in the variable  $z^{1/m}$ ,  $z := t^{-1}$  and the field of locally convergent Puiseux series  $\mathcal{P}_{loc} = \bigcup_{m \geq 1} \mathcal{P}_{loc,m}$ , equipped with the derivation  $d/dt = -z^2 d/dz$ . For any  $m \in \mathbb{N}^\times$ ,  $\mathbb{C}(t) \subsetneq \mathcal{P}_{loc,m} \subsetneq \mathcal{P}_{loc} \subsetneq \mathcal{M}$ . Each element of  $\mathbb{C}(t)$  (resp.  $\mathcal{M}$ ) is a smooth function in the complement of a finite (resp. discrete) subset of  $\mathbb{R}$ , and each element of  $\mathcal{P}_{loc}$  is a smooth function in an interval of the form  $(\tau, +\infty)$  for  $\tau \geq 0$  large enough. The

field  $\mathcal{P}_{loc}$  was introduced in [9] for stability study of LTV continuous-time systems. The Leibniz rule  $\frac{d}{dt}(ab) = a\dot{b} + \dot{a}b$  is a particular case of (1) with  $\dot{a} = a^\delta$ ,  $\dot{b} = b^\delta$  and

$$\alpha := 1_{\mathbf{K}}, \quad a^\delta := da/dt \quad (3)$$

Since  $\delta$  is not induced by  $\alpha$ , it is called an “outer derivation”. Further, the indeterminate  $s$  acts on a signal  $w$  according to

$$sw = dw/dt \quad (4)$$

If  $\mathbf{K} = \mathbb{C}(t)$  (resp.  $\mathbf{K} = \mathcal{M}$ ), a signal is a smooth function in the complement of a finite (resp. discrete) subset of  $\mathbb{R}$  [49], [20], [21]. If  $\mathbf{K} = \mathcal{P}_{loc}$ , a signal is a smooth function in an interval  $(\tau, +\infty)$  of the real line for  $\tau$  large enough [9]. The Leibniz rule  $d/dt(aw) = adw/dt + \dot{a}w$  can also be written  $s(aw) = a(sw) + a^\delta w$  which yields the commutation rule  $sa = as + a^\delta$ , identical to (2) with  $\alpha$  satisfying the first equality of (3).

2. *Discrete-time systems:* For such a system, the time  $t$  belongs to  $\mathbb{Z}$  and  $(\mathbf{K}, \alpha)$  is a difference field ([43], Definition 1.1). The case where  $\mathbf{K} = \mathbb{C}(t)$  (resp.  $\mathbf{K} = \mathcal{P}_{loc}$ ) was considered in [10] (resp. [33]) for stability study of LTV discrete-time systems. The automorphism  $\alpha$  and the  $\alpha$ -derivation  $\delta$  are defined according to

$$a^\alpha(t) := a(t+1), \quad a^\delta(t) := a(t+1) - a(t) \quad (5)$$

hence  $\delta$  is the “inner derivation”  $\alpha - 1_{\mathbf{K}}$  and (1) holds true, as easily seen. Further,  $s$  acts on a signal  $w$  according to  $sw(t) = w(t+1) - w(t)$ . Such a signal is a sequence defined in an interval  $(\tau, +\infty) \cap \mathbb{Z}$  of  $\mathbb{Z}$  for  $\tau$  large enough [10], [33]. We have  $s(aw)(t) = a(t+1)w(t+1) - a(t)w(t) = a(t+1)(w(t+1) - w(t)) + (a(t+1) - a(t))w(t)$  which yields the commutation rule (2).

The transfer matrix of an LTV system (either in the continuous- or discrete-time case) has its entries in  $\mathbf{F}$ . In the LTI case,  $\alpha = 1_{\mathbf{K}}$ ,  $\delta = 0$  and  $\mathbf{A} = \mathbf{K}[s]$ ,  $\mathbf{F} = \mathbf{K}(s)$ , respectively the ring of usual polynomials and the field of usual rational fractions with coefficients in  $\mathbf{K}$  and indeterminate  $s$ .

A nonzero skew polynomial  $f \in \mathbf{A} := \mathbf{K}[s; \alpha, \delta]$  is of the form  $f = \sum_{0 \leq i < n} f_i s^i = \sum_{0 \leq i < n} s^i f'_i$  where the coefficients  $f_i, f'_i$  are uniquely determined and  $f_n, f'_n \neq 0$ . This skew polynomial has degree  $\deg(f) = n$ , and  $\deg(0) := -\infty$ . If  $g \in \mathbf{F} := \mathbf{K}(s; \alpha, \delta)$ , there exist skew polynomials  $n, n', d, d'$  ( $d, d' \neq 0$ ) such that  $g = d^{-1}n = n'd'^{-1}$  and the degree of  $g$  is

$$\deg(g) = \deg(n) - \deg(d) = \deg(n') - \deg(d'). \quad (6)$$

**Lemma 1** *The map  $\deg : \mathbf{F} \rightarrow \mathbb{Z} \cup \{-\infty\}$  has the following properties (where  $g, h \in \mathbf{F}$ ):*

- (i) *It is surjective and  $\deg(g) > -\infty$  if and only if  $g \neq 0$ .*
- (ii)  *$\deg(gh) = \deg(g) + \deg(h)$ .*
- (iii)  *$\deg(g - h) \leq \max\{\deg(g), \deg(h)\}$ .*
- (iv) *Equality holds in (iii) if  $\deg(g) \neq \deg(h)$ .*

**Proof.** Properties (i), (ii) are clear and (iii), (iv) can be proved as in ([2], p. IV.20, Proposition 1). ■

**Definition 2** The subring of  $\mathbf{F}$  consisting of all proper skew rational fractions, i.e. of those fractions  $g$  such that  $\deg(g) \leq 0$ , is denoted as  $\mathbf{F}_{pr}$ .

Let  $G \in \mathbf{F}^{p \times m}$ . Extending a notion introduced by Forney [17] for  $p \times 1$  or  $1 \times m$  matrices with entries in  $\mathbf{K}[s]$ , we define below the *Forney degree*, or *F-degree*, of  $G$ :

**Definition 3** The F-degree of  $G \in \mathbf{F}^{p \times m}$  is

$$\deg_F(G) := \max \{ \deg(g_{ij}) : 1 \leq i \leq p, 1 \leq j \leq m \}$$

where  $g_{ij}$  is the entry with indices  $(i, j)$  of  $G$ .

**Lemma 4** (i)  $\deg_F : \mathbf{F}^{p \times m} \rightarrow \mathbb{Z} \cup \{-\infty\}$  is surjective and  $\deg_F(G) > -\infty$  if and only if  $G \neq 0$ .

(ii) If  $G, H \in \mathbf{F}^{p \times m}$ , then  $\deg_F(G - H) \leq \max \{ \deg_F(G), \deg_F(H) \}$ .

(iii) Equality holds in (ii) if  $\deg_F(G) \neq \deg_F(H)$ .

(iv) If  $G \in \mathbf{F}^{p \times m}$ ,  $H \in \mathbf{F}^{m \times q}$ , then  $\deg_F(GH) \leq \deg_F(G) + \deg_F(H)$ .

(v) If  $g \in \mathbf{F}$ ,  $H \in \mathbf{F}^{p \times m}$ , then  $\deg_F(gH) = \deg(g) + \deg_F(H)$ .

**Proof.** (i) is a consequence of Lemma 1(i), considering a matrix whose entries are all zero except perhaps that with indices  $(1, 1)$ .

(ii) Let  $G = (g_{ij})$ ,  $H = (h_{ij})$ . For any pair of indices  $(i, j)$ ,

$$\begin{aligned} \deg(g_{ij} - h_{ij}) &\leq \max \{ \deg(g_{ij}), \deg(h_{ij}) \} \\ &\leq \max_{k,l} \{ \max \{ \deg(g_{k,l}), \deg(h_{k,l}) \} \} \end{aligned}$$

hence the inequality stated.

(iii) If  $\deg_F(G) > \deg_F(H)$ , there exists a pair of indices  $(i, j)$  such that  $\deg_F(G) = \deg(g_{ij}) > \deg(h_{kl})$  for all pairs of indices  $(k, l)$ , hence

$$\deg_F(G - H) = \max_{k,l} \{ \deg(g_{kl} - h_{kl}) \} = \deg(g_{ij}) = \deg_F(G)$$

(iv) For any pair of indices  $(i, j)$ ,

$$\deg \left( \sum_l g_{il} h_{lj} \right) \leq \max_l \deg(g_{il} h_{lj}) \leq \max_l \deg(g_{il}) + \max_k \deg(h_{kj})$$

and the inequality stated follows.

(v) is obvious. ■

**Remark 5** The inequality in Lemma 4(iv) is not an equality in general, even for diagonal nonsingular matrices, as shown by the example where  $G = \text{diag} \{s, 1\}$ ,  $H = \text{diag} \{1, s\}$ , hence  $GH = \text{diag} \{s, s\}$ . Nevertheless,  $\deg_F$  is a degree on the polynomial algebra  $\mathfrak{M}_n(\mathbf{A})$  ([2], Chapter IV, Section 1.2).

Put  $\sigma = s^{-1}$ . Then (2) can equivalently be written

$$a\sigma = \sigma a^\alpha + \sigma a^\delta \sigma \Leftrightarrow \sigma a = a^\beta \sigma + \sigma a^{\beta\delta} \sigma \quad (\beta = \alpha^{-1}) \quad (7)$$

and  $\mathbf{S} := \mathbf{K}[[\sigma; \alpha, \delta]]$  denotes the ring of skew formal power series with coefficients in  $\mathbf{K}$  and indeterminate  $\sigma$  equipped with the commutation rule (7). This is an Ore domain and its quotient skew field is the field of skew formal Laurent series denoted as  $\mathbf{L} := \mathbf{K}((\sigma; \alpha, \delta))$ . Any nonzero element  $f$  of  $\mathbf{L}$  is of the form

$$f = \sum_{i \geq k} f_i \sigma^i = \sum_{i \geq k} \sigma^i f'_i \quad (8)$$

where the coefficients  $f_i, f'_i$  are uniquely determined and  $f_k, f'_k \neq 0$ ;  $f$  belongs to  $\mathbf{S}$  if and only if  $k \geq 0$ . Since  $\sigma = s^{-1}$ , through its series expansion any element of  $\mathbf{F}_{pr}$  is identified with an element of  $\mathbf{S}$ , hence  $\mathbf{F}_{pr} \subset \mathbf{S}$  and  $\mathbf{F} \subset \mathbf{L}$ . Further, the element  $f$  in (8), where  $f_k \neq 0$  (or equivalently  $f'_k \neq 0$ ), belongs to  $\mathbf{A}$  if and only if  $k \leq 0$  and the index  $i$  varies from  $k$  to an integer  $l \leq 0$ , and then  $\deg(f) = -k$ , thus  $\mathbf{A} \subset \mathbf{L}$ . Since  $\mathbf{F}$  is the smallest skew field containing  $\mathbf{A}$ , this implies  $\mathbf{F} \subset \mathbf{L}$ . To summarize:

$$\begin{array}{ccccc} \mathbf{A} & \subset & \mathbf{F} & \subset & \mathbf{L} \\ & & \mathbf{U} & & \mathbf{U} \\ & & \mathbf{F}_{pr} & \subset & \mathbf{S} \end{array}$$

## 2.2 Normal forms of polynomial and rational matrices

Let  $\mathbf{R}$  be a principal ideal domain and  $N \in \mathbf{R}^{p \times m}$  be a matrix of rank  $r > 0$  over  $\mathbf{Q}(\mathbf{R})$ . The following is classic ([12], Chapter 8): There exist unimodular matrices  $U \in \mathrm{GL}_p(\mathbf{R})$ ,  $V \in \mathrm{GL}_m(\mathbf{R})$  such that

$$U^{-1}NV = \underbrace{\mathrm{diag}\{e_1, \dots, e_r, 0, \dots, 0\}}_S \quad (9)$$

(in other words,  $N$  is equivalent to  $S$ , and we write  $N \sim_{\mathbf{R}} S$ ) where  $e_i \in \mathbf{R}^\times$  ( $i = 1, \dots, r$ ) and for  $i = 1, \dots, r-1$ ,  $e_i$  is a total divisor of  $e_{i+1}$ , written  $e_i \parallel e_{i+1}$ , which means that there exists an invariant element  $c$  (i.e.  $c \neq 0$  and the equality  $\mathbf{R}e = e\mathbf{R}$  holds) such that  $e_i | c | e_{i+1}$  (“|” meaning “divides”). These elements  $e_i$  are uniquely determined up to similarity, i.e. if  $e'_i$  ( $i = 1, \dots, r$ ) are other such elements, then  $\mathbf{R}/\mathbf{R}e_i \cong_{\mathbf{R}} \mathbf{R}/\mathbf{R}e'_i$ , written  $e_i \approx_{\mathbf{R}} e'_i$ . The matrix  $S$  on the right hand-side of (9) is called a Jacobson-Teichmüller normal form of  $N$ . If  $\mathbf{R} = \mathbf{A}$  ( $\mathbf{A} = \mathbf{K}[s; \alpha, \delta]$ ), similarity  $e_i \approx_{\mathbf{R}} e'_i$  implies  $\deg(e_i) = \deg(e'_i)$  since

$$\dim_{\mathbf{K}} \{\mathbf{R}/\mathbf{R}e_i\} = \deg(e_i) \quad (10)$$

([12], Section 8.4).

If  $\mathbf{A} = \mathbf{K}[s]$  (case of constant coefficients), all nonzero elements of  $\mathbf{A}$  are invariant so that  $e_i \parallel e_{i+1}$  if and only if  $e_i | e_{i+1}$ , the polynomials  $e_i$  ( $i = 1, \dots, r$ ) are uniquely determined up to associates and can be assumed to be monic, and



$S$  is the usual Smith normal form of  $N$ . Let  $G = d^{-1}N \in \mathbf{F}^{p \times m}$  be a nonzero transfer matrix where  $d \neq 0$ . Then

$$U^{-1}GV = \text{diag} \{n_1/d_1, \dots, n_r/d_r, 0, \dots, 0\} \quad (11)$$

where  $n_i/d_i = e_i/d$ , the rational fraction  $n_i/d_i$  is irreducible and  $n_i, d_i$  are monic ( $i = 1, \dots, r$ ). The matrix on the right-hand side of (11) is the McMillan normal form of  $G$ , and the transmission poles and zeros of the system with transfer matrix  $G$  (also called the McMillan poles and zeros of  $G$ ) are the roots of the  $d_i$ 's and the  $n_i$ 's respectively (multiplicities accounted for) in an algebraic closure of  $\mathbf{K}$ .

### 2.3 The determinantal degree $\text{deg det}$

Let  $\mathbf{R}$  be a skew field and  $A \in \mathfrak{M}_n(\mathbf{R})$ . The Dieudonné determinant  $\det(A)$  of  $A$  is zero if  $A$  is singular and, if not, belongs to the abelianization  $\mathbf{R}^{\times ab} = \mathbf{R}^{\times} / \mathbf{D}(\mathbf{R}^{\times})$  of the multiplicative group  $\mathbf{R}^{\times}$ , where  $\mathbf{D}(\mathbf{R}^{\times}) = (\mathbf{R}^{\times}, \mathbf{R}^{\times})$  is the derived subgroup of  $\mathbf{R}^{\times}$  ([13], Section 9.2).

Let  $\mathbf{F} = \mathbf{K}(s; \boldsymbol{\alpha}, \boldsymbol{\delta})$  and  $\mathbf{A} = \mathbf{K}[s; \boldsymbol{\alpha}, \boldsymbol{\delta}]$ . The following results are proved in [39]:

1. If  $a \in \mathbf{F}^{\times}$ , then  $\text{deg}(a)$  depends only on the class  $\bar{a}$  of  $a$  in  $\mathbf{F}^{\times ab}$ , therefore the function  $\text{deg} : \mathbf{F}^{\times} \rightarrow \mathbb{Z}$  induces an epimorphism of abelian groups

$$\overline{\text{deg}} : \mathbf{F}^{\times ab} \twoheadrightarrow \mathbb{Z} \quad (12)$$

2. Let us extend the group epimorphism  $\text{deg det} := \overline{\text{deg}} \circ \det : \text{GL}_n(\mathbf{F}) \twoheadrightarrow \mathbb{Z}$  to  $\mathfrak{M}_n(\mathbf{F})$  by putting  $\text{deg det}(A) = -\infty$  if  $A$  is singular; then the mapping  $\text{deg det} : \mathfrak{M}_n(\mathbf{F}) \rightarrow \mathbb{Z} \cup \{-\infty\}$  is surjective and  $\text{deg det}(AA') = \text{deg det}(A) + \text{deg det}(A')$  ( $A, A' \in \mathfrak{M}_n(\mathbf{F})$ ).
3. For  $n = 1$ ,  $\text{deg det} = \text{deg}$ .
4. The restriction of  $\text{deg det}$  to  $\mathfrak{M}_n(\mathbf{A})$  is a surjection onto  $\mathbb{N} \cup \{-\infty\}$ . Furthermore, if  $A \in \mathfrak{M}_n(\mathbf{A})$ , then  $\text{deg det}(A) > -\infty$  if and only if  $A$  is regular (over  $\mathbf{F}$ ), i.e. if the  $\mathbf{A}$ -module  $\text{coker}_{\mathbf{A}}(\bullet A)$  (where  $(\bullet A)$  is the right-multiplication by  $A$ ) is torsion, and in that case  $\text{deg det}(A) = \dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet A)\}$ ; if  $A$  is singular (over  $\mathbf{F}$ ), there exists a free  $\mathbf{A}$ -module  $F \neq 0$  such that  $\text{coker}_{\mathbf{A}}(\bullet A) = F \oplus \text{tor}(\text{coker}_{\mathbf{A}}(\bullet A))$ , thus  $\dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet A)\} = +\infty$ . Therefore,  $\text{deg det}(A)$  is finite if and only if  $\dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet A)\}$  is finite. The matrix  $A \in \mathfrak{M}_n(\mathbf{A})$  is unimodular if and only if  $\dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet A)\} = 0$ , i.e.  $\text{deg det}(A) = 0$ .

If  $G \in \mathbf{F}^{p \times m}$  and  $M \in \mathfrak{M}_k(\mathbf{R})$  ( $k \leq \min(p, m)$ ) is a submatrix of  $G$ , then  $\det(M) \in \mathbf{R}^{\times ab} \cup \{0\}$  is called a *minor* of order  $k$  of  $A$ . The  $k$ -th compound matrix  $G^{(k)}$  of  $G$  is the matrix of dimension  $\binom{p}{k} \times \binom{m}{k}$ , all entries of which are minors  $M_{ij}$  of order  $k$  of  $G$ , written in lexicographic order by rows and columns.

If  $G' \in \mathbf{F}^{p \times m}$ ,  $G'' \in \mathbf{F}^{m \times q}$  and  $k \leq \min\{p, m, q\}$ , we have the generalized Cauchy-Binet identity

$$(G'G'')^{(k)} = G'^{(k)}G''^{(k)} \quad (13)$$

([11], Theorem 5.2).

### 3 Generalized degree of a transfer matrix

#### 3.1 The LTI case

Assume that  $\mathbf{F} = \mathbf{K}(s)$  and let  $G \in \mathbf{F}^{p \times m}$  be a transfer matrix. If  $G \neq 0$ , consider its McMillan normal form (11). We put

$$\deg_M^{\mathbf{P}}(G) = \sum_{i=1}^r \deg(d_i), \quad \deg_M^{\mathbf{Z}}(G) = \sum_{i=1}^r \deg(n_i) \quad (14)$$

$$\deg_M(G) = \deg_M^{\mathbf{Z}}(G) - \deg_M^{\mathbf{P}}(G) = \deg \det \left( \text{diag} \{n_i/d_i\}_{1 \leq i \leq r} \right) \quad (15)$$

Further, if  $G = 0$ , we put  $\deg_M(G) = -\infty$ .

**Definition 6**  $\deg_M(G)$  (resp.  $\deg_M^{\mathbf{P}}(G)$ , resp.  $\deg_M^{\mathbf{Z}}(G)$ ) here above is called the generalized degree (resp. the generalized degree relative to poles, resp. the generalized degree relative to zeros) of  $G$ .

Note that  $\deg_M^{\mathbf{P}}(G)$  is the McMillan degree  $\delta_M(G)$  of  $G$  if this transfer matrix is proper ([23], p. 466) and that  $\deg_M(G) = \deg(G)$  when  $p = m = 1$  (see (6)), which justifies the notation. As shown in Theorem 11 below, the generalized degree enjoys some properties of a usual degree, as do  $\deg_F$  (Remark 5) and  $\delta_M$  ([23], p. 466), but, contrary to  $\deg_F$  and  $\delta_M$ , not all of them (Remark 12). Using the following lemma we will be able to extend (14) and (15) to the LTV case in the next subsection.

**Lemma 7** Let  $D_l^{-1}N_l$  (resp.  $N_r D_r^{-1}$ ) be any left- (resp. right-) coprime factorization over  $\mathbf{A}$  of  $G \in \mathbf{F}^{p \times m}$ , assumed to be of rank  $r > 0$ . The generalized degrees  $\deg_M^{\mathbf{P}}$ ,  $\deg_M^{\mathbf{Z}}$  in (14) satisfy the equalities

$$\deg_M^{\mathbf{P}}(G) = \dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet D_l)\} = \dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet D_r)\}, \quad (16)$$

$$\deg_M^{\mathbf{Z}}(G) = \dim_{\mathbf{K}} \{\text{tor}(\text{coker}_{\mathbf{A}}(\bullet N_l))\} = \dim_{\mathbf{K}} \{\text{tor}(\text{coker}_{\mathbf{A}}(\bullet N_r))\} \quad (17)$$

**Proof.** 1) Let  $U^{-1}GV$  be the McMillan normal form on the right of the equality (11). With  $N_l = \text{diag}\{n_1, \dots, n_r, 0, \dots, 0\}V^{-1}$ ,  $D_l = \text{diag}\{d_1, \dots, d_r, 1, \dots, 1\}U^{-1}$ ,  $D_l^{-1}N_l$  is a left-coprime factorization of  $G$  over  $\mathbf{A}$  and we have  $\text{coker}_{\mathbf{A}}(\bullet D_l) \cong_{\mathbf{A}} \bigoplus_{i=1}^r \text{coker}_{\mathbf{A}}(\bullet d_i)$ ,  $\text{tor}(\text{coker}_{\mathbf{A}}(\bullet N_l)) \cong_{\mathbf{A}} \bigoplus_{i=1}^r \text{coker}_{\mathbf{A}}(\bullet n_i)$  from which the first equality of (16) and that of (17) follow since  $\dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet d_i)\} = \deg(d_i)$ ,  $\dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet n_i)\} = \deg(n_i)$  by (10). The other equalities are established similarly.

2) If  $\tilde{D}_l^{-1}\tilde{N}_l$  (resp.  $\tilde{N}_r\tilde{D}_r^{-1}$ ) is another left- (resp. right-) coprime factorization of  $G$  over  $\mathbf{A}$ , there exist unimodular matrices  $U \in \text{GL}_p(\mathbf{A})$ ,  $V \in \text{GL}_m(\mathbf{A})$  such that  $\begin{bmatrix} \tilde{D}_l & \tilde{N}_l \end{bmatrix} = U \begin{bmatrix} D_l & N_l \end{bmatrix}$  (resp.  $\text{col} \left\{ \tilde{D}_r, \tilde{N}_r \right\} = \text{col} \{D_r, N_r\} V$ ), thus  $\text{coker}_{\mathbf{A}}(\bullet\tilde{D}_l) \cong_{\mathbf{A}} \text{coker}_{\mathbf{A}}(\bullet D_l)$  and  $\text{coker}_{\mathbf{A}}(\bullet\tilde{N}_l) \cong_{\mathbf{A}} \text{coker}_{\mathbf{A}}(\bullet N_l)$  (resp.  $\text{coker}_{\mathbf{A}}(\bullet\tilde{D}_r) \cong_{\mathbf{A}} \text{coker}_{\mathbf{A}}(\bullet D_r)$  and  $\text{coker}_{\mathbf{A}}(\bullet\tilde{N}_r) \cong_{\mathbf{A}} \text{coker}_{\mathbf{A}}(\bullet N_r)$ ). ■

### 3.2 The LTV case

Let  $\mathbf{F} = \mathbf{K}(s; \alpha, \delta)$  and  $\mathbf{A} = \mathbf{K}[s; \alpha, \delta] \neq \mathbf{K}[s]$ . The generalized degrees  $\text{deg}_M$ ,  $\text{deg}_M^{\mathbf{z}}$ ,  $\text{deg}_M^{\mathbf{p}}$  can no longer be defined by (14) and (15) since the McMillan normal form of a matrix  $G \in \mathbf{F}^{p \times m}$  does not exist.

Let us recall the notion of state-space system according to Fliess [15] (see also ([6], Sections 1.6 and 2.1) where the connection – here a duality – between Fliess's approach and the behavioral approach is detailed). A linear system over  $\mathbf{A}$  is (described by) a finitely generated  $\mathbf{A}$ -module  $M = \text{coker}_{\mathbf{A}}(\bullet R)$  where  $R = (r_{ji}) \in \mathbf{A}^{q \times k}$ . Denoting by  $\mathbf{w}_i$  the canonical image of the  $i$ -th element  $\underline{\mathbf{w}}_i$  of the canonical basis of  $\mathbf{A}^{1 \times k}$  in the quotient  $M = \mathbf{A}^{1 \times k} / \mathbf{A}^{1 \times q} R$ , we have  $R\mathbf{w} = 0$  where  $\mathbf{w} = \text{col} \{ \mathbf{w}_i \}_{1 \leq i \leq k}$  and  $M = [\mathbf{w}]_{\mathbf{A}}$ , i.e.  $M$  is the  $\mathbf{A}$ -module generated by the  $\mathbf{w}_i$ . So,  $M$  is determined up to isomorphism to be the  $\mathbf{A}$ -module  $[\mathbf{w}]_{\mathbf{A}}$  where the column  $\mathbf{w}$  satisfies the relation  $R\mathbf{w} = 0$  (and no additional nonredundant one<sup>1</sup>). A control for such a system is a column  $\mathbf{u}$  of  $m$  elements  $\mathbf{u}_i$  such that  $[\mathbf{u}]_{\mathbf{A}}$  is a free submodule of rank  $m$  of  $M$  and the quotient module  $M/[\mathbf{u}]_{\mathbf{A}}$  is torsion; an output is a column  $\mathbf{y}$  of  $p$  elements  $\mathbf{y}_i$  such that  $[\mathbf{y}]_{\mathbf{A}}$  is a submodule of  $M$ . The triple  $(M, \mathbf{u}, \mathbf{y})$  is called a linear control system ([8], Definition 851) (or a linear dynamics in Fliess's terminology [15]). Consider the  $\mathbf{F}$ -vector space  $\hat{M} := \mathbf{F} \otimes_{\mathbf{A}} M$  and let  $\hat{\mathbf{u}}_i, \hat{\mathbf{y}}_j$  be the canonical image of  $\mathbf{u}_i, \mathbf{y}_j$ , respectively, in  $\hat{M}$  ( $i = 1, \dots, m; j = 1, \dots, p$ ), i.e.  $\hat{\mathbf{u}}_i = 1 \otimes \mathbf{u}_i, \hat{\mathbf{y}}_j = 1 \otimes \mathbf{y}_j$ . Then  $\{\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_m\}$  is a basis of  $\hat{M}$ , hence there exists a unique  $\mathbf{F}$ -linear mapping  $G$  such that  $\hat{\mathbf{y}} = G\hat{\mathbf{u}}$  where  $\hat{\mathbf{y}} = \text{col} \{ \hat{\mathbf{y}}_j \}_{1 \leq j \leq p}$ ,  $\hat{\mathbf{u}} = \text{col} \{ \hat{\mathbf{u}}_i \}_{1 \leq i \leq m}$ , and  $G$  is the transfer matrix of  $(M, \mathbf{u}, \mathbf{y})$  [16]. The control system  $(M, \mathbf{u}, \mathbf{y})$  is controllable if and only if  $M$  is free, and is observable if and only if  $M = [\mathbf{u}, \mathbf{y}]_{\mathbf{A}}$  [15]. As shown in the latter reference, the control system  $(M, \mathbf{u}, \mathbf{y})$  has a state-space representation of finite order  $n$

$$s\mathbf{x} = A\mathbf{x} + B\mathbf{u}, \quad \mathbf{y} = C\mathbf{x} + D\mathbf{u} \quad (18)$$

$A \in \mathfrak{M}_n(\mathbf{K})$ ,  $B \in \mathbf{K}^{n \times m}$ ,  $C \in \mathbf{K}^{p \times n}$ ,  $D \in \mathbf{A}^{p \times m}$ . The order of this system is  $n = \dim_{\mathbf{K}} \{ \text{coker}_{\mathbf{A}}(\bullet(sI_n - A)) \} = \dim_{\mathbf{K}} \{ M/[\mathbf{u}]_{\mathbf{A}} \}$  since  $M = [\mathbf{x}, \mathbf{u}]_{\mathbf{A}}$ . Besides, (18) is a state-space realization of  $G$  if and only if  $G = C(sI_n - A)^{-1}B + D$  and as usual, the state-space realization (18) is minimal if and only if it is both observable and controllable.

The following result generalizes that of Kalman [24] regarding Statement (i), and ([46], Section 4.1, Theorem 49) regarding (ii):

<sup>1</sup>Strictly speaking, the module  $M$  is then said to be determined by generators  $\underline{\mathbf{w}}_i$  and relations  $\left\{ \sum_{1 \leq i \leq k} r_{ji} \underline{\mathbf{w}}_i : 1 \leq j \leq q \right\}$  ([37], Section 3.1, p. 106).

**Theorem 8** Let  $G \in \mathbf{F}^{p \times m}$  be a transfer matrix of rank  $r > 0$ , and  $D_l^{-1}N_l$  (resp.  $N_r D_r^{-1}$ ) be any left- (resp. right-) coprime factorization of  $G$  over  $\mathbf{A}$ .  
(i)  $\dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet D_l)\}$  is the order of any minimal state-space realization of  $G$ .  
(ii) We have

$$\text{coker}_{\mathbf{A}}(\bullet D_l) \cong_{\mathbf{A}} \text{coker}_{\mathbf{A}}(\bullet D_r), \quad \text{tor}(\text{coker}_{\mathbf{A}}(\bullet N_l)) \cong_{\mathbf{A}} \text{tor}(\text{coker}_{\mathbf{A}}(\bullet N_r)) \quad (19)$$

therefore the invariant factors of  $D_l$  coincide with those of  $D_r$  up to similarity, and likewise the nonzero invariant factors of  $N_l$  coincide with those of  $N_r$  up to similarity; hence

$$\dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet D_r)\} = \dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet D_l)\}, \quad (20)$$

$$\dim_{\mathbf{K}} \{\text{tor}(\text{coker}_{\mathbf{A}}(\bullet N_r))\} = \dim_{\mathbf{K}} \{\text{tor}(\text{coker}_{\mathbf{A}}(\bullet N_l))\}. \quad (21)$$

**Proof.** (i) From the above, the linear control system  $(M, \mathbf{u}, \mathbf{y})$ , of order  $n$ , is observable if and only if  $\mathbf{y}, \mathbf{u}$  satisfy a relation of the form

$$D_l \mathbf{y} = N_l \mathbf{u} \quad (22)$$

(and no additional nonredundant relation is satisfied, so that  $M = [\mathbf{u}, \mathbf{y}]_{\mathbf{A}}$ ),  $D_l \in \mathfrak{M}_p(\mathbf{A})$  is regular (since  $M/[\mathbf{u}]_{\mathbf{A}}$  is torsion) and  $N_l \in \mathbf{A}^{k \times m}$ ; then,  $G = D_l^{-1}N_l$ . This observable system is controllable if and only if  $\begin{bmatrix} D_l & -N_l \end{bmatrix}$  is right-invertible (freeness condition of  $M$ ), in other words  $D_l^{-1}N_l$  is a left-coprime factorization of  $G$  over  $\mathbf{A}$ . We have  $M/[\mathbf{u}]_{\mathbf{A}} \cong_{\mathbf{A}} \text{coker}_{\mathbf{A}}(\bullet D_l)$ , thus  $n = \dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet D_l)\}$ .

Conversely, if  $D_l^{-1}N_l$  is a left-coprime factorization of a transfer matrix  $G$ , there exists a controllable and observable linear control system  $(M, \mathbf{u}, \mathbf{y})$ , of order  $n$ , with module  $M = [\mathbf{y}, \mathbf{u}]_{\mathbf{A}}$  where  $\mathbf{y}, \mathbf{u}$  are linked by the only nonredundant relation (22), and whose  $G$  is the transfer matrix. Thus,  $M/[\mathbf{u}]_{\mathbf{A}} \cong_{\mathbf{A}} \text{coker}_{\mathbf{A}}(\bullet D_l)$  and  $n = \dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet D_l)\}$ , which proves (i).

(ii) Let the linear control system  $(M, \mathbf{u}, \mathbf{y})$  be controllable and let  $\xi_r$  be a column, the entries of which form a basis of  $M$ . This system has a representation of the form

$$D_r \xi_r = \mathbf{u}, \quad N_r \xi_r = \mathbf{y} \quad (23)$$

The column  $\xi_r$  is called a partial state ([23], Section 6.4.1), and  $D_r$  is regular square. Thus  $M/[\mathbf{u}]_{\mathbf{A}} \cong_{\mathbf{A}} \text{coker}(\bullet D_r)$ . This controllable system is observable if and only if  $N_r D_r^{-1}$  is a right-coprime factorization of  $G$  over  $\mathbf{A}$  (in other words,  $M = [\mathbf{y}, \mathbf{u}]_{\mathbf{A}}$ ).

Conversely, if  $N_r D_r^{-1}$  is a right-coprime factorization of the transfer matrix  $G$ , there exists a controllable and observable control system  $(M, \mathbf{u}, \mathbf{y})$ , with module  $[\xi_r, \mathbf{u}]_{\mathbf{A}}$  and output  $\mathbf{y}$  where  $\xi_r, \mathbf{u}, \mathbf{y}$  are linked by the only relations (23), and whose  $G$  is the transfer matrix, hence  $M/[\mathbf{u}]_{\mathbf{A}} \cong_{\mathbf{A}} \text{coker}(\bullet D_r)$ . Thus the first isomorphism of (19) as well as the equality (20) are proved.

Let the control system  $(M, \mathbf{u}, \mathbf{y})$  be controllable and observable. From (22),  $M/[\mathbf{y}] \cong_{\mathbf{A}} \text{coker}_{\mathbf{A}}(\bullet N_l)$ . From (23),  $M/[\mathbf{y}] = [\bar{\xi}_r, \bar{\mathbf{u}}]_{\mathbf{A}}$  where the entries of the

column  $\bar{\xi}_r$  (resp.  $\bar{\mathbf{u}}$ ) are the canonical images of the entries of the column  $\xi_r$  (resp.  $\mathbf{u}$ ) in the quotient  $M/[\mathbf{y}]$ ; hence we have

$$M/[\mathbf{y}] \cong_{\mathbf{A}} \text{coker}_{\mathbf{A}} \left( \bullet \begin{bmatrix} N_r & 0 \\ D_r & -I_m \end{bmatrix} \right) \cong_{\mathbf{A}} \text{coker}_{\mathbf{A}} (\bullet N_r)$$

which proves the second isomorphism of (19) and the equality (21). ■

We are led to the following definition of the generalized degrees  $\deg_M$ ,  $\deg_M^{\mathbf{P}}$  and  $\deg_M^{\mathbf{Z}}$  which, according to Lemma 7, generalizes (14) and (15):

**Definition 9** Let  $G \in \mathbf{F}^{p \times m}$  be a matrix of rank  $r$  and, if  $r > 0$ , let  $D_l^{-1}N_l$  (resp.  $N_r D_r^{-1}$ ) be a left- (resp. right-) coprime factorization of  $G$  over  $\mathbf{A}$ . We put

$$\deg_M^{\mathbf{P}}(G) = \dim_{\mathbf{K}} \{ \text{coker}_{\mathbf{A}} (\bullet D_r) \} = \dim_{\mathbf{K}} \{ \text{coker}_{\mathbf{A}} (\bullet D_l) \}, \quad (24)$$

$$\deg_M^{\mathbf{Z}}(G) = \dim_{\mathbf{K}} \{ \text{tor}(\text{coker}_{\mathbf{A}} (\bullet N_r)) \} = \dim_{\mathbf{K}} \{ \text{tor}(\text{coker}_{\mathbf{A}} (\bullet N_l)) \} \quad (25)$$

if  $r > 0$  and

$$\deg_M(G) = \begin{cases} \deg_M^{\mathbf{Z}}(G) - \deg_M^{\mathbf{P}}(G) & \text{if } r > 0 \\ -\infty & \text{if } r = 0 \end{cases} \quad (26)$$

The generalized degrees  $\deg_M^{\mathbf{P}}(G)$ ,  $\deg_M^{\mathbf{Z}}(G)$ ,  $\deg_M(G)$  are further detailed in the following lemma:

**Lemma 10** Let  $G \in \mathbf{F}^{p \times m}$  be a matrix of rank  $r > 0$ .

(i) There exist  $U \in \text{GL}_p(\mathbf{A})$ ,  $V \in \text{GL}_m(\mathbf{A})$  such that

$$UGV^{-1} = \begin{bmatrix} \bar{G} & 0 \\ 0 & 0 \end{bmatrix}$$

where  $\bar{G} \in \text{GL}_r(\mathbf{F})$ .

(ii) Let  $\bar{D}^{-1}\bar{N}$  be a left-coprime factorization or  $\bar{N}\bar{D}^{-1}$  be a right-coprime factorization of  $\bar{G}$  over  $\mathbf{A}$ . Then  $\deg_M^{\mathbf{Z}}(G) = \deg \det(\bar{N}) = \deg_M^{\mathbf{Z}}(\bar{G})$  and  $\deg_M^{\mathbf{P}}(G) = \deg \det(\bar{D}) = \deg_M^{\mathbf{P}}(\bar{G})$ , hence  $\deg_M(G) = \deg_M(\bar{G}) = \deg \det(\bar{G})$ .

**Proof.** (i) There exist  $G' \in \mathbf{F}^{p \times r}$  and  $G'' \in \mathbf{F}^{r \times m}$ , both of rank  $r$ , such that  $G = G'G''$ , as a consequence of ([1], Chapter II, §10, Proposition 13). Let  $N'D'^{-1}$  (resp.  $D''^{-1}N''$ ) be a right- (resp. left-) coprime factorization of  $G'$  (resp.  $G''$ ) over  $\mathbf{A}$ . There exist  $U \in \text{GL}_p(\mathbf{A})$ ,  $V \in \text{GL}_m(\mathbf{A})$  such that  $UN' = \text{col}\{\bar{N}', 0\}$  (resp.  $N''V^{-1} = [\bar{N}'' \ 0]$ ), choosing for  $\bar{N}' \in \mathfrak{M}_r(\mathbf{A})$  (resp.  $\bar{N}'' \in \mathfrak{M}_r(\mathbf{A})$ ) a column (resp. row) Hermite form of  $N'$  (resp.  $N''$ ) ([32], Section II.6). Thus  $UGV^{-1} = \text{col}\{\bar{N}', 0\} \Delta [\bar{N}'' \ 0]$  where  $\Delta = (D''D')^{-1} \in \text{GL}_r(\mathbf{F})$ . We get  $UGV^{-1} = \begin{bmatrix} \bar{G} & 0 \\ 0 & 0 \end{bmatrix}$  where  $\bar{G} = \bar{N}' \Delta \bar{N}'' \in \text{GL}_r(\mathbf{F})$ .

(ii) Let  $\bar{D}^{-1}\bar{N}$  be a left-coprime factorization of  $\bar{G}$  over  $\mathbf{A}$ . Then

$$UGV^{-1} = \begin{bmatrix} \bar{D}^{-1}\bar{N} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{D} & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \bar{N} & 0 \\ 0 & 0 \end{bmatrix}$$

and this is a left-coprime factorization of  $UGV^{-1}$  over  $\mathbf{A}$ . Thus  $D^{-1}N$  where

$$D = U \begin{bmatrix} \bar{D} & 0 \\ 0 & I \end{bmatrix}, \quad N = \begin{bmatrix} \bar{N} & 0 \\ 0 & 0 \end{bmatrix} V$$

is a left-coprime factorization of  $G$  over  $\mathbf{A}$ . Hence  $\deg_M^{\mathbf{P}}(G) = \deg \det(D) = \deg \det(\bar{D}) = \deg_M^{\mathbf{P}}(\bar{G})$  and  $\deg_M^{\mathbf{Z}}(G) = \dim_{\mathbf{K}} \{\text{tor}(\text{coker}_{\mathbf{A}}(\bullet N))\} = \dim_{\mathbf{K}} \{\text{coker}_{\mathbf{A}}(\bullet \bar{N})\} = \deg_M^{\mathbf{Z}}(\bar{G})$ , thus  $\deg_M(G) = \deg_M(\bar{G})$ . Since  $\bar{G} = \bar{D}^{-1}\bar{N}$ ,  $\deg \det(\bar{G}) = \deg \det(\bar{N}) - \deg \det(\bar{D}) = \deg_M^{\mathbf{Z}}(\bar{G}) - \deg_M^{\mathbf{P}}(\bar{G}) = \deg_M(\bar{G})$ .

The case where  $G = \bar{N}\bar{D}^{-1}$  is a right-coprime factorization of  $\bar{G}$  over  $\mathbf{A}$  is similar. ■

Statements (i), (ii) of Lemma 1 are extended to the matrix case in the theorem below with  $\deg$  changed to  $\deg_M$ :

**Theorem 11** (i) *The mapping  $\deg_M : \mathbf{F}^{p \times m} \rightarrow \mathbb{Z} \cup \{-\infty\}$  is surjective.*

(ii) *If  $G' \in \mathbf{F}^{p \times r}$ ,  $G'' \in \mathbf{F}^{r \times m}$  are two matrices of rank  $r > 0$ , then  $\deg_M(G')$ ,  $\deg_M(G'')$  are both  $> -\infty$  and  $\deg_M(G'G'') = \deg_M(G') + \deg_M(G'')$ .*

**Proof.** The proof of (i) is identical to that of Lemma 4(i).

(ii) As shown using elementary operations, there exist  $U \in \text{GL}_p(\mathbf{A})$ ,  $V \in \text{GL}_m(\mathbf{A})$  such that  $UG' = \text{col}\{\bar{G}', 0\}$ ,  $G''V^{-1} = \begin{bmatrix} \bar{G}'' & 0 \end{bmatrix}$  where  $\bar{G}', \bar{G}'' \in \text{GL}_r(\mathbf{F})$ , thus

$$U(G'G'')V^{-1} = \begin{bmatrix} \bar{G}'\bar{G}'' & 0 \\ 0 & 0 \end{bmatrix}$$

and by Lemma 10,  $\deg_M(G'G'') = \deg \det(\bar{G}'\bar{G}'') = \deg \det(\bar{G}') + \deg \det(\bar{G}'') = \deg_M(G') + \deg_M(G'')$ . ■

**Remark 12** *Property (iii) in Lemma 1 does not extend to the matrix case with  $\deg$  changed to  $\deg_M$ , as shown by the following example:*

$$\underbrace{\begin{bmatrix} s & 1 \end{bmatrix}}_{\deg_M(G)=0} - \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\deg_M(H)=0} = \underbrace{\begin{bmatrix} s & 0 \end{bmatrix}}_{\deg_M(G-H)=1}$$

### 3.3 Irreducible polynomial matrices

Let us study irreducible polynomial matrices in the sense of ([23], p. 378) in the case where  $\mathbf{A} = \mathbf{K}[s; \boldsymbol{\alpha}, \boldsymbol{\delta}]$ . First note that the restriction of the mapping  $\deg_M$  to  $\mathbf{A}^{p \times m}$  is a surjection onto  $\mathbb{N} \cup \{-\infty\}$ .

**Lemma 13** Let  $H \in \mathbf{A}^{r \times p}$  be a matrix such that  $\text{rk}_{\mathbf{F}}(H) = r > 0$ . The following conditions are equivalent:

- (i)  $\text{deg}_M(H) = 0$ .
- (ii)  $(\bullet H) : \mathbf{A}^{1 \times r} \longrightarrow \mathbf{A}^{1 \times p}$  is a split monomorphism.
- (iii) For any  $y \in (\mathbf{F}^{1 \times r} H) \cap \mathbf{A}^{1 \times p}$ , if  $x \in \mathbf{F}^{1 \times r}$  is such that

$$y = xH \tag{27}$$

then  $x \in \mathbf{A}^{1 \times r}$ .

**Proof.** First note that  $\text{deg}_M(H) = \text{deg}_M^{\mathbf{z}}(H)$  and that the equality  $\text{rk}_{\mathbf{F}}(H) = r$  means that  $(\bullet H)$  is a monomorphism.

(i)  $\Leftrightarrow$  (ii) : If  $\text{deg}_M(H) = 0$ , then the quotient module  $\text{coker}_{\mathbf{A}}(\bullet H)$  of  $\mathbf{A}^{1 \times p}$  is torsion-free, thus free since  $\mathbf{A}$  is a principal ideal domain, therefore the monomorphism  $(\bullet H)$  is split. Conversely, if the monomorphism  $(\bullet H)$  is split,  $\text{coker}_{\mathbf{A}}(\bullet H)$  is torsion-free and  $\text{deg}_M(H) = 0$  by (25).

(ii)  $\Leftrightarrow$  (iii) : (ii) means that  $f := (\bullet H) : \mathbf{A}^{1 \times r} \longrightarrow \mathbf{A}^{1 \times p}$  admits a retraction ([1], Chapter II, §1, Proposition 15), i.e. there exists  $g = (\bullet J) : \mathbf{A}^{1 \times p} \longrightarrow \mathbf{A}^{1 \times r}$  such that  $g \circ f = 1_{\mathbf{A}^{1 \times r}}$ , and since  $g \circ f = (\bullet HJ)$ , this means that  $H$  is right-invertible over  $\mathbf{A}$ , i.e. (iii) holds true. ■

**Definition 14** A polynomial matrix  $H \in \mathbf{A}^{r \times p}$  of rank  $r > 0$  is said to be irreducible if the equivalent conditions of Lemma 13 are satisfied. An irreducible matrix  $K \in \mathbf{A}^{m \times r}$  of rank  $r > 0$  is likewise defined, replacing (27) (where  $x, y$  are rows of  $r$  and  $p$  elements respectively) by  $y' = Kx'$  (where  $x', y'$  are columns of  $m$  and  $r$  elements respectively).

## 4 Structure and valuation at infinity of a transfer matrix

### 4.1 Valuation at infinity of a rational fraction and structure of $\mathbf{F}_{pr}$

As in the case where  $\mathbf{F} = \mathbf{K}(s)$  ([2], p. IV.20), the mapping

$$\nu_{\infty} : \mathbf{F} \longrightarrow \mathbb{Z} \cup \{+\infty\} : g \mapsto -\text{deg}(g) \tag{28}$$

is a discrete valuation on the skew field  $\mathbf{F} = \mathbf{K}(s; \alpha, \delta)$ . The valuation ring of  $\mathbf{F}$ , i.e. the set of all elements  $g \in \mathbf{F}$  such that  $\nu_{\infty}(g) \in \mathbb{N} \cup \{+\infty\}$ , is the ring  $\mathbf{F}_{pr}$  of proper skew rational fractions. An element  $v \in \mathbf{F}_{pr}$  is a unit if and only if  $\nu_{\infty}(v) = 0$  and  $\nu_{\infty}(s^{-1}) = 1$ , thus  $\sigma := s^{-1}$  is a normed uniformizer of  $\nu_{\infty}$  ([3], Chap. VI, §3, Definition 3). It follows that  $\sigma \mathbf{F}_{pr} = \mathbf{F}_{pr} \sigma = (\sigma)$  (where  $(\sigma)$  is the two-sided ideal in  $\mathbf{F}_{pr}$  generated by  $\sigma$ ) is the unique maximal left ideal and the unique maximal right ideal in  $\mathbf{F}_{pr}$ , and that every nonzero element  $g$  of  $\mathbf{F}_{pr}$  can be written in the form

$$g = v\sigma^k = \sigma^k v', \quad k = \nu_{\infty}(g), \quad v, v' \in \mathbf{U}(\mathbf{F}_{pr}) \tag{29}$$

As a result,  $\mathbf{F}_{pr}$  is a local principal ideal domain ([12], section 0.5), ([3], Chap. VI, §3, n°6) and two elements of  $\mathbf{F}_{pr}$  are similar if and only if they are associated. Further, by (29), every nonzero element of  $\mathbf{F}_{pr}$  is invariant and if  $n, n' \in \mathbf{F}_{pr}^\times$ ,

$$n \parallel n' \Leftrightarrow n \mid n' \Leftrightarrow \nu_\infty(n) \leq \nu_\infty(n'). \quad (30)$$

More generally, if  $g, g' \in \mathbf{F}^\times$ , we write

$$g \mid g' \Leftrightarrow \nu_\infty(g) \leq \nu_\infty(g') \quad (31)$$

and notice that if  $\mu_1, \dots, \mu_q \in \mathbb{Z}$ ,

$$\nu_\infty(\gcd\{\sigma^{\mu_1}, \dots, \sigma^{\mu_q}\}) = \min\{\mu_1, \dots, \mu_q\}. \quad (32)$$

Likewise, considering the element  $f$  of  $\mathbf{L} := \mathbf{K}((\sigma; \alpha, \delta))$  defined by (8), let  $\hat{\nu}_\infty(f) = \min\{i \in \mathbb{Z} : f_i \neq 0\} = \min\{i \in \mathbb{Z} : f'_i \neq 0\}$  if  $f \neq 0$  and  $\hat{\nu}_\infty(0) = +\infty$ ; then  $\hat{\nu}_\infty : \mathbf{L} \rightarrow \mathbb{Z} \cup \{+\infty\}$  is a discrete valuation on the skew field  $\mathbf{L}$  and  $\mathbf{S} := \mathbf{K}[[\sigma; \alpha, \delta]]$  is the valuation ring of  $\mathbf{L}$ . The connection between  $\mathbf{F}_{pr}$  and  $\mathbf{S}$  is clarified by the following lemma:

**Lemma 15** *The ring  $\mathbf{S}$  is the completion of  $\mathbf{F}_{pr}$  in the  $(\sigma)$ -adic topology. In particular,  $U(\mathbf{F}_{pr}) \underset{\text{identification}}{=} U(\mathbf{S}) \cap \mathbf{F}$ .*

**Proof.** Let  $f = \sum_{i \geq 0} f_i \sigma^i$  be an element of  $\mathbf{S}$ . Then  $f_\lambda = \sum_{0 \leq i \leq \lambda} f_i \sigma^i$  belongs to  $\mathbf{F}_{pr}$  for any  $\lambda \geq 0$  and  $(f_\lambda) \rightarrow f$  as  $\lambda \rightarrow +\infty$ , thus  $\mathbf{F}_{pr}$  is dense in the complete ring  $\mathbf{S}$  in the  $(\sigma)$ -adic topology which is the topology determined by  $\hat{\nu}_\infty$  ([3], Chap. VI, §5, n°1), ([18], §I.3). Let  $v \in \mathbf{F}$ ; then  $v \in U(\mathbf{F}_{pr})$  if and only if  $\nu_\infty(v) = 0$  which is equivalent to  $\hat{\nu}_\infty(v) = 0$ , i.e.  $v \in U(\mathbf{S})$ . ■

## 4.2 Equivalence of matrices over the ring of proper transfer functions

Let  $\mathbf{R}$  be a principal ideal domain. The Jacobson-Teichmüller form  $S$  of  $N \in \mathbf{R}^{p \times m}$  (see (9)) can be obtained using a finite sequence of elementary and secondary row and column operations ([12], Section 8.1). Recall that elementary row operations are defined as follows where  $A \in \mathbf{R}^{p \times m}$  is the matrix obtained from  $N$  at any step:

1. Replace the row  $l_i$  of  $A$  by  $l_i + \lambda l_j$  ( $i \neq j$ ,  $\lambda \in \mathbf{R}$ ); this is equivalent to left multiplying  $A$  by the matrix  $B_{ij}(\lambda) = I_p + \lambda E_{ij} \in \text{GL}_p(\mathbf{R})$  where  $E_{ij} \in \mathfrak{M}_p(\mathbf{R})$  is the matrix whose all elements are zero except that with indices  $i, j$  which is equal to 1;
2. Replace the row  $l_i$  of  $A$  by  $vl_i$  where  $v \in U(\mathbf{R})$ ; this is equivalent to left multiplying  $A$  by the matrix  $\Delta_i(v) = \text{diag}\{1, \dots, 1, v, 1, \dots, 1\} \in \text{GL}_p(\mathbf{R})$  where  $v$  is at the  $i$ -th place;
3. Exchange  $l_i$  and  $l_j$  ( $i \neq j$ ); this is equivalent to left multiplying  $A$  by a permutation matrix  $P_{ij} \in \text{GL}_p(\mathbf{R})$ .



The matrices  $B_{ij}(\lambda)$ ,  $\Delta_i(v)$ ,  $P_{ij}$  are called elementary matrices. Elementary column operations are likewise defined (with  $p$  changed to  $m$  and left multiplications to right ones). The following result generalizes Corollary 1 of [42] where  $\mathbf{F} = \mathbf{K}(s)$  (see also ([40], Corollary 3.10)):

**Lemma 16** *Every nonzero matrix  $N \in \mathbf{F}_{pr}^{p \times m}$  can be reduced to its Jacobson-Teichmüller normal form using a finite number of elementary row and column operations. Thus the group  $\mathrm{GL}_n(\mathbf{F}_{pr})$  is generated by elementary matrices  $B_{ij}(\lambda), \Delta_i(v), P_{ij} \in \mathrm{GL}_n(\mathbf{F}_{pr})$  ( $\lambda \in \mathbf{F}_{pr}, v \in \mathrm{U}(\mathbf{F}_{pr})$ ).*

**Proof.** According to ([25], Theorem 5.1), to prove the first statement for any  $N \in \mathbf{F}_{pr}^{p \times m}$ , it is sufficient to prove it for  $1 \times 2$ ,  $2 \times 1$  and  $2 \times 2$  matrices. Let  $N = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \in \mathbf{F}_{pr}^{1 \times 2}$ . There exist units  $v_i \in \mathrm{U}(\mathbf{F}_{pr})$  and integers  $\eta_i \in \mathbb{N}$  such that  $a_i = v_i \sigma^{\eta_i}$  ( $i = 1, 2$ ). Using a permutation of columns, we are led to the case where  $\eta_1 \leq \eta_2$ . Then  $N \sim_{\mathbf{F}_{pr}} \begin{bmatrix} \sigma^{\eta_1} & 0 \end{bmatrix}$  where the equivalence is obtained using two additional elementary column operations. Likewise, any  $2 \times 1$  matrix is reduced to its Jacobson-Teichmüller normal form using three elementary row operations and any  $2 \times 2$  matrix is reduced to its Jacobson-Teichmüller normal form using a finite number of elementary row and column operations.

As a result, if  $N \in \mathrm{GL}_n(\mathbf{F}_{pr})$ , there exist matrices  $U, V \in \mathrm{GL}_n(\mathbf{F}_{pr})$ , each of them being a finite product of elementary matrices, such that  $U^{-1}NV = I_n$ , thus  $N = UV^{-1}$ , which proves that the group  $\mathrm{GL}_n(\mathbf{F}_{pr})$  is generated by elementary matrices. ■

Note that the statement of Lemma 16 is still true if  $\mathbf{F}_{pr}$  is replaced by any discrete valuation ring, for example  $\mathbf{S}$  (and as is well known, it is true if  $\mathbf{F}_{pr}$  is replaced by any Euclidean domain, for example  $\mathbf{A}$ ).

**Corollary 17** *Let  $k \in \mathbb{Z}$  and  $U \in \mathrm{GL}_n(\mathbf{F}_{pr})$ . There exist matrices  $U_k, U'_k \in \mathrm{GL}_n(\mathbf{F}_{pr})$  such that  $U\sigma^k = \sigma^k U_k$ ,  $\sigma^k U = U'_k \sigma^k$ .*

**Proof.** We prove this statement for  $k = 1$  since the result for  $k$  arbitrary then follows by induction. By Lemma 16, it is sufficient to prove the stated property for elementary matrices. The case where  $U = P_{ij}$  is trivial. If  $U = \Delta_i(v)$ ,  $v \in \mathrm{U}(\mathbf{F}_{pr})$ , then there exists  $v_1 \in \mathrm{U}(\mathbf{F}_{pr})$  such that  $U\sigma = \sigma \Delta_i(v_1)$  by (29). If  $U = B_{ij}(\lambda) = I_n + \lambda E_{ij}$ , then  $U\sigma = \sigma B_{ij}(\lambda^\alpha + \lambda^\delta \sigma)$  by (7), where  $B_{ij}(\lambda^\alpha + \lambda^\delta \sigma) \in \mathrm{GL}_n(\mathbf{F}_{pr})$ . So, in the three cases,  $U\sigma = \sigma U_1$  for some  $U_1 \in \mathrm{GL}_n(\mathbf{F}_{pr})$ , and this is still true for any  $U \in \mathrm{GL}_n(\mathbf{F}_{pr})$ . Likewise,  $\sigma U = U'_1 \sigma$  for some  $U'_1 \in \mathrm{GL}_n(\mathbf{F}_{pr})$ . ■

As a consequence of the structure of the ring  $\mathbf{F}_{pr}$  and of Corollary 17, we obtain the theorem below which improves ([7], Proposition 2) and generalizes to the case where  $\mathbf{F} = \mathbf{K}(s; \boldsymbol{\alpha}, \boldsymbol{\delta})$  Theorem 2 of [42] (see also ([40], Theorem 3.13)) where  $\mathbf{F} = \mathbf{K}(s)$ .

**Theorem 18** *Let  $G \in \mathbf{F}^{p \times m}$  be a transfer matrix of rank  $r$ . We have*

$$G \sim_{\mathbf{F}_{pr}} \Sigma := \mathrm{diag} \{ \sigma^{\mu_1}, \dots, \sigma^{\mu_r}, 0, \dots, 0 \}, \quad \mu_i = \nu_\infty(\sigma^{\mu_i}) \in \mathbb{Z}, \quad \mu_i \leq \mu_{i+1}$$

i.e. there exist matrices  $U_k \in \mathrm{GL}_p(\mathbf{F}_{pr})$ ,  $V \in \mathrm{GL}_m(\mathbf{F}_{pr})$  such that  $\Sigma = U_k^{-1}GV$ . Equivalence  $\sim_{\mathbf{F}_{pr}}$  is obtained using a finite number of elementary row and column operations.

**Proof.** There exists an integer  $k \geq 0$  such that  $N = \sigma^k G \in \mathbf{F}_{pr}^{p \times m}$  and there exist matrices  $U \in \mathrm{GL}_p(\mathbf{F}_{pr})$ ,  $V \in \mathrm{GL}_m(\mathbf{F}_{pr})$  such that  $U^{-1}NV$  is a Jacobson-Teichmüller form of  $N$  which, by (29), can be assumed to be

$$U^{-1}NV = \mathrm{diag}\{\sigma^{\gamma_1}, \dots, \sigma^{\gamma_r}, 0, \dots, 0\}, \quad \gamma_i = \nu_\infty(\sigma^{\gamma_i}) \in \mathbb{N}, \quad \gamma_i \leq \gamma_{i+1}$$

By Corollary 17,  $U_k^{-1}\sigma^{-k}NV = (\sigma^k U_k)^{-1}NV = (U\sigma^k)^{-1}NV = \sigma^{-k}U^{-1}NV$ , hence  $U_k^{-1}GV = \mathrm{diag}\{\sigma^{\gamma_1-k}, \dots, \sigma^{\gamma_r-k}, 0, \dots, 0\}$  which proves the statement with  $\mu_i = \gamma_i - k$ . ■

The key difference with ([7], Proposition 2) is that the equivalence  $\sim_{\mathbf{S}}$  is replaced by  $\sim_{\mathbf{F}_{pr}}$ .

### 4.3 Extension of the valuation at infinity to transfer matrices

Let  $G \in \mathbf{F}^{p \times m}$ . A nonzero minor of  $G$  is an element of the abelianization  $\mathbf{F}^{\times ab}$  of the multiplicative group  $\mathbf{F}^\times$  (see Subsection 2.3). The following result improves the one given in ([8], lemma 1061):

**Lemma 19** *We have  $\mathrm{D}(\mathbf{F}^\times) \triangleleft \mathrm{U}(\mathbf{F}_{pr})$  (i.e. the derived group  $\mathrm{D}(\mathbf{F}^\times)$  is a normal subgroup of  $\mathrm{U}(\mathbf{F}_{pr})$ ), thus the canonical epimorphism of groups  $\varphi : \mathbf{F}^\times \rightarrow \mathbf{F}^{\times ab}$  is given by  $v\sigma^i \mapsto \bar{v}\sigma^i$  where  $i \in \mathbb{Z}$  and  $\bar{v}$  is the canonical image of  $v \in \mathrm{U}(\mathbf{F}_{pr})$  in the quotient group  $\mathrm{U}(\mathbf{F}_{pr})/\mathrm{D}(\mathbf{F}^\times)$ .*

**Proof.** The derived group  $\mathrm{D}(\mathbf{F}^\times)$  consists of all commutators  $(x_1, x_2) := (x_2 x_1)^{-1} (x_1 x_2)$  of elements  $x_1, x_2 \in \mathbf{F}^\times$ . If  $x_1, x_2 \in \mathbf{F}^\times$ , then  $\nu_\infty(x_1 x_2) = \nu_\infty(x_1) + \nu_\infty(x_2) = \nu_\infty(x_2 x_1)$ , thus  $\nu_\infty((x_1, x_2)) = 0$  and  $(x_1, x_2) \in \mathrm{U}(\mathbf{F}_{pr})$ , which proves that  $\mathrm{D}(\mathbf{F}^\times)$  is a subgroup of  $\mathrm{U}(\mathbf{F}_{pr})$ . In addition, if  $v \in \mathrm{U}(\mathbf{F}_{pr})$  and  $x_1, x_2 \in \mathbf{F}^\times$ , we have  $v(x_1, x_2) = (vx_1 v^{-1}, vx_2 v^{-1})v$ , hence  $\mathrm{D}(\mathbf{F}^\times) \triangleleft \mathrm{U}(\mathbf{F}_{pr})$ . ■

For any  $i \in \mathbb{Z}$  and any  $\bar{v} \in \mathrm{U}(\mathbf{F}_{pr})/\mathrm{D}(\mathbf{F}^\times)$ , we put  $\nu_\infty(\bar{v}\sigma^i) := i$  and call this integer the valuation at infinity of  $\bar{v}\sigma^i$ . This applies when  $\bar{v}\sigma^i$  is a nonzero minor  $\mathbf{m}$  of  $G$  and, if  $\mathbf{m} = 0$ ,  $\nu_\infty(\mathbf{m}) := +\infty$ . Let  $\bar{v}_i \sigma^{\mu_i} \in \mathbf{F}^{\times ab}$  ( $i = 1, 2$ ) and write  $\bar{v}_1 \sigma^{\mu_1} \mid \bar{v}_2 \sigma^{\mu_2}$  if and only if  $\mu_1 \leq \mu_2$  (compare with (31)). This combined with (32) yields the following lemma :

**Lemma 20** *Assume that  $r = \mathrm{rk}_{\mathbf{F}}(G) > 0$  and let  $q$  be an integer such that  $1 \leq q \leq r$ . Then with the notation in Theorem 18*

$$\sum_{i=1}^q \mu_i = \min \{ \nu_\infty(\mathbf{m}) : \mathbf{m} \in \{\text{minors of order } q \text{ of } G\} \} \quad (33)$$

**Proof.** Let  $N = \sigma^k G \in \mathbf{F}_{pr}^{p \times m}$  be as in the proof of Theorem 18 and  $\text{diag}\{e_1, \dots, e_r, 0, \dots, 0\}$  be its Jacobson-Teichmüller form where  $e_i = \sigma^{\mu_i}$ . One has the relation

$$\prod_{1 \leq i \leq q} e_i = \gcd\{\mathbf{m} : \mathbf{m} \in \{\text{minors of order } q \text{ of } N\}\} \quad (1 \leq q \leq r) \quad (34)$$

classic in the commutative case and still valid here (although  $\mathbf{A}$  is a noncommutative) by (30) and the proof of ([22], Theorem 3.9). The equality (33) follows immediately. ■

Assuming  $\text{rk}_{\mathbf{K}}(G) = r$ , we put as in ([23], Section 6.5.3, (30a)):

$$\nu_{\infty}(G) = \begin{cases} \min\{\nu_{\infty}(\mathbf{m}) : \mathbf{m} \in \{\text{minors of order } r \text{ of } G\}\} & \text{if } r > 0 \\ +\infty & \text{if } r = 0 \end{cases} \quad (35)$$

so that, by Lemma 20,  $\nu_{\infty}(G) = \sum_{i=1}^r \mu_i$  if  $r > 0$ . Note that for a full rank square matrix  $G$ ,  $\nu_{\infty}(G) = \nu_{\infty}(\det(G))$ . We modify ([7], Definition 2) as follows, using the terminology in ([23], Section 6.5.3):

**Definition 21** Let  $G \in \mathbf{F}^{p \times m}$ . The matrix  $\Sigma$  in Theorem 18 is the McMillan normal form of  $G$  at infinity and  $\{\mu_i : i = 1, \dots, r\}$  is the set of structural indices of  $G$  at infinity. If there exists  $i \in \{1, \dots, r\}$  such that  $\mu_i > 0$  (resp.  $\mu_i < 0$ ), then  $G$  is said to have a zero (resp. a pole) at infinity and  $\omega_{\infty}^z(G) := \max_{i:\mu_i > 0} \{\mu_i\}$  (resp.  $\omega_{\infty}^p(G) := \max_{i:\mu_i < 0} \{-\mu_i\}$ ) is the order of this zero (resp. this pole) at infinity; furthermore,  $\mathbf{d}_{\infty}^z(G) := \sum_{i:\mu_i > 0} \mu_i$  (resp.  $\mathbf{d}_{\infty}^p(G) := \sum_{i:\mu_i < 0} (-\mu_i)$ ) is the degree of this zero (resp. this pole) at infinity. If  $G$  has no zero (resp. no pole) at infinity, then  $\omega_{\infty}^z(G)$  and  $\mathbf{d}_{\infty}^z(G)$  (resp.  $\omega_{\infty}^p(G)$  and  $\mathbf{d}_{\infty}^p(G)$ ) are defined to be zero.

As a consequence, if  $G \neq 0$

$$\nu_{\infty}(G) = \mathbf{d}_{\infty}^z(G) - \mathbf{d}_{\infty}^p(G) \quad (36)$$

The following definition is adapted from ([41], Definition 3):

**Definition 22** A transfer matrix  $G \in \mathbf{F}_{pr}^{p \times m}$  is called left (resp. right) biproper if it has a left (resp. right) inverse  $G^{-L}$  (resp.  $G^{-R}$ ) which is proper. It is called biproper if it is left and right biproper.

The properties of the valuation  $\nu_{\infty}$  extended as above to  $\mathbf{F}^{p \times m}$  are detailed in the theorem below :

**Theorem 23** (i) A transfer matrix  $G \in \mathbf{F}^{p \times m}$  is proper if and only if  $\mathbf{d}_{\infty}^p(G) = 0$  and this implies  $\nu_{\infty}(G) \geq 0$ .  
(ii) Let  $G' \in \mathbf{F}^{p \times m}$  and  $G'' \in \mathbf{F}^{m \times q}$ . Then  $\nu_{\infty}(G'G'') \leq \nu_{\infty}(G') + \nu_{\infty}(G'')$  with equality if  $G'$  and  $G''$  are both of rank  $r = m$ .  
(iii) Let  $G', G'' \in \mathbf{F}^{p \times m}$ . Then  $\nu_{\infty}(G' + G'') \geq \min\{\nu_{\infty}(G'), \nu_{\infty}(G'')\}$  with equality if  $\nu_{\infty}(G') \neq \nu_{\infty}(G'')$ .

(iv) Assuming that  $G$  has a left inverse  $G^{-L}$  (resp. a right inverse  $G^{-R}$ ), then  $\nu_\infty(G^{-L}) = -\nu_\infty(G)$  (resp.  $\nu_\infty(G^{-R}) = -\nu_\infty(G)$ ),  $G^{-L}$  (resp.  $G^{-R}$ ) is proper if and only if  $\mathbf{d}_\infty^{\mathbf{z}}(G) = 0$ , and  $G$  is left (resp. right) biproper if and only if  $\mathbf{d}_\infty^{\mathbf{z}}(G) = 0$  and  $\mathbf{d}_\infty^{\mathbf{p}}(G) = 0$ , which implies  $\nu_\infty(G) = 0$ .

**Proof.** (i) is clear.

(ii) Thus, assuming  $\text{rk}_{\mathbf{K}}(G'G'') = r$  and taking  $k = r$

$$\begin{aligned} \nu_\infty(G'G'') &= \min_{i,j} \left\{ \nu_\infty \left( (G'G'')_{ij}^{(r)} \right) \right\} = \min_{i,j} \left\{ \nu_\infty \left( (G'^{(r)}G''^{(r)})_{ij} \right) \right\} \\ &\leq \min_{i,j} \left\{ \nu_\infty(G')_{ij}^{(r)} \right\} + \min_{i,j} \left\{ \nu_\infty(G'')_{ij}^{(r)} \right\} \end{aligned}$$

with equality if  $\text{rk}_{\mathbf{F}}(G') = \text{rk}_{\mathbf{F}}(G'') = m$  since in that case  $r = m$  and there exists among all products of minors  $(G'^{(m)}G''^{(m)})_{ij}$  at least one term which is the product of factors, each of them with minimal valuation at infinity.

(iii) Let  $G \in \mathbf{F}^{p \times m}$ ,  $G \neq 0$ . There exists a nonzero matrix  $G_0 \in \mathbf{K}^{p \times m}$  such that  $G = G_0 \sigma^{\nu_\infty(G)} + O_{p,m}(\sigma^{(\nu_\infty(G)+1)})$  where  $O_{p,m}(\sigma^{(\nu_\infty(G)+1)}) \in \mathbf{F}^{p \times m}$  is a matrix, all entries of which belong to the ideal generated by  $\sigma^{(\nu_\infty(G)+1)}$  in  $\mathbf{F}_{pr}$ . With this notation, if  $G', G'' \neq 0$ , there exist nonzero matrices  $G'_0, G''_0 \in \mathbf{K}^{p \times m}$  such that  $G' = G'_0 \sigma^{\nu_\infty(G')} + O_{p,m}(\sigma^{(\nu_\infty(G')+1)})$ ,  $G'' = G''_0 \sigma^{\nu_\infty(G'')} + O_{p,m}(\sigma^{(\nu_\infty(G'')+1)})$ . If  $\nu_\infty(G') < \nu_\infty(G'')$ , then  $G'' \in O_{p,m}(\sigma^{(\nu_\infty(G')+1)})$  and  $\nu_\infty(G' + G'') = \nu_\infty(G')$ . If  $\nu_\infty(G') = \nu_\infty(G'') = \eta$ , we have  $\nu_\infty(G' + G'') = \eta$  if  $G'_0 + G''_0 \neq 0$  and  $\nu_\infty(G' + G'') > \eta$  if  $G'_0 + G''_0 = 0$ .

(iv) If  $G$  has a right inverse  $G^{-R}$ , then by Theorem 18 there exist integers  $\mu_i \in \mathbb{Z}$  ( $i = 1, \dots, p$ ) such that  $G \sim_{\mathbf{F}_{pr}} [\text{diag}\{\sigma^{\mu_i}\}_{1 \leq i \leq p} \ 0]$  and  $G^{-R} \sim_{\mathbf{F}_{pr}} \text{col}\{\text{diag}\{\sigma^{-\mu_i}\}_{1 \leq i \leq p}, 0\}$ , thus  $\nu_\infty(G^{-R}) = -\nu_\infty(G)$ . In addition,  $G^{-R}$  is proper if and only if  $\mu_i \leq 0$  ( $1 \leq i \leq p$ ), i.e.  $\mathbf{d}_\infty^{\mathbf{z}}(G) = 0$ ; and  $G$  is right biproper if and only if  $\mu_i = 0$  ( $1 \leq i \leq p$ ), i.e.  $\mathbf{d}_\infty^{\mathbf{z}}(G) = 0$  and  $\mathbf{d}_\infty^{\mathbf{p}}(G) = 0$ . The rationale is similar when  $G$  has a left inverse  $G^{-L}$ . ■

**Remark 24** 1) In the case where  $\mathbf{F} = \mathbf{K}(s)$ , statements (i), (ii) in Theorem 23 are classic ([36], Theorem 4), ([44], Section I.5.6), ([23], Section 6.5.4).

2) In ([41], Definition 6) and ([40], Section 1.2, Definition 1.1), the “degree” of a nonzero matrix  $G \in \mathbf{K}[s]^{p \times m}$  is the mapping  $\text{deg}_V$  such that  $\text{deg}_V(G) = -\nu_\infty(G)$ . Consider the matrices

$$H_1 = \begin{bmatrix} 1 & 0 & s \\ 0 & 1 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

Then  $\text{deg}_V(H_1) = 1$ ,  $\text{deg}_V(H_2) = 0$  whereas  $\text{deg}_M(H_1) = 0$  and  $\text{deg}_F(H_2) = 1$ , hence  $\text{deg}_V$  coincides neither with  $\text{deg}_F$  nor with  $\text{deg}_M$ .

3) Some errors can be found in the literature: the transfer matrix  $G = \text{diag}\{s, 1/s\}$  is not biproper although  $\nu_\infty(G) = 0$ ; in addition, with  $G' =$

$\text{diag}\{1/s, 0\}$ ,  $G'' = \text{diag}\{1, 1/s\}$ , we have  $\nu_\infty(G') = \nu_\infty(G'') = 1$  and  $\nu_\infty(G'G'') = 1 < \nu_\infty(G') + \nu_\infty(G'')$ ; last,  $G' = N_r D_r^{-1}$  where  $N_r = \text{diag}\{1, 0\}$ ,  $D_r = \text{diag}\{s, s^2\}$ , so  $\deg_V(N_r) = 0$ ,  $\deg_V(D_r) = 3$  and  $\deg_V(D_r) - \deg_V(N_r) = 3 > \nu_\infty(G')$ . These examples contradict respectively Propositions 5, 6 and 7 of [41] which are identical respectively to Propositions 3.81, 3.80, 3.82 of [40]. The two last examples also show that the inequality in Theorem 23(ii) is not an equality in general if the rank condition specified in that statement is not satisfied.

## 5 Defect of a transfer matrix

### 5.1 Forney's predictable property

Recall that a polynomial matrix  $H$  is said to be *row reduced* (or *row proper*) if the coefficient matrix of the highest degree term in each row of  $H$  is full row rank. This definition, classic for a matrix over  $\mathbf{K}[s]$  ([23], p. 384), ([48], Definition 2.5.6), is still valid for a matrix over  $\mathbf{A} = \mathbf{K}[s; \alpha, \delta]$ .

**Lemma 25** *Let  $H = \text{col}\{h_i\}_{1 \leq i \leq r} \in \mathbf{A}^{r \times p}$  be a matrix such that  $\text{rk}_{\mathbf{F}}(H) = r > 0$ .*

(i) *For any  $x \in \mathbf{A}^{1 \times r}$ , let  $y = xH$  as in (27). Then  $H$  is row reduced if and only if Forney's "predictable property" holds, i.e.*

$$\deg_{\mathbf{F}}(y) = \max_{i: x_i \neq 0} \{\deg(x_i) + \eta_i\}, \quad \eta_i := \deg_{\mathbf{F}}(h_i)$$

(ii) *If  $H$  is row reduced, then  $\mathbf{d}_{\infty}^{\mathbf{P}}(H) = \sum_{1 \leq i \leq r} \eta_i$ .*

**Proof.** Statement (i) is a part of Forney's "Main Theorem" ([17], p. 495); its proof is identical to that of ([23], Theorem 6.3-13).

Statement (ii) follows from the equivalence  $H \sim_{\mathbf{F}_{pr}} \text{diag}\{s^{\eta_i}\}_{1 \leq i \leq r}$  obtained using elementary column operations (with elementary matrices whose entries belong to  $\mathbf{F}_{pr}$ ). ■

### 5.2 Definition and elementary properties of the defect

We let  $\mathbf{F} = \mathbf{K}(s; \alpha, \delta)$  and  $\mathbf{A} = \mathbf{K}[s; \alpha, \delta]$ .

**Definition 26** *The defect of  $G \in \mathbf{F}^{p \times m}$  is  $\text{def}(G) := -(\deg_M(G) + \nu_\infty(G))$ .*

By (26), (36),  $\text{def}(G) = \deg_M^{\mathbf{P}}(G) + \mathbf{d}_{\infty}^{\mathbf{P}}(G) - (\deg_M^{\mathbf{Z}}(G) + \mathbf{d}_{\infty}^{\mathbf{Z}}(G))$ , which shows that, roughly speaking,  $\text{def}(G)$  is the difference between the total number of poles and the total number of zeros of  $G$ , as in the LTI case. The following result generalizes the "product formula" (see [17], Appendix, p. 518):

**Lemma 27** *If  $G \in \text{GL}_r(\mathbf{F})$  ( $r > 0$ ), then  $\text{def}(G) = 0$ .*

**Proof.** The equality  $\text{def}(G) = 0$  is clear when  $r = 1$  and the result follows when  $r$  is any positive integer by applying this equality to  $g = \det(G)$ . ■

The following result is classic when  $\mathbf{F} = \mathbf{K}(s)$  ([23], p. 461); it is extended here to the case where  $\mathbf{F} = \mathbf{K}(s; \alpha, \delta)$ :

**Lemma 28** *Let  $G \in \mathbf{F}^{p \times m}$  be a matrix of rank  $r > 0$ . There exist matrices  $G' \in \mathbf{F}^{p \times r}$ ,  $G'' \in \mathbf{F}^{r \times m}$ , both of them of rank  $r$ , such that  $G = G'G''$ , and we have*

$$\text{def}(G) = \text{def}(G') + \text{def}(G'') \quad (37)$$

**Proof.** There exist matrices  $U \in \text{GL}_p(\mathbf{F})$ ,  $V \in \text{GL}_m(\mathbf{F})$  such that ([1], Chapter II, §10, Proposition 13)

$$U^{-1}GV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow G = \underbrace{U \text{col}\{I_r, 0\}}_{G'} \underbrace{\begin{bmatrix} I_r & 0 \end{bmatrix} V^{-1}}_{G''} \quad (38)$$

Thus  $\text{deg}_M(G) = \text{deg}_M(G') + \text{deg}_M(G'')$  by Theorem 11(ii), and (37) follows by Theorem 23(ii) and Definition 26. ■

### 5.3 Minimal polynomial basis of the left- or right-kernel of a rational matrix

This notion is classic and due to Forney [17] for a matrix  $G \in \mathbf{F}^{p \times m}$  when  $\mathbf{F} = \mathbf{K}(s)$  and its properties are well explained in ([23], Section 6.5.4) in that case. We consider below the case where  $\mathbf{F} = \mathbf{K}(s; \alpha, \delta)$ .

The left  $\mathbf{F}$ -vector space  $\ker_{\mathbf{F}}(\bullet G)$  is the space of those rows  $x \in \mathbf{F}^{1 \times p}$  which are such that  $xG = 0$ ;  $\ker_{\mathbf{F}}(\bullet G)$  is of dimension  $p - r$  where  $r = \text{rk}_{\mathbf{F}}(G)$ . Let  $x = [x_1 \ \cdots \ x_p] \in \mathbf{F}^{1 \times p}$ ; since  $\mathbf{A}$  is an Ore domain, the  $x_i$  ( $i = 1, \dots, p$ ) have a left common denominator  $d$ , thus  $x = d^{-1}x'$  where  $x' = \mathbf{A}^{1 \times p}$  and  $x'G = 0$ . Therefore,  $\ker_{\mathbf{F}}(\bullet G)$  admits a basis consisting of  $p - r$  elements of  $\mathbf{A}^{1 \times p}$ , and called a polynomial basis of the left-kernel of  $G$ . We write  $\ker_{\mathbf{F}}(\bullet G) \cap \mathbf{A}^{1 \times p} = \ker_{\mathbf{A}}(\bullet G)$ . A polynomial basis of the right-kernel of  $G$ , i.e. of  $\ker_{\mathbf{A}}(G \bullet) = \{y \in \mathbf{A}^m : Gy = 0\}$ , is likewise defined.

For the reader's convenience, let us recall the construction of a *minimal* polynomial basis: Among all nonzero  $x \in \ker_{\mathbf{A}}(\bullet G)$ , let  $h_1$  be one of minimal  $F$ -degree  $\eta_1$ . Then, among all  $x \in \ker_{\mathbf{A}}(\bullet G)$  which are  $\mathbf{A}$ -linearly independent of  $h_1$ , let  $h_2$  be one of minimal  $F$ -degree  $\eta_2$  ( $\eta_2 \geq \eta_1$ ). Continuing this way we obtain  $p - r$  rows  $h_i$  and a matrix  $H = \text{col}\{h_i\}_{1 \leq i \leq p-r} \in \mathbf{A}^{(p-r) \times p}$  of rank  $p - r$  over  $\mathbf{F}$ . These rows  $h_1, h_2, \dots, h_{p-r}$  have  $F$ -degrees  $\eta_1 \leq \eta_2 \leq \dots \leq \eta_{p-r}$  and constitute a basis  $\mathfrak{B}$  of  $\ker_{\mathbf{A}}(\bullet G)$ . This polynomial basis is called minimal since the  $F$ -degrees  $\eta_i$  ( $i = 1, \dots, p - r$ ) are minimal. As shown in ([23], p. 456), whose rationale is still valid, the set  $\{\eta_1, \dots, \eta_{p-r}\}$  is (contrary to  $H$  and to the minimal basis  $\mathfrak{B}$ ) uniquely determined by  $G$ , and we recall the following definition:

**Definition 29** (i)  $\{\eta_1, \dots, \eta_{p-r}\}$  is the set of left minimal indices of  $G$ . The set of right minimal indices of  $G$  is likewise defined (with rows replaced by columns)

and  $\ker_{\mathbf{A}}(\bullet G)$  by  $\ker_{\mathbf{A}}(G\bullet)$ .

(ii) Given a polynomial basis  $\mathfrak{B}' = \{h'_1, \dots, h'_{p-r}\}$  of  $\ker_{\mathbf{A}}(\bullet G)$ , the integer  $\omega' = \sum_{i=1}^{p-r} \deg_F(h'_i)$  is called the order of  $\mathfrak{B}'$  (or of the matrix  $H' = \text{col}\{h'_i\}_{1 \leq i \leq p-r}$ ). The order of a polynomial basis of  $\ker_{\mathbf{A}}(G\bullet)$  is likewise defined.

Thus, a polynomial basis is minimal if and only if it has minimal order. The following result generalizes ([23], Theorem 6.5-10).

**Lemma 30** *Let  $G \in \mathbf{F}^{p \times m}$  be a matrix of rank  $r > 0$  and  $H \in \mathbf{A}^{(p-r) \times p}$  (resp.  $H' \in \mathbf{A}^{m \times (m-r)}$ ) be a matrix, the rows (resp. the columns) of which form a basis  $\mathfrak{B}$  (resp.  $\mathfrak{B}'$ ) of  $\ker_{\mathbf{A}}(\bullet G)$  (resp.  $\ker_{\mathbf{A}}(G\bullet)$ ). The following conditions are equivalent:*

- (i) *The polynomial basis  $\mathfrak{B}$  (resp.  $\mathfrak{B}'$ ) is minimal.*
- (ii) *The matrix  $H$  (resp.  $H'$ ) is row- (resp. column-) reduced and irreducible.*
- (iii) *The basis  $\mathfrak{B}$  (resp.  $\mathfrak{B}'$ ) has minimal order.*

**Proof.** The rationale is made for  $H$  and applies to  $H'$  by symmetry. It is identical to the proof of ([23], Theorem 6.5-10) (using Lemmas 13 and 25(i)), with one exception: that (i) implies that  $H$  is irreducible, which is a part of (i) $\Rightarrow$ (ii).

So, let us prove that if  $H$  is not irreducible, then  $\mathfrak{B} = \{h_1, \dots, h_{p-r}\}$  is not minimal. Assume that the monomorphism  $(\bullet H) : \mathbf{A}^{1 \times (p-r)} \rightarrow \mathbf{A}^{1 \times p}$  is not split. Then the module  $M = \text{coker}_{\mathbf{A}}(\bullet H)$  contains a torsion element  $\bar{x} \neq 0$  and  $\bar{x}$  is the canonical image in the quotient  $M = \mathbf{A}^{1 \times p} / \mathbf{A}^{1 \times (p-r)} H$  of an element  $x \neq 0$  of  $\mathbf{A}^{1 \times p}$ . There exists a nonunit  $\alpha \in \mathbf{A}$  such that  $\alpha \bar{x} = 0$ , thus  $\alpha x \in \text{im}_{\mathbf{A}}(\bullet H)$  and there exist elements  $\lambda_i \in \mathbf{A}$  ( $i = 1, \dots, p-r$ ) such that  $\alpha x = \sum_{i=1}^{p-r} \lambda_i h_i$ . Since  $\alpha x G = 0$ , we have  $x G = 0$  and  $x \in \ker_{\mathbf{A}}(\bullet G)$ . There exists at least one index  $i \in \{1, \dots, p-r\}$  such that  $\lambda_i \neq 0$ . Among all  $h_i$  for which  $\lambda_i \neq 0$ , let  $h_j$  be a row with highest  $F$ -degree  $\eta_j$ . We get  $\deg_F(x) = \eta_j - \deg_F(\alpha) < \eta_j$  by lemma 4(v), thus, replacing  $h_j$  by  $x$  in the basis  $\mathfrak{B}$ , we obtain a new basis  $\mathfrak{B}$  of  $\ker_{\mathbf{A}}(\bullet G)$ , and a new matrix  $\bar{H}$ , the  $j$ -th row of which has  $F$ -degree  $\bar{\eta}_j < \eta_j$ , so  $\mathfrak{B}$  is not minimal. ■

The following lemma was obtained by Verghese ([44], section I.5.8) from a result due to Forney ([17], Theorem 3) when  $\mathbf{F} = \mathbf{K}(s)$ . We show that it is still valid when  $\mathbf{F} = \mathbf{K}(s; \alpha, \delta)$ :

**Lemma 31** *Let  $G \in \mathbf{F}^{p \times r}$  be a matrix of rank  $r > 0$  and  $H = \text{col}\{h_i\}_{1 \leq i \leq p-r} \in \mathbf{F}^{(p-r) \times p}$  where  $\{h_i : i = 1, \dots, p-r\}$  is a basis of  $\ker_{\mathbf{F}}(\bullet G)$ . Then  $\text{def}(H) = \text{def}(G)$ .*

**Proof.** After a permutation of the rows of  $G$  if necessary, one can assume that the  $r$  first rows of  $G$  are linearly independent over  $\mathbf{F}$ . Then  $G = \text{col}\{R, W\}$  where  $R \in \text{GL}_r(\mathbf{F})$  and  $W \in \mathbf{F}^{(p-r) \times r}$ . As a consequence,  $GR^{-1} = \text{col}\{I_r, Y\}$  where  $Y = WR^{-1} \in \mathbf{F}^{(p-r) \times r}$ . Therefore,  $HGR^{-1} = 0$  where  $H = \begin{bmatrix} Y & -I_{p-r} \end{bmatrix}$  and, since  $\text{rk}_{\mathbf{F}}(H) = p-r$ , the  $p-r$  rows  $h_i$  of  $H$  form

a basis of  $\ker_{\mathbf{F}}(\bullet G)$ . We have  $\text{def}(H) = \text{deg}_M^{\mathbf{P}}(Y) = \text{def}(GR^{-1}) = \text{def}(G)$  by lemmas 27 and 28. In addition, if  $H' \in \mathbf{F}^{(p-r) \times p}$  is another matrix, the  $p-r$  rows  $h'_i$  of which form a basis of  $\ker_{\mathbf{F}}(\bullet G)$ , there exists a matrix  $L \in \text{GL}_{p-r}(\mathbf{F})$  such that  $H' = LH$ . We have as above  $\text{def}(H') = \text{def}(H) + \text{def}(L)$  and  $\text{def}(L) = 0$ . ■

The theorem below is the main result of this section. It generalizes to the case where  $\mathbf{F} = \mathbf{K}(s; \boldsymbol{\alpha}, \boldsymbol{\delta})$  Theorem 2.1 of Kung and Kailath [28] and Theorem 3 of Verghese, van Dooren and Kailath [45] (see also ([23], Theorem 6.5-11)), obtained when  $\mathbf{F} = \mathbf{K}(s)$ .

**Theorem 32** *Let  $G \in \mathbf{F}^{p \times m}$  be a matrix of rank  $r > 0$ , and  $\{\eta_1, \dots, \eta_{p-r}\}$  (resp.  $\{\rho_1, \dots, \rho_{m-r}\}$ ) be the set of its left (resp. right) minimal indices. We have*

$$\text{def}(G) = \sum_{i=1}^{p-r} \eta_i + \sum_{j=1}^{m-r} \rho_j \geq 0$$

**Proof.** As in (38), let  $G' \in \mathbf{F}^{p \times r}$ ,  $G'' \in \mathbf{F}^{r \times m}$  be two matrices of rank  $r$  such that  $G = G'G''$ . Let  $\mathfrak{B} = \{h_i : 1 \leq i \leq r\}$  be a minimal polynomial basis of  $\ker_{\mathbf{A}}(\bullet G') = \ker_{\mathbf{A}}(\bullet G)$  and  $H = \text{col}\{h_i\}_{1 \leq i \leq r} \in \mathbf{A}^{(p-r) \times p}$ . We have  $\text{def}(G') = \text{def}(H)$  by Lemma 31. In addition, according to Definition 26,  $\text{def}(H) = -(\text{deg}_M(H) + \nu_{\infty}(H))$  where  $\nu_{\infty}(H) = \mathbf{d}_{\infty}^z(H) - \mathbf{d}_{\infty}^p(H) = -\sum_{i=1}^{p-r} \eta_i$  by Lemma 30, Definition 14 and Lemma 25(ii). By Lemmas 13 and 30, and again Definition 14,  $\text{deg}_M(H) = \text{deg}_M^z(H) - \text{deg}_M^p(H) = 0 - 0 = 0$ , thus  $\text{def}(G') = \sum_{i=1}^{p-r} \eta_i$ . By a similar rationale, we get  $\text{def}(G'') = \sum_{j=1}^{m-r} \rho_j$ , and the result follows by Lemma 28. ■

## 6 Exact matching problem

Again,  $\mathbf{F} = \mathbf{K}(s; \boldsymbol{\alpha}, \boldsymbol{\delta})$ .

### 6.1 Exact left model-matching

Given two nonzero transfer matrices  $A \in \mathbf{F}^{p \times m}$  and  $B \in \mathbf{F}^{q \times m}$ , the problem considered below is the following:

1. To determine a necessary and sufficient condition for a post-compensator to exist with transfer matrix  $H \in \mathbf{F}^{q \times p}$  such that

$$HA = B \tag{39}$$

2. Assuming that this condition is satisfied, to determine a necessary and sufficient condition for this compensator to be proper, i.e.  $H \in \mathbf{F}_{pr}^{q \times p}$ .

This problem was treated when  $\mathbf{F} = \mathbf{K}(s)$  by Wang and Davison [47], by Forney ([17], Section 8), and by Kung and Kailath [28]; the two latter authors



gave a nice solution; see also ([23], pp. 462-464). The solution of [28] was extended to the case where  $\mathbf{F} = \mathbf{K}(s; \boldsymbol{\alpha}, \boldsymbol{\delta})$  by Marinescu and Bourlès [31], nevertheless with  $\mathbf{S}$  instead of  $\mathbf{F}_{pr}$ , which is artificial. We show below that this change of ring is not necessary.

**Lemma 33** (i) *There exists  $H \in \mathbf{F}^{q \times p}$  satisfying (39) if and only if*

$$\operatorname{rk}_{\mathbf{F}}(A) = \operatorname{rk}_{\mathbf{F}}(G), \quad G := \operatorname{col}\{A, B\} \quad (40)$$

(ii) *Assuming that the necessary and sufficient condition (40) is satisfied, let  $r = \operatorname{rk}_{\mathbf{F}}(G)$ , and let  $\bar{G} = \operatorname{col}\{\bar{A}, \bar{B}\} \in \mathbf{F}^{(p+q) \times r}$ ,  $Q \in \mathbf{F}^{r \times m}$ , both of them of rank  $r$ , be such that  $G = \bar{G}Q$ . The set  $\mathfrak{H}$  of all transfer matrices  $H \in \mathbf{F}^{q \times p}$  satisfying (39) is parametrized as follows:*

$$\mathfrak{H} = \{(\bar{B} + X)(\bar{A}^{-L} + Y) : X \in \mathbf{F}^{q \times r}, XQ = 0, Y \in \mathbf{F}^{r \times p}, Y\bar{A} = 0\} \quad (41)$$

where  $\bar{A}^{-L}$  is a left inverse of  $\bar{A}$ .

**Proof.** (i) Putting  $H = \operatorname{col}\{H_i\}_{1 \leq i \leq q}$ ,  $B = \operatorname{col}\{B_i\}_{1 \leq i \leq q}$ , then (39) is equivalent to  $H_i A = B_i$ ,  $i = 1, \dots, q$ . There exists  $H_i \in \mathbf{F}^{1 \times p}$  such that  $H_i A = B_i$  if and only if the row  $B_i$  belongs to the  $\mathbf{F}$ -vector space  $\mathbf{F}^{1 \times p} A$ , i.e.  $\operatorname{rk}_{\mathbf{F}}(\operatorname{col}\{A, B_i\}) = \operatorname{rk}_{\mathbf{F}}(A)$ . The necessary and sufficient condition (40) follows immediately.

(ii) By Lemma 28, there exist matrices  $\bar{G}$  and  $Q$  as indicated. Since  $r = \operatorname{rk}_{\mathbf{F}}(\bar{A}Q) \leq \operatorname{rk}_{\mathbf{F}}(\bar{A})$ , we have  $\operatorname{rk}_{\mathbf{F}}(\bar{A}) = r$  and  $\bar{A}$  has a left inverse. Given a left inverse  $\bar{A}^{-L} \in \mathbf{F}^{r \times p}$  of  $\bar{A}$ ,  $Z$  is a left inverse of  $\bar{A}$  if and only if  $\bar{A}^{-L}\bar{A} = Z\bar{A}$ , which is equivalent to  $Y\bar{A} = 0$  where  $Y = Z - \bar{A}^{-L}$ , thus the set of all left inverses of  $\bar{A}$  is  $\{\bar{A}^{-L} + Y : Y \in \mathbf{F}^{r \times p}, Y\bar{A} = 0\}$ . Condition (39) is equivalent to  $(H\bar{A} - \bar{B})Q = 0$  which can be written  $H\bar{A} - \bar{B} = X$ ,  $XQ = 0$ . Let  $X$  be such that by the first of these two equalities holds and put  $H = H'(\bar{A}^{-L} + Y)$  where  $Y\bar{A} = 0$ . Then

$$H'(\bar{A}^{-L} + Y)\bar{A} = \bar{B} + X \Rightarrow H' = \bar{B} + X \Rightarrow H = (\bar{B} + X)(\bar{A}^{-L} + Y)$$

Conversely, let  $H = (\bar{B} + X)(\bar{A}^{-L} + Y)$  where  $Y\bar{A} = 0$  and  $XQ = 0$ . It follows that  $\operatorname{col}\{I_p, H\}\bar{A} = \operatorname{col}\{\bar{A}, \bar{B} + X\} = \bar{G} + \operatorname{col}\{0, X\}$  whence  $\operatorname{col}\{I_p, H\}\bar{A}Q = \bar{G}Q = \operatorname{col}\{A, B\}$  and  $HA = B$ . ■

The following theorem is the main result of this section. The necessary and sufficient condition and the parametrization provided improve those in ([31], Theorem 1) and ([30], Proposition 4), respectively. In these references, it is assumed that  $p = r$ , which is no longer the case below.

**Theorem 34** (i) *There exists a matrix  $H \in \mathbf{F}_{pr}^{q \times p}$  satisfying (39) if and only if (40) holds and  $\boldsymbol{\nu}_{\infty}(A) = \boldsymbol{\nu}_{\infty}(G)$ .*

(ii) *There exist matrices  $\bar{G} = \operatorname{col}\{\bar{A}, \bar{B}\} \in \mathbf{F}_{pr}^{(p+q) \times r}$  and  $Q \in \mathbf{F}^{r \times m}$ , both of them of rank  $r = \operatorname{rk}_{\mathbf{F}}(G)$ , such that  $G = \bar{G}Q$ . If the necessary and sufficient condition in (i) is satisfied, the set  $\mathfrak{H}$  of all transfer matrices  $H \in \mathbf{F}_{pr}^{q \times p}$  satisfying (39) is given by (41) where the conditions  $X \in \mathbf{F}^{q \times r}$  and  $Y \in \mathbf{F}^{r \times p}$  are replaced by  $X \in \mathbf{F}_{pr}^{q \times r}$  and  $Y \in \mathbf{F}_{pr}^{r \times p}$  respectively.*

**Proof.** Let  $\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}$  be the McMillan normal form at infinity of  $G$  where  $\Delta = \text{diag}\{\sigma^{\mu_i}\}_{1 \leq i \leq r}$  (Definition 21). Then we have (38) where  $U \in \text{GL}_{p+q}(\mathbf{F}_{pr})$ ,  $V \in \text{GL}_m(\mathbf{F}_{pr})$ , so  $G = \bar{G}Q$  with  $\bar{G} = G'$ ,  $Q = G''$  and  $\bar{G} \sim_{\mathbf{F}_{pr}} \text{col}\{I_r, 0\}$ ,  $Q \sim_{\mathbf{F}_{pr}} \begin{bmatrix} \Delta & 0 \end{bmatrix}$ , hence  $\nu_\infty(Q) = \nu_\infty(G)$  by Theorem 11(ii).

As in the proof of Lemma 33(ii), let  $H = (\bar{B} + X)(\bar{A}^{-L} + Y)$  where  $\bar{G} = \text{col}\{\bar{A}, \bar{B}\} \sim_{\mathbf{F}_{pr}} \text{col}\{I_r, 0\}$ ,  $XQ = 0$  and  $Y\bar{A} = 0$ . Let us show that  $H$  is proper if and only if  $X$  and  $\bar{A}^{-L} + Y$  are both proper. This condition is obviously sufficient. To prove that it is necessary, let us assume that  $H$  is proper. Since  $H\bar{A} = \bar{B} + X$ ,  $X$  is proper. In addition,  $\text{col}\{I_p, H\} = (\bar{G} + \text{col}\{0, X\})(\bar{A}^{-L} + Y)$  where  $\bar{G} + \text{col}\{0, X\} \sim_{\mathbf{F}_{pr}} \text{col}\{I_r, 0\}$ , hence  $\nu_\infty(\bar{G} + \text{col}\{0, X\}) = 0$ . Thus by Theorem 23(ii) we have  $0 = \nu_\infty(\text{col}\{I_p, H\}) = \nu_\infty(\bar{A}^{-L} + Y)$  and, since  $\bar{A}^{-L} + Y$  is a left inverse of  $\bar{A}$ ,  $\nu_\infty(\bar{A}^{-L} + Y) = -\nu_\infty(\bar{A})$  by Theorem 23(iv). Since  $\bar{A}$  is proper and  $\nu_\infty(\bar{A}) = 0$ ,  $\bar{A}$  is left biproper and  $\bar{A}^{-L} + Y$  is proper.

As shown here above,  $\bar{A}^{-L} + Y$  is proper if and only if  $\nu_\infty(\bar{A}) = 0$ . Since  $A = \bar{A}Q$ , by Theorem 23(ii) we have  $\nu_\infty(A) = \nu_\infty(\bar{A}) + \nu_\infty(Q) = \nu_\infty(\bar{A}) + \nu_\infty(G)$ . As a consequence,  $\nu_\infty(\bar{A}) = \nu_\infty(A) - \nu_\infty(G)$  and  $\bar{A}^{-L} + Y$  is proper if and only if  $\nu_\infty(G) - \nu_\infty(A) = 0$ . Thus, the latter condition is necessary and sufficient for any left inverse  $\bar{A}^{-L}$  of  $\bar{A}$  to be proper. Assuming that  $\bar{A}^{-L}$  is proper,  $\bar{A}^{-L} + Y$  is proper if and only if  $Y$  is proper too. ■

## 6.2 Exact right model-matching

The problem considered below is the following: given two nonzero transfer matrices  $A \in \mathbf{F}^{p \times m}$ ,  $B \in \mathbf{F}^{m \times q}$ ,

1. To determine a necessary and sufficient condition for pre-compensator with transfer matrix  $H \in \mathbf{F}_{pr}^{m \times q}$  to exist such that

$$AH = B \quad (42)$$

2. When this necessary and sufficient condition is satisfied, to determine a parametrization of all solutions  $H \in \mathbf{F}_{pr}^{m \times q}$  to this problem.

See [38] in the nonlinear case. We note that  $H \in \mathbf{F}_{pr}^{m \times q}$  is a solution to (42) if and only if  $H^\top A^\top = B^\top$ . Thus Theorem 34 yields immediately the following result :

**Corollary 35** (i) *There exists a matrix  $H \in \mathbf{F}_{pr}^{m \times q}$  satisfying (42) if and only if  $\text{rk}_{\mathbf{F}}(A) = \text{rk}_{\mathbf{F}}\left(\begin{bmatrix} A & B \end{bmatrix}\right)$  and  $\nu_\infty(A) = \nu_\infty\left(\begin{bmatrix} A & B \end{bmatrix}\right)$ .*  
(ii) *If this necessary and sufficient condition is satisfied, there exist matrices  $\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} \in \mathbf{F}_{pr}^{r \times (m+q)}$  and  $Q \in \mathbf{F}^{p \times r}$ , both of them of rang  $r$ , such that  $\begin{bmatrix} A & B \end{bmatrix} = Q \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix}$ . The set  $\mathfrak{H}$  of all matrices  $H \in \mathbf{F}_{pr}^{m \times q}$  satisfying (42) is parametrized as follows :*

$$\mathfrak{H} = \{(\bar{A}^{-R} + Y)(\bar{B} + X) : X \in \mathbf{F}_{pr}^{r \times q}, QX = 0, Y \in \mathbf{F}_{pr}^{m \times r}, \bar{A}Y = 0\}$$

where  $\bar{A}^{-R}$  is a right inverse of  $\bar{A}$ .

**Remark 36** *With a diagonal or block-diagonal matrix  $B$ , the above exact model-matching problems become decoupling problems by post- or pre-compensation [19]. In the case of time-varying coefficients, all solutions  $H$  with entries in the ring  $\mathbf{S}$  to these decoupling problems were obtained by Marinescu ([30], Section 5) when  $A$  is full column- or row-rank. Using Theorem 34 and Corollary 35, it is easy to improve the results in [30] in such a way that the solutions  $H$  be obtained with entries in the ring  $\mathbf{F}_{pr}$  of proper transfer matrices, without assuming  $A$  to be full column- or row-rank. The details are left to the reader.*

## 7 Example: application to a FACTS

### 7.1 Description of the system

The electric circuit in Figure 1 represents the so-called  $\pi$ -model of a high-voltage transmission system. Such a model is used for simulation and control of electric power systems ([27], p. 236) and to represent a Flexible Ac Transmission System (FACTS) to control the power flow on a transmission line ([27], p. 650). It is assumed that the resistance  $R$  is negligible, the inductance  $L$  depends on time in an affine way ( $L = L(t)$ ,  $\dot{L} = \text{const.}$ ) in order to adapt the impedance of the transmission line while the load and the network topology are changing, and the capacitances  $C_1, C_2$  are constant. Put  $u = (i_A, i_B) = (u_1, u_2)$  and  $y = (V_1, V_2) = (y_1, y_2)$ . By (4), the system equations can be written

$$\begin{cases} C_k s y_k = i_k & (k = 1, 2) \\ y_1 - y_2 = L s (u_1 - i_1) \\ u_1 - u_2 = i_1 + i_2 \end{cases}$$

The system has a symmetry and its equations are invariant under substitution  $(y_1, y_2, i_1, i_2, C_1, C_2, u_1, u_2) \mapsto (y_2, y_1, i_2, i_1, C_2, C_1, -u_2, -u_1)$ . Let

$$\begin{aligned} D & : = \left( C_1 + C_2 + \dot{L} C_1 C_2 s + L C_1 C_2 s^2 \right) s \\ N_k & : = 1 + \dot{L} C_k s + L C_k s^2 \quad (k = 1, 2) \end{aligned}$$

Using the commutation rule (2) with  $\mathbf{K} = \mathbb{C}(t)$ ,  $\boldsymbol{\alpha} = \mathbf{1}_{\mathbf{K}}$ , one obtains the following expression for the transfer matrix  $A$  of the control system with input  $u$  and output  $y$ :

$$A = \begin{bmatrix} D^{-1} N_2 & -D^{-1} \\ D^{-1} & -D^{-1} N_1 \end{bmatrix}$$

(this expression corrects those given in [31] and ([8], Section 10.6)).

### 7.2 Defect

Consider the transfer matrix  $G$  of the control system with input  $u$  and output  $y_1$ , i.e.  $G = D^{-1} \begin{bmatrix} N_2 & -1 \end{bmatrix}$ . Since  $\begin{bmatrix} N_2 & -1 \end{bmatrix} \sim_{\mathbf{A}} \begin{bmatrix} 1 & 0 \end{bmatrix}$  and  $\deg(D) = 3$ ,

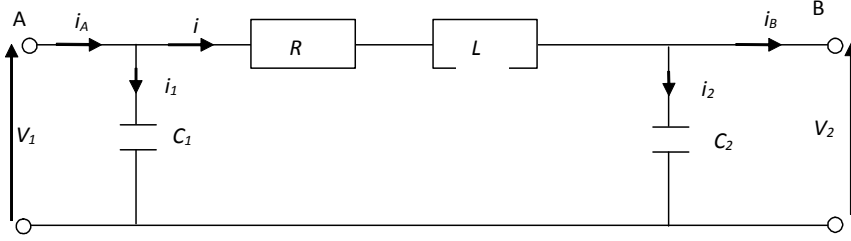


Figure 1: Electric circuit

we have by Definition 9  $\deg_M^z(G) = 0$ ,  $\deg_M^p(G) = 3$  and  $\deg_M(G) = 0 - 3 = -3$ . In addition,  $G \sim_{\mathbf{F}_{pr}} \begin{bmatrix} \sigma & 0 \end{bmatrix}$ , so by Definition 21  $d_\infty^z(G) = 1$ ,  $d_\infty^p(G) = 0$ , and by (36)  $\nu_\infty(G) = 1$ . According to Definition 26,  $\text{def}(G) = 3 - 1 = 2$ .

Let  $\mathbf{F} = \mathbf{K}(s; 1_{\mathbf{K}}, d/dt)$ . Obviously,  $\ker_{\mathbf{F}}(\bullet G) = 0$ . Let us calculate  $\ker_{\mathbf{F}}(G\bullet)$ . A vector  $x = \text{col}\{x_1, x_2\} \in \mathbf{F}^{2 \times 1}$  belongs to  $\ker_{\mathbf{F}}(G\bullet)$  if and only if  $\begin{bmatrix} N_2 & -1 \end{bmatrix} x = 0$ , thus  $\ker_{\mathbf{F}}(G\bullet)$  is the  $\mathbf{K}$ -vector space with basis  $\text{col}\{1, N_2\}$  which has  $F$ -degree  $\deg(N_2) = 2$ . Therefore,  $\rho_1 = 2$  and  $\text{def}(G) = \rho_1$  as expected from Theorem 32.

### 7.3 Exact right model-matching

Using elementary operations (Lemma 16) and the commutation rule (7), one obtains  $A \sim_{\mathbf{F}_{pr}} \text{diag}\{\sigma, \sigma\}$  which is the McMillan normal form at infinity of  $A$  (Definition 21). This calculation is greatly facilitated by the fact that every element of  $\mathbf{F}_{pr}$  is of the form (29). Likewise, with  $B = \text{diag}\{k_i/s^{n_i}\}_{1 \leq i \leq 2}$ , we can easily check that  $\nu_\infty(\begin{bmatrix} A & B \end{bmatrix}) = \nu_\infty(A)$  if and only if  $n_i \geq 1$  ( $i = 1, 2$ ). By Corollary 35, this is the necessary and sufficient condition for a matrix  $H \in \mathfrak{M}_2(\mathbf{F}_{pr})$  to exist such that the equality (42) holds. The physical interpretation of this condition is clear. In this case,  $A$  is invertible over  $\mathbf{F}$ , thus  $H = A^{-1}B$  is the unique solution to the problem.

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