



HAL
open science

Integration of the Baker-Campbell-Hausdorff product

Mario Fuentes

► **To cite this version:**

| Mario Fuentes. Integration of the Baker-Campbell-Hausdorff product. 2024. hal-04611607

HAL Id: hal-04611607

<https://hal.science/hal-04611607>

Preprint submitted on 13 Jun 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the integration of the Baker-Campbell-Hausdorff product

Mario Fuentes

June 13, 2024

Abstract

In an arbitrary complete differential graded Lie algebra, we construct a group operation \bullet on L_1 such that the differential of the product of two elements is the Baker-Campbell-Hausdorff product of their differentials, i.e., $d(x \bullet y) = dx * dy$. We study some properties of this new structure and some applications, especially in homotopy theory, where this operation can be used to construct a Lie model for the 4-simplex. In particular, this solves, in dimension 4, a problem proposed by Lawrence and Sullivan.

Introduction

In a nilpotent Lie algebra L over the field of rational numbers, there is a well-known group structure given by the *Baker-Campbell-Hausdorff product* or BCH product, denoted by $*$ and defined by a formal series whose lower terms are

$$x * y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] - \frac{1}{12}[y, [x, y]] + \dots$$

This operation is associative, has 0 as neutral element, and $x^{-1} = -x$ is the inverse of the element $x \in L$. Therefore, it provides L with a group structure from its original Lie algebra structure.

The condition of being nilpotent can be weakened to being *complete*, this is, that there exists a filtration of subspaces

$$L = F^1 \supset F^2 \supset \dots$$

compatible with the Lie bracket, $[F^i, F^j] \subset F^{i+j}$ for all i, j , for which the natural map

$$L \xrightarrow{\cong} \varprojlim_n L/F^n$$

is an isomorphism of Lie algebras. In that case, the formal series above is a well-defined element in the complete Lie algebra.

The BCH formula initially appeared as a means to understand the lack of commutativity of the exponential map in the context of Lie groups (see [4] or [1] for an introduction

and complete references to the historical aspects). From a more algebraic point of view, Malcev [13] showed that this formula can be used to establish an equivalence between the categories of nilpotent (or complete) rational groups and nilpotent (or complete) Lie algebras over \mathbb{Q} .

In the ‘differential graded’ world, we consider the case of complete differential graded Lie algebras (abbreviated as complete dgl’s or cdgl’s from now on). These are graded Lie algebras of the form

$$L = \bigoplus_n L_n$$

with a differential of degree -1, $d : L_* \rightarrow L_{*-1}$, and a filtration (compatible with the Lie bracket and the differential) making it complete. Then L_0 is automatically a complete Lie algebra and, therefore, it has a well-defined BCH product on it. The complete dgl’s, accompanied by their group structures at degree 0 given by the BCH product, are fundamental in diverse areas of mathematics such as algebraic topology, algebraic geometry, deformation theory or operad theory (see for instance [5, §2.2] or [6] for some examples of the application of the theory of complete differential graded Lie algebras).

In these objects, a natural question arises: can we extend the Baker-Campbell-Hausdorff product to form other analogous group structures on L_n for $n \neq 0$? More specifically, we say that a map $\diamond : L_1 \times L_1 \rightarrow L_1$ is an *integral of the BCH product* if $d(x \diamond y) = dx * dy$ for all $x, y \in L_1$. In any contractible cdgl, this is $H_*(L) = 0$, since $d(dx * dy) = 0$, such an element $x \diamond y$ has to exist. However, it is not unique: it is only uniquely defined up to a boundary. In the case of a non-contractible cdgl L , we can consider a contractible cdgl \mathcal{L} generated by two elements of degree 1 and their differentials. This could work as a kind of ‘universal example’, where we can construct the integral of the BCH product, \diamond , and then define $x \diamond y \in L_1$ as the image of the only morphism $\mathcal{L} \rightarrow L$ sending the generators to x, y respectively.

This strategy is followed in [7, §6.5] or [10], for example, to construct (different) integrals of the BCH product. The problem with this generic approach is that it is difficult to deduce properties of the operation \diamond constructed. For example, we may have an operation \diamond which fails to be associative or to have an inverse.

In this text, we construct a specific integral of the BCH product, denoted by \bullet which inherits some of the nice properties of the BCH product. More concretely, the \bullet operation turns out to be associative, to have a neutral element, and an inverse. This implies that it defines a group (L_1, \bullet) just as the BCH product does at L_0 .

This question has important consequences. For example, in rational homotopy theory, Quillen’s Theory of dgl’s [14] can be extended to a non-connected and non-simply connected setting by means of complete dgl’s.

The theory of complete dgl’s in rational homotopy theory has several equivalent approaches; see, for instance, [2], [11], [16], or [17]. The different functors constructed in these references realize complete dgl’s (or complete L_∞ -algebras) to obtain simplicial sets, which are not, in general, connected, nor simply connected. Moreover, we can relate some

algebraic concepts with their topological counterparts: the Maurer-Cartan elements of a complete dgl correspond to the vertices of its realization, the gauge action of L_0 on the Maurer-Cartan elements corresponds to paths joining two points, and the BCH product on L_0 corresponds to the concatenation of paths in the topological space.

In [7], the authors constructed a functor in the opposite direction: that takes a simplicial set, and obtains a complete dgl ‘modeling’ the space. The main tool of that theory is a cosimplicial object in the category of complete dgl’s denoted by $\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n \geq 0}$, where each complete dgl \mathfrak{L}_n is supposed to work as a ‘model of the n -simplex’, using the terminology of Sullivan’s approach to rational homotopy theory.

Each \mathfrak{L}_n is a (quasi-)free complete dgl generated by the n -simplex. More explicitly, we consider the graded vector space

$$\Delta^n = \text{Span}\{a_{i_1, \dots, i_k} \mid \emptyset \neq I = (i_1 < i_2 < \dots < i_k) \subset \{0, \dots, n\}\}$$

where $|a_{i_1, \dots, i_k}| = k - 2$ (i.e., one degree lower than its geometrical dimension). Then

$$\mathfrak{L}_n = (\widehat{\mathbb{L}}(\Delta^n), \delta),$$

where $\mathbb{L}(-)$ is the free object in the category of graded Lie algebras, $\widehat{\mathbb{L}}(-)$ denotes its completion with respect to the word-length filtration, and δ is the differential. The relevant part of the theory relies on the differential δ , which must satisfy some properties (see Section 5 for more details). With those properties, the cosimplicial object \mathfrak{L}_\bullet is unique up to (cosimplicial) isomorphism. Therefore, rather than *the* model \mathfrak{L}_n of the n -simplex, we should talk about *a* model of the n -simplex.

The existence of such ‘Lie models’ of the simplices was anticipated by Lawrence and Sullivan in their work [19], where they construct an explicit model of the interval (the subsequently called *Lawrence-Sullivan interval*) and proposed as an open question finding explicit models of the simplices in each dimension (see [19, §1]):

PROBLEM. Study this free differential Lie algebra attached to a cell complex, determine its topological and geometric meaning as an intrinsic object. Give closed form formulae for the differential and for the induced maps associated to subdivisions.

Models of the n -simplices, \mathfrak{L}_n , are explicitly known for $n = 0, 1, 2$, and 3. However, a general formula for the differential δ in any other dimension is not known. The proof of [7, Theorem 6.1] provides an existence result but not a constructive formula for these objects. In the explicit formulas of these differentials in low dimensions, some important algebraic concepts of the theory of Lie algebras arise: for example, \mathfrak{L}_0 is constructed by imposing that the only generator a_0 is a *Maurer-Cartan element*. In the formula for \mathfrak{L}_1 the *gauge action* arises. The differential of the top generator of \mathfrak{L}_2 involves the *Baker-Campbell-Hausdorff product*, and for constructing \mathfrak{L}_3 it is necessary to consider an integral of the BCH product.

Even though several homotopical properties can be deduced only by the existence of the cosimplicial cdgl \mathfrak{L}_\bullet , without needing the explicit formulas defining it, the knowledge of the explicit models has important consequences. For example, in [9], it is shown that the realization of a cdgl concentrated in degree 0, $L = L_0$, is isomorphic to the simplicial bar construction of the group $(L_0, *)$. Furthermore, in [10] they show that if L is concentrated in degree 0 and 1, its realization is isomorphic to the classifying simplicial set of the crossed module associated with the pair $(\langle L \rangle, \langle L_0 \rangle)$. For all these results, a complete knowledge of the explicit models of the simplices in low dimensions is fundamental.

In [7, §6.5] or [10], models of the tetrahedron \mathfrak{L}_3 are given using some operations integrating the BCH, i.e., with the property that $d(x \diamond y) = x * y$ for $x, y \in L_1$. All these different operations having that property, have to agree up to a boundary. In this text, we construct such an operation, denoted by the symbol \bullet , and therefore, equal up to boundary to the ones used in the references, but with some nice properties: explicitly, in Theorem 1.6, we prove that \bullet is a group multiplication.

For constructing the complete dgl \mathfrak{L}_3 the operation \diamond chosen may not seem very important: different operations (agreeing up to a boundary) will give different differentials, but in the end, by uniqueness, all of them will be isomorphic. However, if we want to go further and get higher models of simplices, we need to have a very explicit and well-behaved operation \diamond , and precisely this is what happens with the operation \bullet .

And thank to that, in Section 5, we solved the problem stated above in dimension 4 by constructing an explicit model for the hypertetrahedron or the pentahedroid

$$\mathfrak{L}_4 = (\widehat{\mathbb{L}}(\Delta^4), \delta),$$

and again, new operations of complete dgl's arise in this construction. For example, in Section 3, we are interested in the conjugation of elements, i.e., in obtaining an expression for $x \bullet y \bullet x^{-1}$. For this goal, we develop in Sections 3 and 4 two maps $\sigma, \tau : L_1 \times L_1 \rightarrow L_2$ that measure the failure to commute of the operation \bullet .

By its own definition as an operation 'integrating' the BCH product, when applied to two cycles, the \bullet operation simply becomes the sum

$$x \bullet y = x + y, \text{ if } dx = 0, dy = 0.$$

In particular, in the homology of any complete dgl $H_*(L, d)$ this operation disappears and becomes the abelian operation of the sum. However, before homology, using this operation produces a very interesting effect: it makes the differential d to be a group morphism, in the sense of respecting all the products. This is explored in Section 2 and more explicitly in Theorem 2.2; it is proved that, with the correct operations (L_*, d) becomes a chain complex of (possibly non-abelian) groups. In particular, we interpret the operation \bullet as a kind of middle step between the abelian operation of the sum and the non-abelian operation of the BCH product.

Acknowledgments: I would like to thank Ricardo Campos, Aníbal Medina-Mardones, and Aniceto Murillo for their useful comments and feedback about this text. The author has been partially supported by Labex CIMI under grant agreement ANR-11-LABX-0040, the MICINN grant PID2020-118753GB-I00 of the Spanish Government, and the Junta de Andalucía grant ProyExcel-00827.

1 The operation \bullet on L_1

For each $k \geq 1$, we define the complete graded differential Lie algebra $\mathcal{L}^{(k)}$, or simply \mathcal{L} if there is no confusion, as $\mathcal{L} = \widehat{\mathbb{L}}(u_1, \dots, u_k, v_1, \dots, v_k)$ with $|u_i| = 1$, $|v_i| = 0$, $du_i = v_i$ for $i = 1, \dots, k$. Here $\mathbb{L}(-)$ denotes the free object in the category of graded Lie algebras (forgetting about the differential) and $\widehat{\mathbb{L}}(-)$ denotes its completion by the filtration by length of words.

Proposition 1.1. *The complete dgl \mathcal{L}^k is contractible.*

Proof. If we write $U = \text{Span}\{u_1, \dots, u_k\}$, $V = \text{Span}\{v_1, \dots, v_k\}$ and $d^1 : U \oplus V \rightarrow U \oplus V$ denotes the linear part of the differential in \mathcal{L} , it is clear that $H_*(U \oplus V, d^1) = 0$. Consider the trivial morphism $\varphi : 0 \rightarrow \mathcal{L}$, then its linear part $\varphi^1 : 0 \rightarrow U \oplus V$ is a quasi-isomorphism and by [7, Proposition 3.12] φ is a quasi-isomorphism. \square

In this contractible Lie algebra \mathcal{L} , the cycle $v_1 * v_2 * \dots * v_k$ has an ‘integral’, this means that there exists an element ω of degree 1 with $d\omega = v_1 * v_2 * \dots * v_k$. However, this element is not unique (since the space is acyclic, $\omega + d\mathcal{L}_2$ is exactly the set of elements whose differential is $v_1 * v_2 * \dots * v_k$). We need a canonical way of defining this element ω .

Let θ be the degree 1 derivation of \mathcal{L} acting on the generators as

$$\theta(v_i) = u_i, \quad \theta(u_i) = 0,$$

and consider the map $\tilde{\theta} : \mathcal{L}_0 \rightarrow \mathcal{L}_1$ defined as follows: if $x \in \widehat{\mathbb{L}}^n(U \oplus V)$ is a word of length n then we define

$$\tilde{\theta}(x) = \frac{1}{n}\theta(x).$$

This is a well-defined linear map since we can decompose $\widehat{\mathbb{L}}(U \oplus V)$ as a product

$$\widehat{\mathbb{L}}(U \oplus V) = \prod_{n \geq 1} \mathbb{L}^n(U \oplus V).$$

Moreover, it is a section of the differential $d : \mathcal{L}_1 \rightarrow \mathcal{L}_0$.

Proposition 1.2. $d \circ \tilde{\theta} = \text{id}_{\mathcal{L}_0}$

Proof. On generators, we have $d(\tilde{\theta}(v_i)) = d(\theta(v_i)) = du_i = v_i$. Suppose that $x \in \mathcal{L}_0$ is a word of the form

$$x = [v_{i_1}, [v_{i_2}, \dots, [v_{i_{n-1}}, v_{i_n}] \dots]]$$

for some indices $i_1, \dots, i_n \in \{1, \dots, k\}$. Then

$$\tilde{\theta}(x) = \frac{1}{n} \sum_{j=1}^n [v_{i_1}, [v_{i_2}, \dots, [u_{i_j}, \dots [v_{i_{n-1}}, v_{i_n}] \dots] \dots]].$$

Clearly on any of these terms, d sends any element v_i to zero, thus a non trivial term only appears when d sends u_{i_j} to v_{i_j} . Therefore

$$d(\tilde{\theta}(x)) = \frac{1}{n} \sum_{j=1}^n [v_{i_1}, [v_{i_2}, \dots, [v_{i_j}, \dots [v_{i_{n-1}}, v_{i_n}] \dots] \dots]] = \frac{1}{n} \sum_{j=1}^n x = x.$$

By linearity, the result holds for any arbitrary element in \mathcal{L}_0 . □

As a consequence, we get that the section $\tilde{\theta} : \mathcal{L}_0 \rightarrow \mathcal{L}_1$ gives rise to a bijection

$$\mathcal{L}_0 \xrightleftharpoons[d]{\tilde{\theta}} \tilde{\theta}(\mathcal{L}_0) \subset \mathcal{L}_1.$$

Now, we define the \bullet operation as the image by $\tilde{\theta}$ of the BCH product $*$.

Definition 1.3. Given two elements $x, y \in \tilde{\theta}(\mathcal{L}_1)$ we define

$$x \bullet y = \tilde{\theta}(dx * dy).$$

In particular,

$$u_1 \bullet u_2 = \tilde{\theta}(v_1 * v_2).$$

Given an arbitrary cdgl L and two elements $\alpha, \beta \in L_1$, there exists only one cdgl morphism $\varphi : \mathcal{L}^{(2)} \rightarrow L$ with $\varphi(u_1) = \alpha$ and $\varphi(u_2) = \beta$. In particular, $\varphi(v_1) = d\varphi(u_1) = d\alpha = a$ and $\varphi(v_2) = d\beta = b$.

Definition 1.4. On L_1 , we define the \bullet product as

$$\alpha \bullet \beta = \varphi(u_1 \bullet u_2),$$

where $\varphi : \mathcal{L}^{(2)} \rightarrow L$ is the only cdgl morphism sending u_1, u_2 to α, β respectively.

Using the explicit formula of the Baker-Campbell-Hausdorff product, we can write the first terms of this expression:

$$\begin{aligned} \alpha \bullet \beta &= \alpha + \beta + \frac{1}{4}[\alpha, \beta] + \frac{1}{4}[a, \beta] + \\ &+ \frac{1}{36}[\alpha, [a, b]] + \frac{1}{36}[a, [\alpha, b]] + \frac{1}{36}[a, [a, \beta]] - \frac{1}{36}[\beta, [a, b]] - \frac{1}{36}[b, [\alpha, b]] - \frac{1}{36}[b, [a, \beta]] + \dots \end{aligned} \quad (1)$$

for $d\alpha = a$ and $d\beta = b$.

Remark 1.5. Intuitively we can think of this product as considering the BCH of the degree 0 elements a and b and, in any word of the expression $a * b$, replacing one of the letters a or b appearing there by α or β , and doing so in all the possible ways (and adding a factor of $1/n$ to account for repetitions).

However, this way of conceptualizing this operation might be a bit tricky: it has to be done with the ‘formal’ expression of $a * b$, and after doing the replacing, to compute all the possible cancellations that could appear, not in the reverse order. For example, if $d\beta = b = 0$, then $a * b = a$; but $\alpha \bullet \beta \neq \alpha$ when β is a cycle, as we will see in Section 4.

The following theorem claims that, with this definition, the \bullet product defines a group structure on L_1 .

Theorem 1.6. *For any cdgl L , the operation $\bullet : L_1 \times L_1 \rightarrow L_1$ is associative, has 0 as neutral element and the inverse of an element α is given by $\alpha^{-1} = -\alpha$. In particular, (L_1, \bullet) is a group.*

Proof. That 0 is the neutral element can be directly checked from the expression (1): if $\beta = 0$, then $b = 0$ and any term in this sum will cancel except for α , so $\alpha \bullet 0 = \alpha$. Analogously $0 \bullet \beta = \beta$.

Now we want to prove that $\alpha \bullet (-\alpha) = 0$. We know that there exists a morphism $\varphi : \mathcal{L}^{(2)} \rightarrow L$ with $\varphi(u_1) = \alpha$, $\varphi(u_2) = -\alpha$. This morphism factors through $\mathcal{L}^{(1)}$. Consider $\rho : \mathcal{L}^{(2)} \rightarrow \mathcal{L}^{(1)} = \widehat{\mathbb{L}}(u, v)$ defined by

$$u_1 \mapsto u, \quad u_2 \mapsto -u, \quad v_1 \mapsto v, \quad v_2 \mapsto -v$$

and $\psi : \mathcal{L}^{(1)} \rightarrow L$ with

$$u \mapsto \alpha, \quad v \mapsto a.$$

Thus, we have that $\psi \circ \rho = \varphi$. We need to check that ρ commutes with the map $\tilde{\theta}$. For the generators we have that $\rho(\theta(v_i)) = \rho(u_i) = u = \theta(v) = \theta(\rho(v_i))$ for $i = 1, 2$. Since θ is a derivation, this implies that $\rho \circ \theta = \theta \circ \rho$.

For x a word of length n in $\mathcal{L}^{(2)}$, note that $\rho(x)$ could be zero, but if it is not, its length would be also n . Thus, in both cases, we have

$$\rho(\tilde{\theta}(x)) = \frac{1}{n}\rho(\theta(x)) = \frac{1}{n}\theta(\rho(x)) = \tilde{\theta}(\rho(x)),$$

from where we deduce that ρ commutes with $\tilde{\theta}$.

$$\begin{array}{ccc} \mathcal{L}_1^{(2)} & \xrightarrow{\rho} & \mathcal{L}_1^{(1)} \\ \tilde{\theta} \uparrow & & \tilde{\theta} \uparrow \\ \mathcal{L}_0^{(2)} & \xrightarrow{\rho} & \mathcal{L}_0^{(1)} \end{array}$$

We finally conclude that

$$\alpha \bullet (-\alpha) = \varphi(u_1 \bullet u_2) = \psi(\rho(\tilde{\theta}(v_1 * v_2))) = \psi(\tilde{\theta}(\rho(v_1 * v_2))) = \psi(\tilde{\theta}(v * (-v))) = \psi(\tilde{\theta}(0)) = 0$$

and analogously for $(-\alpha) \bullet \alpha = 0$.

For proving the associativity of the product we follow a similar strategy.

For $k = 3$, the associativity of the BCH product $(v_1 * v_2) * v_3 = v_1 * v_2 * v_3 = v_1 * (v_2 * v_3)$ implies that

$$\tilde{\theta}((v_1 * v_2) * v_3) = \tilde{\theta}(v_1 * v_2 * v_3) = \tilde{\theta}(v_1 * (v_2 * v_3))$$

are the same element in $\mathcal{L}^{(3)}$, which we denote by $u_1 \bullet u_2 \bullet u_3$.

Deducing the associativity of the operation in a general cdgl L is more subtle. Consider 3 elements α, β, γ in L_1 . We are going to define cdgl morphisms that make the following diagram commutative

$$\begin{array}{ccc}
 \mathcal{L}^{(2)} & & L \\
 \searrow \iota & \xrightarrow{\phi} & \nearrow \psi \\
 & \mathcal{L}^{(3)} & \\
 \nearrow \rho & \xrightarrow{\psi} & \nearrow \\
 \mathcal{L}^{(2)} & & L \\
 & \xrightarrow{\varphi} & \nearrow
 \end{array}$$

Note that it is only necessary to specify the image of u_i for defining these cdgl morphisms. We define

$$\begin{aligned}
 \iota(u_1) &= u_2, & \iota(u_2) &= u_3 \\
 \phi(u_1) &= \beta, & \phi(u_2) &= \gamma \\
 \rho(u_1) &= u_1, & \rho(u_2) &= 0 \bullet u_2 \bullet u_3 \\
 \varphi(u_1) &= \alpha, & \varphi(u_2) &= \beta \bullet \gamma \\
 \psi(u_1) &= \alpha, & \psi(u_2) &= \beta, & \psi(u_3) &= \gamma.
 \end{aligned}$$

The triangle above is commutative by a direct computation:

$$\psi(\iota(u_1)) = \psi(u_2) = \beta = \phi(u_1) \text{ and } \psi(\iota(u_2)) = \psi(u_3) = \gamma = \phi(u_2).$$

By an straightforward verification, we see that ι commutes with $\tilde{\theta}$. In particular this implies that

$$\iota(u_1 \bullet u_2) = \tilde{\theta}(\iota(v_1 * v_2)) = \tilde{\theta}(v_2 * v_3) = \tilde{\theta}(0 * v_2 * v_3) = 0 \bullet u_2 \bullet u_3.$$

For the triangle below we have

$$\psi(\rho(u_1)) = \psi(u_1) = \alpha,$$

and using the commutativity of the first triangle and the equality above we have

$$\psi(\rho(u_2)) = \psi(0 \bullet u_2 \bullet u_3) = \psi(\iota(u_2 \bullet u_3)) = \phi(u_2 \bullet u_3) = \beta \bullet \gamma = \varphi(u_2).$$

Finally note that by definition $\varphi(u_1 \bullet u_2) = \alpha \bullet (\beta \bullet \gamma)$, and this is equal to $\psi(\rho(u_1 \bullet u_2)) = \psi(u_1 \bullet (0 \bullet u_2 \bullet u_3)) = \psi(u_1 \bullet u_2 \bullet u_3)$, where we have used the associativity of \bullet in $\mathcal{L}^{(3)}$. A completely dual argument would also show that

$$\psi(u_1 \bullet u_2 \bullet u_3) = (\alpha \bullet \beta) \bullet \gamma$$

and that proves that \bullet is associative. □

Furthermore, this structure is preserved by cdgl morphisms.

Proposition 1.7. *If $f : L \rightarrow L'$ is a cdgl morphism, then $f(\alpha \bullet \beta) = f(\alpha) \bullet f(\beta)$ for all $\alpha, \beta \in L_1$.*

Proof. Immediate from the construction. □

Since $-x$ is the inverse element of x for both products $*$, \bullet on L , we write $x^{-1} = -x$ for any element $x \in L$.

2 The chain complex of groups (L_*, d)

The significance of the product \bullet on L_1 is that it is an integral of the BCH product.

Proposition 2.1. *For $\alpha, \beta \in L_1$, where L is a cdgl,*

$$d(\alpha \bullet \beta) = d\alpha * d\beta.$$

Proof. This result follows directly from the construction of $\alpha \bullet \beta$. If $\varphi : \mathcal{L}^{(2)} \rightarrow L$ is a cdgl morphism sending u_1 and u_2 to α and β respectively, then

$$d(\alpha \bullet \beta) = d\varphi(\tilde{\theta}(v_1 * v_2)) = \varphi(d\tilde{\theta}(v_1 * v_2)) = \varphi(v_1 * v_2) = \varphi(v_1) * \varphi(v_2) = d\alpha * d\beta.$$

□

If we consider the case of α and β being cycles, then it is immediate from equation (1) to see that

$$\alpha \bullet \beta = \alpha + \beta.$$

This easy result has a fundamental consequence:

Theorem 2.2. *Let $L = L_{\geq 0}$ be a non-negatively graded cdgl then, for each n , the differential $d : L_n \rightarrow L_{n-1}$ is a morphism of groups, where we consider the following group structures: $(L_0, *)$ with the BCH product, (L_1, \bullet) with the \bullet product and $(L_n, +)$ with the (abelian) group structure of the sum, for $n \geq 2$.*

In particular, there is a chain complex of (non-abelian) groups of the form

$$\cdots \rightarrow (L_3, +) \xrightarrow{d} (L_2, +) \xrightarrow{d} (L_1, \bullet) \xrightarrow{d} (L_0, *).$$

This theorem invites one to think of \bullet as an intermediate step between two very different structures: the sums which make $(L_n, +)$ vector spaces and $(L_0, *)$ a (possibly non-abelian) Malcev complete group.

When we compute the homology groups (L, d) , then the \bullet operation of L_1 becomes the sum on $H_1(L)$ since we are working with cycles in the homology. This implies that the operation \bullet is a kind of ‘non-homological’ operation in L that gives L_1 a non-abelian structure, that disappears after computing the homology.

It is insightful to compare this with the topological situation. The topological spaces have abelian higher homotopy groups and a non-abelian fundamental group, as happens with $H_*(L)$.

Let $X = \langle L \rangle$ be the realization of a non-negatively graded cdgl, which is a connected space. Then it is known [7, Theorem 7.18] that there are natural isomorphisms

$$\pi_{n+1}(X) \cong H_n(L)$$

for all n , where, for $n = 0$, we consider the BCH product on $H_0(L)$. In particular, for $n = 1$, we only see the abelian group $H_1(L)$ with $+$ as operation, which is isomorphic to the abelian group $\pi_2(X)$. Therefore, under this correspondence, \bullet represents an operation that disappears after homotopy.

3 Conjugation in (L_1, \bullet)

In the group $(L_0, *)$, we can express the conjugation of an element y by x using the exponential map (see [7, Proposition 4.13]):

$$x * y * x^{-1} = e^{\text{ad}_x}(y). \tag{2}$$

Here ad_x is the *adjoint operator*, i.e., the derivation that sends y to $[x, y]$ and e^{ad_x} is the Lie morphism defined by the series

$$e^{\text{ad}_x} = \sum_{n \geq 0} \frac{\text{ad}_x^n}{n!}.$$

We would like to have a similar equation for our new product; however, the situation is not going to be so suitable in this case, as we will see. Given $\alpha, \beta \in L_1$ and $a = d\alpha, b = d\beta$, note that

$$d(e^{\text{ad}_a}(\beta)) = e^{\text{ad}_a}(d\beta) = a * b * a^{-1}.$$

which has the same differential as $\alpha \bullet \beta \bullet \alpha^{-1}$. In the equation above we have used that a is a cycle and that, therefore, e^{ad_a} commutes with the differential.

Suppose that we are working in a contractible cdgl (for example \mathcal{L}); this implies that the difference between $\alpha \bullet \beta \bullet \alpha^{-1}$ and $e^{\text{ad}_a}(\beta)$ is a boundary. Our goal is to find that boundary and prove that such element is the difference between $\alpha \bullet \beta \bullet \alpha^{-1}$ and $e^{\text{ad}_a}(\beta)$, even if L is not contractible.

Definition 3.1. Let $\alpha, \beta \in L_1$ be two elements of degree 1 in a cdgl L . Write $a = d\alpha$ and $b = d\beta$. Define $\sigma(\alpha, \beta)$ as the element of degree 2

$$\sigma(\alpha, \beta) = \sum_{i \geq 0} \sum_{j \geq 0} \frac{1}{(i+j+2)!} \text{ad}_a^i \circ \text{ad}_\alpha \circ \text{ad}_a^j(\beta) \in L_2.$$

In particular $\sigma(\alpha, \beta) = \frac{1}{2}[\alpha, \beta] +$ terms of higher length containing a .

The differential of this element precisely gives the difference between the two elements presented above.

Theorem 3.2. Let $\alpha, \beta \in L_1$ be two elements of degree 1 in a cdgl L . Then

$$\alpha \bullet \beta \bullet \alpha^{-1} = e^{\text{ad}_a}(\beta) - d\sigma(\alpha, \beta). \quad (3)$$

Proof. By the proof of Theorem 1.6 we know that we can compute a triple \bullet product using the contractible cdgl $\mathcal{L}^{(3)}$. Consider the morphism $\psi : \mathcal{L}^{(3)} \rightarrow L$ defined by

$$u_1 \mapsto \alpha, \quad u_2 \mapsto \beta, \quad u_3 \mapsto \alpha^{-1}.$$

Then $\alpha \bullet \beta \bullet \alpha^{-1} = \psi(u_1 \bullet u_2 \bullet u_3)$.

This morphism factors through $\mathcal{L}^{(2)}$ by the following morphisms:

$$\begin{array}{ccc} & \mathcal{L}^{(2)} & \\ \rho \nearrow & & \searrow \varphi \\ \mathcal{L}^{(3)} & \xrightarrow{\psi} & L \end{array}$$

defined by

$$\rho(u_1) = u_1, \quad \rho(u_2) = u_2, \quad \rho(u_3) = -u_1$$

$$\varphi(u_1) = \alpha, \quad \varphi(u_2) = \beta.$$

We can directly check that ρ commutes with $\tilde{\theta}$, and therefore

$$\rho(u_1 \bullet u_2 \bullet u_3) = \rho(\tilde{\theta}(v_1 * v_2 * v_3)) = \tilde{\theta}(\rho(v_1 * v_2 * v_3)) = \tilde{\theta}(v_1 * v_2 * v_1^{-1}) = \tilde{\theta}(e^{\text{ad}_{v_1}}(v_2)),$$

which implies that

$$\alpha \bullet \beta \bullet \alpha^{-1} = \psi(u_1 \bullet u_2 \bullet u_3) = \varphi(\tilde{\theta}(e^{\text{ad}_{v_1}}(v_2))).$$

Our first step is to compute $\tilde{\theta}(e^{\text{ad}_{v_1}}(v_2))$:

$$e^{\text{ad}_{v_1}}(v_2) = \sum_{n \geq 0} \frac{1}{n!} \text{ad}_{v_1}^n(v_2) \implies \tilde{\theta}(e^{\text{ad}_{v_1}}(v_2)) = \sum_{n \geq 0} \frac{1}{(n+1)!} \theta(\text{ad}_{v_1}^n(v_2)),$$

where the extra factor $1/(n+1)$ comes from the difference between θ and $\tilde{\theta}$. For a fixed $n \geq 0$ we have

$$\theta(\text{ad}_{v_1}^n(v_2)) = \sum_{\ell=0}^{n-1} \text{ad}_{v_1}^{n-1-\ell} \circ \text{ad}_{u_1} \circ \text{ad}_{v_1}^\ell(v_2) + \text{ad}_{v_1}^n(u_2),$$

where we consider that the summation is zero if $n = 0$. Applying φ to this expression we get:

$$\alpha \bullet \beta \bullet \alpha^{-1} = \sum_{n \geq 1} \frac{1}{(n+1)!} \sum_{\ell=0}^{n-1} \text{ad}_a^{n-1-\ell} \circ \text{ad}_\alpha \circ \text{ad}_a^\ell(\beta) + \sum_{n \geq 0} \frac{1}{(n+1)!} \text{ad}_a^n(\beta). \quad (4)$$

What remains now is the direct computation that $e^{\text{ad}_a}(\beta) - d\sigma$ agrees with (4). From the definition of σ , we see that its differential has two terms: one corresponding to differentiating α , and the other one to differentiating β . The first one is

$$\sum_{i \geq 0} \sum_{j \geq 0} \frac{1}{(i+j+2)!} \text{ad}_a^{i+j+1}(\beta) = \sum_{n \geq 1} \sum_{i+j+1=n} \frac{1}{(n+1)!} \text{ad}_a^n(\beta) = \sum_{n \geq 0} \frac{n}{(n+1)!} \text{ad}_a^n(\beta),$$

and the second one (note that a sign -1 appears since α is an element of odd degree) is

$$-\sum_{i \geq 0} \sum_{j \geq 0} \frac{1}{(i+j+2)!} \text{ad}_a^i \circ \text{ad}_\alpha \circ \text{ad}_a^j(\beta) = -\sum_{n \geq 1} \sum_{\ell=0}^{n-1} \frac{1}{(n+1)!} \text{ad}_a^{n-\ell-1} \circ \text{ad}_\alpha \circ \text{ad}_a^\ell(\beta),$$

where we have made the substitution $j \mapsto \ell$ and $i \mapsto n - \ell - 1$. Therefore, we have:

$$e^{\text{ad}_a}(\beta) - d\sigma = \sum_{n \geq 0} \frac{1}{n!} \text{ad}_a^n(\beta) - \sum_{n \geq 0} \frac{n}{(n+1)!} \text{ad}_a^n(\beta) + \sum_{n \geq 1} \sum_{\ell=0}^{n-1} \frac{1}{(n+1)!} \text{ad}_a^{n-\ell-1} \circ \text{ad}_\alpha \circ \text{ad}_a^\ell(\beta).$$

Since

$$\frac{1}{n!} - \frac{n}{(n+1)!} = \frac{1}{(n+1)!},$$

we can sum the two first terms to obtain

$$e^{\text{ad}_a}(\beta) - d\sigma = \sum_{n \geq 0} \frac{1}{(n+1)!} \text{ad}_a^n(\beta) + \sum_{n \geq 1} \sum_{\ell=0}^{n-1} \frac{1}{(n+1)!} \text{ad}_a^{n-\ell-1} \circ \text{ad}_\alpha \circ \text{ad}_a^\ell(b)$$

which agrees with (4) and finalizes the proof. \square

In the case of β being a cycle, the conjugation formula is simpler:

Corollary 3.3. *If $d\beta = 0$ then*

$$\alpha \bullet \beta \bullet \alpha^{-1} = \varepsilon(\text{ad}_a)(\beta)$$

for

$$\varepsilon(t) = \frac{e^t - 1}{t} = \sum_{n \geq 0} \frac{t^n}{(n+1)!}$$

Proof. Direct by writing $b = 0$ in (4). \square

4 Multiplication by a cycle

Take two elements $\alpha, \beta \in L_1$ such that $d\alpha = a$ and $d\beta = 0$. We announced in Remark 1.5 that $\alpha \bullet \beta$ cannot be computed as $(d\alpha) * (d\beta) = a * 0 = a$ and then substituting a by α .

Instead, we will use the following result from [15, §3.4] that asserts that we can express the BCH product of two elements as

$$v_1 * v_2 = v_1 + \frac{\text{ad}_{v_1}}{1 - e^{-\text{ad}_{v_1}}}(v_2) + (\text{terms with more than one } v_2).$$

When we apply $\tilde{\theta} : \mathcal{L}_0 \rightarrow \mathcal{L}_1$ to this expression, in any term containing more than one v_2 , at least one letter v_2 has to survive. When we apply the morphism $\varphi : \mathcal{L} \rightarrow L$ sending u_1 to α and u_2 to β , then $\varphi(v_2) = d\beta = 0$. Therefore, the whole term between the parenthesis disappears.

To describe the function above as a series, we need to use the *Bernoulli numbers* B_n , whose defining property is that

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n}{n!} t^n.$$

The first Bernoulli numbers are

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}$$

where all the odd Bernoulli numbers are zero, except for $B_1 = -\frac{1}{2}$. Substituting t with $-t$, we get

$$\frac{t}{1 - e^{-t}} = \sum_{n \geq 0} (-1)^n \frac{B_n}{n!} t^n,$$

where the only term affected by the sign is $-B_1 t$. Using this expression, we can rewrite the BCH product as

$$v_1 * v_2 = v_1 + \sum_{n \geq 0} (-1)^n \frac{B_n}{n!} \text{ad}_{v_1}^n(v_2) + \dots$$

When we apply $\tilde{\theta}$ to this expression, we are only interested in the terms where v_2 is sent to u_2 ; otherwise, we will get 0 when we apply φ .

$$u_1 \bullet u_2 = \tilde{\theta}(v_1 * v_2) = u_1 + \sum_{n \geq 0} (-1)^n \frac{B_n}{(n+1)!} \text{ad}_{v_1}^n(u_2) + \dots$$

and this proves the following proposition.

Proposition 4.1. *For $\alpha, \beta \in L_1$ with $a = d\alpha$ and $d\beta = 0$,*

$$\alpha \bullet \beta = \alpha + \sum_{n \geq 0} \frac{(-1)^n B_n}{(n+1)!} \text{ad}_a^n(\beta).$$

Proof.

□

Note that this series has as generating function

$$f(t) = \frac{1}{t} \int_{s=0}^t \frac{s}{1 - e^{-s}} ds = \sum_{n \geq 0} \frac{(-1)^n B_n}{(n+1)!} t^n.$$

By an analogous argument (see again [15, §3.4]) if α is a cycle and $d\beta = b$, then

$$\alpha \bullet \beta = \beta + \sum_{n \geq 0} \frac{B_n}{(n+1)!} \text{ad}_b^n(\alpha).$$

We can use this result to reformulate the conjugation law of Theorem 3.2, to express the conjugate element $\alpha \bullet \beta \bullet \alpha^{-1}$ as a \bullet product instead of as a sum.

Consider the following problem: we have elements $\alpha, \gamma \in L_1$, with $a = d\alpha$ and γ a cycle. Is there an element $\beta \in L_1$ such that $\alpha \bullet \beta = \alpha + \gamma$? Note that in that case $d\alpha * d\beta = d\alpha + d\gamma = d\alpha$ implies that β is also a cycle.

The answer is yes and you can get the element β by formally inverting the function f . Define $\xi(t)$ as:

$$\xi(t) = \frac{1}{f(t)} = \frac{t}{\int_{s=0}^t \frac{s}{1-e^{-s}} ds} = \sum_{n \geq 0} A_n t^n$$

where the coefficients of this series can be calculated by the usual formulas of inverting formal series:

$$1 = \xi(t)f(t) = \sum_{i \geq 0} \sum_{j \geq 0} \frac{A_i (-1)^j B_j}{(j+1)!} t^{i+j}$$

which implies that $A_0 = 1$ and for $n \geq 1$,

$$A_n = - \sum_{\ell=1}^n \frac{A_{n-\ell} (-1)^\ell B_\ell}{(\ell+1)!},$$

$$A_0 = 1, A_1 = -\frac{1}{4}, A_2 = \frac{5}{144}, A_3 = -\frac{1}{576}, A_4 = -\frac{4}{15829}, \dots$$

The following proposition answers the question above.

Proposition 4.2. *Given $\alpha, \gamma \in L_1$ with $a = d\alpha$ and γ a cycle, then $\alpha + \gamma = \alpha \bullet \beta$ for*

$$\beta = \xi(\text{ad}_a)(\gamma) \in L_1.$$

Here $\xi(\text{ad}_a)$ means the operator

$$\xi(\text{ad}_a) = \sum_{n \geq 0} A_n \text{ad}_a^n = \text{id} - \frac{1}{4} \text{ad}_a + \frac{5}{144} \text{ad}_a^2 + \dots$$

Proposition 4.2 along with Theorem 3.2 imply that

$$\alpha \bullet \beta \bullet \alpha^{-1} = e^{\text{ad}_a}(\beta) - d\sigma(\alpha, \beta) = e^{\text{ad}_a}(\beta) \bullet \xi(\text{ad}_{a*b*a^{-1}})(-d\sigma(\alpha, \beta)),$$

where we have used that $d(e^{\text{ad}_a}(\beta)) = e^{\text{ad}_a}(d\beta) = e^{\text{ad}_a}(b) = a * b * a^{-1}$. Recall that, since $a * b * a^{-1}$ is a cycle, $\text{ad}_{a*b*a^{-1}}$ commutes with the differential, and so does $\xi(\text{ad}_{a*b*a^{-1}})$. Therefore

$$\xi(\text{ad}_{a*b*a^{-1}})(-d\sigma(\alpha, \beta)) = -d\xi(\text{ad}_{a*b*a^{-1}})(\sigma(\alpha, \beta)).$$

This motivates the following definition.

Definition 4.3. For $\alpha, \beta \in L_1$ with $d\alpha = a$ and $d\beta = b$, let $\tau(\alpha, \beta) \in L_2$ be the element

$$\tau(\alpha, \beta) = \xi(\text{ad}_{a*b*a^{-1}})(\sigma(\alpha, \beta)).$$

Up to length 4, $\tau(\alpha, \beta)$ is equal to

$$\tau(\alpha, \beta) = \frac{1}{2}[\alpha, \beta] + \frac{1}{6}[a, [\alpha, \beta]] + \frac{1}{6}[\alpha, [a, \beta]] - \frac{1}{8}[b, [\alpha, \beta]] + \dots$$

Then, the conjugation law of Theorem 3.2 becomes

$$\alpha \bullet \beta \bullet \alpha^{-1} = e^{\text{ad}_a}(\beta) \bullet d\tau(\alpha, \beta)^{-1}. \quad (5)$$

Note that if $d\beta = b = 0$ then $a * b * a^{-1} = 0$ and $\tau(\alpha, \beta) = \sigma(\alpha, \beta)$.

5 Models of simplices

The theory of complete differential Lie algebras plays an important role in rational homotopy theory. The *Deligne-Getzler-Hinich ∞ -groupoid* (see [12] for the original article or [11] for a profound study of this topic) is a functor that transforms a cdgl into a simplicial set:

$$\mathrm{MC}_\bullet(L) = \mathrm{MC}(\mathrm{A}_{\mathrm{PL}}(\Delta^\bullet) \widehat{\otimes} L)$$

where $\mathrm{MC}(-)$ denotes the set of Maurer-Cartan elements, $\mathrm{A}_{\mathrm{PL}}(\Delta^\bullet)$ is the simplicial commutative differential graded algebra of polynomial linear forms on the simplices, constructed by Sullivan [18], and $\widehat{\otimes}$ denotes the completion of the tensor product between a commutative algebra and a complete Lie algebra.

This construction was extended in [11] to nilpotent L_∞ -algebras through a deformation retract $\gamma_\bullet(L)$ of $\mathrm{MC}_\bullet(L)$ called the *nerve* of L .

The homotopical properties of this object have been deeply studied. For example, Berglund [2, Theorem 1.1] proved that the homology groups of L are isomorphic to the homotopy groups of $\mathrm{MC}_\bullet(L)$, for any connected nilpotent dgl or L_∞ -algebra, via an explicit isomorphism

$$H_n(L) \xrightarrow{\cong} \pi_{n+1}(\mathrm{MC}_\bullet(L)),$$

for $n \geq 0$. Moreover, in the case of L being positively graded, the MC_\bullet groupoid agrees (up to homotopy) with the classical Quillen's realization of a dgl [14] (see [8, Theorem 2.1] and [3, Theorem 8.1]).

In [7] the authors constructed a cosimplicial cdgl \mathfrak{L}_\bullet that makes the nerve functor corepresentable; this is

$$\langle L \rangle = \mathrm{hom}_{\mathrm{cdgl}}(\mathfrak{L}_\bullet, L) \cong \gamma_\bullet(L) \simeq \mathrm{MC}_\bullet(L).$$

We call the functor $\langle L \rangle = \mathrm{hom}_{\mathrm{cdgl}}(\mathfrak{L}_\bullet, L)$ the *realization* functor. The cosimplicial cdgl object \mathfrak{L}_\bullet also allows the construction of a left adjoint to the realization functor, which we denote by $\mathfrak{L}(-)$ and call the *model* functor, following Quillen's notation (since they both extend, up to homotopy, the Quillen's realization and model functors). The model functor is constructed as a Kan extension, and on objects, it can be computed as the colimit

$$\mathfrak{L}(X) = \varinjlim_{\sigma \in X} \mathfrak{L}_{|\sigma|}.$$

These two functors, form a pair of adjoint functors

$$\mathbf{sset} \begin{array}{c} \xrightarrow{\mathfrak{L}} \\ \xleftarrow{\langle - \rangle} \end{array} \mathbf{cdgl}.$$

As we have seen in the introduction, the cdgl's \mathfrak{L}_n , the central objects in this theory are characterized by the following properties: as graded Lie algebras, they are the (completion)

of the free graded Lie algebra $\widehat{\mathbb{L}}(\Delta^n)$ on the sets of generators $\{a_I\}_I$ with I a non-empty subset of $\{0, 1, \dots, n\}$ of degree $|a_I| = \#I - 2$. The differential δ on $\widehat{\mathbb{L}}(\Delta^n)$ has to satisfy 3 conditions: *a)* the elements of degree -1, corresponding to geometrical vertices, have to be Maurer-Cartan elements, i.e.,

$$\delta a_i = -\frac{1}{2}[a_i, a_i], \quad \text{for } i = 0, \dots, n$$

b) the linear part of the differential has to agree with the ‘cellular boundary operator’. This means

$$\delta^1 a_{i_0, \dots, i_k} = \sum_{j=0}^k (-1)^j a_{i_0, \dots, \widehat{i}_j, \dots, i_k}$$

where, on generators, $\delta = \delta^1 +$ terms of length 2 or greater. And *c)* the faces and degeneracies of the cosimplicial object are induced by those of the cosimplicial graded vector space Δ^\bullet , which in turn are induced by the faces and degeneracies of the cosimplicial simplicial set Δ^\bullet with $\Delta_n^m = \text{hom}_\Delta([n], [m])$.

It is proved in [7, Theorem 6.1] that such object exists (and it is unique, up to complete dgl isomorphism, with the properties *a), b)* and *c)*) and its homotopical and homological properties are deeply studied.

If we modify the condition *c)* to forget about the degeneracies (and only ask that the cofaces are complete dgl morphisms), we obtain the weaker concept of a *sequence of models*. We say that a sequence of models is *inductive* if the differential of the top generator $a_{0, \dots, n}$, for $n \geq 2$, is of the form

$$\delta a_{0, \dots, n} = [a_0, a_{0, \dots, n}] + \Phi, \quad \text{with } \Phi \in \widehat{\mathbb{L}}(\dot{\Delta}^n) \quad (6)$$

where $\dot{\Delta}^n$ is the boundary of Δ^n , i.e.,

$$\dot{\Delta}^n = \text{Span}\{a_I \mid I \subset \{0, \dots, n\}, \emptyset \neq I \neq \{0, \dots, n\}\}.$$

In other words, in an inductive sequence of models the top generator does not appear in the expression of its own differential, except for the term $[a_0, a_{0, \dots, n}]$. Note this kind of requirement only makes sense in dgl’s with negative elements, in the classical setting, it is automatically satisfied.

We are going to work with inductive sequence of models of the simplices instead of with the cosimplicial cdgl satisfying *a), b)* and *c)*. This means that we forget about the codegeneracies and impose the extra condition (6). We do not lose information with this simplification since, by [7, Theorem 6.13], we can define some codegeneracies (agreeing with the canonical ones only up to terms of length greater than or equal to 2) that make the sequence of models an actual cosimplicial cdgl. Moreover, by the uniqueness up to simplicial isomorphisms of these sequence of models [7, Theorem 6.7], this approach gives a cosimplicial object isomorphic to the one in [7, Theorem 6.1].

Remark 5.1. Note that for a sequence of models \mathfrak{L}_\bullet , the requirement that the faces are cdgl morphisms imposes that the differentials of any generator (except for the top one) in \mathfrak{L}_n are already determined by the cdgl's \mathfrak{L}_m with $m < n$. For this, take $I = (i_0 < \dots < i_k) \subsetneq \{0, \dots, n\}$ and consider, in the simplex category Δ , the increasing map

$$f : \{0, \dots, k\} \rightarrow \{0, \dots, n\}, \quad j \mapsto i_j$$

which is a composition of cofaces in Δ . Then, by our hypothesis, the induced map

$$f_* : \mathfrak{L}_k \rightarrow \mathfrak{L}_n, \quad a_{j_0, \dots, j_\ell} \mapsto a_{f(j_0), \dots, f(j_\ell)}$$

is a cdgl morphism and, in particular, it commutes with the differential

$$\delta a_{i_0, \dots, i_k} = \partial(f_*(a_{0, \dots, k})) = f_*(\partial a_{0, \dots, k}).$$

This means that $\delta a_{i_0, \dots, i_k}$ in \mathfrak{L}_n is the same as $\delta a_{0, \dots, k}$ in \mathfrak{L}_k up to a reassignment of the indices.

Therefore, determining an inductive sequence of models consists of defining, recursively for $n = 0, 1, \dots$, an element Φ in $(\widehat{\mathbb{L}}(\dot{\Delta}^n), \partial)$ (with the differential on any generator a_I already defined by the remark above), with the condition that its linear part Φ_1 is of the form

$$\Phi_1 = a_{1, \dots, n} - a_{0, 2, \dots, n} + \dots \pm a_{0, 1, \dots, n-1}$$

and that Φ is a cycle with respect to the perturbed differential $\delta_0 = \delta + [a_0, -]$ (this is a well-defined differential since a_0 is a MC-element, see [7, §4.1] for more details). With these properties, define

$$\delta a_{0, \dots, n} = [a_0, a_{0, \dots, n}] + \Phi,$$

and it is easy to check that this define a cdgl \mathfrak{L}_n where all the conditions of being an inductive sequence of models are satisfied.

Now, we briefly recall the explicit models of \mathfrak{L}_n for $n = 0, 1, 2$ and 3.

- For $n = 0$, we have $\mathfrak{L}_0 = (\widehat{\mathbb{L}}(a_0), \delta)$ with

$$\delta a_0 = -\frac{1}{2}[a_0, a_0]$$

since it has to be a MC-element.

- For $n = 1$, $\mathfrak{L} = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), \delta)$ with a_0, a_1 being MC-elements. This imposes that

$$\delta a_{01} = [a_{01}, a_1] + \sum_{k \geq 0} \frac{B_k}{k!} \text{ad}_{a_{01}}^k (a_1 - a_0).$$

Here B_k are the Bernoulli numbers, which we have already seen in section 4. This complete dgl is known as the *Lawrence-Sullivan interval* due to the authors that originally defined it [19] (see also [7, §5]).

One important property of the Lawrence-Sullivan interval is that the two MC-elements a_0 and a_1 are related by the gauge action of a_{01} . In particular, this implies [7, Proposition 4.24] that

$$\delta_0 e^{01}(x) = e^{01}(\delta_1 x) \quad (7)$$

for all $x \in \mathfrak{L}_1$. Here we are using the (abusing) notation $e^{ij} = e^{\text{ad}_{a_{ij}}}$. We also introduce the notation $\delta_i = \delta_{a_i}$ for $i = 0, \dots, n$. Do not confuse δ_1 with the linear part of the differential δ^1 .

Another important property of the exponential maps that we will use is that [7, Corollary 4.12]

$$e^{\text{ad}_x} \circ e^{\text{ad}_y} = e^{\text{ad}_{x*y}}$$

for all $x, y \in L_0$.

- For $n = 2$, we have that (one) model of the triangle is given by the formula

$$\delta_0(a_{012}) = a_{01} * a_{12} * a_{02}^{-1}.$$

Note that the linear part of the BCH product is just the sum, so we recover the correct linear part. Moreover, using the properties of the Lawrence-Sullivan interval, it can be proved that it is a δ_0 -cycle [7, Proposition 5.14].

- For $n = 3$, in [7, §6.5], an explicit model for the tetrahedron $\mathfrak{L}_3 = (\widehat{\mathbb{L}}(\Delta^3), \delta)$ is built by means of a generic integral of the BCH product, that is, any operation \diamond in L_1 with $d(x \diamond y) = dx * dy$. However, for our goals (going further until \mathfrak{L}_4) we need to slightly modify that formula. Firstly, we fix the operation \bullet of section §1 as our integration of the BCH. In particular, since this operation is associative, there is no need of writing parenthesis.

Proposition 5.2. *There is a model for the tetrahedron $\mathfrak{L}_3 = (\widehat{\mathbb{L}}(\Delta^3), \delta)$ with*

$$\delta_0(a_{0123}) = e^{01}(a_{123}) \bullet a_{013} \bullet a_{023}^{-1} \bullet a_{012}^{-1}.$$

The \bullet product is performed in the complete dgl $(\widehat{\mathbb{L}}(\Delta^3), \delta_0)$, this means that $\delta_0(x \bullet y) = \delta_0 x \bullet \delta_0 y$.

Proof. We have to check that $\Phi = e^{01}(a_{123}) \bullet a_{013} \bullet a_{023}^{-1} \bullet a_{012}^{-1}$ has the correct linear part and that it is a δ_0 -cycle. By the expression of the \bullet product as a formal series (1), we see that the linear part is the sum of the linear part of the terms. Moreover, the linear part of the exponential map is the identity, so we conclude that

$$\Phi_1 = a_{123} + a_{013} - a_{023} - a_{012}$$

which was the desired expression.

Now we compute $\delta_0\Phi$. Since the \bullet product is an integral of the BCH product (Proposition 2.1), then

$$\delta_0(\Phi) = \delta_0(e^{01}(a_{123})) \bullet \delta_0(a_{013}) \bullet \delta_0(a_{023}^{-1}) \bullet \delta_0(a_{012}^{-1}) = e^{01}(\delta_1 a_{123}) \bullet \delta_0 a_{013} \bullet (\delta_0 a_{023})^{-1} \bullet (\delta_0 a_{012})^{-1},$$

where we have used (7) to permute the differential and the exponential. As we saw in Remark 5.1, the differential of any element (except for a_{0123}) is determined by the models of the subsimplices. In particular, using the model of the triangle presented above, we see that, for $i < j < k$ in $\{0, 1, 2, 3\}$,

$$\delta_i a_{ijk} = a_{ij} \bullet a_{jk} \bullet a_{ik}^{-1}.$$

Replace this in the expression above to obtain

$$\delta_0(\Phi) = e^{01}(a_{12} \bullet a_{23} \bullet a_{13}^{-1}) \bullet a_{01} \bullet a_{13} \bullet a_{23}^{-1} \bullet a_{12}^{-1} \bullet a_{01}^{-1}.$$

Finally, use (2) to see that this term cancels. □

- For $n = 4$, we construct a model for the pentahedroid. The key results for obtaining them are the conjugation laws (3) and (5).

Theorem 5.3. *There is a model for the pentahedroid $\mathfrak{L}_4 = (\widehat{\mathbb{L}}(\Delta^4), \delta)$ with*

$$\begin{aligned} \delta_0(a_{01234}) &= \tau(a_{012}, a_{024} \bullet a_{034}^{-1} \bullet a_{023}^{-1}) + e^{01 \bullet 12 \bullet 20}(a_{0234}^{-1}) + e^{01}(a_{1234}) + \\ &+ a_{0123} + \varepsilon^{01 \bullet 12 \bullet 23 \bullet 31 \bullet 10}(a_{0134}) + \varepsilon^{01 \bullet 12 \bullet 23 \bullet 34 \bullet 42 \bullet 21 \bullet 10}(a_{0124}^{-1}). \end{aligned}$$

In this theorem and its proof we are using the following simplifications on the notation. We write $e^x = e^{\text{ad}_x}$ for any element x of degree 0, $e^{ij} = e^{a_{ij}}$, $e^{ij \bullet jk} = e^{a_{ij} \bullet a_{jk}}$, and if $i < j$ then a_{ji} denotes a_{ij}^{-1} . We use analogous notation for the function $\varepsilon^x = \varepsilon(\text{ad}_x)$. The \bullet products are supposed to be performed with respect to the differential δ_0 ; otherwise we write \bullet_i to indicate that they performed with respect to the differential δ_i .

Proof. Call Φ our candidate to be the differential of a_{01234} . We start computing the differential of Φ . Recall that on cycles the \bullet product becomes the sum. If we write the six terms of Φ as $\chi_1 + \chi_2 + \dots$, then

$$\delta_0(\chi_1 + \chi_2 + \dots) = \delta_0\chi_1 \bullet \delta_0\chi_2 \bullet \dots$$

or in any other order that we desire (since the sum is commutative). We analyze separately the differential of the first and of the last 3 terms.

The differential of the first 3 terms is equal to

$$\delta_0 \tau(a_{012}, a_{024} \bullet a_{034}^{-1} \bullet a_{023}^{-1}) \bullet \delta_0 e^{01*12*20}(a_{0234}^{-1}) \bullet \delta_0 e^{01}(a_{1234}).$$

We now study the second term: $e^{01*12*20}$ commutes with the differential (see (7) and that $\delta_0 a_{012} = a_{01} * a_{12} * a_{02}^{-1}$ is a cycle, by the model of the triangle). So using the model of the tetrahedron we get

$$\delta_0 e^{01*12*20}(a_{0234}^{-1}) = e^{01*12*20}(\delta_0 a_{0234}^{-1}) = e^{01*12*20}(a_{023} \bullet a_{034} \bullet a_{024}^{-1} \bullet e^{02}(a_{234}^{-1})).$$

Since $e^{01*12*20}$ is a cdgl morphism and the \bullet product is preserved by morphisms we deduce that this is equal to

$$e^{01*12*20}(a_{023} \bullet a_{034} \bullet a_{024}^{-1}) \bullet e^{01*12}(a_{234}^{-1}).$$

Here, we have used that $e^{01*12*20} \circ e^{02} = e^{01*12*20*02}$. Using the (inverse) of (5) we see that the sum of the first two terms is

$$\begin{aligned} \delta_0 \tau(a_{012}, a_{024} \bullet a_{034}^{-1} \bullet a_{023}^{-1}) \bullet e^{01*12*20}(a_{023} \bullet a_{034} \bullet a_{024}^{-1}) \bullet e^{01*12}(a_{234}^{-1}) = \\ = a_{012} \bullet a_{023} \bullet a_{034} \bullet a_{024}^{-1} \bullet a_{012}^{-1} \bullet e^{01*12}(a_{234}^{-1}). \end{aligned}$$

We now analyze the third term

$$\delta_0(e^{01}(a_{1234})) = e^{01}(\delta_1 a_{1234}) = e^{01}(e^{12}(a_{234}) \bullet_1 a_{124} \bullet_1 a_{134}^{-1} \bullet_1 a_{123}^{-1})$$

where we have used the equation (7) and the Remark 5.1. Since $e^{01} : (\mathfrak{L}_4, \delta_0) \rightarrow (\mathfrak{L}_4, \delta_1)$ is a cdgl morphism [7, Proposition 4.24] and the \bullet product commutes with morphisms, we deduce that this is equal to

$$e^{01*02}(a_{234}) \bullet e^{01}(a_{124}) \bullet e^{01}(a_{134}^{-1}) \bullet e^{01}(a_{123}^{-1}).$$

Therefore, the sum of the 3 first terms is

$$a_{012} \bullet a_{023} \bullet a_{034} \bullet a_{024}^{-1} \bullet a_{012}^{-1} \bullet e^{01}(a_{124}) \bullet e^{01}(a_{134}^{-1}) \bullet e^{01}(a_{123}^{-1}). \quad (*)$$

We now consider the rest of the terms. The differential of the fourth one is simply

$$\delta_0 a_{0123} = e^{01} a_{123} \bullet a_{013} \bullet a_{023}^{-1} \bullet a_{012}^{-1}.$$

In the fifth term, apply Corollary 3.3 to obtain

$$\begin{aligned} \delta_0 \varepsilon^{01*12*23*31*10}(a_{0134}) = \varepsilon^{01*12*23*31*10}(\delta_0 a_{0134}) = e^{01}(a_{123}) \bullet (\delta_0 a_{0134}) \bullet e^{01}(a_{123}^{-1}) = \\ = e^{01}(a_{123}) \bullet e^{01}(a_{134}) \bullet a_{014} \bullet a_{034}^{-1} \bullet a_{013}^{-1} \bullet e^{01}(a_{123}^{-1}). \end{aligned}$$

Note that $x = a_{01} * a_{12} * a_{23} * a_{31} * a_{10}$ is a cycle and that, therefore, δ_0 commutes with ε^x . Analogously for the sixth term we get

$$\begin{aligned} & \varepsilon^{01*12*23*34*42*21*10}(\delta_0 a_{0124}^{-1}) = \\ & = e^{01}(a_{123}) \bullet e^{01}(a_{134}) \bullet e^{01}(a_{124}^{-1}) \bullet \delta_0 a_{0124}^{-1} \bullet e^{01}(a_{124}) \bullet e^{01}(a_{134}^{-1}) \bullet e^{01}(a_{123}^{-1}) = \\ & = e^{01}(a_{123}) \bullet e^{01}(a_{134}) \bullet e^{01}(a_{124}^{-1}) \bullet a_{012} \bullet a_{024} \bullet a_{014}^{-1} \bullet e^{01}(a_{134}^{-1}) \bullet e^{01}(a_{123}^{-1}). \end{aligned}$$

Now sum the sixth, fifth and fourth terms (in that order) to obtain

$$e^{01}(a_{123}) \bullet e^{01}(a_{134}) \bullet e^{01}(a_{124}^{-1}) \bullet a_{012} \bullet a_{024} \bullet a_{034}^{-1} \bullet a_{023}^{-1} \bullet a_{012}^{-1}. \quad (**)$$

Compare this expression (**) with the sum of the 3 first terms computed above (*) to conclude that both expressions cancel: $\delta_0 \Phi = (*) \bullet (**) = 0$.

We finalize the proof by checking that Φ has the correct linear part: note that $\tau(\alpha, \beta)$ has $\frac{1}{2}[\alpha, \beta]$ as the term of lowest length and, thus, it does not contribute to the linear part of Φ . On the other hand, the linear parts of both the exponential $e^{\text{ad}_x}(\alpha)$ and $\varepsilon^{\text{ad}_x}(\alpha)$ are α . Then, we conclude that

$$\Phi_1 = a_{0234}^{-1} + a_{1234} + a_{0134} + a_{0123} + a_{0124}^{-1},$$

which is the desired linear term. □

References

- [1] Rüdiger Achilles and Andrea Bonfiglioli. The early proofs of the theorem of Campbell, Baker, Hausdorff, and Dynkin. *Archive for History of Exact Sciences*, 66, 2012.
- [2] Alexander Berglund. Rational homotopy theory of mapping spaces via Lie theory for L_∞ -algebras. *Homology, Homotopy and Applications*, 17:343–369, 2015.
- [3] Alexander Berglund. On exponential groups and Maurer-Cartan spaces. *arXiv*, 2023, 2305.06904.
- [4] Andrea Bonfiglioli and Roberta Fulci. *Topics in Noncommutative Algebra: The Theorem of Campbell, Baker, Hausdorff and Dynkin*, volume 2034. 2012.
- [5] Ricardo Campos and Albin Grataloup. Operadic deformation theory. *arXiv*, 2024, 2307.11187.
- [6] Vladimir Dotsenko, Sergey Shadrin, and Bruno Vallette. *Maurer-Cartan Methods in Deformation Theory: The Twisting Procedure*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2023.
- [7] Yves Félix, Urtzi Buijs, Aniceto Murillo, and Daniel Tanré. *Lie Models in Topology*. 2020.
- [8] Yves Félix, Mario Fuentes, and Aniceto Murillo. All known realizations of complete lie algebras coincide. *arXiv*, 2023, 2207.10886.
- [9] Yves Félix and Daniel Tanré. Spatial realization of a Lie algebra and Bar construction of a group. *arXiv*, 2021, 1905.11347.
- [10] Yves Félix and Daniel Tanré. Realization of Lie algebras and classifying spaces of crossed modules. *Algebraic & Geometric Topology*, 24:141–157, 2024.
- [11] Ezra Getzler. Lie theory for nilpotent L_∞ -algebras. *Annals of Mathematics*, 170:271–301, 2009.
- [12] Vladimir Hinich. Descent of Deligne groupoids. *International Mathematics Research Notices*, 1997(5):223–239, 1997.
- [13] Anatoly I. Maltsev. On a class of homogeneous spaces. *Izv. Akad. Nauk SSSR Ser. Mat.*, 13:9–32, 1949.
- [14] Daniel Quillen. Rational homotopy theory. *Annals of Mathematics*, 90(2):205–295, 1969.
- [15] Christophe Reutenauer. *Free Lie Algebras*. Oxford University Press, 1993.

- [16] Daniel Robert-Nicoud. Representing the deformation ∞ -groupoid. *Algebraic & Geometric Topology*, 19:1453–1476, 2019.
- [17] Daniel Robert-Nicoud and Bruno Vallette. Higher Lie theory. *arXiv*, 2020, 2010.10485.
- [18] Dennis Sullivan. Infinitesimal computations in topology. *Publications Mathématiques de l’IHÉS*, 47(1):269–331, 1977.
- [19] Dennis Sullivan and Ruth Lawrence. A formula for topology/deformations and its significance. *Fundamenta Mathematicae*, 225(0):229–242, 2014.

UNIVERSITÉ PAUL SABATIER, INSTITUT DE MATHÉMATIQUES DE TOULOUSE, 118,
ROUTE DE NARBONNE, F-31062 TOULOUSE CEDEX, FRANCE
E-mail address: `mario.fuentes.rumi@gmail.com`