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# Sparse Tensors and Subdivision Methods for Finding the Zero Set of Polynomial Equations

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**Abstract.** Finding the solutions to a system of multivariate polynomial equations is a fundamental problem in mathematics and computer science. It involves evaluating the polynomials at many points, often chosen from a grid. In most current methods, such as subdivision, homotopy continuation, or marching cube algorithms, polynomial evaluation is treated as a black box, repeating the process for each point. We propose a new approach that partially evaluates the polynomials, allowing us to efficiently reuse computations across multiple points in a grid. Our method leverages the Compressed Sparse Fiber data structure to efficiently store and process subsets of grid points. We integrated our amortized evaluation scheme into a subdivision algorithm. Experimental results show that our approach is efficient in practice. Notably, our software `voxelize` can successfully enclose curves defined by two trivariate polynomial equations of degree 100, a problem that was previously intractable.

**Keywords:** Subdivision, sparse tensor, polynomials, root finding

## 1 Introduction

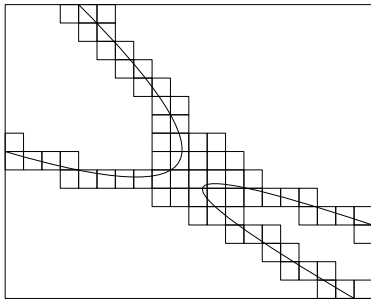
Subdivision algorithms are widely used to enclose the zero set of a function  $F$  ([17, 18, 21, 24, 27] among others). They roughly consist in evaluating  $F$  on boxes created along a subdivision tree. If the input function is a high degree polynomial, one of the bottlenecks of those algorithms is the time required to evaluate  $F$ . We propose a new approach that amortizes the evaluation cost over the boxes created in a subdivision algorithm. It combines on the one hand partial evaluations of the input polynomial with interval arithmetics, and on the other hand sparse tensors [5, 26] to store the boxes created during the subdivision algorithm. This approach was implemented in the software `voxelize`, and the source code is available on gitlab<sup>1</sup>. Experimental results show that this software can enclose the zero set of polynomial systems that were not reachable with state-of-the-art software.

After giving an overview of our main results in the introduction, we present in Section 2.1 the Compressed Sparse Fiber data structure and we show in Section 2.2 how it can be used to evaluate efficiently a polynomial on a subset

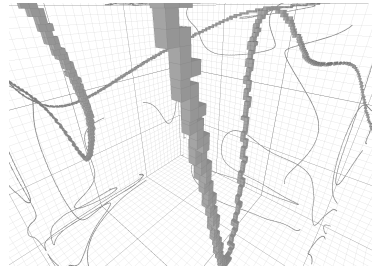
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<sup>1</sup> <https://gitlab.inria.fr/gmoro/voxelize>

of a grid of boxes. Then in Section 3.1, we show how our new evaluation scheme yields a quasi-linear time algorithm to compute a discrete Fourier transform. We show in Section 3.2 how to integrate our evaluation scheme into a subdivision algorithm to enclose the zero set of a polynomial system. Finally, in Section 4, we present the timing results of `voxelize` on several polynomial systems, including random polynomial systems (Section 4.1), and systems coming from applications (Section 4.2).



**Fig. 1.** Boxes on the same level of the subdivision tree



**Fig. 2.** Enclosing of a curve defined by 2 trivariate polynomials of degree 100

### 1.1 Amortized Evaluation on a Grid of Boxes

The first idea to reduce the evaluation redundancies is to use partial evaluation. Assume that  $F(x_1, x_2)$  is a bivariate polynomial of degree  $d$ . Moreover, let  $(I_i)_{0 \leq i < n}$  and  $(J_j)_{0 \leq j < n}$  be two sequences of real intervals. Using the Hörner scheme, evaluating  $F$  on a box requires  $O(d^2)$  arithmetic operations, and evaluating  $F$  on all the boxes  $I_i \times J_j$  for  $0 \leq i, j < n$  requires  $O(d^2 n^2)$  arithmetic operations. By reorganizing the operations using partial evaluations, the number of arithmetic operations can be reduced to  $O(dn(d+n))$ . This idea is well known and was used for example to speed up the multiplication of polynomials [23]. It is also currently implemented in the well-spread library NumPy to evaluate polynomials in 2 and 3 variables [9].

More precisely the operations are reordered as follows. For a given  $I_i$ , the partial evaluation of  $F$  in  $I_i$  results in a univariate polynomial  $f_i$  of degree  $d$ . This step requires  $O(d^2)$  arithmetic operations. Then evaluating  $f_i$  on  $n$  intervals requires  $O(dn)$  arithmetic operations. Finally, repeating these operations for all the  $n$  intervals  $I_i$ , this allows us to evaluate  $F$  on all the boxes of the grid with a total number of arithmetic operations in  $O(dn(d+n))$ . More generally, for higher dimensions, this leads to the following result.

*Property 1* ([23]). Let  $F$  be a polynomial in  $k$  variables and of degree at most  $d-1$  in each variable. Let  $X_1, \dots, X_k$  be  $k$  sets of  $n$  real intervals each. Then it is

possible to evaluate  $F$  on all the boxes of  $X_1 \times \cdots \times X_k$  in  $O(kdn \max(n, d)^{k-1})$  arithmetic operations.

In the case where  $n > d$ , this approach results in a significant speedup since the amortized number of arithmetic operations to evaluate  $F$  on each box of the grid is  $O(kd)$  instead of  $O(d^k)$ .

## 1.2 Amortized Evaluation on a Sparse Subset of a Grid

For the simple subdivision algorithm mentioned at the beginning of the introduction, if the boxes created are never discarded, then each level of the subdivision tree forms a dense grid of boxes. In this case, the partial evaluation approach shown in the previous section can be applied directly to reduce the total number of arithmetic operations required to evaluate  $F$  on each box with interval methods. In the general case though, many boxes are discarded, and the boxes appearing in a given level of the subdivision tree form a subset of a grid, as shown in Figure 1. The boxes created in the subdivision algorithm can be handled in different orders. Using a breadth-first walk on the subdivision, the boxes on the same level are a subset of a grid. In this case, we need to evaluate a polynomial on a sparse subset of a grid.

To evaluate a polynomial on a general set of points, the case of a univariate polynomial is well understood [6, 10, 16, 20]. For multivariate polynomials, there are fewer results that are efficient in practice when the points are not arranged as a grid. A breakthrough, that was recently improved, is a quasi-linear algorithm to evaluate a polynomial of degree  $d$  in  $k$  variables on  $d^k$  points in a finite field [1, 2, 12, 19, 28]. For multipoint evaluation with real numbers, the only subquadratic algorithms are for bivariate polynomials [22], or require precomputation more than quadratic in the number of points [13, 14]. Finally, a recent work addresses the case of approximate numerical evaluation [7]. Unfortunately, those approaches are not yet efficient in practice. Our main result is a practical improvement to amortize multipoint evaluations in the case where the points or boxes that we consider are a sparse subset of a grid.

Boxes in a sparse subset of a grid can be gathered and stored as a sparse tensor in the Compressed Sparse Fiber (CSF) format [5, 26]. The CSF is a generalization of the Compressed Row Format used to store the entries of a sparse matrix. Then,  $F$  can be evaluated efficiently on these boxes (see Section 2.2 for more details). This approach was implemented in the library `voxelize`. Figure 2 shows the output boxes of the software `voxelize` enclosing an algebraic curve defined by two polynomial equations of degree 100, where the coefficients are randomly drawn from a normal law centered at zero. Performing the partial evaluation approach on a set of boxes in a CSF format leads to Theorem 1.

## 1.3 Notations

For a set  $E$ , we denote by  $|E|$  its number of elements. Then, we define the notations for the size of the projection of a subset  $E$  of a grid. In particular, the

size of the projection is smaller when the elements of  $E$  are aligned within the grid.

**Definition 1.** Given a finite set  $E \subset \mathbb{N}^k$ , and an integer  $i$  between 1 and  $k$ , we denote by  $N_i(E)$  (resp.  $\tilde{N}_i(E)$ ) the number of elements in the projection  $E$  on the first (resp. last)  $i$  coordinates, counting repeated projections only once.

Even though this definition holds for a set of integer tuples, it can be naturally extended for multivariate polynomials. Indeed, for each monomial, we can associate its vector of exponents. If  $F$  is a polynomial in  $k$  variables, for a given integer  $i$ , we can define  $N_i(F)$  (resp.  $\tilde{N}_i(F)$ ) as the size of the projections of the set of vectors of exponents of  $F$  to their first (resp. last)  $i$  coordinates.

For  $S$  a set of points or boxes that is a subset of a grid, we can also extend the definition of  $N_i$  by simply indexing the elements of  $S$  by their integer positions in the grid. Letting  $S_{ind}$  be the set of integer indices of the boxes of  $S$ , we can define  $N_i(S)$  by  $N_i(S_{ind})$ .

#### 1.4 Main Result

We can now state our main theorem to evaluate a multivariate polynomial on a set of boxes that is a sparse subset of a grid of boxes  $G$  that is the Cartesian product of  $k$  sets of intervals  $X_1 \times \cdots \times X_k$ .

**Theorem 1.** Let  $F$  be a polynomial in  $k$  variables, and  $S$  be a subset of boxes of  $G$ . It is possible to evaluate  $F$  on all the boxes of  $S$  in  $O(\sum_{i=0}^{k-1} \tilde{N}_{k-i}(F)N_{i+1}(S))$  arithmetic operations.

When the set of boxes enclose a variety of dimension  $j$ , the projection of  $S$  on the first  $j$  coordinates is often a dense grid. In this case, we have the following corollary.

**Corollary 1.** For  $1 \leq j \leq k - 1$ , assume that the projection of  $S$  on the first  $j$  coordinates is:

- i. a dense grid, denoted by  $X_1 \times \cdots \times X_j$
- ii.  $|X_i| > d$  for all  $1 \leq i \leq j$ .

Then we can evaluate each box of  $S$  in  $O(j(d+1)^{k-j+1})$  arithmetic operations on average, instead of  $O((d+1)^k)$  operations.

*Proof (Corollary 1).* First, if  $F$  has degree at most  $d$  in each variable, then  $\tilde{N}_{k-i}(F)$  is less than  $d^{k-i}$  for all non-negative integers less or equal to  $k$ .

For  $0 \leq i < j$ , Assumption i implies that  $N_{i+1}(S) = N_i(S)|X_{i+1}|$ . Then we deduce with Assumption ii. that  $(d+1)N_i(S) \leq N_{i+1}(S)$ . This implies that:

$$\begin{aligned} \tilde{N}_{k-i}(F)N_{i+1}(S) &\leq (d+1)^{k-i}N_{i+1}(S) \\ &\leq (d+1)^{k+1-j} \\ N_j(S) &\leq (d+1)^{k+1-j}|S|. \end{aligned}$$

For  $i \geq j$  we have  $\tilde{N}_{k-i}(F) \leq (d+1)^{k-i}$ , such that  $\tilde{N}_{k-i}(F)N_{i+1}(S) \leq (d+1)^{k-i}|S|$ . Thus, the evaluation of  $F$  on all the boxes of  $S$  is in

$$O(j(d+1)^{k-j+1}|S| + \sum_{i=j}^{k-1} (d+1)^{k-i}|S|) = O(j(d+1)^{k-j+1}).$$

In particular, the amortized cost of evaluating each box is in  $O(jd^{k-j+1})$  arithmetic operations instead of  $O(d^k)$  with a direct algorithm.

## 2 Evaluating Polynomials with Compressed Sparse Fibers

### 2.1 Sparse Tensor Data Structure

The main data structure used in our algorithms is the Compressed Sparse Fiber, as described in [5, 26]. This data structure is well suited to store a subset of a grid in high dimension. It can be seen as a generalization of the classical Compressed Sparse Row data structure used to store the entries of a sparse matrix as in Figure 3.

	0	1	2	3	4	5
0	5	1				
1	7	3				
2						
3	8			4	9	

**Fig. 3.** Numbers stored in a sparse matrix

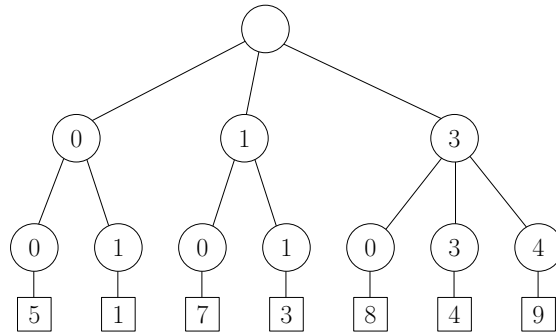
For a subset of a 2D grid, the data structure is a labeled tree that stores the positions of the non-empty rows in the children of the root node, and then in each row, the position of the non-empty entries are stored in the children of the corresponding node (Figure 4). In higher dimension  $k$ , this idea is applied recursively. Let  $E$  be a subset of points in  $\mathbb{N}^k$ . For  $t = (t_1, \dots, t_\ell) \in \mathbb{N}^\ell$  a tuple of size  $\ell < k$ , we denote by  $\pi_t(E)$  the subset of  $\mathbb{N}$  defined by:

$$\pi_t(E) = \{i \in \mathbb{N} \mid \exists y_{\ell+2}, \dots, y_k \in \mathbb{N} \text{ such that } (t_1, \dots, t_\ell, i, y_{\ell+2}, \dots, y_k) \in E\}.$$

Then the Compressed Sparse Fiber (or CSF) data structure associated to  $E$  is a labeled tree of depth  $k$  defined recursively as follows. The root of the tree

is at depth 0 and its children are the nodes labeled by the elements of  $\pi_\emptyset(E)$ , where  $\emptyset$  denotes the empty tuple. Consider now a node  $N$  of the tree at depth  $1 \leq \ell < k$ . Let  $t(N)$  be the tuple of size  $\ell$ , where the  $i$ -th coordinate is the label of the  $i$ -th node on the path from the root to  $N$ . Then the children of  $N$  are the nodes labeled by the elements of  $\pi_{t(N)}(E)$ . Finally, for a node  $N$  at depth  $k$ , it is possible to add a leaf that can be labeled with the value of the entry associated to the tuple  $t(N)$ . Given a CSF data structure, the corresponding set of tuple  $E$  is unique and is called its *support*.

As an example, using the compressed sparse data structure to store the sparse matrix given in Figure 3, we get the tree shown in Figure 4, and its support is  $\{(0, 0), (0, 1), (1, 0), (1, 1), (3, 0), (3, 3), (3, 4)\}$ .



**Fig. 4.** Compressed Sparse Fiber associated to the sparse matrix

*Remark 1.* Given  $T$  a CSF data structure associated to  $E$ , remark the size of the projection on the first  $i$  coordinates is the number of nodes of depth  $i$  in  $T$ . In particular, we have  $N_i(E) = \sum_{t \in \mathbb{N}^{i-1}} |\pi_t(E)|$ .

**Representing a Multivariate Polynomial.** A natural application of the Compressed Sparse Fiber data structure is to encode the monomials of a sparse polynomial. Given a polynomial in  $k$  variables, the exponents of each monomial can be represented as a  $k$ -tuple of integers in  $\mathbb{N}^k$ , and the coefficients can be represented as the entries associated to each tuple. Using this representation, it can be directly encoded in a CSF data structure. Given a polynomial  $F(x_1, \dots, x_k)$ , we denote by  $T_F$  the CSF tree associated to  $F$ . By extension of Definition 1, we define the size  $N_i(F)$  as the size  $N_i(T_F)$  of its corresponding CSF tree truncated to depth  $i$ .

For example the following polynomial in two variables would be encoded with the CSF tree in Figure 4:

$$5 + x_2 + 7x_1 + 3x_1x_2 + 8x_1^3 + 4x_1^3x_2^3 + 9x_1^3x_2^4$$

**Representing a Set of Boxes.** A sparse subset of a grid of boxes can also be represented with the Compressed Sparse Fiber data structure, by applying it to the indexes of the boxes within the grid. Without loss of generality, consider a subdivision of the unit box  $[0, 1]^k$  into  $n^k$  smaller boxes, where each box is a product of intervals of the form  $\prod_{i=1}^k [a_i, b_i]$ , where  $a_i$  and  $b_i$  are real numbers. In the case where the subdivision is uniform, let  $G_n$  be the set of these  $n^k$  boxes. Each cube from  $G_n$  can be indexed by a  $k$ -tuple of integers in  $\mathbb{N}^k$ . In particular, for a sparse subset of  $G_n$ , we can associate the set  $B$  of the indices of its boxes. Then,  $B$  can be encoded in a CSF data structure. In this case, the tree we construct won't have leaves since there is no entry associated to each box.

## 2.2 Evaluation Algorithm

**One Variable.** A classical way to evaluate a univariate polynomial on a point is the Hörner algorithm that we recall in Algorithm 1 for the evaluation of a sparse polynomial on an interval.

---

### Algorithm 1: Hörner algorithm

---

**Input:** An interval  $I$  and a polynomial  $F(x) = a_0x^{e_0} + \dots + a_\ell x^{e_\ell}$  where:  
 $e_0 < \dots < e_\ell$  are integers  
 $a_0, \dots, a_\ell$  are real numbers or intervals.

**Output:** The interval obtained by evaluating  $F$  on  $I$  with the Hörner scheme.

```

1  $J \leftarrow a_\ell$ 
2 for  $j$  from  $\ell - 1$  to 0 do
3    $J \leftarrow J \times I^{e_{j+1} - e_j} + a_j$ 
4 return  $J$ 

```

---

**Several Variables.** For multivariate polynomials  $F$ , we can use the Hörner scheme recursively. Moreover, if we want to evaluate  $F$  on a set of boxes, Algorithm 2 generalizes the Hörner scheme to the case where  $F$  and the boxes are stored in a CSF data structure. The key idea in Algorithm 2 is that for boxes that share the same coordinate, we only evaluate the polynomial partially on those coordinates. Then we reuse those partially evaluated polynomials to evaluate the boxes on the remaining coordinates.

The advantage of using the approach in Algorithm 2 is that it allows us to amortize the cost of the evaluation when several boxes have the same projection. In the following, we will prove that the complexity of Algorithm 2 is in  $O(\sum_{i=0}^{k-1} \tilde{N}_{k-i}(F)N_{i+1}(S))$  arithmetic operations, which will prove Theorem 1.

*Proof (Theorem 1).* Since Algorithm 2 is recursive, we will prove its complexity by recurrence. Algorithm 2 is a loop over the nodes of the root of  $T$ . In particular,



---

**Algorithm 2:** Evaluation on a set of boxes
 

---

**Input:**  $F$  a polynomial in  $k$  variables  
 $T$  a CSF tree representing the indices of a subset  $S$  of boxes of a grid  
 $X_1 \times \cdots \times X_k$ , where  $X_i$  is a set of intervals.  
**Output:** A CSF data structure representing the evaluation of the polynomial  
 represented by  $F$  on all the boxes in  $S$ .

```

1 Function EvaluationCSF( $F, T$ ):
2    $X_1 \leftarrow$  the list of intervals of the first coordinate in the grid  $G$ 
3    $L \leftarrow$  empty list
4   for  $i$  in  $\pi_0(T)$  do
5      $I \leftarrow$  the interval of index  $i$  in  $X_1$ 
6      $F_I \leftarrow F(I, x_2, \dots, x_n)$ 
7     if  $F$  is univariate
8        $\lfloor$  Append  $F_I$  to  $L$ 
9     else
10       $T_i \leftarrow$  the subtree of  $T$  rooted at the node at depth 1 with label  $i$ 
11       $L_i \leftarrow$  EvaluationCSF( $F_I, T_i$ )
12       $\lfloor$  Append  $L_i$  to  $L$ 
13  return  $L$ 

```

---

this loop will be called  $N_1(S)$ . In each loop, the dominating complexities are in line 6 and 11. In line 6, the complexity of evaluating partially  $F$  in one variable  $x_1$  is  $\tilde{N}_k(F)$ . Thus, the total complexity carried by line 6 is in  $O(\tilde{N}_k(F)N_1(S))$ . And if  $F$  is univariate, the complexity of Algorithm 2 is in  $O(\tilde{N}_1(F)N_1(S))$ .

Then, if  $F$  is a polynomial in  $k$  variables with  $k > 1$ , the number of operations is again carried by lines 6 and 11. Let  $S_I$  be the set of boxes represented by the tree  $T_I$ . By recurrence the number of operations in line 11 is in

$$O\left(\sum_{i=0}^{k-2} \tilde{N}_{k-1-i}(F_I)N_{i+1}(S_I)\right).$$

In particular, remark that  $\tilde{N}_{k-1-i}(F_I) = \tilde{N}_{k-1-i}(F)$ . And using Remark 1, the sum of the  $N_{i+1}(S_I)$  on all the intervals  $I$  children of the root of  $T$  is equal to  $N_{i+2}(S)$ . Thus, the complexity of Algorithm 2 carried by line 11 is  $O\left(\sum_{i=0}^{k-2} \tilde{N}_{k-1-i}(F)N_{i+2}(S)\right)$ . By changing the index of the sum, this complexity becomes

$$O\left(\sum_{i=1}^{k-1} \tilde{N}_{k-i}(F)N_{i+1}(S)\right).$$

Since the complexity carried by line 6 is  $O(\tilde{N}_k(F)N_1(S))$ , this concludes the proof.

### 3 Applications

#### 3.1 The Fast Fourier Transform Revisited

Given a vector  $u$  of  $d+1$  complex numbers  $u_0, \dots, u_d$ , its discrete Fourier Transform is the vector  $v$  of  $d+1$  complex numbers  $v_0, \dots, v_d$  such that:

$$v_k = \sum_{j=0}^d u_j e^{-i2\pi \frac{k}{d+1} j} \quad (1)$$

The fast Fourier Transform algorithm returns the vector  $v$  using  $O(d \log d)$  arithmetic operations. If we reinterpret Equation (1) as the evaluation of a multivariate polynomial on a set of points stored with a CSF tree data structure, we can use Algorithm 2 to compute the discrete Fourier transform in  $O(d \log d)$  arithmetic operations.

Without restriction of generality, assume that there exists an integer  $k$  such that  $d+1 = 2^k$  is a power of two. Let  $F$  be the polynomial in  $k$  variables defined by:

$$F = \sum_{(i_1, \dots, i_k) \in \{0,1\}^k} u_{i_1 + \dots + i_k 2^{k-1}} x_1^{i_1} \cdots x_k^{i_k}$$

Moreover, let  $w$  be the  $(d+1)$ -th root of unity  $e^{-i2\pi/(d+1)}$ . For  $1 \leq j \leq k$ , let  $X_j = \{1, w^{2^{k-j}}\}$ , and let  $G$  be the grid of points  $g_{i_1, \dots, i_k}$  in  $\mathbb{C}^k$  for  $(i_1, \dots, i_k) \in \{0,1\}^k$ , defined by:

$$g_{i_1, \dots, i_k} = (w^{i_1 2^{k-1}}, \dots, w^{i_k}) \in X_1 \times \cdots \times X_k$$

Then, using the notations of Equations (1), for an integer  $j = i_1 2^{k-1} + \dots + i_k$  we have  $v_j = F(g_{i_1, \dots, i_k})$ . The polynomial  $F$  has a degree at most 1 in each variable and the set of points on which  $F$  is evaluated is a Cartesian product  $X_1 \times \cdots \times X_k$  where  $X_j$  has size 2 for all  $1 \leq j \leq k$ . Then, using Claim 1, this evaluation can be done using  $O(k2^k)$ , that is  $O(d \log d)$  arithmetic operations.

#### 3.2 Subdivision Algorithm

A classical approach to find the zero locus of a set of a polynomial equation is to use a subdivision algorithm. Given a polynomial equation  $F$  and a box  $B$ , assume that we have two criteria  $C_0(F, B)$  and  $C_1(F, B)$  such that:

- if  $C_0(F, B)$  is true, then  $F$  doesn't vanish in  $B$
- if  $C_1(F, B)$  is true, then  $F$  vanishes in  $B$

The idea of a subdivision algorithm is to start with a set of boxes, and to bisect them recursively until the criterion  $C_0$  is true, or  $C_1$  is true and the size is smaller than a given threshold. Recall that the grid  $G_n$  is the set of  $n^k$  boxes obtained by subdividing uniformly  $[0, 1]^k$  in  $n$  boxes in all the directions. Given a box  $B$  from the grid  $G_n$ , if we bisect it uniformly in 2 in all the directions, we

end up with a set of  $2^k$  boxes, all of them included in  $G_{2n}$ . In particular, if we bisect a set of boxes in  $G_n$ , we end up with a set of boxes in  $G_{2n}$ . Moreover, if the criteria  $C_0$  and  $C_1$  are based on polynomial evaluations, we can use Algorithm 2 to amortize the evaluation. This leads to Algorithm 3, that computes a set of boxes that enclose the zero-set of a polynomial equation. If we want to compute the zero set of a system of polynomial equations and inequalities, Algorithm 3 can be used unchanged, and the criteria  $C_1$  and  $C_0$  can be easily adapted to detect if a system of equations has solutions or not in a given box. To ensure that Algorithm 3 terminates, it is necessary that for boxes small enough, either criterion  $C_0$  or  $C_1$  succeed.

---

**Algorithm 3:** Simple subdivision algorithm to enclose the zero locus of a polynomial equation

---

**Input:**  $F$  a multivariate polynomial

$\varepsilon$  a positive threshold real number

**Output:** A CSF data structure representing the boxes of size at most  $\varepsilon$ , such that  $F$  vanishes in all the boxes, and doesn't vanish outside the boxes.

```

1  $S \leftarrow \{[0, 1]^k\}$ 
2  $R \leftarrow \{\}$ 
3  $size \leftarrow 1$ 
4 while  $S$  is not empty do
5    $S \leftarrow$  set of boxes  $B$  in  $S$  not satisfying  $C_0(F, B)$ 
6   if  $size < \varepsilon$ 
7      $R \leftarrow R$  union the set of boxes  $B$  in  $S$  satisfying  $C_1(F, B)$ 
8      $S \leftarrow$  set of boxes  $B$  in  $S$  not satisfying  $C_1(F, B)$ 
9    $S \leftarrow$  set of boxes bisected from the boxes in  $S$ 
10   $size \leftarrow size/2$ 
11 return  $R$ 

```

---

### Criteria for Exclusion and Inclusion

*Exclusion Criterion.* A simple exclusion criterion  $C_0(F, B)$  consists in evaluating  $F$  on  $B$  using interval arithmetic. Interval arithmetic is the generalization of standard arithmetic operations to the case where numbers are replaced by intervals. If  $[a, b]$  and  $[c, d]$  are two intervals, the result of  $[a, b] + [c, d]$  is the interval  $[a + c, b + d]$ . If  $F$  is a polynomial in  $k$  variables and  $B$  is a product of  $k$  intervals, we denote by  $\square F(B)$  the interval returned when  $F$  is evaluated on  $B$  using interval arithmetic. The main property of interval arithmetic is that the interval  $\square F(B)$  satisfies  $\{F(x) \mid x \in B\} \subset \square F(B)$ . In particular, if  $0 \notin \square F(B)$ , then  $F$  does not vanish in  $B$ . Thus, we can define  $C_0(F, B)$  as the predicate  $0 \notin \square F(B)$ .

The exclusion criterion can also be computed using other schemes to evaluate  $F$  on  $B$ , such as the Taylor form, which can reduce the overestimation near the zeros of  $F$  [11, §3.5].

**Definition 2 (Taylor Form [25, Definition 3.3], [15]).** *If  $c$  is the middle point of  $B$ , for a given integer  $m$ , the Taylor form of order  $m$  of the polynomial  $F$  in  $k$  variables is defined by:*

$$T_m(F, x) = F(c) + \dots + \frac{F^{(m-1)}(c)}{(m-1)!}(x-c)^{m-1} + \frac{\square F^{(m)}(B)}{m!}(x-c)^m$$

where  $x = (x_1, \dots, x_k)$  is a tuple of symbolic variables.

This evaluation scheme satisfies the property  $\{F(x) \mid x \in B\} \subset T_m(F, B)$ , such that  $0 \notin T_m(F, B)$  implies that  $F$  does not vanish in  $B$ . In the case of a system of several equations, we can simply test if any of the input polynomial does not contain 0.

*Inclusion Criterion.* For the inclusion criterion  $C_1(F, B)$  to detect if  $F$  vanishes in  $B$ , a simple test consists in evaluating  $F$  on all the vertices of  $B$  and returning **true** if two of them have different signs, and **false** if all the signs are the same. Remark that the set of all the vertices of all the boxes are a subset of a grid, and thus we can also use Algorithm 2 to amortize the cost of their evaluation.

The inclusion criterion  $C_1(F, B)$  can also be based on the Taylor form if we computed it with order  $m$ , where  $m$  is an integer greater or equal to 2. Let  $\ell(x)$  be the linear part of  $T_m(F, x)$ . Let  $V_{min}$  be a vertex of  $B$  that minimizes  $\ell$  and  $V_{max}$  one that maximizes  $\ell$ . Then we can reduce the evaluation of  $F$  to the vertices  $V_{min}$  and  $V_{max}$ . We can also use the Taylor form to evaluate lower and upper bounds of the values of  $F$  at  $V_{min}$  and  $V_{max}$ . In this case, our predicate will return **true** if the lower bound on  $F(V_{max})$  is positive and the upper bound on  $F(V_{min})$  is negative.

If  $F$  is a vector of multiple polynomials, and if we want to test if they vanish simultaneously inside a box, we can use a criterion  $C_1$  derived from the Newton Interval criterion [8, 21]. First, when the number of input equations  $F_1 = 0, \dots, F_k = 0$  is equal to the number of variables, the Newton Interval criterion can be seen as a fixed-point theorem. Letting  $S$  be the  $k \times k$  matrix defined by

$$S_{ij} = \begin{cases} \frac{F_i(x_1, \dots, x_{j-1}, x_j, c_{j+1}, \dots, c_k) - F_i(x_1, \dots, x_{j-1}, c_j, c_{j+1}, \dots, c_k)}{x_j - c_j} & \text{if } x_j \neq c_j \\ \frac{dF}{dx_j}(x_1, \dots, x_{j-1}, c_j, \dots, c_k) & \text{if } x_j = c_j \end{cases},$$

and  $c$  be the center of the box  $B$ , we define the formula  $N(x) = c - S(x)^{-1}F(c)$ . If  $N(B) \subset B$ , the fixed-point theorem ensures that there exists a point  $x_0$  in  $B$  such that  $N(x_0) = x_0$ , which is equivalent to  $F_1(x_0) = 0, \dots, F_k(x_0) = 0$ . Otherwise, when the number of polynomial equations is less than the number of variables, we can intersect the box with the linear space spanned by the gradient

vectors of the input polynomial at the center of the box  $B$ . Then we can use the Newton Interval criterion on the resulting system that has as many equations as variables.

## 4 Experiments

Algorithm 2 and 3 have been implemented in C++ in the software `voxelize`. This software can take as input a list of polynomial equations and polynomial inequalities, and it returns a list of boxes enclosing the set of points where the input system has solutions. Furthermore, if the input is a single polynomial equation, then it is guaranteed to vanish in each box returned by `voxelize` that are larger than a threshold given by the user. The software can be used as a standalone program, taking one file per polynomial, or it can be used through a python interface.

The criterion  $C_0$  used to exclude boxes is based on the Taylor form evaluation scheme described in Definition 2. The criterion  $C_1$  is implemented in the case where the input is a single polynomial equation, and it follows the approach based on the Taylor form detailed at the end of Section 3.2. In the case of multiple input polynomial equations, the subdivision process stops when the boxes are smaller than a threshold given by the user.

### 4.1 Random Polynomials

In Table 1, we show the time to enclose the zero-set of polynomial equations in  $k$  variables where  $k$  is either 2, 3 or 4. In each case, we consider three cases: a hypersurface defined by one equation, a curve defined by  $k - 1$  equations, points defined by  $k$  equations. And for each case, we generated random polynomials of total degree either 20 or 100, except for  $k = 4$  where `voxelize` could not handle polynomials in 4 variables and total degree 100. The random coefficients are floating-point numbers with double precision uniformly sampled between  $-10$  and  $10$ .

The computation have been done on a laptop with a 1.9GHz CPU and 16G of RAM. The tests have been done with one thread, for easier comparison with other single-thread programs. Note that `voxelize` is also implemented with the multi-thread library `openmp` and it can distribute the computations on several threads. Up to our knowledge, `voxelize` is the only available software that can handle the systems with polynomials of degree 100 in 3 variables presented in Table 1.

### 4.2 Polynomials Coming from Applications

We also used the software on two polynomial systems coming from robotics and automatic applications. In these cases, we compared our software with the state-of-the-art subdivision software `ibex`. The `ibex` software is a general subdivision software including a specific feature called contractors [3]. A contractor is an

**Table 1.** Timing in seconds for computing enclosing boxes in the cube  $[-2, 2]^k$ . For points and curves, the subdivision process stopped for boxes smaller than  $2^{-8} \simeq 0.004$ . For hypersurfaces, the subdivision process stopped when either the criterion  $C_0$  or  $C_1$  was satisfied on all the boxes, and the boxes had a size smaller than  $2^{-5} \simeq 0.03$ .

dimension $k$	2D		3D		4D
degree $d$	20	100	20	100	20
points <sup>2</sup> ( $k$ equations)	0.006	0.32	0.5	273	56
curves <sup>2</sup> ( $k - 1$ equations)	0.062	0.31	1.3	270	91
hypersurfaces (1 equation)	0.062	0.31	1.1	412	373

<sup>2</sup> Only the exclusion criterion was implemented for this case, and not the inclusion criterion.

operator that takes as input a function  $F$  and a box  $B$ , and that returns a smaller box  $B'$  such that the intersection of  $B'$  with the zero set  $Z$  of  $F$  is the same as the intersection of  $B$  with  $Z$ .

*Robotics.* In robotics, a classical problem is to compute the parallel singularities of a robot. That is the set of control parameters around which the robot can be assembled in two nearby configurations. In particular, the following set of equations defines the singularities in the orientation space of the 3-PPPS manipulator [4]. The orientation space is modeled with 4 quaternion variables, commonly used to parametrize the rotation matrices in  $3D$ . The sum of the squares of the quaternion variables is constrained to be 1.

$$(R) \begin{cases} 0 &= -6 Q_2^2 Q_3 Q_1 + 6 Q_2 Q_3^2 Q_4 + 3 \sqrt{3} Q_2^2 Q_3 Q_4 \\ &\quad - 6 Q_2 Q_1^2 Q_4 + 6 Q_1 Q_4^2 Q_3 \\ &\quad - 3 \sqrt{3} Q_2 Q_1 Q_4^2 + 3 \sqrt{3} Q_2 Q_3^2 Q_1 \\ &\quad - 3 \sqrt{3} Q_3 Q_1^2 Q_4 + \sqrt{3} Q_2^3 Q_1 \\ &\quad - \sqrt{3} Q_2 Q_1^3 + Q_4 \sqrt{3} Q_3^3 - Q_3 \sqrt{3} Q_4^3 \\ 1 &= Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 \end{cases}$$

*Automatic.* In control theory, a common problem is to decide if it is possible to add a controller to a dynamic system such that it becomes stable. In some case, this problem can be reduced to decide if a polynomial system does not vanish on complex numbers of modulus less than one. For example, the following system in 3 complex variables was communicated by Thomas Cluzeau and Alban Quadrat. If it has no solution where  $z_1, z_2$  and  $z_3$  have a modulus less than 1, then it is possible to design a stable controller for the corresponding dynamic system.

$$(A) \left\{ \begin{array}{l} |z_1| \leq 1 \\ |z_2| \leq 1 \\ |z_3| \leq 1 \\ 0 = z_1 z_2^2 - z_1 z_3 - 2 \\ 0 = 12z_2^3 z_3^3 - 2z_1^2 z_2 z_3^2 + z_2^3 z_3^2 - 2z_2^2 z_3^3 - 12z_2 z_3^4 + 2z_1^2 z_3^2 - z_2 z_3^3 \\ \quad - 2z_1 z_2 z_3 - 7z_2^3 - 10z_2^2 z_3 + 14z_1 z_3 - 8z_2^2 + 9z_2 z_3 + 12z_3^2 + 30z_3 + 2 \\ 0 = z_1^3 z_3^3 + z_1 z_2 z_3^4 - z_1^3 z_3^2 + z_1 z_3^4 - 12z_2^2 z_3^3 - 6z_1^2 z_2 z_3 + 3z_1^2 z_3^2 - z_1 z_2 z_3^2 \\ \quad - z_2^2 z_3^2 - 10z_2 z_3^3 - 7z_1^2 z_3 - 12z_1 z_2 z_3 - 2z_1 z_3^2 - z_2 z_3^2 + 2z_3^3 - z_1 z_2 \\ \quad - 9z_1 z_3 + 7z_2^2 + 10z_2 z_3 - z_1 + 15z_2 + 8z_3 + 8 \\ 0 = z_1^3 z_2 z_3^2 - z_1^3 z_3^2 + z_1 z_3^4 + z_1^2 z_2 z_3 - 12z_2 z_3^3 - 7z_1^2 z_3 - z_1 z_2 z_3 - z_1 z_3^2 \\ \quad - z_2 z_3^2 + 2z_3^3 - 11z_1 z_3 - z_1 + 7z_2 + 10z_3 + 8 \end{array} \right.$$

By using the change of variable  $z_j = x_j + iy_j$ , we get 8 polynomial equations in 6 variables, with the additional inequalities  $x_i^2 + y_i^2 \leq 1$ .

*Experiences.* We used `voxelize` and `ibex` on those two system of polynomial equations and inequalities. The timings and the number of boxes returned for the two software are presented in Table 2.

**Table 2.** Subdivision solvers used to enclose the zero-set of the systems (R) and (A). For the system (R), the subdivision process was stopped when boxes were smaller than  $2^{-4} \simeq 0.06$ , both in `ibexsolve` and `voxelize`.

Software	Robotics (R)		Automatic (A)	
	Time	Number of boxes	Time	Number of boxes
<code>ibexsolve</code>	103s	29871	2.5s	0
<code>voxelize</code>	0.1s	7228	1.2s	0

We can see that both solvers could detect that the system (A) has no complex solutions of moduli less than 1. In both cases, `voxelize` was faster than `ibexsolve`, and significantly faster for the system (R). This shows that the amortized evaluation scheme based on the CSF data structure is efficient not only in theory, but also in practice. On the other hand, `ibexsolve` and `voxelize` solve the system (A) with a time within the same order of magnitude, despite the fact that `ibexsolve` does not used amortized evaluations. This might be due to the fact that the contractors used by `ibexsolve` work well for this system. Remark that it could be possible to combine contractors and amortized evaluation scheme. The main issue is that after applying a contractor, the boxes are not anymore aligned on a grid. This could be solved by snapping the boxes to expanded boxes from a refined grid after applying the contractors.

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