Solving a singular limit problem arising with Euler-Korteweg dispersive waves

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Abstract

Phase transition in compressible flows involves capillarity effects, described by the Euler-Korteweg equations. Far from phase transition, the dispersion terms vanish and one should recover the hyperbolic Euler equations of fluid dynamics. However, the solution of Euler-Korteweg equations does not converge towards the solution of Euler equations when dispersion tends towards zero while being non-null: it is a singular limit problem. To avoid this issue, a Navier-Stokes-Korteweg model is considered, whose viscosity is chosen to counterbalance exactly the dispersive terms. In the limit of small viscosity and small dispersion, the Euler model is recovered. Numerically, an extended Lagrangian method is used to integrate the Navier-Stokes-Korteweg equations so-obtained. Doing so allows to use classical numerical schemes of Godunov type with source term. Numerical results for a Riemann problem illustrate the convergence properties with vanishing dispersion.

Keywords: nonlinear dispersive waves, Euler-Korteweg equations, capillary effects, extended Lagrangian method, hyperbolic systems

1. Introduction

The motivation of this work is to simulate the dynamics of a fluid near the critical point, where the liquid-vapor phase transition occurs. This topic is of major interest in many application, such as transport of cryogenic fluids in turbopumps, polymer solvent-mixture, expansion during pipe breaks under pressure for CO2, etc.

Near the critical point, it is important to account for the capillary forces. In lines with Korteweg theory and the kinetic theory [1, 2, 3, 4], an interfacial energy is added to the energy of the Euler model. This additional term depends on the density gradient ρ_x and on a capillarity coefficient $\mathcal{K}(\rho)$. Away from the phase transition, the interfacial energy vanishes. However, the solution of Euler-Korteweg equations does not converge towards the solution of

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the Euler equations when dispersion tends towards zero (but is non null).

Such a singular limit problem is usual with nonlinear dispersive waves, and is induced by the high-order spatial derivatives involved by dispersion [5, 6]. This issue can be highlighted in the case of a Riemann problem with very small dispersion: the intermediate state differs from the one of the Euler equations, which results in large errors of shock speeds. The main objective of this article is to fix this issue.

For this purpose, the proposed strategy is to introduce optimally an artificial viscosity $\mu(\rho)$ in the low-dispersion regime. The viscosity of this Navier-Stokes-Korteweg model is chosen to counterbalance exactly the dispersion effects. This choice is driven by a linear dispersion analysis and by the study of travelling waves. In the limit of small dispersion and small viscosity, the solution of Euler model is recovered [7].

However, the mixed hyperbolic-elliptic structure of a Navier-Stokes-Korteweg model raises new numerical difficulties [8]. To overcome them, an extended Lagrangian method is used [9, 10, 11, 12, 13]. In brief, the Lagrangian of Navier-Stokes-Korteweg model is modified by a penalisation term (through a parameter λ), and by an inertia term (through a parameter β). Governing equations are then derived using Hamilton's principle [14]. In this formulation, no high-order terms are introduced, so that the solution of the extended model recovers formally the solution of the original model when $(\lambda, \beta) \to (+\infty, 0)$.

The sketch of the article is then as follows. Section 2 illustrates the singular limit problem identified in [15]. Differences between the structure of the dispersive waves and of the hyperbolic waves are established. Section 3 deals with the influence of the viscosity on the dispersion analysis and on the travelling wave. The choice of the viscosity is motivated and argumented. Section 4 presents the extended Lagrangian formulation of Navier-Stokes-Korteweg model, leading to a hyperbolic system with source term. Dispersion analysis of the extended model is given. In Section 5, the numerical strategy is shortly presented, based on splitting and on a second-order finite-volume scheme for hyperbolic systems. Numerical examples of Riemann problems show the evolution of the solution with the viscosity. When the optimal viscosity is chosen, convergence towards the solution of Euler equations is observed with vanishing viscosity.

2. Singular limit problem for vanishing dispersion

2.1. Euler-Korteweg model

We consider the 1D Euler-Korteweg (EK) model of capillary fluid. In this model, the total energy depends on the density and on the gradient of the density. The volumic Lagrangian writes:

$$\mathcal{L} = \frac{1}{2}\rho u^2 - \rho\, e_h(\rho) - \mathcal{K}(\rho)\frac{(\varepsilon\rho_x)^2}{2},$$

where $u^2/2$ is the specific kinetic energy, $e_h(\rho)$ is the specific hydrodynamic energy, $\mathcal{K}(\rho)$ is the capillarity coefficient [16], and ε is a parameter controlling the magnitude of dispersion.

Hamilton's principle of stationary action yields the EK system:

$$\text{EK}: \begin{cases} \rho_t + (\rho u)_x = 0, & \text{(1a)} \\ (\rho u)_t + \left(\rho u^2 + \rho^2 e_h' - \varepsilon^2 \left[(\rho \mathcal{K}' - \mathcal{K}) \frac{{\rho_x}^2}{2} + \rho \mathcal{K} \rho_{xx} \right] \right)_x = 0. & \text{(1b)} \end{cases}$$

Depending on ε, e_h, and $\mathcal{K}$, the EK system recovers various known systems (Table 1). When $\varepsilon = 0$, one obtains non-dispersive systems, notably the Euler equations of compressible flows with a convex hydrodynamic energy e_h: the relation between the thermodynamic pressure $\rho^2 {e_h}'(\rho)$ and the specific volume $1/\rho$ is a convex function. Particular values of e_h and γ lead to the system S_0 of shallow water equations.

Table 1: Various forms of the Euler-Korteweg system. $\gamma > 1$ is the polytropic coeficient

dispersion	e_h	$\mathcal{K}$	Name	Abreviation
0	$\rho^{\gamma-1}/(2(\gamma-1))$		Euler equations	
0	$\rho/2$		Shallow water ($\gamma = 2$)	S_0
ε	$\rho/2$	$1/(4\rho)$	Defocusing Nonlinear Schrödinger	S_ε

When $\varepsilon \neq 0$, dispersive systems are obtained, which are not hyperbolic in the sense of Lax [17]. A particular form of $\mathcal{K}$ leads to the hydrodynamic form of the defocusing nonlinear Schrödinger equation [18, 10, 19], denoted by S_ε. This system is integrable, with known exact solutions.

2.2. Illustration of the singular limit problem

The objective of this part is to highlight a singular limit problem of the system S_ε when $\varepsilon \to 0$. The two systems studied here are:

$$S_0: \begin{cases} \rho_t + (\rho u)_x = 0, & \text{(2a)} \\ (\rho u)_t + \left(\rho u^2 + \dfrac{\rho^2}{2} \right)_x = 0, & \text{(2b)} \end{cases}$$

and

$$S_\varepsilon: \begin{cases} \rho_t + (\rho u)_x = 0, & \text{(3a)} \\ (\rho u)_t + \left(\rho u^2 + \dfrac{\rho^2}{2} - \dfrac{\varepsilon^2 \rho}{4} \left(\dfrac{\rho_x}{\rho} \right)_x \right)_x = 0. & \text{(3b)} \end{cases}$$

A Riemann problem (RP) is considered, with x_c the position of the initial discontinuity and $\rho_L > \rho_R$:

$$u(x,0) = 0, \quad \rho(x,0) = \begin{cases} \rho_L, & x < x_c, \\ \rho_R, & x > x_c. \end{cases} \tag{4}$$

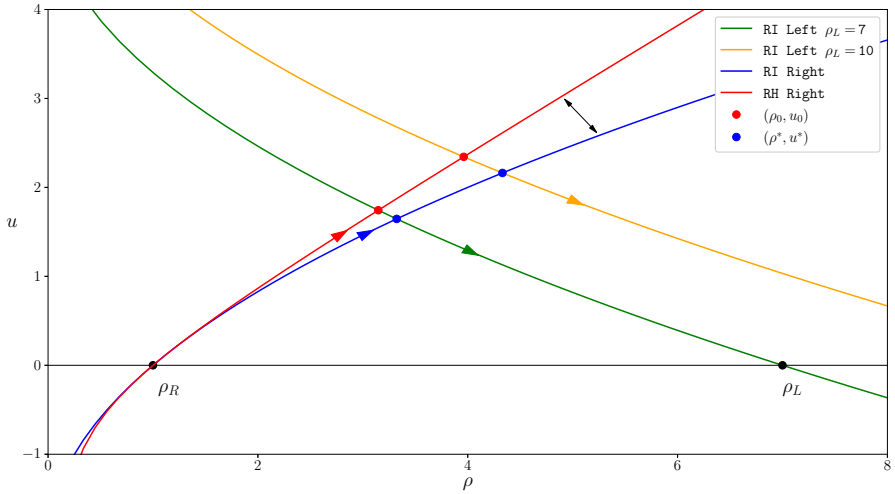


Figure 1: RI curves and RH curve in the (ρ, u) plane. Intersections between the curves give the plateau values. The black arrow shows the increasing gap between (ρ^*, u^*) and (ρ_0, u_0) when ρ_L increases.

Solution of system S_0. The complete derivation of the analytical solution can be found in many reference books [20, 21]. Here we only give the main results for completeness. For $t > 0$, the initial discontinuity splits into a right-going shock wave (SW), a left-going centered rarefaction wave and an intermediate plateau (ρ_0, u_0). The shock intensity and the shock speed are determined by the Rankine-Hugoniot jump conditions:

$$[\![\rho]\!]\, D = [\![\rho u]\!]\,, \tag{5}$$

$$[\![\rho u]\!]\, D = \left[\!\!\left[\rho u^2 + \frac{\rho^2}{2}\right]\!\!\right], \tag{6}$$

where D is the shock speed and $[\![f]\!] = f_R - f$ is the jump of a given f across the SW. Eliminating D in (6) by using (5) gives the "Right Rankine-Hugoniot Locus" (RH Right) in the (ρ, u) plane:

$$u = (\rho - \rho_R)\sqrt{\frac{1}{2}\left(\frac{1}{\rho} + \frac{1}{\rho_R}\right)}. \tag{7}$$

In a left-going rarefaction wave, the Riemann invariant $r = u + 2\sqrt{\rho}$ is conserved. This equation gives the "Left Riemann invariant Locus" (RI Left) in the (ρ, u) plane:

$$u = 2\left(\sqrt{\rho_L} - \sqrt{\rho}\right). \tag{8}$$

Intersection of (7) and (8) gives the plateau value (ρ_0, u_0):

$$2(\sqrt{\rho_L} - \sqrt{\rho_0}) - (\rho_0 - \rho_R)\sqrt{\frac{1}{2}\left(\frac{1}{\rho_0} + \frac{1}{\rho_R}\right)} = 0, \tag{9}$$

$$u_0 = 2(\sqrt{\rho_L} - \sqrt{\rho_0}), \tag{10}$$

represented in Figure 1 for various values of ρ_L and ρ_R.

Solution of system S_ε. The RP problem for S_ε is better known as a "Gurevich-Pitaevskii problem" [22]. The exact solution has been constructed in [19]. The asymptotic solution in time is composed of a left-going centered rarefaction wave (same as system S_0), a right-going dispersive shock wave (DSW) and a plateau (ρ^*, u^*). The DSW is a wave train modulated in amplitude and wavenumber. Modulation theory gives its analytical expression and its Whitham envelope [23, 19, 10]. The plateau is given by the intersection of a "RI Left" and a "RI Right" curve:

$$u = 2\left(\sqrt{\rho_L} - \sqrt{\rho}\right), \quad u = 2\left(\sqrt{\rho} - \sqrt{\rho_R}\right), \tag{11}$$

leading to:

$$u^* = \sqrt{\rho_L} - \sqrt{\rho_R}, \tag{12}$$

$$\rho^* = \frac{1}{4}\left(\sqrt{\rho_L} + \sqrt{\rho_R}\right)^2. \tag{13}$$

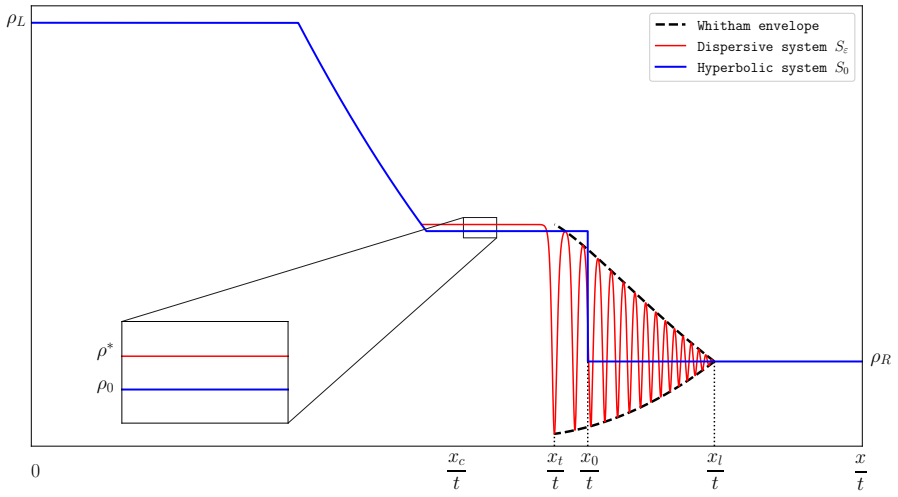


Figure 2: Exact solutions of S_0 and S_ε in x/t. The parameters are $\rho_L = 5$, $\rho_R = 1$. Particular points are: x_t the position of the trailing edge, x_l the position of the leading edge, and x_0 the position of the shock wave.

Comparison between both solutions. The solutions of S_0 and S_ε are compared in Figure 2. Three main differences are observed. Firstly, the SW is a discontinuity at x_0, whereas the DSW has a finite length $x_t - x_l$. This length increases with time and is independent of ε [19]:

$$x_t - x_l = \left(\sqrt{\rho^*} - \frac{8\rho^* - 8\sqrt{\rho^*\rho_R} + \rho_R}{2\sqrt{\rho^*} - \sqrt{\rho_R}}\right) t.$$

The only influence of ε is on the wavenumber k of the DSW:

$$k(m, \varepsilon) = \frac{2\pi}{J(m)} \frac{\sqrt{\rho_1 - \rho_3}}{\varepsilon}, \quad m = \frac{\rho_2 - \rho_3}{\rho_1 - \rho_3},$$

with m the modulation parameter, $\rho_1 > \rho_2 > \rho_3$ modulated amplitudes, and $J(m)$ the elliptic integral of the first kind [19]. Secondly, the difference of plateau (ρ_0, u_0) and plateau (ρ^*, u^*)

comes from the difference between (9) and (13), and it is highlighted in Figure 1. When $\rho_L - \rho_R$ is large, the gap between (ρ_0, u_0) (red dot) and (ρ^*, u^*) (blue dot) increases, as highlighted by the black arrow in Figure 1. Moreover, (9) and (13) do not depend on ε. Therefore:

$$\lim_{\varepsilon \to 0} \rho^* \neq \rho_0, \quad \lim_{\varepsilon \to 0} u^* \neq u_0.$$

To summarize this section, the solutions of S_0 and S_ε have different structures, induced by the difference between SW and DSW. The solutions are self-similar with respect to the change of variable $(x,t) \to (x/\varepsilon, t/\varepsilon)$. Consequently, the differences between SW and DSW never vanish when $\varepsilon \to 0$.

3. Strategy to fix the singular limit problem

3.1. Navier-Stokes-Korteweg model

To by-pass this singular limit problem, introduction of an optimal viscosity $\tau\mu(\rho)$ is considered. The parameter τ controls the effect of viscosity and will be compared to the dispersive parameter ε. EK system (1) becomes the Navier-Stokes-Korteweg (NSK) system:

$$\text{NSK}: \begin{cases} \rho_t + (\rho u)_x = 0, & \text{(14a)} \\ (\rho u)_t + \left(\rho u^2 + \rho^2 e_h{}' - \varepsilon^2 \left[(\rho \mathcal{K}' - \mathcal{K}) \dfrac{{\rho_x}^2}{2} + \rho \mathcal{K} \rho_{xx} \right] - \tau \mu \, u_x \right)_x = 0. & \text{(14b)} \end{cases}$$

The idea is to find the functions $\tau := \hat{\tau}$ and $\mu := \hat{\mu}$ which cancel the dispersion introduced by $\mathcal{K}$ and ε (the hat refers to the optimal value, to be found). Those functions are obtained in two ways: performing the dispersion analysis, and exhibiting a travelling wave solution. It will be the subject of the two next subsections.

3.2. Dispersion analysis

Here, we prove by a dispersion analysis that a particular choice of $\hat{\tau}\hat{\mu}$ can lead to dispersiveless properties: wave's speed of propagation can be rendered independent of the wavenumber κ whatever ε. Galilean invariance of NSK system (14) allows to choose a uniform state $(\overline{\rho}, \bar{u} = 0)$. Linearization of the NSK around this uniform states leads to:

$$\begin{cases} \tilde{\rho}_t + \overline{\rho} \tilde{u}_x = 0, & \text{(15a)} \\ \overline{\rho} \tilde{u}_t + \overline{c^2} \tilde{\rho}_x - \varepsilon^2 \overline{\rho} \, \mathcal{K}(\overline{\rho}) \tilde{\rho}_{xxx} - \tau \mu(\overline{\rho}) \tilde{u}_{xx} = 0, & \text{(15b)} \end{cases}$$

where $(\tilde{\rho}, \tilde{u})$ are fluctuations around the uniform state, and $\bar{c} = \sqrt{(\overline{\rho}^2 \overline{e}')'}$ is the sound speed of the uniform state. Assuming harmonic solution of the form $\exp(\mathrm{i}(\omega t - \kappa x))$, where ω is the angular frequency, the phase velocity $c_p = \omega/\kappa$ is:

$$c_p(\kappa) = \mathrm{i} \, \tau \kappa \, \frac{\mu(\overline{\rho})}{2\overline{\rho}} \pm \sqrt{\bar{c}^2 + \kappa^2 \left(\varepsilon^2 \overline{\rho} \, \mathcal{K}(\overline{\rho}) - \tau^2 \frac{\mu(\overline{\rho})^2}{4\overline{\rho^2}} \right)}. \quad \text{(16)}$$

The real part of (16) depends on κ, which is the signature of a dispersive wave. If we choose

$$\tau := \hat{\tau} = \varepsilon, \tag{17}$$

$$\mu := \hat{\mu} = 2\,\overline{\rho}\sqrt{\overline{\rho}\,\mathcal{K}(\overline{\rho})}, \tag{18}$$

then (16) degenerates into

$$\hat{c}_p(\kappa) = \mathrm{i}\,\varepsilon\kappa\sqrt{\overline{\rho}\,\mathcal{K}(\overline{\rho})} \pm \overline{c}. \tag{19}$$

The real part of the phase velocity no longer depends on κ, and thus dispersion is canceled. In line with the "viscosity method" [24, 25, 26], the solutions of NSK tend towards the solutions of the Euler equations when viscosity vanishes: no singular limit problem occurs. It leads to conjecture that the solution of NSK tends towards the solution of the Euler equations when $\varepsilon \to 0$. Numerical simulations will confirm this hypothesis.

3.3. Travelling wave

In this section, $\tau := \hat{\tau} = \varepsilon$ and μ is arbitrary. Using the change of variables $(x,t) \to \xi = (x - D\,t)/\varepsilon$, (14a) becomes:

$$q = \rho\,(u - D), \tag{20}$$

where q is the constant flow rate of fluid across the wave and is chosen to be negative. Integration of (14b) and use of (20) gives

$$\frac{q^2}{\rho} + \rho^2 {e_h}'(\rho) + \mu(\rho)\frac{q}{\rho^2}\left(\frac{d\rho}{d\xi}\right) - \rho^2\left[\left(\frac{\mathcal{K}}{\rho}\right)'\frac{1}{2}\left(\frac{d\rho}{d\xi}\right)^2 + \frac{\mathcal{K}}{\rho}\left(\frac{d^2\rho}{d\xi^2}\right)\right] = C_1, \tag{21}$$

where C_1 describes the constant right-state R at infinity:

$$C_1 = \frac{q^2}{\rho_R} + {\rho_R}^2 {e_h}'(\rho_R).$$

Equation (21) is divided by ρ^2 and then integrated, yielding a Lyapounov function E

$$E\left(\rho, \frac{d\rho}{d\xi}\right) = \frac{1}{2}\frac{\mathcal{K}(\rho)}{\rho}\left(\frac{d\rho}{d\xi}\right)^2 - e_h(\rho) - \frac{C_1}{\rho} + \frac{q^2}{2\rho^2},$$

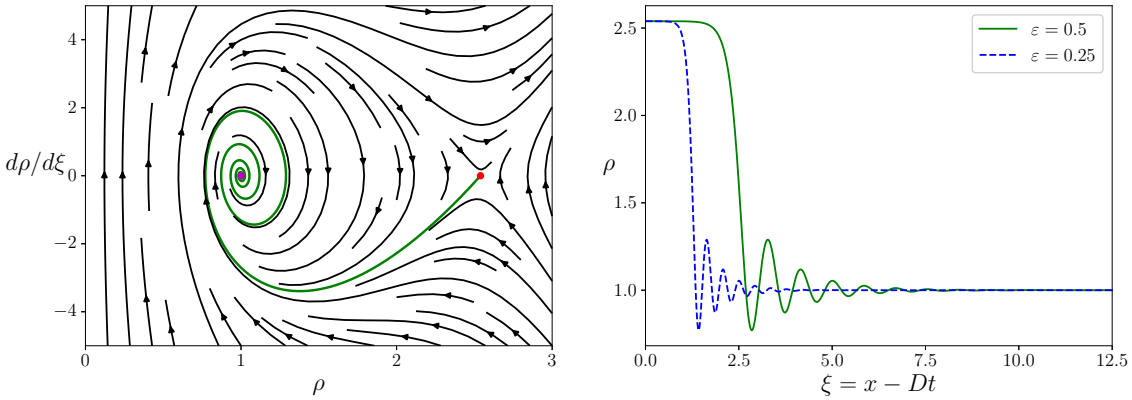
which decreases along trajectories of the system in the phase space:

$$\frac{dE}{d\xi} = q\,\frac{\mu(\rho)}{\rho^4}\left(\frac{d\rho}{d\xi}\right)^2 < 0. \tag{22}$$

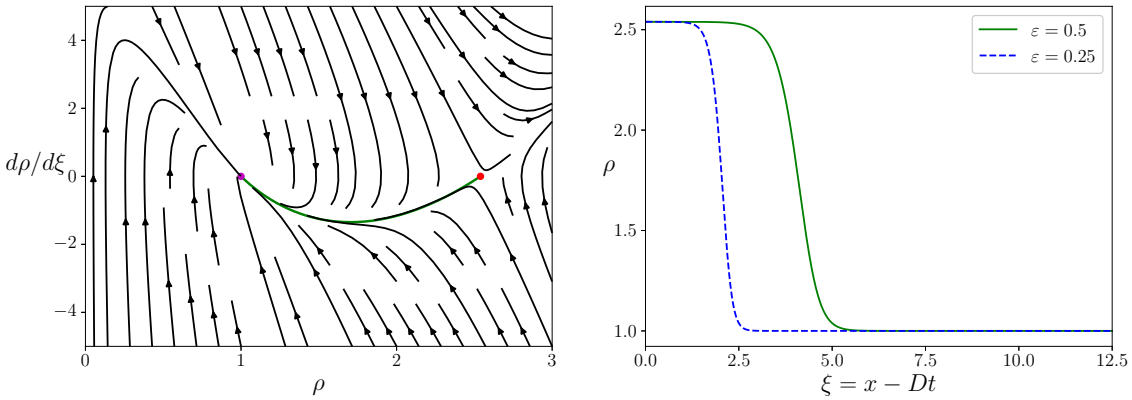
The local extrema ρ_c of E verify the following conditions:

$$\left.\frac{d\rho}{d\xi}\right|_{\rho_c} = 0, \quad \frac{q^2}{\rho_R} + {\rho_R}^2 {e_h}'(\rho_R) = {\rho_c}^2 {e_h}'(\rho_c) + \frac{q^2}{\rho_c}. \tag{23}$$

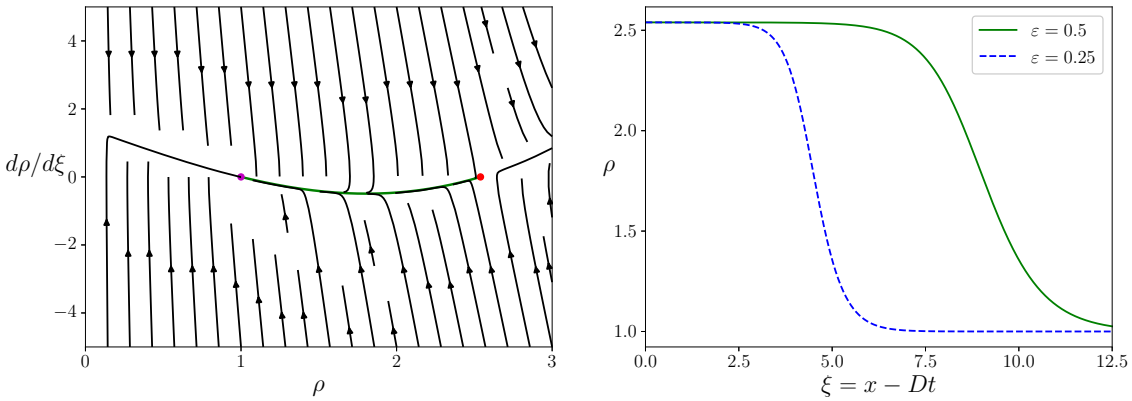
Additional details about the extrema are given in Appendix Appendix A. Setting $c = \sqrt{(\rho^2\,{e_h}')'}$



(a) Pseudo-periodic regime ($\theta = 0.1 < 1$)



(b) Critical regime ($\theta = 1$)



(c) Overdamped regime ($\theta = 3$)

Figure 3: Left: the phase portrait and one solution for $\varepsilon = 0.5$. Right: density for various ε.

the sound speed and $M = (u - D)/c$ the Mach number, the linearized version of (21) around the extrema is:

$$\frac{d^2\Delta\rho}{d\xi^2} - 2\theta\frac{c_c M_c}{\sqrt{\rho_c \mathcal{K}_c}}\frac{d\Delta\rho}{d\xi} + \frac{{c_c}^2({M_c}^2 - 1)}{\rho_c \mathcal{K}_c}\Delta\rho = 0, \quad \theta = \frac{\mu_c}{2\rho_c\sqrt{\rho_c \mathcal{K}_c}}, \tag{24}$$

where $\Delta\rho = \rho - \rho_c$. The analysis is reduced to the study of a damped harmonic oscillator with θ the damping ratio. Eigenwavenumbers are:

$$k_c = \frac{c_c M_c}{\sqrt{\rho_c \mathcal{K}_c}}\left[\theta \pm \sqrt{(\theta^2 - 1) + \frac{1}{{M_c}^2}}\right]. \tag{25}$$

Non-oscillating solutions of (24) correspond to real-valued eigenwavenumber. Assuming that c_c is a real-valued function, we have to choose $\theta \geq 1$. This condition is also known as the *tame capillarity* condition [7]. The choice $\theta^2 = 1 - 1/{M_c}^2$ is not adequate since θ is complex-valued in region where ${M_c}^2 < 1$.

Numerical simulations of the ODE (21) is considered. Initial conditions are:

$$\text{IC} \ : \begin{cases} \rho(0) = \rho_0 - 10^{-5}, & \text{(26a)} \\ \left.\dfrac{d\rho}{d\xi}\right|_{\xi=0} = 0. & \text{(26b)} \end{cases}$$

Figure 3 illustrates various regimes obtained for different values of θ. The red dot corresponds to the state ρ_0 and the purple dot to the state ρ_R. The numerical simulations are obtained by the RK4 method. Numerical results in Figure 3-(b,c) confirm our choice for θ. Comparing the solution at $\varepsilon = 0.5$ confirms that the thickness of the wave is large when θ is large ($\theta \gg 1$). As a consequence, $\theta = 1$ will be the choice in forthcoming numerical simulations.

4. Extended Lagrangian formulation

4.1. Principle

Numerical scheme for NSK-type systems are not classical due to the "mixed hyperbolic-elliptic" structure of the system [8]. One way is to solve a discrete elliptic/parabolic operator on the whole computational domain and to solve the hyperbolic part locally. An example of such of scheme is given in [27] for the Serre-Green-Naghdi model. This method provides good results but at a high computational cost. To overcome this issue, a extended Lagrangian formulation was proposed in [9]. The original system, named the *master system*, is replaced by an approximate hyperbolic system, named the *extended system*, composed of a new variable η close to ρ. The extended solution recovers the behavior of the master solution in some limit.

This formulation was successfully applied to a wide list of nonlinear wave phenomena, to cite a few: DNS equation in quantum mechanics [10], thin films down an inclined plane [11], Ericksen bar model in solid mechanics [12], and BBM equations [13]. The detailed description of this formulation can be found in the aforementioned articles and is shortly presented in the next subsection.

4.2. Extended Euler-Korteweg model

We recall the master Lagrangian of the EK system (1)

$$\mathcal{L} = \frac{1}{2}\rho\, u^2 - \left(\rho\, e_h(\rho) + \mathcal{K}(\rho)\frac{(\varepsilon\rho_x)^2}{2}\right).$$

This Lagrangian is extended by adding new terms:

- a penalisation term: $\lambda\,\dfrac{\rho}{2}\left(\dfrac{\eta}{\rho}-1\right)^2$, where η is a new variable;
- a kinetic energy term: $\dfrac{1}{2}\,\rho\,\beta\left(\dfrac{d\eta}{dt}\right)^2$, analogous to a term involved in bubbly flows [14].

The parameter λ (resp. β) controls the penalisation (resp. the inertia) term. Replacing ρ_x by η_x and setting new variables $p = \eta_x$ and $w = d\eta/dt$, the extended Lagrangian $\mathcal{L}_e$ writes:

$$\mathcal{L}_e = \frac{1}{2}\rho\,\beta\, w^2 + \frac{1}{2}\rho\, u^2 - \left(\rho\, e_h(\rho) + \mathcal{K}(\rho)\frac{(\varepsilon p)^2}{2} + \lambda\,\frac{\rho}{2}\left(\frac{\eta}{\rho}-1\right)^2\right).$$

By applying twice a variational principle (see the methodology in the appendix of [10]) and adding evolution equations for η,w and p, we obtain the extended EK system:

$$\mathrm{EK}_e : \begin{cases} \rho_t + (\rho u)_x = 0, & \text{(27a)}\\ (\rho u)_t + \left(\rho u^2 + \rho^2\, e' + \varepsilon^2\dfrac{p^2}{2}\,(\rho\,\mathcal{K}' + \mathcal{K}) + \lambda\eta\left(1-\dfrac{\eta}{\rho}\right)\right)_x = 0, & \text{(27b)}\\ (\rho\eta)_t + (\rho u\eta)_x = \rho w, & \text{(27c)}\\ (\rho w)_t + \left(\rho u w - \varepsilon^2\dfrac{\mathcal{K}p}{\beta}\right)_x = \dfrac{\lambda}{\beta}\left(1-\dfrac{\eta}{\rho}\right), & \text{(27d)}\\ p_t + (pu - w)_x = 0. & \text{(27e)} \end{cases}$$

We expect that $\eta = \rho + \mathcal{O}(\beta/\lambda)$, so that the extended EK system recovers the master system (1) when $\lambda \to +\infty$ and $\beta \to 0$. This property has been justified rigorously in [28] in the case of Serre-Green-Naghdi model with $\beta = 1$, and we suppose it to be valid here. Injecting (27d) into (27b) leads to:

$$(\rho u)_t + \left(\rho u^2 + \rho^2\, e' + \beta\rho\,\eta\frac{dw}{dt} - \varepsilon^2\left[((2\eta-\rho)\mathcal{K}' - \mathcal{K})\,\frac{p^2}{2} + \eta\,\mathcal{K}p_x\right]\right)_x = 0,$$

where $dw/dt = w_t + uw_x$. When $(\lambda,\beta) \to (\infty, 0)$, and assuming that $\eta \to \rho$ and $p \to \rho_x$, then one recovers (14b) of the EK system.

4.3. Extended Navier-Stokes-Korteweg model

As proposed in [12], the viscosity is introduced as an additional source term in (27d). This leads to:

$$\mathrm{NSK}_e : \begin{cases} \rho_t + (\rho u)_x = 0, & \text{(28a)} \\ (\rho u)_t + \left(\rho u^2 + \rho^2 e' + \varepsilon^2 \dfrac{p^2}{2} (\rho \mathcal{K}' + \mathcal{K}) + \lambda \eta \left(1 - \dfrac{\eta}{\rho} \right) \right)_x = 0, & \text{(28b)} \\ (\rho \eta)_t + (\rho u \eta)_x = \rho w, & \text{(28c)} \\ (\rho w)_t + \left(\rho u w - \varepsilon^2 \dfrac{\mathcal{K} p}{\beta} \right)_x = \dfrac{\lambda}{\beta} \left(1 - \dfrac{\eta}{\rho} \right) - \tau \dfrac{\mu(\rho)}{\beta \rho^2} w, & \text{(28d)} \\ p_t + (pu - w)_x = 0. & \text{(28e)} \end{cases}$$

Injecting (28d) into (28b) and using (28a), we recover NSK model when $\lambda \to \infty$ and $\beta \to 0$.

4.4. Characteristic speeds

NSK_e system (28) can be put in a quasi-linear form:

$$\mathbf{W}_t + \mathbf{A}\mathbf{W}_x = \mathbf{B}(\mathbf{W})$$

where $\mathbf{A}$ is a 5×5 matrix given in Appendix Appendix B and $\mathbf{W} = (\rho, u, \eta, w, p)^\top$ is the vector of primitive (non-conservative) variable and $\mathbf{B}(\mathbf{W})$ a vector that contains the source term. The five eigenvalues/characteristic speeds α_i of the system are:

$$\alpha^1 = u, \quad \alpha^{2,3} = u \pm c_1, \quad \alpha^{4,5} = u \pm c_2, \tag{29}$$

with c_1 and c_2 given in Appendix Appendix B. In general, the hyperbolicity for such a general case is not proven. If we consider the case of the extended DNS system (Table 1), hyperbolicity has been proved in [10] when $\lambda \geq 0$ and $\beta > 0$. The five eigenvalues are then:

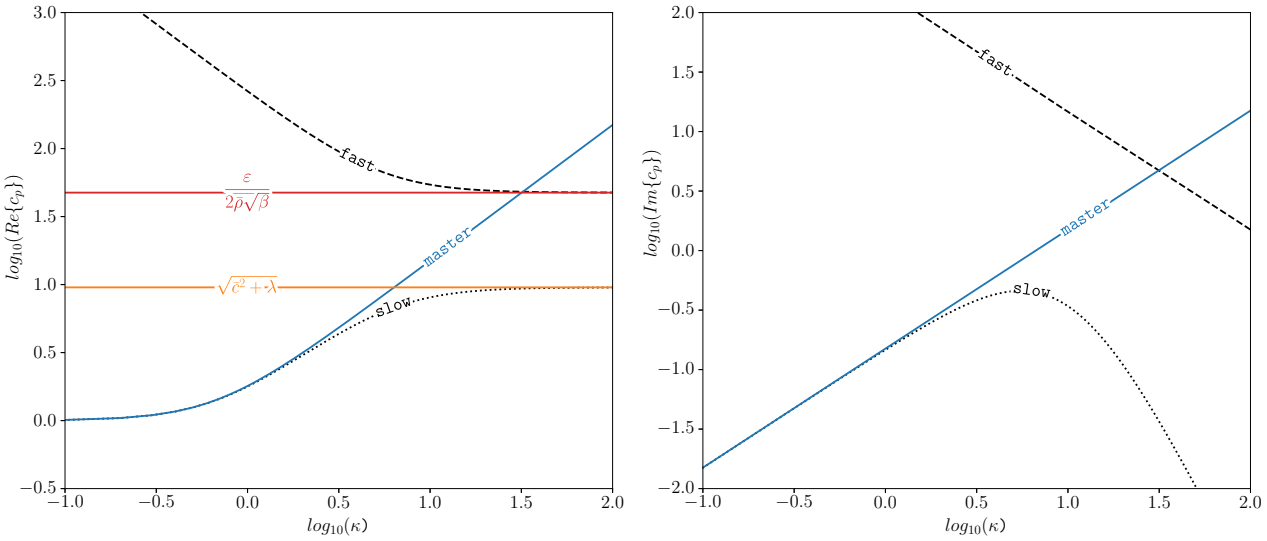
$$\alpha^1 = u, \quad \alpha^{2,3} = u \pm \sqrt{\rho + \lambda \frac{\eta^2}{\rho^2}}, \quad \alpha^{4,5} = u \pm \sqrt{\frac{\varepsilon^2}{4\,\beta\,\rho^2}}. \tag{30}$$

These eigenvalues will be useful for the numerical method detailed in Section 5.1.

4.5. Choice of (λ, β)

When using the extended Lagrangian method, it is a important to estimate the error of this formulation. To do so, a dispersion analysis of the linearized extended system is usually carried out [9, 10, 12, 11, 13]. We provide a similar analysis here for the NSK_e system in the case of DNS ($e_h(\rho) = \rho/2$ and $\mathcal{K}(\rho) = 1/(4\rho)$). Using the same notation as in Section 3.2, we obtain a quartic expression for the phase velocity c_p:

$$(c_p)^4 + (c_p)^3 \frac{\mathrm{i}\bar{\mu}\tau}{\beta \bar{\rho}^3 \kappa} - (c_p)^2 \left(\frac{\lambda}{\beta \bar{\rho}^2 \kappa^2} + \lambda + \bar{c}^2 + \frac{\varepsilon^2}{4\beta\bar{\rho}^2} \right) - (c_p) \frac{\mathrm{i}\bar{\mu}\tau(\lambda + \bar{c}^2)}{\beta \bar{\rho}^3 \kappa} + (\lambda + \bar{c}^2) \frac{\varepsilon^2}{4\beta\bar{\rho}^2} + \frac{\lambda \bar{c}^2}{\beta \bar{\rho}^2 \kappa^2} = 0,$$



(a) (b)

Figure 4: Dispersion curve of the linearized NSK$_e$ system, with $\lambda = 200$, $\beta = 0.003$, $\bar{\mu} = 0.1$, $\varepsilon = \tau = 3$, $\bar{\rho} = 1$.

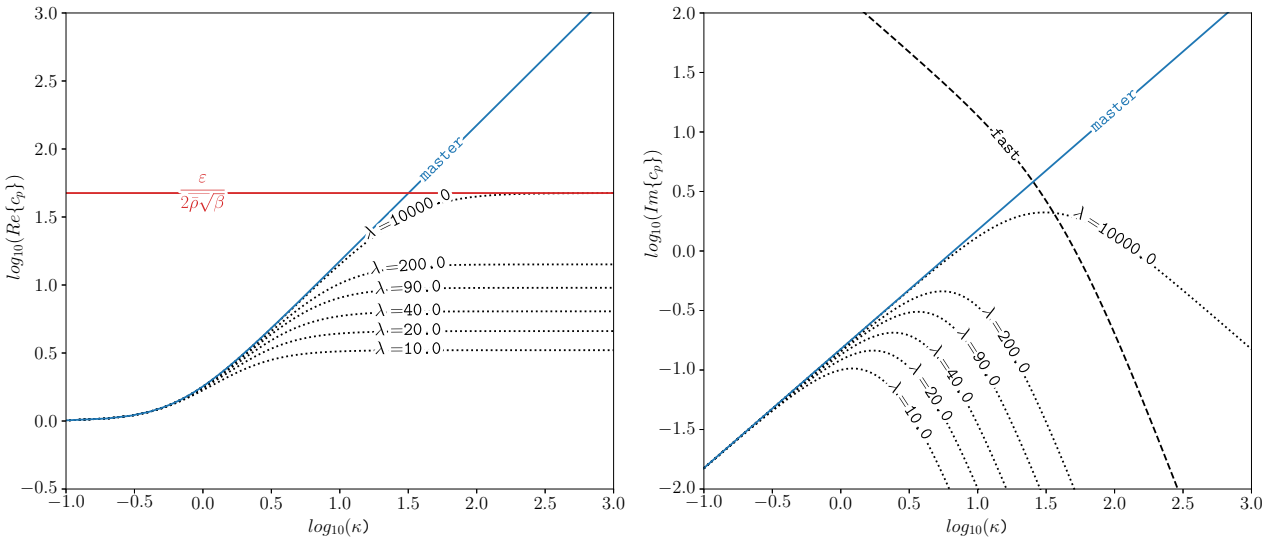
where the uniform state is $(\bar{\rho}, \bar{u} = 0, \bar{\eta} = \bar{\rho}, \bar{p} = 0, \bar{w} = 0)$. Assuming a positive phase velocity, this equation have two solutions namely the *slow wave* ${c_p}^-$ and the *fast wave* ${c_p}^+$. Comparison with the phase speed of the *master wave* (Section 3.2) is given in Figure 4. The slow wave is related to the master wave since it shares the same limit when $\kappa \to 0$. The fast wave is a spurious wave induced by the modification of the Lagrangian. The limits of the slow wave and of the fast wave when $\kappa \to \infty$ are:

$$\lim_{k\to+\infty} \Re\{{c_p}^{(+)}\} = \max\left(\sqrt{\bar{\rho}+\lambda}, \sqrt{\frac{\varepsilon^2}{4\,\beta\,\bar{\rho}^2}}\right), \quad \lim_{k\to+\infty} \Re\{{c_p}^{(-)}\} = \min\left(\sqrt{\bar{\rho}+\lambda}, \sqrt{\frac{\varepsilon^2}{4\,\beta\,\bar{\rho}^2}}\right). \tag{31}$$

In the inviscid case ($\mu = 0$), one condition has to be fulfilled to ensure that the slow wave mimics the master wave [12]. This condition is due to the dispersive nature of the master system. In the case of DNS, the dispersion is positive [19, 15, 29]: c_p increases with κ. Therefore, the following condition has to be fulfilled by $\bar{\rho}$:

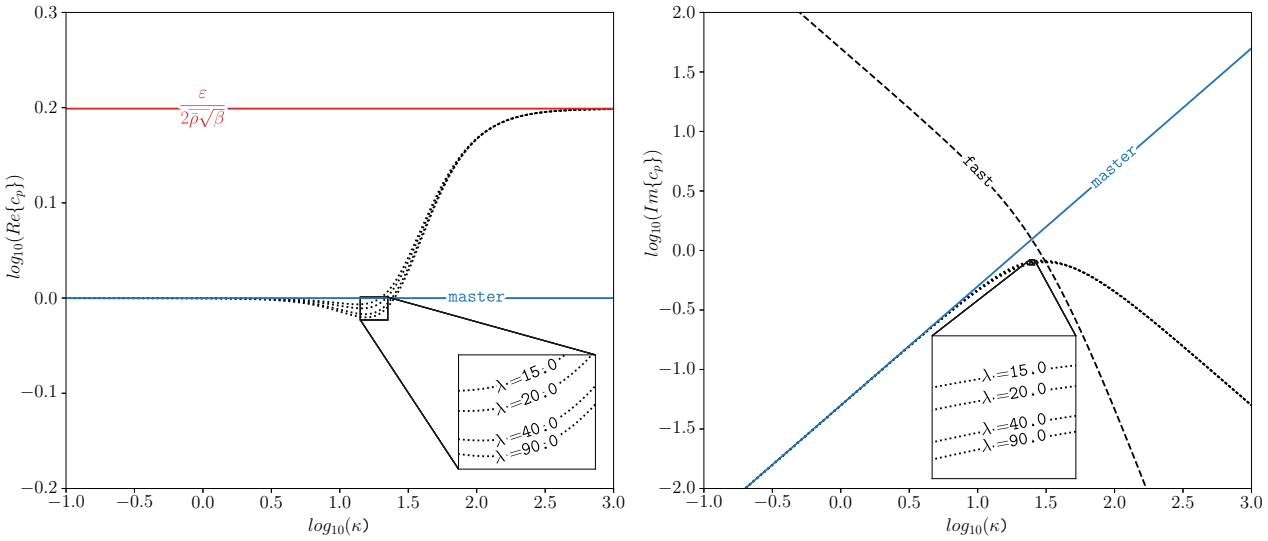
$$\min\left(\bar{\rho}+\lambda, \frac{\varepsilon^2}{4\,\beta\,\bar{\rho}^2}\right) > \bar{\rho}. \tag{32}$$

From (16), the dispersion is still positive when $\mu \leq \hat{\mu}$. Consequently, we suppose that the condition (32) holds in this case. Figure 5 shows the behavior of the slow wave for variable λ and a given β. As λ increases, the slow wave mimics the master wave. For a critical value of λ, the slow wave is bounded by the limit of the fast wave (see the slow wave for $\lambda = 10^4$ in Figure 5) , as described by (31)). To avoid this bound, β has to be decreased. The case where $\bar{\mu} = \hat{\mu}(\bar{\rho})$ is shown in Figure 6. The master wave speed is constant and equal to the local sound speed. From the zoom, it is clear that λ has no influence on the behavior of the slow wave when



(a) (b)

Figure 5: Dispersion curve of the linearized NSK_e system for various λ. Parameters: $\beta = 0.003$, $\bar{\mu} = 0.1$, $\varepsilon = \tau = 3$, $\bar{\rho} = 1$.



(a) (b)

Figure 6: Dispersion curve of the linearized NSK_e system for various λ when $\bar{\mu} = \hat{\mu}(\bar{\rho})$. Parameters: $\beta = 0.003$, $\varepsilon = \tau = 0.1$, $\bar{\rho} = 1$.

$\varepsilon \to 0$. The relevant parameter is then β, and it can be chosen properly. Indeed, the condition (32) becomes:

$$\frac{\varepsilon^2}{4\beta\,\bar{\rho}^2} \geq \bar{\rho} \iff \beta \leq \frac{\varepsilon^2}{4\,\bar{\rho}^3}. \tag{33}$$

To summarize, the slow wave mimics the behavior of the master wave for a limited range of wavenumbers, when $\lambda \to \infty$ and $\beta \to 0$. In the case where $\mu = \hat{\mu}(\bar{\rho})$ and $\varepsilon \to 0$, β has to satisfy (33).

5. Numerical results

5.1. Numerical methods

The NSK_e system can be recast as:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}), \tag{34}$$

with $\mathbf{U} = (\rho, \rho u, \rho\eta, \rho w, p)^\top$ the vector of conserved quantities, $\mathbf{F}(\mathbf{U})$ a vector of physical fluxes and $\mathbf{S}(\mathbf{U})$ a vector that contains the source term. To simulate numerically (34), we apply a Godunov splitting [21, 20]:

$$\text{Step 1 : compute } \mathbf{U}^* \text{ :} \begin{cases} \text{PDE} : \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}, & \text{(35a)} \\ \text{IC} : \mathbf{U}^n. & \text{(35b)} \end{cases}$$

$$\text{Step 2 : compute } \mathbf{U}^{n+1} \text{ :} \begin{cases} \text{ODE} : \mathbf{U}_t = \mathbf{S}(\mathbf{U}), & \text{(36a)} \\ \text{IC} : \mathbf{U}^*. & \text{(36b)} \end{cases}$$

The spatial domain $[0, L]$ is discretized into N cells. The volume of each cells is $\Delta x = \frac{L}{N}$ and the central position of a cell i is set at x_i. The time step is $\Delta t = t^{n+1} - t^n$. We define by $\mathbf{U}_i^n$ the average of $\mathbf{U}$ over a cell at t^n and $\mathbf{F}_{i\pm\frac{1}{2}}$ the numerical fluxes at the boundary of a cell. At the beginning of Step 1, characteristics speed ${\alpha_i}^1$,${\alpha_i}^2$, ... ,${\alpha_i}^p$ are computed based on $\mathbf{U}_i^n$. The time step is obtained from the Courant-Friedrichs-Lewy condition:

$$\text{CFL} = \frac{\alpha_{max}\,\Delta t}{\Delta x} \leq 1, \quad \alpha_{max} = \max_{i,p}(|\alpha_i^p|),$$

where α_{max} is the maximum of all the characteristic speeds over the spatial domain. The second-order TVD MUSCL-Hancock method [21] applied on $\mathbf{U}_i^n$ gives boundary extrapolated values $\mathbf{U}_i^{L,R}$ based on limited slopes; the MINMOD limiter is used for this purpose. Numerical fluxes $\mathbf{F}_{i+\frac{1}{2}}$ are estimated using Rusanov flux:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{F}(\mathbf{U}_{i+1}^L) + \mathbf{F}(\mathbf{U}_i^R)\right) - \alpha_{i+\frac{1}{2}}\left(\mathbf{U}_{i+1}^L - \mathbf{U}_i^R\right), \quad \alpha_{i+\frac{1}{2}} = \max_p(|{\alpha_i}^p|, |{\alpha_{i+1}}^p|),$$

where $\alpha_{i+\frac{1}{2}}$ is the maximum of the characteristic speeds between the cell i and cell $i+1$. The time derivative is approximated by a first-order Euler method. Averaging (35a) over a cell volume gives:

$$\mathbf{U}_i^* = \mathbf{U}_i^n + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i-\frac{1}{2}} - \mathbf{F}_{i+\frac{1}{2}}\right).$$

Step 2 corresponds to the following system:

$$\begin{cases} \dfrac{d\rho}{dt} = 0, \quad \dfrac{d\rho u}{dt} = 0, \quad \dfrac{dp}{dt} = 0, & \text{(37a)} \\ \dfrac{d\rho\eta}{dt} = \rho w, & \text{(37b)} \\ \dfrac{d\rho w}{dt} = \dfrac{\lambda}{\beta}\left(1 - \dfrac{\eta}{\rho}\right) - \dfrac{\tau\mu(\rho)}{\beta\rho^2} w. & \text{(37c)} \end{cases}$$

This system has exact solutions given in Appendix Appendix C. For numerical implementation, the solutions are discretized in space between the time t^n and t^{n+1}. Since the source term is solved exactly, no numerical issues are encountered for low value of β. This might not be the same with an approximate solver. From condition (33), the ratio $\tau/\beta \to \infty$ as $\varepsilon \to 0$.

5.2. Numerical set-up

The RP problem of Section 2 is considered. The extended system of DNS with viscosity (vDNS$_e$) is compared with the hyperbolic system S_0. Initial conditions for the vDNS$_e$ are:

$$\rho|_{t=0} = \eta|_{t=0} = \begin{cases} \rho_L, & x < x_c \\ \rho_R, & x > x_c \end{cases}, \quad u|_{t=0} = w|_{t=0} = 0, \quad p|_{t=0} = \frac{\rho_R - \rho_L}{2}\delta(x - x_c).$$

The initial total energy of vDNS$_e$ is:

$$\mathcal{E}_e|_{t=0} = \frac{(\rho|_{t=0})^2}{2} + \frac{1}{4\,\rho|_{t=0}} \frac{(\varepsilon\, p|_{t=0})^2}{2}.$$

This energy is initially unbounded at $x = x_c$ because p does not belong to L^2. This will yield a localised spurious solution.

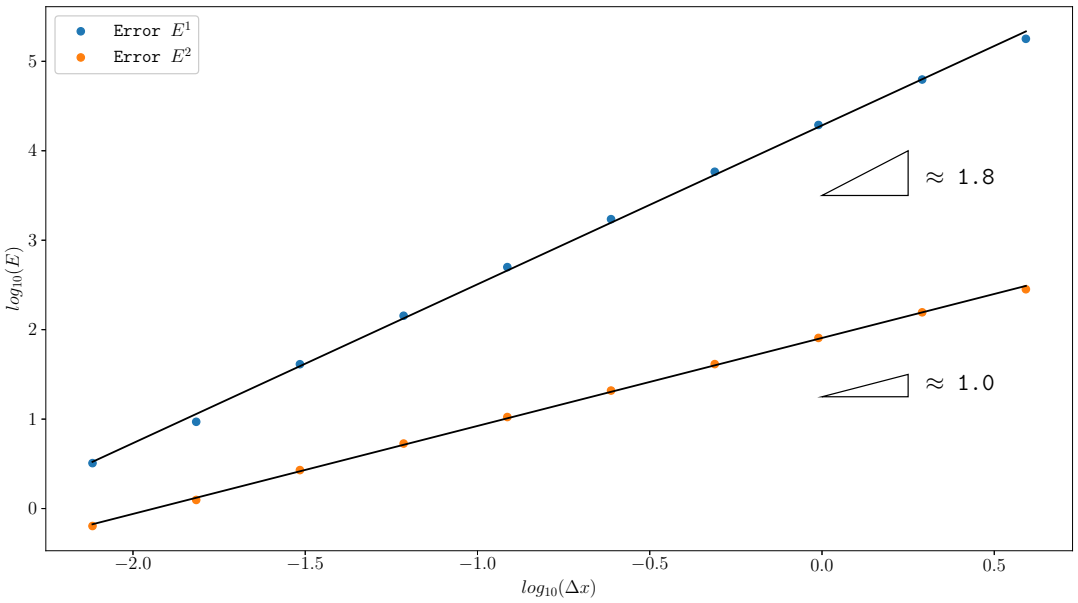
The spatial domain x is $[0; 500]$. The parameters are $\rho_L = 5$, $\rho_R = 1$, $x_c = 250$, CFL $= 0.9$. From Section 3.3, the viscosity function is $\mu = 2\rho\sqrt{\rho\,\mathcal{K}}$. The parameters of the extended Lagrangian (λ, β) are chosen in accordance with Section 4.5.

5.3. Numerical convergence

Figure 7 shows convergence results of the numerical scheme when $\lambda = 500$, $\beta = 10^{-7}$ and $\tau = \varepsilon = 10^{-2}$. The parameters are the same as those used in forthcoming Figure 10 (d), where no dispersion occurs. Doing so allows to focus purely on the properties of the numerical scheme. The errors E^1 and E^2 are defined by:

$$E^1 = \Delta x \sum_{i=0}^{N} |u_i - u_{e,i}| \approx \int_0^L |u(x) - u_e(x)|\, dx,$$

$$E^2 = \sqrt{\Delta x \sum_{i=0}^{N} (u_i - u_{e,i})^2} \approx \sqrt{\int_0^L (u(x) - u_e(x))^2 dx},$$



(a)

Figure 7: Global error E as a function of Δx. $N \in \{2^7; 2^8; ...; 2^{17}\}$. Parameters: $\lambda = 500$, $\beta = 10^{-7}$ and $\tau = \varepsilon = 10^{-2}$ at $t = 40$.

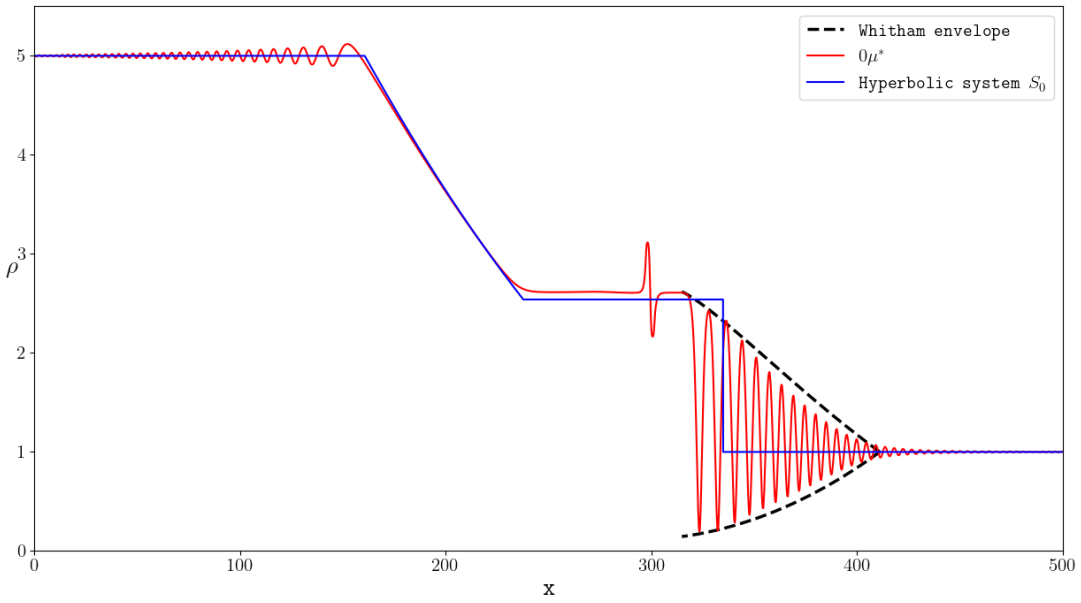
where u_i is the numerical solution at x_i and $u_{e,i}$ is the reference solution at x_i. The latter is the numerical solution for $N = 2^{18}$. Almost second-order accuracy is obtained in L^1 norm, and first-order accuracy is obtained in L^2 norm. From now on, $N = 2^{15}$ grid points are used.

5.4. Influence of the viscosity

Case $\tau = 0$. Figure 8 shows the numerical solution of the extended DNS when $\lambda = 500$, $\beta = 10^{-3}$ and $\varepsilon = 3$. The main features of the exact dispersive solution of S_ε, shown in Figure 2, are recovered: the dispersive shock lies inside the Whitham envelope, the plateau value ρ^* and the rarefaction wave are well captured.

Compared with Figure 2, additional features are observed. First, oscillations exist at the leading edge of the rarefaction wave. These oscillations decays when t increases [10]. Second, a spurious solution is localized at $x \approx 300$. It is induced by the discontinuous initial data (Section 5.2). As seen further, the optimal choice of τ will cancel this spurious solution.

Case $\tau \neq 0$. Figure 9 illustrates the effect of the viscosity on the solution of vDNS$_e$ with the parameters of the previous case. For small viscosity, the amplitude of oscillations and the plateau value decrease. When the viscosity is even stronger that of the dispersion (d), that is when $\tau = \varepsilon$, the dispersive nature of the solution is removed. Indeed, the amplitude of the oscillations is null and the plateau value is close to the one of the hyperbolic system S_0. This result validates the analytical predictions of Section 3. The parasite solution is still present due to the dispersion parameter $\varepsilon \neq 0$.



(a)

Figure 8: Comparison between the exact solution of S_0 and the solution of vDNS$_e$ with $\lambda = 500$, $\beta = 10^{-3}$, $\varepsilon = 3$ and $\tau = 0$.

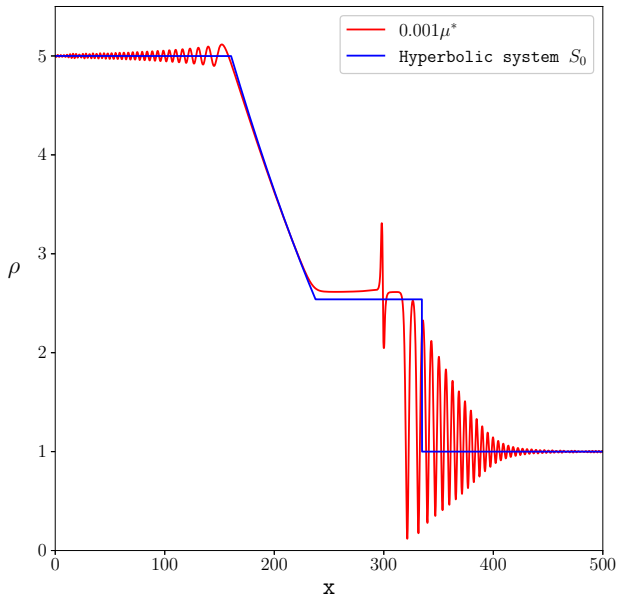
Case $\tau = \varepsilon \to 0$. Figure 10 illustrates the evolution of the solution of vDNS$_e$ when $\varepsilon \to 0$, while keeping the equality $\tau = \varepsilon$ and respecting (33) for β. The viscous nature of the solution is removed when $\varepsilon \to 0$. The solution of the hyperbolic system is recovered in (c) and (d). The spurious solution has also disappeared. The only differences, around the edges of the rarefaction wave and around the shock, are due to the numerical dissipation of the scheme.

6. Conclusion

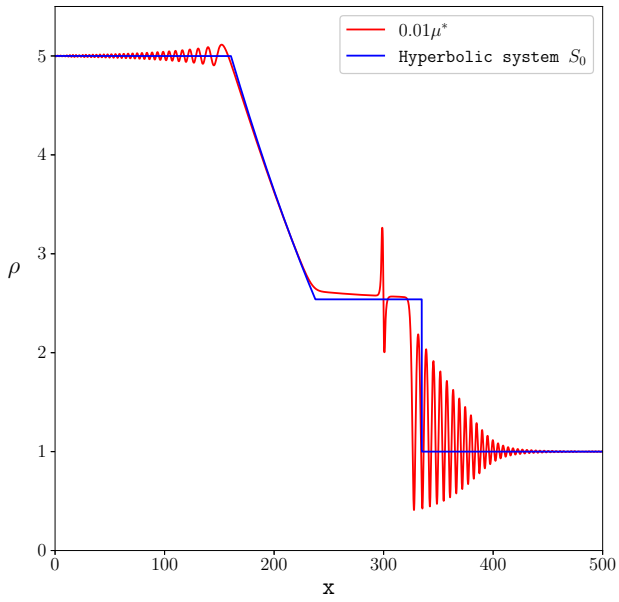
This work was focused on a singular limit problem issued from Euler-Korteweg model. Based on the dispersion analysis and on the study of a travelling wave, an optimal viscosity $\mu(\rho)$ is introduced in the model. Doing so allows to recover the solutions of an Euler model in the limit of small dispersion and small viscosity. Numerical results, through the extended Lagrangian method, confirm the analysis. Classical shock waves and rarefaction waves are then recovered with vanishing dispersion.

The theoretical part of this work was essentially formal and based on: linear dispersion analysis and construction of travelling wave. More sophisticated mathematical tools are required to prove the convergence of NSK solution towards the solution of EK with vanishing dispersion and the optimal viscosity. Another mathematical analysis is required to prove the convergence of the extended Lagrangian method, in lines with [28]. Lastly, a natural extension of this work concerns the case of nonconvex equation of state.

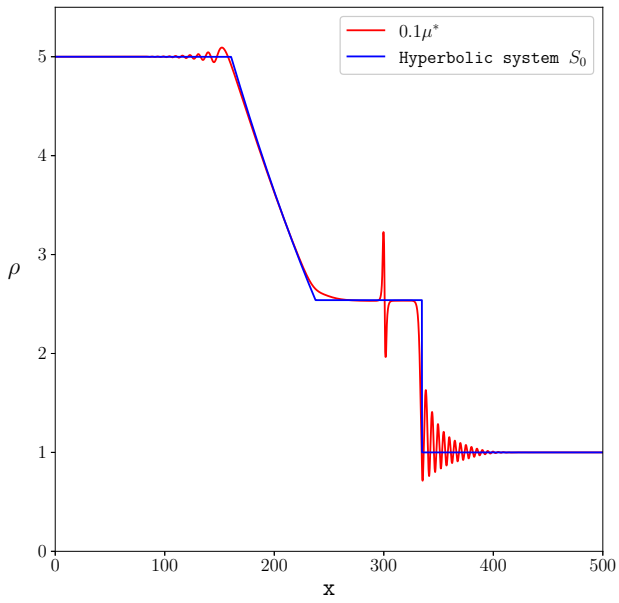
Acknowledgements. The authors are very grateful to Sergey Gavrilyuk for many advices and



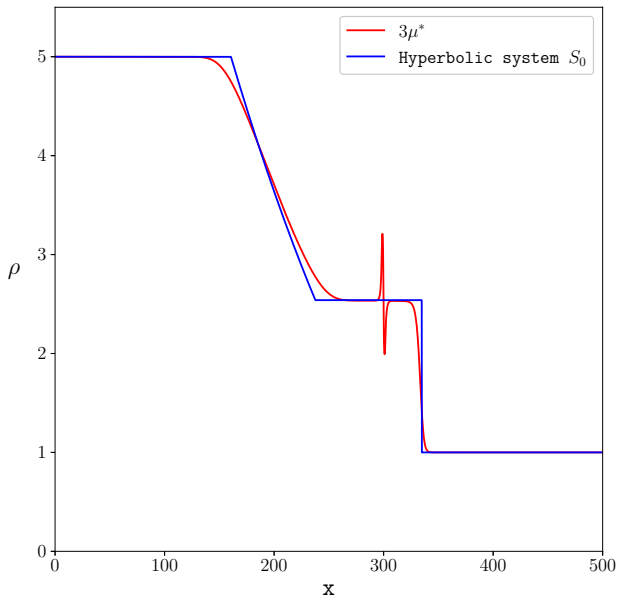
(a) $\tau = 0.001$



(b) $\tau = 0.01$

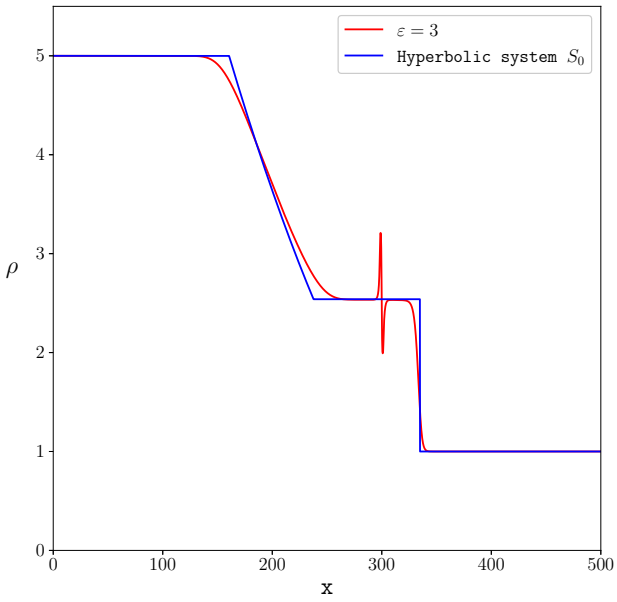


(c) $\tau = 0.1$

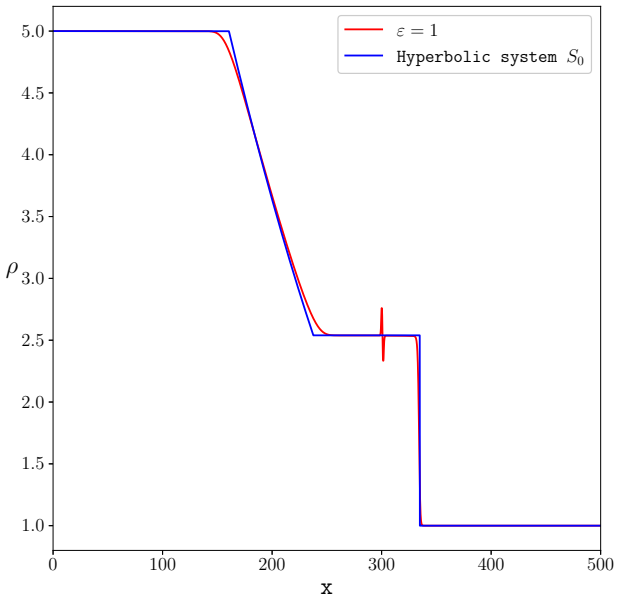


(d) $\tau = \hat{\tau} = 3$

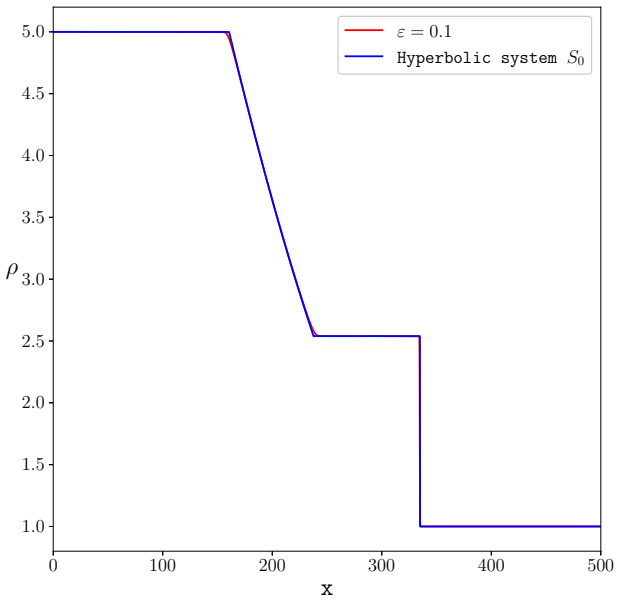
Figure 9: Effect of the viscosity on the solution of vDNS$_e$. $\tau \in \{0.001; 0.01; 0.1; 3\}$ with $\lambda = 500$, $\beta = 0.001$.



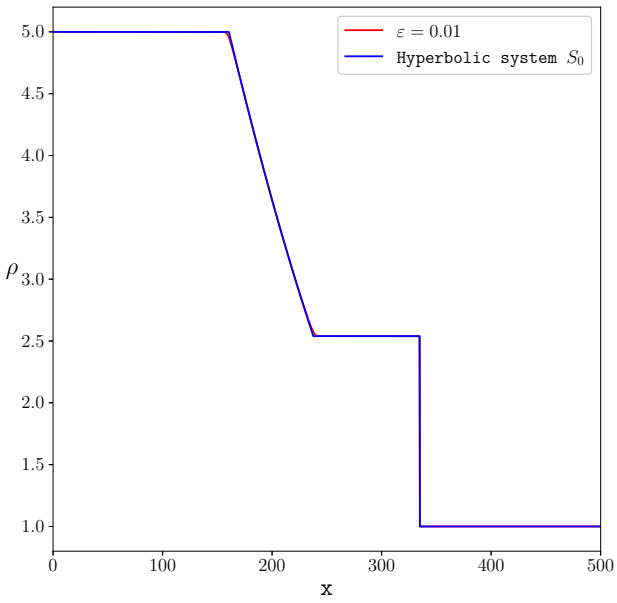
(a) $\tau = 3$, $\beta = 10^{-3}$



(b) $\tau = 1$, $\beta = 10^{-3}$



(c) $\tau = 0.1$, $\beta = 10^{-5}$



(d) $\tau = 0.01$, $\beta = 10^{-7}$

Figure 10: Small dispersion limit and small viscosity limit of the solution of vDNS$_e$. $\tau = \varepsilon \to 0$ with $\lambda = 500$ and variable β.

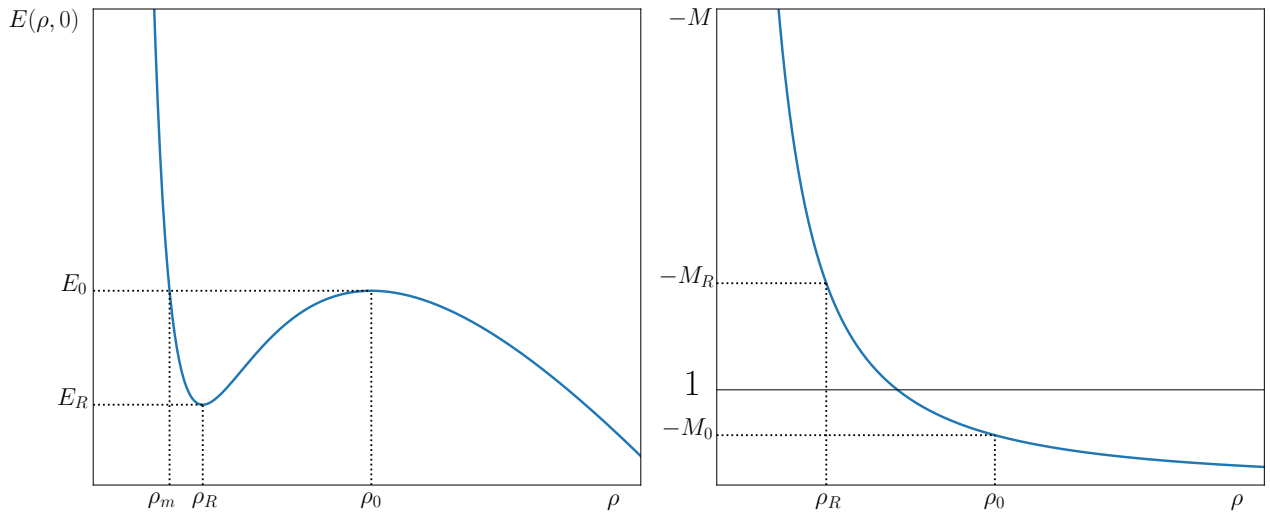
insights about nonlinear waves. This work was partially supported by ANR-ASTRID project SNIP ANR19ASTR001601.

Appendix A. Travelling wave: additional details

When $e_h = \rho/2$, the conditions (23) recover (5) and (6) for S_0 (Table 1): thus (23) amount to the Rankine-Hugoniot conditions. Therefore, we consider the RP problem (4) for which ρ_R and ρ_0 automatically verify (23) with D the shock wave's speed. The structure of the travelling wave is related to the structure of the extrema (e.g. stable foci). Those information are obtained by solving an eigenvalue problem for the linearized version of the nonlinear ODE (21) and by looking at second derivatives of E. Setting $c = \sqrt{(\rho^2\, e_h{}')'}$ the local sound speed and $M = (u - D)/c$ the Mach number, the second derivative of E with respect to ρ at the extrema is:

$$\left.\frac{\partial^2 E}{\partial \rho^2}\right|_{\rho_c} = \frac{{c_c}^2}{{\rho_c}^2}\left(M_c^2 - 1\right). \tag{A.1}$$

The thermodynamic pressure $\rho^2 {e_h}'$ is assumed to be a convex function of $1/\rho$. Therefore, c^2 is always positive. From (A.1), the state R corresponds to the supersonic region ($|M_R| > 1$) and E admits a local minimum at this state. The state 0 is the subsonic region ($|M_0| < 1$) and E admits a local maximum at this state. In the interval $I =]\rho_m; \rho_0[$, where ρ_m has to verify $E_0 = E(\rho_m, 0)$, E admits a unique minimum at $\rho_R \in I$ (Figure A.11-(a)). If $\rho \in I$ at $\xi = 0$



(a) $E(\rho, 0)$ (b) $-M(\rho)$

Figure A.11: Function $E(\rho, 0)$ and $-M(\rho)$, when $e(\rho) = \rho/2$, $\rho_R = 1$, $\rho_L = 5$.

and condition (22) is satisfied, the value $\rho(\xi)$ for any $\xi \neq 0$ remains in I. Therefore, $\rho(\xi) \to \rho_R$ when $\xi \to \infty$. The point ρ_R is asymptotically stable.

Appendix B. Matrix A and characteristic speeds

The NSK_e system in primitive variable is:

$$\begin{pmatrix}\rho\\ u\\ \eta\\ w\\ p\end{pmatrix}_t + \begin{pmatrix}u & \rho & 0 & 0 & 0\\ a & u & b & 0 & d\\ 0 & 0 & u & 0 & 0\\ e & 0 & 0 & u & h\\ 0 & p & 0 & -1 & u\end{pmatrix}\begin{pmatrix}\rho\\ u\\ \eta\\ w\\ p\end{pmatrix}_x = \begin{pmatrix}0\\ 0\\ w\\ \frac{\lambda}{\beta\rho}\left(1-\frac{\eta}{\rho}\right)-\tau\frac{\mu(\rho)}{\beta\rho^3}w\\ 0\end{pmatrix}, \tag{B.1}$$

with

$$\mathbf{A} = \begin{pmatrix}u & \rho & 0 & 0 & 0\\ a & u & b & 0 & d\\ 0 & 0 & u & 0 & 0\\ e & 0 & 0 & u & h\\ 0 & p & 0 & -1 & u\end{pmatrix}.$$

The coefficients are

$$a = 2{e_h}' + \rho {e_h}'' + \varepsilon^2 p^2\left(\frac{\mathcal{K}'}{\rho}+\frac{1}{2}\mathcal{K}''\right)+\lambda\frac{\eta^2}{\rho^3},\quad b=\frac{\lambda}{\rho}\left(1-\frac{2\eta}{\rho}\right),\quad e=-\frac{\varepsilon^2 p\mathcal{K}'}{\beta\rho},$$

$$h=-\frac{\varepsilon^2\mathcal{K}}{\rho\beta},\quad d=\frac{p\,\varepsilon^2}{\rho}\left(\mathcal{K}+\rho\mathcal{K}'\right).$$

The characteristic equation is:

$$(u-\alpha)\left[(u-\alpha)^4+(h-a\rho-pd)\,(u-\alpha)^2+\rho(ed-ha)\right]=0.$$

The characteristic speeds are:

$$\alpha^1 = u,\quad \alpha^{2,3}=u\pm c_1,\quad \alpha^{4,5}=u\pm c_2,$$

with

$$c_1=\sqrt{\frac{1}{2}\left(c_e+\sqrt{(c_s)^2+4\frac{d^2}{\beta}}\right)},$$

$$c_2=\sqrt{\frac{1}{2}\left(c_e-\sqrt{(c_s)^2+4\frac{d^2}{\beta}}\right)},$$

$$c_e=(\rho^2{e_h}')'+\varepsilon^2p^2\left(\frac{\mathcal{K}}{\rho}+2\mathcal{K}'+\frac{1}{2}\rho\mathcal{K}''\right)+\frac{\varepsilon^2\mathcal{K}}{\rho\beta}+\lambda\left(\frac{\eta}{\rho}\right)^2,$$

$$c_s=(\rho^2{e_h}')'+\varepsilon^2p^2\left(\frac{\mathcal{K}}{\rho}+2\mathcal{K}'+\frac{1}{2}\rho\mathcal{K}''\right)-\frac{\varepsilon^2\mathcal{K}}{\rho\beta}+\lambda\left(\frac{\eta}{\rho}\right)^2.$$

Appendix C. Exact solution for the relaxation step of the splitting

From equations (37a), it follows $\rho = \rho^*$, $u = u^*$, $p = p^*$. Equations (37b) and (37c) are equivalent to the second-order ODE:

$$\frac{d^2\eta}{dt^2} + \frac{\tau\mu^*}{\beta\rho^{*3}}\frac{d\eta}{dt} + \frac{\lambda}{\beta\rho^{*2}}\eta = \frac{\lambda}{\beta\rho^*},$$

where $\mu^* = \mu(\rho^*)$. One can distinguish three regimes:

- Overdamped solution if $\left(\frac{\tau\mu^*}{\beta\rho^{*2}} > 2\sqrt{\frac{\lambda}{\beta}}\right)$:

$$r_{1,2} = \left(-\frac{\tau\mu^*}{2\beta\rho^{*3}} \pm \frac{1}{2}\sqrt{\left(\frac{\tau\mu^*}{\beta\rho^{*3}}\right)^2 - \frac{4\lambda}{\beta\rho^{*2}}}\right) \leq 0, \tag{C.1}$$

$$\eta(t) = \rho^* + (\eta^* - \rho^*)\frac{r_2 e^{r_1 t} - r_1 e^{r_2 t}}{r_2 - r_1} + w^*\frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1}, \tag{C.2}$$

$$w(t) = (\eta^* - \rho^*)\frac{r_1 r_2(e^{r_1 t} - e^{r_2 t})}{r_2 - r_1} + w^*\frac{r_2 e^{r_2 t} - r_1 e^{r_1 t}}{r_2 - r_1}. \tag{C.3}$$

- Critically damped solution if $\left(\frac{\tau\mu^*}{\beta\rho^{*2}} = 2\sqrt{\frac{\lambda}{\beta}}\right)$:

$$r_0 = -\frac{\tau\mu^*}{2\beta\rho^{*3}} \leq 0, \tag{C.4}$$

$$\eta(t) = \rho^* + (\eta^* - \rho^*)(1 - r_0 t)e^{r_0 t} + w^* t e^{r_0 t}, \tag{C.5}$$

$$w(t) = w^*(1 + r_0 t)e^{r_0 t} - r_0(\eta^* - \rho^*)r_0 t e^{r_0 t}. \tag{C.6}$$

- Underdamped solution if $\left(\frac{\tau\mu^*}{\beta\rho^{*2}} < 2\sqrt{\frac{\lambda}{\beta}}\right)$:

$$r_c = -\frac{\tau\mu^*}{2\beta\rho^{*3}} \leq 0, \quad \omega_c = \frac{1}{2}\sqrt{\frac{4\lambda}{\beta\rho^{*2}} - \left(\frac{\tau\mu^*}{\beta\rho^{*3}}\right)^2}, \tag{C.7}$$

$$\eta(t) = \rho^* + (\eta^* - \rho^*)\left(\cos(\omega_c t) - \frac{r_c}{\omega_c}\sin(\omega_c t)\right)e^{r_c t} + \frac{w^*}{\omega_c}\sin(\omega_c t)e^{r_c t}, \tag{C.8}$$

$$w(t) = -\omega_c(\eta^* - \rho^*)\left(1 + \left(\frac{r_c}{\omega_c}\right)^2\right)\sin(\omega_c t)e^{r_c t} + w^*\left(\cos(\omega_c t) + \frac{r_c}{\omega_c}\sin(\omega_c t)\right)e^{r_c t}. \tag{C.9}$$

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