



HAL
open science

Numerical Simulation of Polarized Light and Temperature with a Refractive Interface

Olivier Pironneau

► **To cite this version:**

Olivier Pironneau. Numerical Simulation of Polarized Light and Temperature with a Refractive Interface. 2024. hal-04610801v2

HAL Id: hal-04610801

<https://hal.science/hal-04610801v2>

Preprint submitted on 29 Jun 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Numerical Simulation of Polarized Light and Temperature with a Refractive Interface

Olivier Pironneau¹

Abstract

In this article we propose a numerical algorithm to compute the intensity and polarization of a polychromatic electromagnetic radiation crossing a medium with graded refractive index and modeled by the Vector Radiative Refractive Transfer Equations (VRRTE). Special attention is given to the case where the refractive index has a discontinuity for which the Fresnel conditions are necessary. We assume that the only spatial variable of interest is the altitude (stratified medium). An algorithm based on iterations of the sources is shown to be monotone and convergent. Numerical examples are given with highly varying absorption coefficient κ and Rayleigh scattering as in the Earth atmosphere. To study the effect of CO₂ in the atmosphere κ is changed in the frequency ranges where CO₂ is absorbing.

Keywords: Radiative transfer, Polarization, Fresnel's law, Integral equation, Numerical analysis, Climate modeling. MSC classification 3510, 35Q35, 35Q85, 80A21, 80M10

Contents

1	Introduction	2
2	Fundamental Equations	3
3	The Stratified Case	5
4	A Stratified Medium with a Discontinuous Refractive Index	6
5	Stratified VRRTE with a Planar Discontinuity	8

¹olivier.pironneau@sorbonne-universite.fr, LJLL, Sorbonne Université, Paris, France.

6	Application to Fresnel's Conditions	12
6.1	Computation of $\Delta(z, \mu)$	12
6.2	Implementation	13
7	Integral representation of the Problem	14
8	Convergence of the Iteration on the Sources	15
8.1	Principle	15
8.2	Monotony	16
9	Numerical Results	18
10	Conclusion	19

1. Introduction

Understanding and computing a polychromatic electromagnetic radiation crossing a medium with non constant refractive index and non constant absorption and scattering is a challenge for which the works of S. Chandrasekhar [6] and G. Pomraning [16] are fundamental. Applications are numerous in astrophysics, nuclear engineering, atmospheric sciences [7] and more recently image synthesis [1].

The medium is too large and the wavelengths too small to use Maxwell's equations, so the Radiative Transfer Equation (RTE) is used. It is an integro partial differential system in 6 dimensions, 3 for space, 2 for ray directions, 1 for frequencies; so it is a serious numerical challenge. With his coauthors the present author proposed to analyze the coupling of RTE with the temperature equation to understand the effect of the numerous physical parameters on the temperature in the Earth atmosphere. An algorithm based on iterations on the source was shown to be monotone and convergent for stratified media in [10] and in 3D for a general topography in [9]. The method was generalized to polarized light by using the vector RTE (VRTE) for stratified media [14]. Then it was further generalized to VRRTE (the second R is for refractive) for media with non-constant but smooth refractive index [13].

There are numerous methods to solve numerically the VRTE, based on Fourier or Chebyshev expansion [11], [5], discrete ordinate [19], lattice Boltzmann [2], Finite Elements [18], etc. They aim at giving a complete description of the Stokes vector as a function of spatial and ray directional variables. Here we are interested in the temperature and this does not require to solve the full VRTE, it is enough to compute the directional average of the Stokes

vector. Solving the VRTE coupled with the temperature equation has not received much attention, to our knowledge.

In this article the method is generalized to VRRTE with discontinuous refractive index for which the Fresnel conditions are necessary to match the Stokes vector on both sides of the discontinuity. We have used a formulation of Fresnel's conditions given by A. Garcia [8] which is well adapted to the VRRTE.

While Fresnel's conditions are natural jump conservations for the Maxwell equations, they are not natural to the VRRTE. Hence we had to include them in the integral semi-analytic solution of the VRRTE rather than in the partial differential equations (see sections 5 and 6).

By coupling the VRRTE with the temperature equation the problem becomes nonlinear. Iterations on the sources is a very simple idea in which the equations are solved with given right hand side (the sources) and then the sources are updated with the new solution. The temperature equation is solved with Newton iterations as explained in [10]. We prove here that the sequences are monotonous and that the solution can be approached from above and below, at least when the jump in the refractive index is not too large.

The method is tested numerically on two set of cases, one in which the light comes from the Sun and the other where the (infrared) light comes from Earth. These waves cross a medium which has a layer of large refractive index (like the sea) and the atmosphere above it which has a refractive index close to unity.

We intend to generalize the method to non stratified atmosphere as in [9].

2. Fundamental Equations

Light in a medium Ω is an electromagnetic radiation satisfying Maxwell's equations. The electric field $\mathbf{E} = \mathbf{E}_0 \exp(i(\mathbf{k} \cdot \mathbf{x} - \nu t))$ of a monochromatic plane wave of frequency ν propagating in direction \mathbf{k} , is a solution to Maxwell equations which is suitable to describe the propagation of a ray for which ν is very large.

Such radiations are characterized either by \mathbf{k} and \mathbf{E}_0 or equivalently [6][8] by their Stokes vectors $\mathbf{I} = [I, Q, U, V]^T$, made of the irradiance I and 3 functions Q, U, V to define its state of polarization.

The radiation comes from the boundary but also from a the Planck law, a volume source $\mathbf{F} = [\kappa_a B_\nu(T), 0, 0, 0]^T$ for an unpolarized-emitting black-body (for example due to the black-body radiation of air or water). It

is defined in terms of the rescaled Planck function $B_\nu(T) = \nu^3(e^{\frac{\nu}{T}} - 1)^{-1}$, and the rescaled temperature T . The range of frequencies of interest is $\nu \in (0.01, 20) \times 10^{14}$, hence a scaling is applied (see [10]): ν is divided by 10^{14} and T_K in Kelvin is divided by 4798: $T = 10^{-14} \frac{k}{h} T_K = \frac{T_K}{4798}$, where k and h are the Boltzmann and Planck constants. The parameter κ_a is related to absorption and scattering (see (3.3) below), which, by the way, are quantum effects, not described by Maxwell's equations.

Absorption and scattering are modeled by a system of integro-differential partial differential equations, known as VRRTE (short for Vector Radiative Refractive Transfer Equations) [16] p152, [3].

With $\tilde{\mathbf{I}} := \mathbf{I}/n^2$,

$$\frac{n}{c} \partial_t \tilde{\mathbf{I}} + \boldsymbol{\omega} \nabla_{\mathbf{x}} \tilde{\mathbf{I}} + \frac{\nabla_{\mathbf{x}} n}{n} \cdot \nabla_{\boldsymbol{\omega}} \tilde{\mathbf{I}} + \kappa_\nu \tilde{\mathbf{I}} = \int_{\mathbb{S}_2} \mathbb{Z}(\mathbf{x}, \boldsymbol{\omega}' : \boldsymbol{\omega}) \tilde{\mathbf{I}} d\boldsymbol{\omega}' + \tilde{\mathbf{F}}, \quad (2.1)$$

for all $\mathbf{x} \in \Omega, \boldsymbol{\omega} \in \mathbb{S}_2$, where c is the speed of light, n the refractive index of the medium, \mathbb{S}_2 the unit sphere, κ the absorption and \mathbb{Z} the phase scattering matrix for rays $\boldsymbol{\omega}'$ scattered in direction $\boldsymbol{\omega}$ for each frequency ν . It is assumed that n depends continuously on position $\mathbf{x} \in \Omega$ except on surfaces of discontinuities on which additional jump conditions will be applied (Fresnel's conditions); κ depends on \mathbf{x} and strongly on ν , except at interfaces of strong discontinuities where the transmission, reflection and refraction are subject to Fresnel's conditions.

Because c is very large, the term $\frac{1}{c} \partial_t \tilde{\mathbf{I}}$ is neglected. The thermal conductivity is also small so that "Thermal Equilibrium" is assumed:

$$\nabla_{\mathbf{x}} \cdot \int_{\mathbb{R}_+} \int_{\mathbb{S}_2} \tilde{\mathbf{I}} \boldsymbol{\omega} d\boldsymbol{\omega} d\nu = 0. \quad (2.2)$$

Notation 1. *On all variables, the tilde indicates a division by n^2 . Arguments of functions are sometimes written as indices like κ_ν and n_z .*

Following [12], given a cartesian frame $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the third term on the left in (2.1) is computed in polar coordinates, with

$$\boldsymbol{\omega} := \mathbf{i} \sin \theta \cos \varphi + \mathbf{j} \sin \theta \sin \varphi + \mathbf{k} \cos \theta, \quad \mathbf{s}_1 := -\mathbf{i} \sin \varphi + \mathbf{j} \cos \varphi,$$

$$\nabla_{\mathbf{x}} \log n \cdot \nabla_{\boldsymbol{\omega}} \tilde{\mathbf{I}} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left\{ \tilde{\mathbf{I}} (\cos \theta \boldsymbol{\omega} - \mathbf{k}) \cdot \nabla_{\mathbf{x}} \log n \right\} + \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \left\{ \tilde{\mathbf{I}} \mathbf{s}_1 \cdot \nabla_{\mathbf{x}} \log n \right\}.$$

When n does not depend on x, y but only on z , it simplifies to

$$\nabla_{\mathbf{x}} \log n \cdot \nabla_{\boldsymbol{\omega}} \tilde{\mathbf{I}} = (\partial_z \log n) \cdot \partial_\mu \left\{ (1 - \mu^2) \tilde{\mathbf{I}} \right\} \quad \text{where } \mu = \cos \theta.$$

3. The Stratified Case

For an atmosphere of thickness Z over a flat ground, the spatial domain is $\Omega = \mathbb{R}^2 \times (0, Z)$, but if all variables are independent of x, y it reduces to $(0, Z)$. In that case, in [6] p40-53, expressions for the phase matrix \mathbb{Z} are given for Rayleigh and isotropic scattering for $[I, Q]^T$,

$$\mathbb{Z}_R = \frac{3}{2} \begin{bmatrix} 2(1 - \mu^2)(1 - \mu'^2) + \mu^2 \mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{bmatrix}, \quad \mathbb{Z}_I = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

For a given $\beta \in [0, 1]$, we shall consider a combination of $\beta \mathbb{Z}_R$ (Rayleigh scattering) plus $(1 - \beta) \mathbb{Z}_I$ (isotropic scatterings) as in [6],[16],[17].

The two other components of the Stokes vectors have autonomous equations,

$$\mu \partial_z \tilde{U} + \partial_z \log n \cdot \partial_\mu \{(1 - \mu^2) \tilde{U}\} + \kappa \tilde{U} = 0, \quad (3.1)$$

$$\mu \partial_z \tilde{V} + \partial_z \log n \cdot \partial_\mu \{(1 - \mu^2) \tilde{V}\} + \kappa \tilde{V} = \frac{\mu}{2} \int_{-1}^1 \mu' \tilde{V}(z, \mu') d\mu'. \quad (3.2)$$

Notation 2. Denote the scattering coefficient $a_s \in [0, 1]$, which, as κ , is a function of altitude z and frequency ν . Define

$$\kappa_s = \kappa a_s, \quad \kappa_a = \kappa - \kappa_s = \kappa(1 - a_s). \quad (3.3)$$

From (3.1),(3.2) we see that, if the light source at the boundary is unpolarized then $U = V = 0$ and the light can be described either by I and Q or two orthogonal components I_l, I_r , such that $I = I_l + I_r$ and $Q = I_l - I_r$ (see [6]):

$$\left\{ \begin{array}{l} \mu \partial_z \tilde{I}_l + \partial_z \log n \cdot \partial_\mu \{(1 - \mu^2) \tilde{I}_l\} + \kappa \tilde{I}_l \\ \quad = \frac{3\beta\kappa_s}{8} \int_{-1}^1 ([2(1 - \mu'^2)(1 - \mu^2) + \mu'^2 \mu^2] \tilde{I}_l + \mu'^2 \tilde{I}_r) d\mu' \\ \quad + \frac{(1 - \beta)\kappa_s}{4} \int_{-1}^1 [\tilde{I}_l + \tilde{I}_r] d\mu' + \frac{\kappa_a}{2} \tilde{B}_\nu(T(z)), \\ \mu \partial_z \tilde{I}_r + \partial_z \log n \cdot \partial_\mu \{(1 - \mu^2) \tilde{I}_r\} + \kappa \tilde{I}_r = \frac{3\beta\kappa_s}{8} \int_{-1}^1 (\mu'^2 \tilde{I}_l + \tilde{I}_r) d\mu' \\ \quad + \frac{(1 - \beta)\kappa_s}{4} \int_{-1}^1 [\tilde{I}_l + \tilde{I}_r] d\mu' + \frac{\kappa_a}{2} \tilde{B}_\nu(T(z)), \end{array} \right. \quad (3.4)$$

Using an appropriate linear combination of (3.4), the system for \tilde{I} and \tilde{Q} is

derived,

$$\begin{cases} \mu \partial_z \tilde{I} + \partial_z \log n \cdot \partial_\mu \{(1 - \mu^2) \tilde{I}\} + \kappa \tilde{I} \\ \quad = \kappa_a \tilde{B}_\nu + \frac{\kappa_s}{2} \int_{-1}^1 \tilde{I} d\mu' + \frac{\beta \kappa_s}{4} P_2(\mu) \int_{-1}^1 [P_2 \tilde{I} - (1 - P_2) \tilde{Q}] d\mu', \\ \mu \partial_z \tilde{Q} + \partial_z \log n \cdot \partial_\mu \{(1 - \mu^2) \tilde{Q}\} + \kappa \tilde{Q} \\ \quad = -\frac{\beta \kappa_s}{4} (1 - P_2(\mu)) \int_{-1}^1 [P_2 \tilde{I} - (1 - P_2) \tilde{Q}] d\mu', \end{cases} \quad (3.5)$$

where $P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$. The temperature $T(z)$ is linked to I by (2.2) which, in the case of (3.5) is as follows.

Proposition 1. *Thermal equilibrium for (3.4) or (3.5) is*

$$\int_{\mathbb{R}_+} \kappa_a [\tilde{B}_\nu(T) - \frac{1}{2} \int_{-1}^1 \tilde{I} d\mu] d\nu = 0.$$

Proof Averaging in μ the first equation of (3.5) leads to

$$\begin{aligned} \nabla_{\mathbf{x}} \cdot \int_{\mathbb{S}_2} \boldsymbol{\omega} \tilde{I} = \partial_z \left(\frac{1}{2} \int_{-1}^1 \mu \tilde{I} d\mu \right) &= -\frac{1}{2} \partial_z \log n \cdot \int_{-1}^1 \partial_\mu \{(1 - \mu^2) \tilde{I}\} d\mu \\ &\quad - \frac{1}{2} \kappa \int_{-1}^1 \tilde{I} d\mu + \frac{1}{2} \int_{-1}^1 \kappa_a \tilde{B}_\nu d\mu + \frac{\kappa_s}{2} \int_{-1}^1 \tilde{I} d\mu', \end{aligned}$$

because $\int_{-1}^1 P_2(\mu) d\mu = 0$. Now the first term on the right integrates to zero and $\kappa - \kappa_s = \kappa_a$. \square

For the numerical simulations (3.5) is more appropriate, but to derive energy estimates (3.4) is better. The differences are in the source terms and the boundary conditions, so we can easily switch from one to the other.

4. A Stratified Medium with a Discontinuous Refractive Index

Systems (3.4) and (3.5) are not valid across a discontinuity of $z \mapsto n(z)$, but the Fresnel Conditions give the needed jump conditions to patch the solutions. Consider 3 parallel planes at $z = 0$, $z = Y > 0$ and $z = Z > Y$. The refractive index of the medium is n^- when $z < Y$ and n^+ when $z > Y$. Denote, when the roots exist,

$$n_\mp = \frac{n_-}{n_+}, \quad n_\pm = \frac{n_+}{n_-}, \quad \eta(n, \mu) = \sqrt{1 - n^2(1 - \mu^2)}, \quad \mu_c(n) = \sqrt{1 - \frac{1}{n^2}}. \quad (4.1)$$

Fresnel's refraction conditions are written in [8] for \mathbf{I} . Rewritten for $\tilde{\mathbf{I}}$, they are,

$$\begin{aligned}\tilde{\mathbf{I}}(Y^-, -\mu) &= \mathbf{X}(n_{\mp}, \mu)\tilde{\mathbf{I}}(Y^-, \mu) + \mathbf{Y}(n_{\mp}, \mu)\tilde{\mathbf{I}}(Y^+, -\eta(n_{\mp}, \mu)), \quad \mu \in (0, 1), \\ \tilde{\mathbf{I}}(Y^+, \mu) &= \mathbf{X}(n_{\pm}, \mu)\tilde{\mathbf{I}}(Y^+, -\mu) + \mathbf{Y}(n_{\pm}, \mu)\tilde{\mathbf{I}}(Y^-, \eta(n_{\pm}, \mu)), \quad \mu \in (0, 1).\end{aligned}\tag{4.2}$$

$$\mathbf{X}(n, \mu) = \begin{cases} \mathbf{G}(n, \mu), & n \leq 1, \\ \mathbf{G}(n, \mu)H[\mu - \mu_c(n)] + \mathbf{\Gamma}(n, \mu)\{1 - H[\mu - \mu_c(n)]\}, & n \geq 1, \end{cases}\tag{4.3}$$

$$\mathbf{Y}(n, \mu) = \begin{cases} \mathbf{D}(n, \mu), & n \leq 1, \\ \mathbf{D}(n, \mu)H[\mu - \mu_c(n)], & n \geq 1.\end{cases}\tag{4.4}$$

Here $\tilde{\mathbf{I}} \in \mathbb{R}^4$, \mathbf{X}, \mathbf{Y} are 4×4 matrices given in terms of H , the Heaviside function and 3 matrices $\mathbf{G}, \mathbf{D}, \mathbf{\Gamma}$, for which the non-zero terms are ,

$$\begin{aligned}\Gamma_{11} &= \Gamma_{22} = 1, \\ G_{11} = G_{22} &= \frac{1}{2} \left\{ \left[\frac{\mu - n\eta(n, \mu)}{\mu + n\eta(n, \mu)} \right]^2 + \left[\frac{n\mu - \eta(n, \mu)}{n\mu + \eta(n, \mu)} \right]^2 \right\} \\ G_{12} = G_{21} &= \frac{1}{2} \left\{ \left[\frac{\mu - n\eta(n, \mu)}{\mu + n\eta(n, \mu)} \right]^2 - \left[\frac{n\mu - \eta(n, \mu)}{n\mu + \eta(n, \mu)} \right]^2 \right\} \\ D_{11} = D_{22} &= 2n\mu\eta(n, \mu) \left\{ \frac{1}{[\mu + n\eta(n, \mu)]^2} + \frac{1}{[n\mu + \eta(n, \mu)]^2} \right\} \\ D_{12} = D_{21} &= 2n\mu\eta(n, \mu) \left\{ \frac{1}{[\mu + n\eta(n, \mu)]^2} - \frac{1}{[n\mu + \eta(n, \mu)]^2} \right\} \\ \Gamma_{33} = \Gamma_{44} &= 1 - \frac{2(1 - \mu^2)^2}{1 - (1 + n^{-2})\mu^2}, \quad \Gamma_{43} = -\Gamma_{34} = \frac{2\mu(1 - \mu^2)(\mu_c^2 - \mu^2)^{\frac{1}{2}}}{1 - (1 + n^{-2})\mu^2} \\ G_{33} = G_{44} &= \left(\frac{\mu - n\eta(n, \mu)}{\mu + n\eta(n, \mu)} \right) \left(\frac{n\mu - \eta(n, \mu)}{n\mu + \eta(n, \mu)} \right), \\ D_{33} = D_{44} &= \frac{4n\mu\eta(n, \mu)}{(n\eta(n, \mu) + \mu)(n\mu + \eta(n, \mu))}.\end{aligned}$$

Remark 1. Notice that

- System (4.2) is compatible with a Stokes vector like $\tilde{\mathbf{I}} = [I, Q, 0, 0]^T$: the last 2 components of $\tilde{\mathbf{I}}$ on the left and right side of the equations can be zero. Therefore, when the polarization is with $U = V = 0$, we can work with $\tilde{\mathbf{I}} = [\tilde{I}, \tilde{Q}]^T$ and the 2×2 matrices obtained from the left upper part of the full matrices.
- Notice that if $n \equiv 1$ then $\mathbf{X} = 0$ and $\mathbf{Y} = \mathbf{1}$.
- Finally notice that the eigenvalues of the 2×2 matrices \mathbf{X} and \mathbf{Y} are real and less or equal to 1.

Proof

$$\mathbf{X} = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix} \quad \text{with } a = \frac{1}{2} \left[\frac{\mu-n\eta(n,\mu)}{\mu+n\eta(n,\mu)} \right]^2, \quad b = \frac{1}{2} \left[\frac{n\mu-\eta(n,\mu)}{n\mu+\eta(n,\mu)} \right]^2$$

The eigenvalues λ are solutions of

$$\lambda^2 - 2\lambda(a+b) + 4ab = 0 \quad \Rightarrow \quad \lambda_1 = 2a, \quad \lambda_2 = 2b.$$

It is similar for \mathbf{Y} with $a = \frac{2n\mu\eta(n,\mu)}{[\mu+n\eta(n,\mu)]^2} \leq \frac{1}{2}$ and $b = \frac{2n\mu\eta(n,\mu)}{[n\mu+\eta(n,\mu)]^2} \leq \frac{1}{2}$. \square

Remark 2. The Fresnel conditions written for $\tilde{\mathbf{I}} = [\tilde{I}_l, \tilde{I}_r]^T$ have \mathbf{X} and \mathbf{Y} given by the same formulae but with the matrices changed to

$$\begin{pmatrix} \mathbf{X}_{11} + \mathbf{X}_{12} & 0 \\ 0 & \mathbf{X}_{11} - \mathbf{X}_{12} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{Y}_{11} + \mathbf{Y}_{12} & 0 \\ 0 & \mathbf{Y}_{11} - \mathbf{Y}_{12} \end{pmatrix}.$$

Notation 3. From now on the tildes are dropped and \mathbf{I} etc are understood as $\tilde{\mathbf{I}}$ etc.

5. Stratified VRRTE with a Planar Discontinuity

Consider the partition $(0, Z) = (0, Y] \cup [Y, Z)$. Assume that the refractive index is $n_z = n^-(z)$ in $(0, Y)$ and $n_z = n^+(z)$ in (Y, Z) . We assume that $\{n^+, n^-\}$ are smooth functions of z . The convective part of the vector radiative transfer equations for $\mathbf{I} = [I, Q]^T$ is (tildes are dropped),

$$\mu \partial_z \mathbf{I} + \partial_z \log n \cdot \partial_\mu \{(1 - \mu^2) \mathbf{I}\} + \kappa(z) \mathbf{I} = \mathbf{S}, \quad (5.1)$$

with the source terms $\mathbf{S} = [S(z, \mu), S'(z, \mu)]^T$. The characteristics of (5.1) divided by μ are defined by

$$\dot{\xi}(s) = 1, \quad \dot{\omega}(s) = \partial_z \log n(z(s)) \frac{1 - \omega^2(s)}{\omega(s)}$$

So the characteristic curve passing through z and μ is

$$z' \mapsto \omega(z') = \text{sign}(\mu) \sqrt{1 - \left(\frac{n(z)}{n(z')} \right)^2 (1 - \mu^2)}.$$

The solution of (5.1) is obtained by the method of characteristics, which can be adapted to the case of a discontinuity at $z = Y$,

$$\begin{aligned} \mathbf{I}(z, \mu)|_{z < Y} = & \mathbf{1}_{\mu > 0} \left[e^{-\int_0^z \frac{\kappa(z')}{\omega(z')} dz'} \mathbf{I}(0, \omega(0)) + \int_0^z \frac{e^{-\int_{z'}^z \frac{\kappa(z'')}{\omega(z'')} dz''}}{\omega(z')} \mathbf{S}(z', \omega(z')) dz' \right] \\ & + \mathbf{1}_{\mu < 0} \left[e^{\int_z^Y \frac{\kappa(z')}{\omega(z')} dz'} \mathbf{I}(Y^-, \omega(Y)) - \int_z^Y \frac{e^{\int_z^{z'} \frac{\kappa(z'')}{\omega(z'')} dz''}}{\omega(z')} \mathbf{S}(z', \omega(z')) dz' \right] \end{aligned} \quad (5.2)$$

$$\begin{aligned} \mathbf{I}(z, \mu)|_{z > Y} = & \mathbf{1}_{\mu > 0} \left[e^{-\int_Y^z \frac{\kappa(z')}{\omega(z')} dz'} \mathbf{I}(Y^+, \omega(Y)) + \int_Y^z \frac{e^{-\int_{z'}^z \frac{\kappa(z'')}{\omega(z'')} dz''}}{\omega(z')} \mathbf{S}(z', \omega(z')) dz' \right] \\ & + \mathbf{1}_{\mu < 0} \left[e^{\int_z^Z \frac{\kappa(z')}{\omega(z')} dz'} \mathbf{I}(Z, \omega(Z)) - \int_z^Z \frac{e^{\int_z^{z'} \frac{\kappa(z'')}{\omega(z'')} dz''}}{\omega(z')} \mathbf{S}(z', -\omega(z')) dz' \right] \end{aligned} \quad (5.3)$$

Notation 4. Define, when possible,

$$\boxed{\begin{aligned} \eta(n) &:= \sqrt{1 - n^2(1 - \mu^2)}, & \phi(z', z'')|_{z'' \geq z'} &= \exp \left\{ - \int_{z'}^{z''} \frac{\kappa(y)}{\eta(\frac{n_z}{n_y})} dy \right\}, \\ \mathbb{I}(y) &= \mathbf{I}(y, \text{sign}(\mu) \eta(\frac{n_z}{n_y})), & \mathbb{S}(y) &:= \mathbf{S}(y, \eta(\frac{n_z}{n_y})) / \eta(\frac{n_z}{n_y}) \end{aligned}}$$

All are functions of z and μ which are the reference point and direction to define the characteristic.

Lemma 1. Assume that $\mathbf{S}(z, -\mu) = \mathbf{S}(z, \mu)$ for all μ . If, for some function $\Delta(z, \mu)$,

$$\mathbb{I}(Y^+) = \mathbb{I}(Y^-) + \Delta(z, \mu), \quad (5.4)$$

then the solution of (5.1) is

$$\begin{aligned} \mathbf{I}(z, \mu) &= \mathbf{1}_{\mu>0} \left[\phi(0, z) \mathbb{I}(0) + \int_0^z \phi(z', z) \mathbb{S}(z') dz' \right] \\ &\quad + \mathbf{1}_{\mu<0} \left[\phi(z, Z) \mathbb{I}(Z) + \int_z^Z \phi(z, z') \mathbb{S}(z') dz' \right] \\ &\quad + [\mathbf{1}_{z>Y} \mathbf{1}_{\mu>0} \phi(Y, z) - \mathbf{1}_{z<Y} \mathbf{1}_{\mu<0} \phi(z, Y)] \Delta(z, \mu). \end{aligned} \quad (5.5)$$

Proof

With these notations (5.2) and (5.3) are

$$\begin{aligned} \mathbf{I}(z, \mu)|_{z<Y} &= \mathbf{1}_{\mu>0} \left[\phi(0, z) \mathbb{I}(0) + \int_0^z \phi(z', z) \mathbb{S}(z') dz' \right] \\ &\quad + \mathbf{1}_{\mu<0} \left[\phi(z, Y) \mathbb{I}(Y^-) + \int_z^Y \phi(z, z') \mathbb{S}(z') dz' \right]. \end{aligned} \quad (5.6)$$

$$\begin{aligned} \mathbf{I}(z, \mu)|_{z>Y} &= \mathbf{1}_{\mu>0} \left[\phi(Y, z) \mathbb{I}(Y^+) + \int_Y^z \phi(z', z) \mathbb{S}(z') dz' \right] \\ &\quad + \mathbf{1}_{\mu<0} \left[\phi(z, Z) \mathbb{I}(Z) + \int_z^Z \phi(z, z') \mathbb{S}(z') dz' \right]. \end{aligned} \quad (5.7)$$

Using (5.4) in (8.2),

$$\begin{aligned} \mathbf{I}(z, \mu)|_{z<Y} &= \mathbf{1}_{\mu>0} \left[\phi(0, z) \mathbb{I}(0) + \int_0^z \phi(z', z) \mathbb{S}(z') dz' \right] \\ &\quad + \mathbf{1}_{\mu<0} \left[\phi(z, Y) [\mathbb{I}(Y^+) - \Delta(z, \mu)] + \int_z^Y \phi(z, z') \mathbb{S}(z') dz' \right]. \end{aligned}$$

Now by (5.7) used with $z = Y^+$, $\mu < 0$,

$$\mathbb{I}(Y^+) |_{\mu<0} = \phi(Y, Z) \mathbb{I}(Z) + \int_Y^Z \phi(Y, z') \mathbb{S}(z') dz'.$$

Consequently,

$$\begin{aligned}
& \mathbf{I}(z, \mu)|_{z < Y} = \mathbf{1}_{\mu > 0} \left[\phi(0, z) \mathbb{I}(0) + \int_0^z \phi(z', z) \mathbb{S}(z') dz' \right] \\
& + \mathbf{1}_{\mu < 0} \left[\phi(z, Y) \left[\phi(Y, Z) \mathbb{I}(Z) + \int_Y^Z \phi(Y, z') \mathbb{S}(z') dz' - \Delta(z, \mu) \right] \right. \\
& \left. + \int_z^Y \phi(z, z') \mathbb{S}(z') dz' \right] \quad \Rightarrow \\
& \mathbf{I}(z, \mu)|_{z < Y} = \mathbf{1}_{\mu > 0} \left[\phi(0, z) \mathbb{I}(0) + \int_0^z \phi(z', z) \mathbb{S}(z') dz' \right] \\
& + \mathbf{1}_{\mu < 0} \left[\phi(z, Z) \mathbb{I}(Z) + \int_z^Z \phi(z, z') \mathbb{S}(z') dz' - \phi(z, Y) \Delta(z, \mu) \right],
\end{aligned}$$

because $\phi(z, Y)\phi(Y, Z) = \phi(z, Z)$ and $\phi(z, Y)\phi(Y, z') = \phi(z, z')$ when $z < Y$ and $z' > Y$. Now, if $z > Y$ we use (5.4) in (5.7)

$$\begin{aligned}
& \mathbf{I}(z, \mu)|_{z > Y} = \mathbf{1}_{\mu > 0} \left[\phi(Y, z) [\mathbb{I}(Y^-) + \Delta(z, \mu)] + \int_Y^z \phi(z', z) \mathbb{S}(z') dz' \right] \\
& + \mathbf{1}_{\mu < 0} \left[\phi(z, Z) \mathbb{I}(Z) + \int_z^Z \phi(z, z') \mathbb{S}(z') dz' \right],
\end{aligned}$$

and (8.2) at $z = Y$, with $\mu > 0$,

$$\mathbb{I}(Y^-)|_{\mu > 0} = \phi(0, Y) \mathbb{I}(0) + \int_0^Y \phi(z', Y, \mu) \mathbb{S}(z') dz'.$$

It shows that

$$\begin{aligned}
& \mathbf{I}(z, \mu)|_{z > Y} = \mathbf{1}_{\mu > 0} \left[\phi(Y, z) \left[\phi(0, Y) \mathbb{I}(0) + \int_0^Y \phi(z', Y) \mathbb{S}(z') dz' + \Delta(z, \mu) \right] \right. \\
& \left. + \int_Y^z \phi(z', z) \mathbb{S}(z') dz' \right] + \mathbf{1}_{\mu < 0} \left[\phi(z, Z) \mathbb{I}(Z) + \int_z^Z \phi(z, z') \mathbb{S}(z') dz' \right] \quad \Rightarrow \\
& \mathbf{I}(z, \mu)|_{z > Y} = \mathbf{1}_{\mu > 0} \left[\phi(0, z) \mathbb{I}(0) + \int_0^z \phi(z', z) \mathbb{S}(z') dz' + \phi(Y, z) \Delta(z, \mu) \right] \\
& + \mathbf{1}_{\mu < 0} \left[\phi(z, Z) \mathbb{I}(Z) + \int_z^Z \phi(z, z') \mathbb{S}(z') dz' \right].
\end{aligned}$$

Therefore, the additional term due to the discontinuity is

$$[\mathbf{1}_{z > Y} \mathbf{1}_{\mu > 0} \phi(Y, z) - \mathbf{1}_{z < Y} \mathbf{1}_{\mu < 0} \phi(z, Y)] \Delta(z, \mu).$$

□

6. Application to Fresnel's Conditions

Notation 5. Define, when possible,

$$\boxed{\begin{aligned} \eta_{\pm}(n) &= (1 - n_{\pm}^2(1 - \eta(n)^2)^{\frac{1}{2}}), & \phi_{\pm}(z', z'') &= \exp\left\{-\int_{z'}^{z''} \frac{\kappa(y)}{\eta_{\pm}(\frac{n_z}{n_y})} dy\right\} \\ \mathbb{I}_{\pm}(y) &= \mathbf{I}(y, \text{sign}(\mu)\eta_{\pm}(\frac{n_z}{n_y})) & \mathbb{S}_{\pm}(y) &= \mathbf{S}(y, \eta_{\pm}(\frac{n_z}{n_y}))/\eta_{\pm}(\frac{n_z}{n_y}) \end{aligned}}$$

and similarly with \mp for η_{\mp} etc. Let

$$\mathbf{X}_{\pm} := \mathbf{X}(n_{\pm}, |\mu|), \quad \mathbf{X}_{\mp} := \mathbf{X}(n_{\mp}, |\mu|) \text{ and similarly with } \mathbf{Y}.$$

6.1. Computation of $\Delta(z, \mu)$

From (5.7) and (8.2) we obtain

$$\begin{aligned} \mathbb{I}(Y^+) |_{\mu < 0} &= \phi(Y, Z)\mathbb{I}(Z) + \int_Y^Z \phi(Y, z')\mathbb{S}(z')dz', \\ \mathbf{I}(Y^-) |_{\mu > 0} &= \phi(0, Y)\mathbb{I}(0) + \int_0^Y \phi(z', Y)\mathbb{S}(z')dz'. \end{aligned}$$

And (8.2) and (5.7) plugged in the Fresnel conditions (4.2) yield,

$$\begin{aligned} \mathbb{I}(Y^-) |_{\mu < 0} &= \mathbf{X}_{\mp} \left[\phi(0, Y)\mathbb{I}(0, -\mu) + \int_0^Y \phi(z', Y)\mathbb{S}(z')dz' \right] \\ &\quad + \mathbf{Y}_{\mp} \left[\phi_{\mp}(Y, Z)\mathbb{I}_{\mp}(Z) + \int_Y^Z \phi_{\mp}(Y, z')\mathbb{S}_{\mp}(z')dz' \right], \\ \mathbb{I}(Y^+) |_{\mu > 0} &= \mathbf{X}_{\pm} \left[\phi(Y, Z)\mathbb{I}(Z, -\mu) + \int_Y^Z \phi(Y, z')\mathbb{S}(z')dz' \right] \\ &\quad + \mathbf{Y}_{\pm} \left[\phi_{\pm}(0, Y)\mathbb{I}_{\pm}(0) + \int_0^Y \phi_{\pm}(z', Y)\mathbb{S}_{\pm}(z')dz' \right]. \end{aligned}$$

We have added the dependency on μ on some of the functions because of the change of sign requested by the Fresnel conditions.

Therefore,

$$\begin{aligned}
\Delta(z, \mu)|_{\mu>0} &= \mathbb{I}(Y^+)|_{\mu>0} - \mathbb{I}(Y^-)|_{\mu>0} \\
&= \mathbf{X}_\pm \left[\phi(Y, Z)\mathbb{I}(Z, -\mu) + \int_Y^Z \phi(Y, z')\mathbb{S}(z')dz' \right] \\
&\quad + \mathbf{Y}_\pm \left[\phi_\pm(0, Y)\mathbb{I}_\pm(0) + \int_0^Y \phi_\pm(z', Y)\mathbb{S}_\pm(z')dz' \right] \\
&\quad - \phi(0, Y)\mathbb{I}(0) - \int_0^Y \phi(z', Y)\mathbb{S}(z')dz', \\
\Delta(z, \mu)|_{\mu<0} &= \mathbb{I}(Y^+)|_{\mu<0} - \mathbb{I}(Y^-)|_{\mu<0} \\
&= \phi(Y, Z)\mathbb{I}(Z) + \int_Y^Z \phi(Y, z')\mathbb{S}(z')dz' \\
&\quad - \mathbf{X}_\mp \left[\phi(0, Y)\mathbb{I}(0, -\mu) + \int_0^Y \phi(z', Y)\mathbb{S}(z')dz' \right] \\
&\quad - \mathbf{Y}_\mp \left[\phi_\mp(Y, Z)\mathbb{I}_\mp(Z) + \int_Y^Z \phi_\mp(Y, z')\mathbb{S}_\mp(z')dz' \right]. \quad (6.1)
\end{aligned}$$

6.2. Implementation

To implement the iterative algorithm, the only functions needed are $\int_{-1}^1 \mathbf{I}(z, \mu)\mu^k d\mu$, $k = 0, 2$ and that means that we need to compute (see below)

$$\int_0^1 \mu^k \phi(Y, z)\Delta(z, \mu)d\mu, \quad z > Y \quad \text{and} \quad \int_{-1}^0 \mu^k \phi(z, Y)\Delta(z, \mu)d\mu, \quad z < Y.$$

Consequently, with $\mathbf{S} = \mathbf{S}_0 + \mu^2\mathbf{S}_2$,

$$\begin{aligned}
&\int_{-1}^1 \mu^k [\Delta(z, \mu)[\phi(Y, z)\mathbf{1}_{z>Y}\mathbf{1}_{\mu>0} - \phi(z, Y)\mathbf{1}_{z<Y}\mathbf{1}_{\mu<0}]]d\mu \\
&= \boldsymbol{\alpha}^k(z) + \sum_{i=0,2} \int_0^Z \mathbf{Z}^{k,i-1}(z, z')\mathbf{S}_i(z')dz', \quad (6.2)
\end{aligned}$$

with the 2×2 matrices, $i, k = 0, 2$,

$$\begin{aligned}
\mathbf{Z}^{k,i}(z, z') &:= \int_0^1 \mu^k \left(\mathbf{1}_{z>Y}\phi(Y, z) \left\{ \mathbf{X}_\pm\phi(Y, z')\eta^i\left(\frac{n_z}{n_{z'}}\right)\mathbf{1}_{z'>Y} \right. \right. \\
&\quad \left. \left. + [\mathbf{Y}_\pm\phi_\pm(z', Y)\eta_\pm^i\left(\frac{n_z}{n_{z'}}\right) - \mathbf{1}\phi(z', Y)\eta^i\left(\frac{n_z}{n_{z'}}\right)]\mathbf{1}_{z'<Y} \right\} \right. \\
&\quad \left. + \mathbf{1}_{z<Y}\phi(z, Y) \left\{ \mathbf{X}_\mp\phi(z', Y)\eta^i\left(\frac{n_z}{n_{z'}}\right)\mathbf{1}_{z'<Y} \right. \right. \\
&\quad \left. \left. + [\mathbf{Y}_\mp\phi_\mp(Y, z')\eta_\mp^i\left(\frac{n_z}{n_{z'}}\right) - \mathbf{1}\phi(Y, z')\eta^i\left(\frac{n_z}{n_{z'}}\right)]\mathbf{1}_{z'>Y} \right\} \right) d\mu \quad (6.3)
\end{aligned}$$

and the vectors, $k = 0, 2$,

$$\begin{aligned}\boldsymbol{\alpha}^k(z) = & \mathbf{1}_{z>Y} \int_0^1 \mu^k \phi(Y, z) [\mathbf{X}_{\pm} \phi(Y, Z) \mathbb{I}(Z, -\mu) + \mathbf{Y}_{\pm} \phi_{\pm}(0, Y) \mathbb{I}_{\pm}(0) \\ & - \phi(0, Y) \mathbb{I}(0)] d\mu \\ & + \mathbf{1}_{z<Y} \int_0^1 \mu^k \phi(z, Y) [\mathbf{X}_{\mp} \phi(0, Y) \mathbb{I}(0) + \mathbf{Y}_{\mp} \phi_{\mp}(Y, Z) \mathbb{I}_{\mp}(Z) \\ & - \phi(Y, Z) \mathbb{I}(Z, -\mu)] d\mu\end{aligned}$$

Remark 3. *Whenever feasible it is computationally advantageous to separate in $\boldsymbol{\alpha}^k$ the part containing ν from the one containing z .*

For our purpose

$$\begin{aligned}\mathbf{I}(0, \mu) &= [\mu c_E B_{\nu}(T_E), 0]^T, \quad \mathbf{I}(Z, -\mu) = [\mu c_S B_{\nu}(T_S), 0]^T \\ \Rightarrow \boldsymbol{\alpha}^k(z) &= [\alpha_E^k(z) B_{\nu}(T_E) + \alpha_S^k(z) B_{\nu}(T_S), 0]^T\end{aligned}$$

with

$$\begin{aligned}\alpha_E^k(z) &= c_E \int_0^1 \mu^k \left[\mathbf{1}_{z>Y} \phi(Y, z) [\mathbf{Y}_{\pm 11} \phi_{\pm}(0, Y) \eta_{\pm}(\frac{nz}{n_0}) - \phi(0, Y) (\frac{nz}{n_0})] \right. \\ & \quad \left. + \mathbf{1}_{z<Y} \phi(z, Y) \mathbf{X}_{\mp 11} \phi(0, Y) \eta(\frac{nz}{n_0}) \right] d\mu \\ \alpha_S^k(z) &= c_S \int_0^1 \mu^k \left[\mathbf{1}_{z>Y} \phi(Y, z) \mathbf{X}_{\pm 11} \phi(Y, Z) \eta(\frac{nz}{n_Z}) \right. \\ & \quad \left. + \mathbf{1}_{z<Y} \phi(z, Y) [\mathbf{Y}_{\mp 11} \phi_{\mp}(Y, Z) \eta_{\mp}(\frac{nz}{n_Z}) - \phi(Y, Z) \eta(\frac{nz}{n_Z})] \right] d\mu\end{aligned}$$

7. Integral representation of the Problem

With the main purpose of computing the temperature, let us denote

$$J_k(z) = \frac{1}{2} \int_{-1}^1 \mu^k I d\mu, \quad K_k(z) = \frac{1}{2} \int_{-1}^1 \mu^k Q d\mu. \quad k = 0, 2.$$

Consider the system for the irradiance I and the polarization Q ,

$$\begin{aligned}\mu \partial_z I + \partial_z \log n \cdot \partial_{\mu} \{(1 - \mu^2) I\} + \kappa I &= \kappa_a B_{\nu} + \kappa_s J_0 \\ & \quad + \frac{\beta \kappa_s}{4} P_2(\mu) (3J_2 - J_0 - 3K_0 + 3K_2) \\ \mu \partial_z Q + \partial_z \log n \cdot \partial_{\mu} \{(1 - \mu^2) Q\} + \kappa Q &= \\ & \quad - \frac{\beta \kappa_s}{4} (1 - P_2(\mu)) (3J_2 - J_0 - 3K_0 + 3K_2)\end{aligned}$$

where $P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$. Hence

$$\begin{aligned}
\mathbf{S}(z, \mu) &= [S, S']^T = [S_0 + \mu^2 S_2, S'_0 + \mu^2 S'_2]^T \text{ with} \\
S &= \kappa_a B_\nu + \kappa_s J_0 + \frac{\beta \kappa_s}{4} P_2(\mu) (3J_2 - J_0 - 3K_0 + 3K_2) \\
S' &= -\frac{\beta \kappa_s}{4} (1 - P_2(\mu)) (3J_2 - J_0 - 3K_0 + 3K_2), \quad \Rightarrow \\
S_0 &= \kappa_a B_\nu + \kappa_s J_0 - \frac{3}{2} H, \quad S_2 = H, \quad S'_0 = H, \quad S'_2 = -H \\
\text{with } H &= \frac{9\beta \kappa_s}{8} (J_2 - \frac{1}{3} J_0 - K_0 + K_2)
\end{aligned} \tag{7.1}$$

Define

$$\psi(z, z') = \mathbf{1}_{z > z'} \phi(z', z) + \mathbf{1}_{z < z'} \phi(z, z')$$

Let (5.5) be multiplied by μ^k and integrated. Then, by (6.2),

$$\begin{aligned}
J_k(z) &= \frac{1}{2} \int_0^Z \left[\left(\int_0^1 \mu^k \psi(z, z') d\mu + \mathbf{Z}_{11}^{k,0}(z, z') \right) S_0(z') \right. \\
&\quad \left. + \mathbf{Z}_{12}^{k,0}(z, z') S'_0(z') \right] dz' \\
&+ \frac{1}{2} \int_0^Z \left[\left(\int_0^1 \mu^k \psi(z, z') \eta^2 \left(\frac{n_z}{n'_z} \right) d\mu + \mathbf{Z}_{11}^{k,2}(z, z') \right) S_2(z') \right. \\
&\quad \left. + \mathbf{Z}_{12}^{k,2}(z, z') S'_2(z') \right] dz' + \frac{1}{2} \alpha_1^k(z)
\end{aligned} \tag{7.2}$$

Similarly (recall that Q is zero at $z = 0$),

$$\begin{aligned}
K_k(z) &= \frac{1}{2} \int_0^Z \left[\left(\int_0^1 \mu^k \psi(z, z') d\mu + \mathbf{Z}_{22}^{k,0} \right) S'_0(z') + \mathbf{Z}_{21}^{k,0} S_0(z') \right) \\
&\quad + \left(\int_0^1 \mu^k \psi(z, z') \eta^2 \left(\frac{n_z}{n'_z} \right) d\mu + \mathbf{Z}_{22}^{k,2} \right) S'_2(z') \\
&\quad \left. + \mathbf{Z}_{21}^{k,2} S_2(z') \right] dz' + \frac{1}{2} \alpha_2^k(z)
\end{aligned} \tag{7.3}$$

8. Convergence of the Iteration on the Sources

8.1. Principle

In earlier studies, on simpler systems [10], convergence was shown by using the monotony of operators. Here too the same arguments are used, not on

$[I, Q]^T$ but on $\mathbf{I} = [I_l, I_r]^T$. The iterative system is

$$\begin{aligned}
\mathbf{I}^{n+1}(z, \mu) = & \mathbf{1}_{z < Y} \left\{ \mathbf{1}_{\mu > 0} \left[\phi(0, z) \mathbb{I}(0) + \int_0^z \phi(z, z') \mathbb{S}^n(z') dz' \right] \right. \\
& + \mathbf{1}_{\mu < 0} \left[\phi(z, Z) \mathbb{I}(Z) + \int_z^Z \phi(z, z') \mathbb{S}^n(z') dz' \right] \left. \right\} \\
& + \mathbf{1}_{z > Y} \left\{ \mathbf{1}_{\mu > 0} \left[\phi(z, Y) \mathbb{I}(Y^-) + \int_Y^z \phi(z, z') \mathbb{S}^n(z') dz' \right] \right. \\
& + \mathbf{1}_{\mu < 0} \left[\phi(z, Z) \mathbf{I}(Z) + \int_z^Z \phi(z, z') \mathbb{S}^n(z') dz' \right] \left. \right\} \\
& + (\mathbf{1}_{z > Y} \mathbf{1}_{\mu > 0} \phi(Y, z) - \mathbf{1}_{z < Y} \mathbf{1}_{\mu < 0} \phi(z, Y)) \Delta^n(\mu, z) \quad (8.1)
\end{aligned}$$

with \mathbf{S}^n given by (7.1)(7.2)(7.3) and $\Delta(\mu, z)$ given by (6.1). Consequently

$$\begin{aligned}
& \mathbf{I}^{n+1}(z, \mu) - \mathbf{I}^n(z, \mu) \\
& = \mathbf{1}_{z < Y} \left\{ \mathbf{1}_{\mu > 0} \int_0^z \phi(z, z') (\mathbb{S}^n(z') - \mathbb{S}^{n-1}(z')) dz' \right. \\
& \quad + \mathbf{1}_{\mu < 0} \int_z^Z \phi(z, z') (\mathbb{S}^n(z') - \mathbb{S}^{n-1}(z')) dz' \left. \right\} \\
& + \mathbf{1}_{z > Y} \left\{ \mathbf{1}_{\mu > 0} \int_Y^z \phi(z, z') (\mathbb{S}^n(z') - \mathbb{S}^{n-1}(z')) dz' \right. \\
& \quad + \mathbf{1}_{\mu < 0} \int_z^Z \phi(z, z') (\mathbb{S}^n(z') - \mathbb{S}^{n-1}(z')) dz' \left. \right\} \\
& + (\mathbf{1}_{z > Y} \mathbf{1}_{\mu > 0} \phi(Y, z) - \mathbf{1}_{z < Y} \mathbf{1}_{\mu < 0} \phi(z, Y)) (\Delta^n(z, \mu) - \Delta^{n-1}(z, \mu)). \quad (8.2)
\end{aligned}$$

On this system it is seen that $\mathbf{S}^n \geq \mathbf{S}^{n-1}$ and $\Delta^n \geq \Delta^{n-1}$ implies $\mathbf{I}^{n+1} \geq \mathbf{I}^n$ because ϕ is always positive. Conversely if $\mathbf{S}^n \leq \mathbf{S}^{n-1}$ and $\Delta^n \leq \Delta^{n-1}$ then $\mathbf{I}^{n+1} \leq \mathbf{I}^n$. However the monotony is true only for the system $[I_l, I_r]$ and not true for $[I, Q]$ (but true for I). furthermore we can prove that Δ^n is monotone only when the variations of n are small.

8.2. Monotony

For $\mathbf{I} = [I_l, I_r]^T$ in (3.4) the source terms are

$$\begin{aligned}
S & = \frac{3\beta\kappa_s}{8} \int_{-1}^1 ([2(1 - \mu'^2)(1 - \mu^2) + \mu'^2\mu^2] \tilde{I}_l + \mu^2 \tilde{I}_r) d\mu' \\
& \quad + \frac{(1 - \beta)\kappa_s}{4} \int_{-1}^1 [\tilde{I}_l + \tilde{I}_r] d\mu' + \frac{\kappa_a}{2} \tilde{B}_\nu(T(z)), \\
S' & = \frac{3\beta\kappa_s}{8} \int_{-1}^1 (\mu'^2 \tilde{I}_l + \tilde{I}_r) d\mu' \\
& \quad + \frac{(1 - \beta)\kappa_s}{4} \int_{-1}^1 [\tilde{I}_l + \tilde{I}_r] d\mu' + \frac{\kappa_a}{2} \tilde{B}_\nu(T(z)),
\end{aligned}$$

Denote $D_{l,r}^n = I_{r,l}^n - I_{l,r}^{n-1}$. Obviously,

$$\begin{aligned} S_l^n - S_l^{n-1} &= \frac{3\beta\kappa_s}{8} \int_{-1}^1 ([2(1-\mu'^2)(1-\mu^2) + \mu'^2\mu^2]D_l^n + \mu^2 D_r^n) d\mu' \\ &+ \frac{(1-\beta)\kappa_s}{4} \int_{-1}^1 [D_l^n + D_r^n] d\mu' + \frac{\kappa_a}{2} [B_\nu(T^n) - B_\nu(T^{n-1})] \\ S_r^n - S_r^{n-1} &= \frac{3\beta\kappa_s}{8} \int_{-1}^1 (\mu'^2 D_l^n + D_r^n) d\mu' \\ &+ \frac{(1-\beta)\kappa_s}{4} \int_{-1}^1 [D_l^n + D_r^n] d\mu' + \frac{\kappa_a}{2} [B_\nu(T^n) - B_\nu(T^{n-1})] \end{aligned}$$

The terms under the integrals are positive and $T \rightarrow B_\nu(T)$ is monotone in the sense that $T^n \geq T^m$ implies $B_\nu(T^n) \geq B_\nu(T^m)$. Consequently $T^n \geq T^{n-1}$ and $I_{l,r}^n \geq I_{l,r}^{n-1}$ implies $S_{l,r}^n \geq S_{l,r}^{n-1}$. Then, provided that $\Delta^n > \Delta^{n-1}$, it implies by (8.2) that $\mathbf{I}^{n+1} \geq \mathbf{I}^n$.

Finally the temperature equation implies

$$\int_{\mathbb{R}_+} \kappa_a B_\nu(T^{n+1}) d\nu = \int_{\mathbb{R}_+} \kappa_a J_0^n d\nu \geq \int_{\mathbb{R}_+} \kappa_a J_0^{n-1} d\nu = \int_{\mathbb{R}_+} \kappa_a B_\nu(T^{n+1}) d\nu.$$

which implies that $T^{n+1} \geq T^{n+1}$. Let us apply this argument to $\{T^{n-1}, I_{i,r}^{n-1}\}$ instead of $\{T^n, I_{i,r}^n\}$. It shows that

$$T^n \geq T^{n-1}, I_{l,r}^n \geq I_{i,r}^{n-1} \Rightarrow T^{n+1} \geq T^n, I_{l,r}^{n+1} \geq I_{i,r}^n.$$

To start the iterations appropriately, simply set $T^0 = 0$, $I_{l,r}^0 = 0$, then by the positivity of the coefficients $I_{l,r}^1 \geq 0$ and $T^1 \geq 0$.

The same argument works with $T^n \leq T^m$, $I_{l,r}^n \leq I_{i,r}^m$ implying that $I_{l,r}^{n+1} \leq I_{i,r}^{m+1}$ and then $T^{n+1} \leq T^{m+1}$. Hence starting with $T^1 < T^0$, $I_{l,r}^1 \leq I_{i,r}^0$ leads to a decreasing sequence toward the solution. For the scalar model, it is shown that it suffices to take $T^0(z) > T_M, \forall z$ where T_M is the solution of $B_\nu(T_M) = P_E B_\nu(T_E)$. For the present vector model it is not clear that it is sufficient.

Note that the decreasing solution, $\{\mathbf{I}^n\}_n$ is bounded by zero. Consequently a solution exists.

Now we return to the monotony of \mathbf{S} of $\Delta^n(z, \mu)$. instead, let us show that $\int_0^1 \mu^k \Delta^n$ is increasing with n .

In (6.3), with i even, there are 2 negative contributions to $\mathbf{Z}_{ii}^{k,j}$:

$$\begin{aligned} & \int_0^1 \mu^k [\mathbf{1}_{z>Y} \mathbf{1}_{z'<Y} \phi(Y, z) \phi(z', Y) + \mathbf{1}_{z<Y} \mathbf{1}_{z'>Y} \phi(z, Y) \phi(Y, z')] \eta^j \left(\frac{n_z}{n_{z'}} \right) d\mu \\ &= \int_0^1 \mu^k [\mathbf{1}_{z>Y} \mathbf{1}_{z'<Y} \phi(z', z) + \mathbf{1}_{z<Y} \mathbf{1}_{z'>Y} \phi(z, z')] \eta^j \left(\frac{n_z}{n_{z'}} \right) d\mu \\ &= \int_0^1 \mu^k \psi(z, z') \eta^j \left(\frac{n_z}{n_{z'}} \right) d\mu. \end{aligned}$$

But these in (7.2) and (7.3) are cancelled by the same with positive sign. Consequently only positive quantities remain in (7.2) and (7.3). The above results are summarized in the following proposition.

Proposition 2. *If the solution $T^*, I_{l,r}^*$ exists, then it can be reached numerically from above or below by iterations (8.1) and these are monotone increasing and decreasing respectively. If there is an initial guess such that $T^0 \geq T^1, I_{l,r}^0 \geq I_{l,r}^1$, then a solution exists.*

9. Numerical Results

The computer program is written in C++ and opensource. The functions \mathbf{E} , \mathbf{Z} and $\boldsymbol{\alpha}$ are tabulated to speed up the runtime which is around 2 seconds on a high end Apple Macbook.

We investigated 2 cases:

- Case 1: Visible light coming from the Sun through the top of the troposphere at $z = Z$: $I(0, \mu) = 0, I(Z, -\mu) = c_S B_\nu(T_S) \mu, \mu > 0$.
- Case 2: Infrared light coming from Earth and escaping freely at $z = Z$ with and without CO₂ effects modeled by a change in κ_ν taken from the Gemini experiment ²: $I(0, \mu) = c_E B_\nu(T_E) \mu, I(Z, -\mu) = 0, \mu > 0$.

In both cases there is a change of refractive index at $z = Y := Z/2$.

For all tests the following is used:

- $n(z) = 1 + \epsilon \mathbf{1}_{z>Y}$,
- $a_s = a_1 \mathbf{1}_{z \in (z_1, z_2)} + a_2 \mathbf{1}_{z > z_2} \mathbf{1}_{\nu \in (\nu_1, \nu_2)} \left(\frac{\nu}{\nu_2} \right)^4$,
- $\epsilon = -0.01$ or $-0.3, a_1 = 0.7, a_2 = 0.3, z_1 = 0.4, z_2 = 0.8, \nu_1 = 0.6, \nu_2 = 1.5$.

²www.gemini.edu/observing/telescopes-and-sites/sites#Transmission

- $c_S = 2.0e - 5, T_S = 5700/4798, c_E = 2.5, T_E = 300./4798.$

According to [4] the variation of the refractive index in the atmosphere is quite small ~ 0.003 . To enhance the effect we use 3 times this value.

The monotony of the iterative process is displayed in Figure 2. It is clear that by starting below (resp. above) the solution the values of the temperature at $z = 300\text{m}$ are increasing (resp. decreasing). Note that 15 iterations are sufficient to obtain a 3 digit precision.

To study the effect of n on a simple case we ran the program with $\kappa = 0.5, n$ as above, with $\epsilon = 0.01$ and the data of Case 2. The temperatures are shown in Figure 3 in red. The computations are done with and without Fresnel conditions at $z = Y$.

A similar computation is done for Case 1 with $\kappa = 0.5, \epsilon = -0.3$ with and without Fresnel conditions. The temperatures are displayed in blue in Figure 5. The average light and polarization intensities are shown on Figure 4.

Finally, Case 2 was run with $\epsilon = -0.3$ with κ_ν read from the Gemini website and shown on Figure 1. Then this κ_ν was increased in the frequency range where CO_2 is absorbent, shown in red in Figure 1. The corresponding temperatures and average light intensities are shown in Figures 5 and 6.

The main points are

- For Case 1 (Visible light crossing the atmosphere downward and passing through a refractive medium for $z < Y$) Fresnel's conditions have a drastic effect on the results.
- For Case 2 (IR light coming from Earth and the refractive index decreases when $z > Y$) the refraction makes the medium much more absorbing. Furthermore, with a Fresnel interface, an increase in CO_2 decreases the temperature at high altitude.

10. Conclusion

In this article the methodology developed in [10] for the numerical solution of the VRTE has been extended to include the Fresnel conditions at an interface of discontinuity of the refractive index. While the solution of the equation (2.1) given by Pomraning and Chandrasekhar in [6] [16] do give a jump of the Stokes vector and the temperature at the discontinuity, the

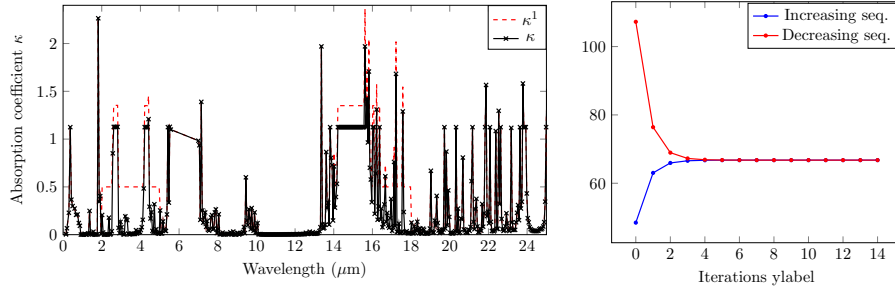


Figure 1: Absorption κ from the Gemini experiment, versus wavenumber ($3/\nu$). In dotted lines, the modification to construct κ_1 to account for the opacity of CO_2 .

Figure 2: Convergence of the temperature at altitude 300m during the iterations. In solid line when it is started with $T^0 = 0$, in dashed line when the initial temperature is 180°C . Notice the monotonicity of both curves.

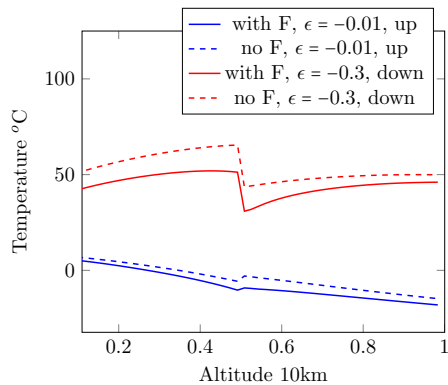


Figure 3: Temperatures versus altitude with $\kappa = 0.5$. In blue with $\epsilon = -0.01$ for **Case 2** and in red with $\epsilon = -0.3$ for **Case 1**. The solid curves are computed with Fresnel's conditions at $z = Y = 0.5$ and the dashed curves are computed without them.

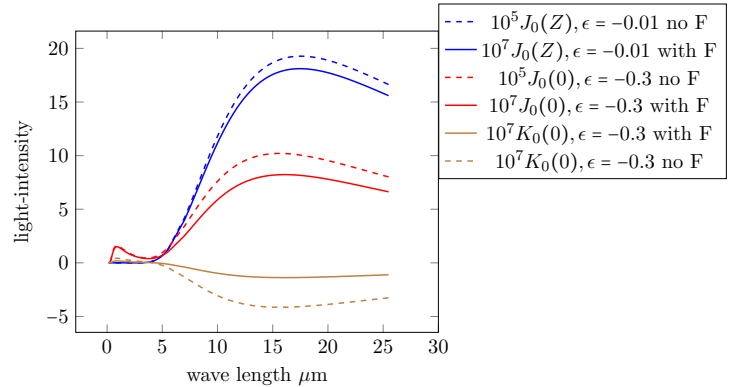


Figure 4: Total light intensity J_0 and polarized K_0 versus wave length at ground level or at altitude $Z=10\text{km}$ for **Case 2** with $\epsilon = -0.01$ and for **Case 1** with $\epsilon = -0.3$. The solid curves are computed with Fresnel's conditions and the dashed one without.

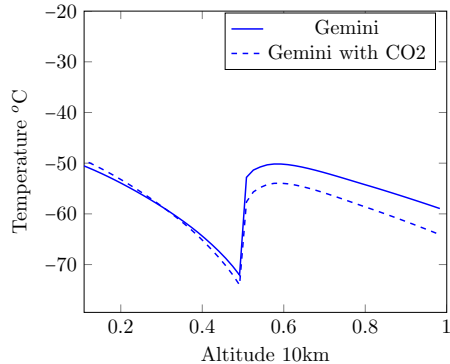


Figure 5: **Case 2.** Effect of CO₂ on the temperature in the presence of a Fresnel interface with $\epsilon = -0.3$.

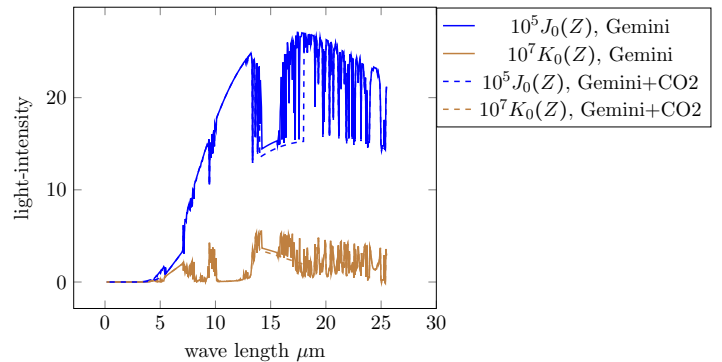


Figure 6: **Case 2.** Total light intensity J_0 and polarized K_0 versus wave length at altitude $Z=10\text{km}$.

amplitude of the jump is not the same as the one given by the Fresnel Conditions.

In principle the method is not hard to program (500 lines of C++) and the execution time is a few seconds; however the formulas are complex and the probability of having bugs cannot be ruled out.

As before we have measured numerically the effect of a change on the absorption due to CO₂. Although preliminary, the conclusion reached in our earlier studies are also valid here when there is a change of refraction index as in water/air: the effect of CO₂ on the infrared radiation from Earth heats up the region near the ground and cools it in high altitude.

Generalization to 3D as in [9]) and [15] for a non-stratified atmosphere looks possible.

References

- [1] M. Ament, C. Bergmann, and D. Weiskopf. Refractive radiative transfer equation. *ACM Transactions on Graphics*, 33(2):2, March 2014.
- [2] Yong Zhang andf Hong-Liang Yi and He-Ping Tan. Lattice boltzmann method for short-pulsed laser transport in a multi-layered medium. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 155(April):75–89, 2015.

- [3] Ph. Ben-Abdallah. When the space curvature dopes the radiant intensity. *J. Opt. Soc. Am. B*, 19(8), 2002.
- [4] H. Bussey and G. Birnbaum. Measurement of variations in atmospheric refractive index with an airborne microwave refractometer. *Journal of National Research of the National Bureau of Standards*, 51(4):171–178, 1953.
- [5] Y.H. Yang Y. Zhang K. Yue X.X. Zhang C.H. Wang, Y.Y. Feng. Chebyshev collocation spectral method for vector radiative transfer equation and its applications in two-layered media. *J. Quant. Spectrosc. Radiat. Transf.*, 243:106822, 2020.
- [6] S. Chandrasekhar. *Radiative Transfer*. Clarendon Press, Oxford, 1950.
- [7] J. Dufresne, V. Eymet, C. Crevoisier, and J. Grandpeix. Greenhouse effect: The relative contributions of emission height and total absorption. *Journal of Climate, American Meteorological Society*, 33(9):3827–3844, 2020.
- [8] R. D.M. Garcia. Radiative transfer with polarization in a multi-layer medium subject to fresnel boundary and interface conditions. *Journal of Quantitative Spectroscopy and Radiative Transfer*, (115):28–45, 2013.
- [9] F. Golse, F. Hecht, O. Pironneau, D. Smetz, and P.-H. Tournier. Radiative transfer for variable 3d atmospheres. *J. Comp. Physics*, 475(111864):1–19, 2023.
- [10] F. Golse and O. Pironneau. Stratified radiative transfer in a fluid and numerical applications to earth science. *SIAM Journal on Numerical Analysis*, 60(5):2963–3000, 2022.
- [11] J. De Haan, P. Bosma, and J. Hovenier. The adding method for multiple scattering computations of polarized light. *Astron Astrophys*, 183:371–391, 1987.
- [12] L.-H. Liu. Finite volume method for radiation heat transfer in graded index medium. *Journal Of Thermophysics And Heat Transfer*, 20(1):59–66, Jan 2006.
- [13] O. Pironneau. Numerical simulation of polarized light and temperature in a stratified atmosphere with a slowly varying refractive index. *Pure and Applied Functional Analysis*, Special issue dedicated to Roger Temam, 2024.

- [14] O. Pironneau. Numerical simulation of polarized light with Rayleigh scattering in a stratified atmosphere. *Pure and Applied Functional Analysis*, Special issue dedicated to Luc Tartar, 2024.
- [15] O. Pironneau and P.-H. Tournier. Reflective conditions for radiative transfer in integral form with h-matrices. *Journal of Computational Physics*, 495(112531):1–14, 2023.
- [16] G. Pomraning. *The equations of Radiation Hydrodynamics*. Pergamon Press, NY, 1973.
- [17] G. Pomraning and B. Ganapol. Simplified radiative transfer for combined rayleigh and isotropic scattering. *The Astrophysical Journal*, 498:671–688, 1998.
- [18] C.H. Wang and H.P. Tan. Discontinuous finite element method for vector radiative transfer. *J Quant Spectrosc Radiat Transf.*, 189:383–397, 2017.
- [19] F. Weng. A multi-layer discrete-ordinate method for vector radiative transfer in a vertically-inhomogeneous, emitting and scattering atmosphere–i. theory. *J Quant Spectrosc Radiat Transfer*, 47:19–33, 1992.
- [20] G. Ymeli, Xiaochuan Liu, A. Fokou, and Cun-Hai Wang. Lattice boltzmann method for radiative transfer in two-layered slab with graded-index and fresnel reflecting surfaces. *International Communications in Heat and Mass Transfer*, 148(107025):1–19, 2023.