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# GAGLIARDO-NIRENBERG INEQUALITIES IN FRACTIONAL COULOMB-SOBOLEV SPACES FOR RADIAL FUNCTIONS

ARKA MALLICK AND HOAI-MINH NGUYEN

ABSTRACT. We extend the range of parameters associated with the Gagliardo-Nirenberg interpolation inequalities in the fractional Coulomb-Sobolev spaces for radial functions. We also study the optimality of this newly extended range of parameters.

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## 1. INTRODUCTION

In this article we continue our study of Gagliardo-Nirenberg(GN) interpolation inequality in the Coulomb-Sobolev spaces [18]. Developing the method in [23], we proved the following result in [18].

**Theorem 1.1.** ([18, Theorem 1.1]) *Let  $d \geq 1$ ,  $\gamma > 1$ ,  $1 \leq p, q < +\infty$ ,  $0 \leq s \leq 1$ ,  $\alpha \in (0, d)$  and  $0 \leq \beta_1, \beta_2 < +\infty$  be such that*

$$(1.1) \quad \beta_1 p + 2\beta_2 q = 1, \quad (d - sp)\beta_1 + (d + \alpha)\beta_2 = d/\gamma,$$

and

$$(1.2) \quad \beta_1 \gamma + \beta_2 \gamma \geq 1.$$

There exists a constant  $C > 0$  such that

$$(1.3) \quad \|g\|_{L^\gamma(\mathbb{R}^d)} \leq C \|g\|_{\dot{W}^{s,p}(\mathbb{R}^d)}^{\beta_1 p} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x)|^q |g(y)|^q}{|x-y|^{d-\alpha}} dx dy \right)^{\beta_2} \text{ for all } g \in C_c^1(\mathbb{R}^d).$$

In Theorem 1.1 and in what follows, for any open set  $\Omega \subset \mathbb{R}^d$ , the following notation is used:

$$(1.4) \quad \|g\|_{\dot{W}^{s,p}(\Omega)} = \begin{cases} \left( \int_{\Omega} \int_{\Omega} \frac{|g(x) - g(y)|^p}{|x - y|^{d+sp}} dx dy \right)^{\frac{1}{p}} & \text{for } 0 < s < 1, \\ \left( \int_{\Omega} |\nabla g(x)|^p dx \right)^{1/p} & \text{for } s = 1, \\ \left( \int_{\Omega} |g(x)|^p dx \right)^{1/p} & \text{for } s = 0. \end{cases}$$

Condition (1.1) is due to the scaling invariance of (1.3) which automatically makes them optimal whereas the optimality of condition (1.2) was proved in [18]. Set

$$(1.5) \quad \mathcal{D} = p(d + \alpha) - 2q(d - sp).$$

In the case  $\mathcal{D} \neq 0$ , one can compute  $\beta_1$  and  $\beta_2$  as a function of  $d, \alpha, p, q, s, \gamma$  by

$$(1.6) \quad \beta_1 = \frac{\gamma(d + \alpha) - 2qd}{\gamma(p(d + \alpha) - 2q(d - sp))} \quad \text{and} \quad \beta_2 = \frac{pd - \gamma(d - sp)}{\gamma(p(d + \alpha) - 2q(d - sp))}.$$

Gagliardo-Nirenberg inequalities in different function spaces play a crucial role in the study of non-linear PDEs. First instance of such an inequality appeared in works of Gagliardo and Nirenberg [11, 26]. In the context of Coulomb-Sobolev spaces, which are relevant in Thomas-Fermi-Dirac-von Weizsäcker models of density functional theory [4, 12, 13] or in Hartree-Fock theory [9, 14, 8], the study of GN type inequalities was initiated by Lions [15, 16] in connection to the study of the Hartree-Fock equation. In fact, Lions proved (1.3) for  $\gamma = 3, p = q = \alpha = 2, d = 3$  and  $s = 1$ . Subsequently many extensions of (1.3) have been established. One was derived by Bellazzini, Frank, Visciglia [1] where they proved (1.3) in the case  $p = 2, q = 2$ , and  $0 < s < 1$  (see also [17, (21)] for the case  $p = q = 2, \alpha = d - 2s$ ). This was extended by Mercuri, Moroz, and Van Schaftingen [20] to the case  $p = 2$  and  $s = 1$ . Bellazzini, Ghimenti, Mercuri, Moroz, and Van Schaftingen [2] then extended to the case  $p = 2$  and  $0 < s < 1$ .

A related context is the one of Caffarelli-Kohn-Nirenberg (CKN) inequalities. More precisely, let  $d \geq 1, 0 < s < 1, p \geq 1, q \geq 1, \tau \geq 1, 0 < a \leq 1$ , and  $\alpha_1, \alpha_2, \beta, \gamma \in \mathbb{R}$ . Set  $\tilde{\alpha} = \alpha_1 + \alpha_2$  and define  $\sigma$  by  $\gamma = a\sigma + (1 - a)\beta$ . Assume that

$$(1.7) \quad \frac{1}{\tau} + \frac{\gamma}{d} = a \left( \frac{1}{p} + \frac{\tilde{\alpha} - s}{d} \right) + (1 - a) \left( \frac{1}{q} + \frac{\beta}{d} \right),$$

and the following conditions hold

$$(1.8) \quad 0 \leq \tilde{\alpha} - \sigma$$

and

$$(1.9) \quad \tilde{\alpha} - \sigma \leq s \text{ if } \frac{1}{\tau} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\tilde{\alpha} - s}{d}.$$

Nguyen and Squassina [24] proved that if  $\frac{1}{\tau} + \frac{\gamma}{d} > 0$ , then there exists some positive constant  $C$  such that

$$(1.10) \quad \begin{aligned} & \| |x|^\gamma g \|_{L^\tau(\mathbb{R}^d)} \\ & \leq C \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p |x|^{\alpha_1 p} |y|^{\alpha_2 p}}{|x - y|^{d+sp}} dx dy \right)^{\frac{a}{p}} \left\| |x|^\beta g \right\|_{L^q(\mathbb{R}^d)}^{1-a} \quad \forall g \in C_c^1(\mathbb{R}^d). \end{aligned}$$

This extends the full range of parameters of the well-known CKN inequalities due to Caffarelli, Kohn, and Nirenberg [7] (see also [6]) for  $s = 1$  to the fractional Sobolev spaces ( $0 < s < 1$ ).

Concerning the radial case, the range of parameters are larger. Let  $C_{c,\text{rad}}^1(\mathbb{R}^d)$  denotes the space of all radial, continuously differentiable and compactly supported functions defined in  $\mathbb{R}^d$  with  $d \geq 1$ . We prove in [19] the following result.

**Theorem 1.2.** ([19, Theorem 1.1]) *Let  $d \geq 2$  and assume (1.7),  $\frac{1}{\tau} + \frac{\gamma}{d} > 0$ , and*

$$(1.11) \quad -(d-1)s \leq \tilde{\alpha} - \sigma < 0.$$

*Then*

$$(1.12) \quad \| |x|^\gamma g \|_{L^\tau(\mathbb{R}^d)} \leq C \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p |x|^{\alpha_1 p} |y|^{\alpha_2 p}}{|x - y|^{d+sp}} dx dy \right)^{\frac{a}{p}} \| |x|^\beta g \|_{L^q(\mathbb{R}^d)}^{1-a}, \quad \forall g \in C_{c,\text{rad}}^1(\mathbb{R}^d).$$

**Remark 1.1.** Related results of (1.10) and (1.12) in the case  $\frac{1}{\tau} + \frac{\gamma}{d} \leq 0$  are also studied in [24, 19].

In this article, we are in pursuit of an optimal version of the inequality (1.3) for radial functions in the spirit of [19]. Recall that  $\mathcal{D}$  is defined in (1.5). Here is the first result of the paper.

**Theorem 1.3.** *Let  $d \geq 2$ ,  $0 < s \leq 1$ ,  $1 < \gamma < +\infty$ ,  $1 \leq p, q < +\infty$ ,  $1 < \alpha < d$ ,  $0 < \beta_1, \beta_2 < +\infty$  be such that (1.1) holds and  $\mathcal{D} \neq 0$ . Assume that either*

$$(1.13) \quad \beta_1 \gamma + \frac{d + \alpha - 2}{d - 1} \beta_2 \gamma > 1$$

*or*

$$(1.14) \quad \left( \beta_1 \gamma + \frac{d + \alpha - 2}{d - 1} \beta_2 \gamma = 1 \text{ and } q(1 - sp) = p \right).$$

*Then*

$$(1.15) \quad \|g\|_{L^\gamma(\mathbb{R}^d)} \leq C \|g\|_{\dot{W}^{s,p}(\mathbb{R}^d)}^{\beta_1 p} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x)|^q |g(y)|^q}{|x - y|^{d-\alpha}} dx dy \right)^{\beta_2}, \quad \text{for all } g \in C_{c,\text{rad}}^1(\mathbb{R}^d).$$

**Remark 1.2.** Theorem 1.3 deals with the case  $1 < \alpha < d$  and extends Theorem 1.1 in this case for radial functions under the assumption that  $\mathcal{D} \neq 0$ . One cannot extend Theorem 1.1 for radial functions in the case  $0 < \alpha \leq 1$  (see Theorem 1.5 below).

**Remark 1.3.** Condition (1.14) implicitly implies that  $sp < 1$ .

Our next result addresses the optimality of the range (1.13)-(1.14).

**Theorem 1.4.** *Let  $d \geq 2$ ,  $0 < s \leq 1$ ,  $1 < \gamma < +\infty$ ,  $1 \leq p, q < +\infty$ ,  $1 < \alpha < d$ ,  $0 < \beta_1, \beta_2 < +\infty$  be such that (1.1) holds and  $\mathcal{D} \neq 0$ . Assume that either*

$$(1.16) \quad \beta_1 \gamma + \frac{d + \alpha - 2}{d - 1} \beta_2 \gamma < 1$$

*or*

$$(1.17) \quad \left( \beta_1 \gamma + \frac{d + \alpha - 2}{d - 1} \beta_2 \gamma = 1 \text{ and } q(1 - sp) \neq p \right).$$

*Then (1.15) does not hold.*

We also obtain the following result which is on the optimality of (1.2) when  $0 < \alpha \leq 1$ .

**Theorem 1.5.** *Let  $d \geq 2$ ,  $0 < s \leq 1$ ,  $1 < \gamma < +\infty$ ,  $1 \leq p, q < +\infty$ ,  $0 < \alpha \leq 1$ ,  $0 < \beta_1, \beta_2 < +\infty$  be such that (1.1) holds and  $\mathcal{D} \neq 0$ . Assume that*

$$(1.18) \quad \beta_1 \gamma + \beta_2 \gamma < 1.$$

Then (1.15) fails to hold.

**Remark 1.4.** Theorem 1.5 confirms the optimality of (1.2) even for radial functions.

Various special cases of Theorem 1.3 are known in the literature where the assumptions were written in a quite involved manner and only for the case  $p = 2$ . More precisely, when  $p = 2$ , the conclusion of Theorem 1.3 was proved under the following assumption on  $\gamma$  instead of the assumption (1.13)-(1.14)

$$(1.19) \quad \left\{ \begin{array}{ll} q + \frac{(q(2s-1)+2)(d-\alpha)}{2s(d+\alpha-2)+(d-\alpha)} < \gamma < \infty, & \text{if } s \geq \frac{d}{2}, \\ \gamma \in \left( q + \frac{(q(2s-1)+2)(d-\alpha)}{2s(d+\alpha-2)+(d-\alpha)}, \frac{2d}{d-2s} \right], & \text{if } s < \frac{d}{2} \text{ and } \frac{1}{q} > \frac{d-2s}{d+\alpha}, \\ \gamma \in \left[ \frac{2d}{d-2s}, q + \frac{(q(2s-1)+2)(d-\alpha)}{2s(d+\alpha-2)+(d-\alpha)} \right), & \text{if } s < \frac{d}{2} \text{ and } \frac{1}{q} < \frac{d-2s}{d+\alpha} \text{ and } \frac{1}{q} \neq \frac{1-2s}{2}, \\ \frac{2d}{d-2s} \leq \gamma \leq q, & \text{if } s < \frac{1}{2} \text{ and } \frac{1}{q} = \frac{1-2s}{2}. \end{array} \right.$$

It can be shown, with the help of Proposition 2.1 in Section 2, that (1.13)-(1.14) is equivalent to (1.19) when  $p = 2$ . Our results are new even in the case  $s = 1$  and  $p \neq 2$ , to our knowledge, and as it is evident from (1.19), the assumptions given in this paper have a simple form than known ones. Concerning known results, Ruiz [27] established the result for the case  $s = 1$ ,  $d = 3$ ,  $\alpha = 2$ , and  $p = q = 2$ . When  $d \geq 2$ , Mercuri, Moroz, and Van Schaftingen [20] obtained the result in the case  $1 < \alpha < d$ ,  $q \geq 1$ ,  $p = 2$ , and  $s = 1$ . The result was later established to the case  $1/2 < s < 1$ , for  $d = 3$  with the same ranges of  $\alpha$ ,  $q$ , and  $p$  by Bellazzini, Ghimenti and Ozawa [3]. Finally, Bellazzini, Ghimenti, Mercuri, Moroz, Van Schaftingen obtained Theorem 1.3 in the case  $p = 2$ . The optimality discussed in Theorem 1.4 and Theorem 1.5 is known in the case  $p = 2$ , a result due to Bellazzini, Ghimenti, Mercuri, Moroz, and Van Schaftingen [2].

We next briefly describe the ideas of the proof of Theorem 1.3. The idea is to derive useful consequences of the radial improvements of the CKN inequalities Theorem 1.2 and a point wise estimate of radial functions related to Strauss' lemma. These results are given in Section 2.1. We then apply these results to some suitable approximation  $\gamma_\varepsilon$  of  $\gamma$  and then interpolation inequalities are involved. At some point, we also use an estimate of the Coulomb energy due to Ruiz [27] (see Lemma 2.5).

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.3 and consists of two subsections. In the first one, we derive various versions of CKN inequalities for radial functions which will be useful in the proof of Theorem 1.3. The proof of Theorem 1.3 is given in the second one. In Section 3, we discuss the optimality of the parameters. Theorem 1.4 and Theorem 1.5 are established there.

## 2. THE COULOMB-SOBOLEV INEQUALITY FOR RADIAL FUNCTIONS

This section consists of two subsections and is devoted to prove Theorem 1.3.

**2.1. Preliminaries.** In this section, we establish various results used in the proof of Theorem 1.3. Recall that  $\mathcal{D}$  is defined in (1.5). The following first result is a consequence of Theorem 1.2 and is the main ingredient for the proof of Theorem 1.3.

**Lemma 2.1.** Let  $d \geq 2$ ,  $0 < s \leq 1$ ,  $1 \leq p, q < \infty$ , and  $0 < \alpha < d$ . There exists a constant  $C > 0$  such that the following two assertions hold.

- (i) If  $\mathcal{D} > 0$ , then for all  $\varepsilon > 0$  small enough such that  $\mathcal{D} - 2\varepsilon p = p(d + \alpha - 2\varepsilon) - 2q(d - sp) > 0$ , it holds

$$(2.1) \quad \|g\|_{L^{\gamma_\varepsilon}(\mathbb{R}^d)} \leq C \|g\|_{\dot{W}^{s,p}(\mathbb{R}^d)}^{a_\varepsilon} \left\| |x|^{-\left(\frac{d-\alpha}{2q} + \frac{\varepsilon}{q}\right)} g \right\|_{L^q(\mathbb{R}^d)}^{1-a_\varepsilon}, \quad \forall g \in C_{c,\text{rad}}^1(\mathbb{R}^d),$$

where  $\gamma_\varepsilon$  and  $a_\varepsilon$  are defined by

$$(2.2) \quad \gamma_\varepsilon = q + \frac{(q(sp-1) + p)(d - \alpha + 2\varepsilon)}{sp(d + \alpha - 2\varepsilon - 2) + (d - \alpha + 2\varepsilon)} \quad \text{and} \quad a_\varepsilon = \frac{p(\gamma_\varepsilon(d + \alpha - 2\varepsilon) - 2qd)}{\gamma_\varepsilon(p(d + \alpha - 2\varepsilon) - 2q(d - sp))},$$

- (ii) and if  $\mathcal{D} < 0$  then for all  $\varepsilon > 0$  small enough such that  $\mathcal{D} + 2\varepsilon p = p(d + \alpha + 2\varepsilon) - 2q(d - sp) < 0$ , it holds

$$(2.3) \quad \|g\|_{L^{\gamma_\varepsilon}(\mathbb{R}^d)} \leq C \|g\|_{\dot{W}^{s,p}(\mathbb{R}^d)}^{a_\varepsilon} \left\| |x|^{-\left(\frac{d-\alpha}{2q} - \frac{\varepsilon}{q}\right)} g \right\|_{L^q(\mathbb{R}^d)}^{1-a_\varepsilon}, \quad \forall g \in C_{c,\text{rad}}^1(\mathbb{R}^d),$$

where  $\gamma_\varepsilon$  and  $a_\varepsilon$  are defined by

$$(2.4) \quad \gamma_\varepsilon = q + \frac{(q(sp-1) + p)(d - \alpha - 2\varepsilon)}{sp(d + \alpha + 2\varepsilon - 2) + (d - \alpha - 2\varepsilon)} \quad \text{and} \quad a_\varepsilon = \frac{p(\gamma_\varepsilon(d + \alpha + 2\varepsilon) - 2qd)}{\gamma_\varepsilon(p(d + \alpha + 2\varepsilon) - 2q(d - sp))}.$$

**Remark 2.1.** The signs in front of  $2\varepsilon$  of the corresponding terms in (2.2) and (2.4) are opposite.

*Proof.* We only prove the assertion (i). Assertion (ii) follows similarly.

We have, by (2.2),

$$\begin{aligned} \gamma_\varepsilon - \frac{2qd}{d + \alpha - 2\varepsilon} &= q + \frac{(q(sp-1) + p)(d - \alpha + 2\varepsilon)}{sp(d + \alpha - 2\varepsilon - 2) + (d - \alpha + 2\varepsilon)} - \frac{2qd}{d + \alpha - 2\varepsilon} \\ &= \frac{(qsp - q + p)(d - \alpha + 2\varepsilon)}{sp(d + \alpha - 2\varepsilon - 2) + (d - \alpha + 2\varepsilon)} - \frac{q(d - \alpha + 2\varepsilon)}{d + \alpha - 2\varepsilon}. \end{aligned}$$

This implies

$$(2.5) \quad \gamma_\varepsilon - \frac{2qd}{d + \alpha - 2\varepsilon} = \frac{(p(d + \alpha - 2\varepsilon) - 2q(d - sp))(d - \alpha + 2\varepsilon)}{(d + \alpha - 2\varepsilon)(sp(d + \alpha - 2\varepsilon - 2) + (d - \alpha + 2\varepsilon))} > 0,$$

by the smallness of  $\varepsilon$ .

Using the definition of  $a_\varepsilon$  in (2.2), we derive from (2.5) and the smallness of  $\varepsilon$  that

$$(2.6) \quad a_\varepsilon > 0.$$

We have, by (2.2),

$$(2.7) \quad 1 - a_\varepsilon = 1 - \frac{p(\gamma_\varepsilon(d + \alpha + 2\varepsilon) - 2qd)}{\gamma_\varepsilon(p(d + \alpha + 2\varepsilon) - 2q(d - sp))} = \frac{2q(dp - \gamma_\varepsilon(d - sp))}{\gamma_\varepsilon(p(d + \alpha - 2\varepsilon) - 2q(d - sp))},$$

which yields

$$a_\varepsilon < 1 \quad \text{if} \quad sp \geq d.$$

We next deal with the case  $sp < d$ . From (2.2), we derive that

$$(2.8) \quad \gamma_\varepsilon = \frac{2qsp(d-1) + p(d - \alpha + 2\varepsilon)}{sp(d + \alpha - 2\varepsilon - 2) + (d - \alpha + 2\varepsilon)}.$$

Using (2.8), we obtain

$$\begin{aligned}
\frac{dp}{d-sp} - \gamma_\varepsilon &= \frac{dsp^2(d+\alpha-2\varepsilon-2) + dp(d-\alpha+2\varepsilon)}{(d-sp)(sp(d+\alpha-2\varepsilon-2) + (d-\alpha+2\varepsilon))} \\
&\quad - \frac{2qsp(d-1)(d-sp) + p(d-sp)(d-\alpha+2\varepsilon)}{(d-sp)(sp(d+\alpha-2\varepsilon-2) + (d-\alpha+2\varepsilon))} \\
&= \frac{dsp^2(d+\alpha-2\varepsilon-2) + sp^2(d-\alpha+2\varepsilon) - 2qsp(d-1)(d-sp)}{(d-sp)(sp(d+\alpha-2\varepsilon-2) + (d-\alpha+2\varepsilon))} \\
&= \frac{sp^2(d-1)(d+\alpha-2\varepsilon-2) + 2sp^2(d-1) - 2qsp(d-1)(d-sp)}{(d-sp)(sp(d+\alpha-2\varepsilon-2) + (d-\alpha+2\varepsilon))} \\
&= \frac{sp(d-1)(p(d+\alpha-2\varepsilon-2) + 2p - 2q(d-sp))}{(d-sp)(sp(d+\alpha-2\varepsilon-2) + (d-\alpha+2\varepsilon))},
\end{aligned}$$

which yields, for  $sp < d$ ,

$$(2.9) \quad \frac{dp}{d-sp} - \gamma_\varepsilon = \frac{sp(d-1)(p(d+\alpha-2\varepsilon) - 2q(d-sp))}{(d-sp)(sp(d+\alpha-2\varepsilon-2) + (d-\alpha+2\varepsilon))} > 0,$$

by the smallness of  $\varepsilon$ , which implies, by (2.2), that

$$a_\varepsilon < 1.$$

We thus established

$$0 < a_\varepsilon < 1.$$

In light of (1.12) under the assumptions (1.7) and (1.11), it suffices to verify these conditions for the following parameters  $(s, p, q, \tau, \gamma, a, \alpha_1, \alpha_2, \beta) = (s, p, q, \gamma_\varepsilon, 0, a_\varepsilon, 0, 0, -\frac{d-\alpha+2\varepsilon}{2q})$ . One can check that

$$(2.10) \quad \tilde{\alpha} = \alpha_1 + \alpha_2 = 0$$

and

$$(2.11) \quad \sigma = \frac{1}{a}(\gamma - (1-a)\beta) = \frac{1-a_\varepsilon}{a_\varepsilon} \frac{d-\alpha+2\varepsilon}{2q}.$$

From (2.2) and (2.7), we deduce that

$$\begin{aligned}
&\frac{d-sp}{d} \frac{\gamma_\varepsilon a_\varepsilon}{p} + \frac{d+\alpha-2\varepsilon}{d} \frac{\gamma_\varepsilon(1-a_\varepsilon)}{2q} \\
&= \frac{(d-sp)(\gamma_\varepsilon(d+\alpha-2\varepsilon) - 2qd) + (d+\alpha-2\varepsilon)(dp - \gamma_\varepsilon(d-sp))}{d(p(d+\alpha-2\varepsilon) - 2q(d-sp))} = 1.
\end{aligned}$$

This implies

$$\frac{1}{\gamma_\varepsilon} = \frac{a_\varepsilon(d-sp)}{dp} + \frac{(1-a_\varepsilon)(d+\alpha-2\varepsilon)}{2dq}$$

and thus (1.7) holds.

We have

$$(2.12) \quad \tilde{\alpha} - \sigma \stackrel{(2.10), (2.11)}{=} -\frac{1-a_\varepsilon}{a_\varepsilon} \frac{d-\alpha+2\varepsilon}{2q} \stackrel{(2.2), (2.7)}{=} -\frac{(dp - \gamma_\varepsilon(d-sp))(d-\alpha+2\varepsilon)}{p(\gamma_\varepsilon(d+\alpha-2\varepsilon) - 2qd)}.$$

Plugging (2.5) and (2.9) in (2.12), we get

$$\tilde{\alpha} - \sigma = -(d-1)s.$$

Therefore, (1.11) is satisfied.

Thus, the assumptions (1.7) and (1.11) are checked. The proof is complete.  $\square$

Here is another consequence of the CKN inequalities (1.12) for radial functions, which is used in the proof of Theorem 1.3.

**Lemma 2.2.** Let  $d \geq 2$ ,  $1 \leq p < \infty$ ,  $0 < s \leq 1$  be such that  $sp < d$ , and  $\beta \in \mathbb{R}$ . Assume that

- (a)  $\frac{1}{r} = \frac{1}{p} + \frac{\beta-s}{d}$ ,
- (b)  $-(d-1)s \leq \beta \leq s$ .

There exists a positive constant  $C$  such that

$$(2.13) \quad \left\| |x|^{-\beta} g \right\|_{L^r(\mathbb{R}^d)} \leq C \|g\|_{\dot{W}^{s,p}(\mathbb{R}^d)}, \text{ for all } g \in C_{c,\text{rad}}^1(\mathbb{R}^d).$$

*Proof.* First of all notice that, since  $\beta \leq s$  we must have  $r \geq p$ . In light of (1.12) under the assumptions (1.7), (1.11), and (1.9), it suffices to verify these conditions for the following parameters  $(a, s, p, \tau, \gamma, \alpha_1, \alpha_2) = (1, s, p, r, -\beta, 0, 0)$ . We compute  $\tilde{\alpha} = \alpha_1 + \alpha_2 = 0$  and  $\sigma = -\beta$ . Note that by the assumption (a) we have  $\frac{1}{r} + \frac{\gamma}{d} = \frac{d-sp}{dp} > 0$ . So, (1.7) is satisfied. On the other hand, (1.11) and (1.9) are consequences of the assumptions (b). We thus have the desired inequality.  $\square$

**Remark 2.2.** Particular versions of (2.13) can be found in [29] (see also of [2, Theorem 4.3] and the discussion afterwards for more references).

As a consequence of (2.13), we obtain the following useful estimate.

**Lemma 2.3.** Let  $d \geq 2$ ,  $1 \leq p < \infty$ , and  $0 < s \leq 1$  be such that  $sp \leq 1$ . Assume that  $\frac{1}{p} - s \leq \frac{1}{\gamma} \leq \frac{1}{p} - \frac{s}{d}$ . Then there exists a positive constant  $C$  such that for any  $R > 0$ , we have

$$(2.14) \quad \left( \int_{|x|>R} |g(x)|^\gamma dx \right)^{\frac{1}{\gamma}} \leq CR^{\frac{d}{\gamma} - (\frac{d}{p} - s)} \|g\|_{\dot{W}^{s,p}(\mathbb{R}^d)}, \text{ for all } g \in C_{c,\text{rad}}^1(\mathbb{R}^d).$$

*Proof.* Inequality (2.14) is a consequence of (2.13) with  $r = \gamma$  and  $\beta = \frac{d}{\gamma} - (\frac{d}{p} - s)$  after noting that  $\beta \leq 0$  by the assumptions.  $\square$

We next present a point wise estimate of radial functions related to Strauss' lemma.

**Lemma 2.4.** Let  $0 < s \leq 1$ ,  $1 \leq p < \infty$  and  $d' > 0$  be such that  $1 < sp < d'$  and assume that  $\Lambda > 1$ . Then there exist a constant  $C = C(d', s, p, \Lambda) > 0$  such that if  $0 < s < 1$ , then for any  $g \in W^{s,p}(\mathbb{R})$  it holds

$$(2.15) \quad |g(x)| \leq \frac{C}{|x|^{\frac{d'-sp}{p}}} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|g(r) - g(\rho)|^p |r|^{d'-1}}{|r - \rho|^{1+sp}} \chi_\Lambda(|r|, |\rho|) dr d\rho \right)^{\frac{1}{p}}, \forall x \neq 0,$$

where, for  $r_1, r_2 \geq 0$ , we denote

$$(2.16) \quad \chi_\Lambda(r_1, r_2) = \begin{cases} 1 & \text{for } \Lambda^{-1}r_1 \leq r_2 \leq \Lambda r_1, \\ 0 & \text{otherwise,} \end{cases}$$

and if  $s = 1$ , then for any  $g \in W^{1,p}(\mathbb{R})$  it holds

$$(2.17) \quad |g(x)| \leq \frac{C}{|x|^{\frac{d'-p}{p}}} \left( \int_{\mathbb{R}} |g'(r)|^p |r|^{d'-1} dr \right)^{\frac{1}{p}}, \forall x \neq 0.$$



*Proof.* First we consider  $0 < s < 1$ . Since  $1 < sp$ ,  $g$  is Hölder continuous by [22, Theorem 8.1]. Without loss of generality, we can assume that  $g$  has compact support and  $\text{supp } g \subset (-2^m, 2^m)$  for some  $m \in \mathbb{N}$ . Since, (2.15) is invariant under scalar multiplication, so we could assume that

$$(2.18) \quad \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|g(r) - g(\rho)|^p |r|^{d'-1}}{|r - \rho|^{1+sp}} \chi_{\Lambda}(|r|, |\rho|) dr d\rho \right)^{\frac{1}{p}} = 1.$$

Using a scaling argument, it suffices to prove (2.15) for  $x = 1$ .

By the embedding for the fractional Sobolev spaces, see e.g., [22, estimates 8.4 and 8.8], we have

$$(2.19) \quad \begin{aligned} |g(1) - g(2)| &\leq C \left( \int_{\frac{1}{2} \leq |r| \leq 4} \int_{\frac{1}{2} \leq |\rho| \leq 4} \frac{|g(r) - g(\rho)|^p}{|r - \rho|^{1+sp}} \chi_{\Lambda}(|r|, |\rho|) dr d\rho \right)^{\frac{1}{p}} \\ &\leq C \left( \int_{\frac{1}{2} < |r| < 4} \int_{\frac{1}{2} < |\rho| < 4} \frac{|g(r) - g(\rho)|^p |r|^{d'-1}}{|r - \rho|^{1+sp}} \chi_{\Lambda}(|r|, |\rho|) dr d\rho \right)^{\frac{1}{p}}. \end{aligned}$$

Here and in what follows,  $C > 0$  is a constant depends on  $s, d', p$  and  $\Lambda$ . Let  $\lambda > 0$  and denote  $g_{\lambda}(x) := g(\lambda x)$ . Applying (2.19) to  $g_{\lambda}$ , we obtain

$$(2.20) \quad |g(\lambda) - g(2\lambda)| \leq \frac{C}{\lambda^{\frac{d'-sp}{p}}} \left( \int_{\frac{\lambda}{2} < |r| < 4\lambda} \int_{\frac{\lambda}{2} < |\rho| < 4\lambda} \frac{|g(r) - g(\rho)|^p |r|^{d'-1}}{|r - \rho|^{1+sp}} \chi_{\Lambda}(|r|, |\rho|) dr d\rho \right)^{\frac{1}{p}}.$$

Using the fact

$$|g(1)| = |g(1) - g(2^m)| \leq \sum_{k=1}^m |g(2^{k-1}) - g(2^k)|,$$

we derive from (2.20) that

$$(2.21) \quad \begin{aligned} |g(1)| &\leq C \sum_{k=1}^m \left( \frac{1}{2^{\frac{k-1}{p}}} \right)^{d'-sp} \\ &\quad \times \left( \int_{2^{k-2} \leq |r| \leq 2^{k+1}} \int_{2^{k-2} \leq |\rho| \leq 2^{k+1}} \frac{|g(r) - g(\rho)|^p |r|^{d'-1}}{|r - \rho|^{1+sp}} \chi_{\Lambda}(|r|, |\rho|) dr d\rho \right)^{\frac{1}{p}}. \end{aligned}$$

Since  $sp < d'$ , so  $\sum_{k \geq 1} 2^{-\frac{(k-1)(d'-sp)}{p}} < \infty$ , using (2.18) in (2.21) we establish (2.15). This completes the proof in the case  $0 < s < 1$ . The proof in the case  $s = 1$  follows similarly. The details are omitted.  $\square$

As a consequence of Lemma 2.4, we obtain the following result.

**Corollary 2.1.** *Let  $0 < s \leq 1$ ,  $1 \leq p < \infty$  and  $d \geq 2$  be such that  $1 < sp < d$ . Then there exists a constant  $C > 0$  depending on  $s, p$  and  $d$ , such that for any  $g \in C_{c,\text{rad}}^1(\mathbb{R}^d)$  we have*

$$(2.22) \quad |g(x)| \leq \frac{C}{|x|^{\frac{d-sp}{p}}} \|g\|_{\dot{W}^{s,p}(\mathbb{R}^d)}, \quad \forall x \in \mathbb{R}^d \setminus \{0\}.$$

*Proof.* Corollary 2.1 is a direct consequence of Lemma 2.4 and [19, Lemma 3.1].  $\square$

**Remark 2.3.** The inequality (2.22) is an extension of the radial lemma of Strauss [28] to the fractional Sobolev spaces. This type of inequalities has been derived previously in [21, Theorem 3.1] and [10] for  $p = 2$  and the known proofs are limited to the case  $p = 2$ .

**2.2. Proof of Theorem 1.3.** We start by proving the following proposition which allow us to divide the proof of Theorem 1.3 into three different sub cases.

**Proposition 2.1.** Let  $d \geq 2$ ,  $0 < s \leq 1$ ,  $1 \leq \gamma, p, q < +\infty$ ,  $1 < \alpha < d$ , and  $0 \leq \beta_1, \beta_2 < +\infty$ , and assume (1.1) and  $(d + \alpha)p - 2q(d - sp) \neq 0$ . Set

$$(2.23) \quad \gamma_{\text{rad}} := q + \frac{(q(sp - 1) + p)(d - \alpha)}{sp(d + \alpha - 2) + (d - \alpha)} = \frac{2qsp(d - 1) + p(d - \alpha)}{sp(d + \alpha - 2) + (d - \alpha)}.$$

Then (1.13) is equivalent to the following:

$$(2.24) \quad \begin{cases} \gamma_{\text{rad}} < \gamma \leq \frac{pd}{d - sp}, & \text{if } \mathcal{D} > 0 \text{ and } sp < d, \\ \gamma_{\text{rad}} < \gamma < \infty, & \text{if } sp \geq d, \\ \frac{pd}{d - sp} \leq \gamma < \gamma_{\text{rad}}, & \text{if } \mathcal{D} < 0, \end{cases}$$

and (1.14) is equivalent to the fact

$$(2.25) \quad \gamma = \gamma_{\text{rad}} \quad \text{and} \quad q = \frac{p}{1 - sp}.$$

*Proof.* We have, by (1.6),

$$\begin{aligned} \beta_1 \gamma + \frac{\beta_2 \gamma (d + \alpha - 2)}{d - 1} - 1 &= \frac{(\gamma(d + \alpha) - 2qd)\gamma}{\gamma(p(d + \alpha) - 2q(d - sp))} + \frac{(pd - \gamma(d - sp))\gamma(d + \alpha - 2)}{(d - 1)\gamma(p(d + \alpha) - 2q(d - sp))} - 1 \\ &= \frac{(d - 1)(\gamma(d + \alpha) - 2qd) + (pd - \gamma(d - sp))(d + \alpha - 2)}{(d - 1)(p(d + \alpha) - 2q(d - sp))} \\ &\quad - \frac{(d - 1)(p(d + \alpha) - 2q(d - sp))}{(d - 1)(p(d + \alpha) - 2q(d - sp))}. \end{aligned}$$

Simplifying the expression after noting  $(d - 1)(d + \alpha) - d(d + \alpha - 2) = d - \alpha$ , we obtain

$$(2.26) \quad \beta_1 \gamma + \frac{\beta_2 \gamma (d + \alpha - 2)}{d - 1} - 1 = \frac{\gamma(sp(d + \alpha - 2) + (d - \alpha)) - (2qsp(d - 1) + p(d - \alpha))}{(d - 1)(p(d + \alpha) - 2q(d - sp))},$$

which yields, by (2.23),

$$(2.27) \quad \beta_1 \gamma + \frac{\beta_2 \gamma (d + \alpha - 2)}{d - 1} - 1 = \frac{sp(d + \alpha - 2) + (d - \alpha)}{(d - 1)} \frac{(\gamma - \gamma_{\text{rad}})}{(p(d + \alpha) - 2q(d - sp))}.$$

On the other hand, since  $\beta_1, \beta_2 \geq 0$ ,  $\gamma > 0$ ,  $d + \alpha - 2 > 0$ , we derive from (1.6) that the fact  $\beta_1, \beta_2 \geq 0$  is equivalent to the one that

$$(2.28) \quad \frac{2qd}{d + \alpha} \leq \gamma \leq \frac{pd}{d - sp} \text{ if } p(d + \alpha) - 2q(d - sp) > 0 \text{ and } d - sp > 0,$$

$$(2.29) \quad \frac{2qd}{d + \alpha} \leq \gamma < +\infty \text{ if } p(d + \alpha) - 2q(d - sp) > 0 \text{ and } d - sp \leq 0,$$

and

$$(2.30) \quad \frac{pd}{d - sp} \leq \gamma \leq \frac{2qd}{d + \alpha} \text{ if } p(d + \alpha) - 2q(d - sp) < 0 \text{ and } d - sp > 0.$$

(Note that the case  $p(d + \alpha) - 2q(d - sp) < 0$  and  $d - sp \leq 0$  does not occur.)

We next claim that

$$(2.31) \quad \gamma_{\text{rad}} - \frac{2qd}{d+\alpha} = \frac{(d-\alpha)}{(d+\alpha)(sp(d+\alpha-2) + (d-\alpha))} (p(d+\alpha) - 2q(d-sp)).$$

Indeed, we have, by (2.23),

$$\begin{aligned} \gamma_{\text{rad}} - \frac{2qd}{d+\alpha} &= q + \frac{(q(sp-1) + p)(d-\alpha)}{sp(d+\alpha-2) + (d-\alpha)} - \frac{2qd}{d+\alpha} \\ &= \frac{(q(sp-1) + p)(d-\alpha)}{sp(d+\alpha-2) + (d-\alpha)} - \frac{q(d-\alpha)}{d+\alpha} \\ &= \frac{(d-\alpha) \left( qsp(d+\alpha) + (p-q)(d+\alpha) - qsp(d+\alpha-2) - q(d-\alpha) \right)}{(d+\alpha)(sp(d+\alpha-2) + (d-\alpha))}, \end{aligned}$$

which yields claim (2.31).

In the case

$$\beta_1 \gamma + \frac{\beta_2 \gamma (d+\alpha-2)}{d-1} > 1,$$

using from (2.27)-(2.31), we have the following three situations:

- if  $\mathcal{D} > 0$  and  $sp < d$ , then

$$\gamma_{\text{rad}} < \gamma \leq \frac{pd}{d-sp}$$

- if  $\mathcal{D} > 0$  and  $sp \geq d$  (this is equivalent to the fact  $sp \geq d$ ), then

$$\gamma_{\text{rad}} < \gamma < +\infty.$$

- if  $\mathcal{D} < 0$  and  $sp < d$  (this is equivalent to the fact  $\mathcal{D} < 0$ ), then

$$\frac{pd}{d-sp} \leq \gamma < \gamma_{\text{rad}}.$$

In the case

$$\beta_1 \gamma + \frac{\beta_2 \gamma (d+\alpha-2)}{d-1} = 1 \quad \text{and} \quad q = \frac{p}{1-sp}$$

we have

$$\gamma = \gamma_{\text{rad}}.$$

The conclusion follows. The proof is complete.  $\square$

Depending on the range where  $\gamma$  belongs, we intend to employ different techniques to prove Theorem 1.3. Using Proposition 2.1, we divide the range of  $\gamma$  into three main sub-ranges.

$$(Range A) \quad \begin{cases} \gamma_{\text{rad}} < \gamma \leq \frac{pd}{d-sp}, & \text{if } \mathcal{D} > 0 \text{ and } sp < d, \\ \gamma_{\text{rad}} < \gamma < \infty, & \text{if } sp \geq d. \end{cases}$$

$$(Range B) \quad \frac{pd}{d-sp} \leq \gamma < \gamma_{\text{rad}}, \quad \text{if } \mathcal{D} < 0.$$

$$(Range C) \quad \gamma = \gamma_{\text{rad}} \text{ and } q = \frac{p}{1-sp}.$$

We next recall the following result from [18].

**Lemma 2.2.** ([18, Lemma 2.5]) *Let  $d \geq 2$ ,  $0 \leq s \leq 1$ ,  $1 \leq \gamma, p, q < \infty$ ,  $0 < \alpha < d$ , and  $0 \leq \beta_1, \beta_2 < +\infty$ . Set*

$$(2.32) \quad \gamma_{cs} := \frac{p(\alpha + 2qs)}{\alpha + sp}.$$

and assume that (1.1) holds and  $(d + \alpha)p - 2q(d - sp) \neq 0$ . Then (1.2) is equivalent to the fact

$$(2.33) \quad \begin{cases} \gamma_{cs} \leq \gamma \leq \frac{pd}{d - sp} & \text{if } \mathcal{D} > 0 \text{ and } sp < d, \\ \gamma_{cs} \leq \gamma < \infty & \text{if } sp \geq d, \\ \frac{pd}{d - sp} \leq \gamma \leq \gamma_{cs} & \text{if } \mathcal{D} < 0. \end{cases}$$

**Remark 2.1.** Let  $d \geq 2$ ,  $0 < s \leq 1$ ,  $1 < \gamma < +\infty$ ,  $1 \leq p, q < +\infty$ ,  $1 < \alpha < d$ ,  $0 < \beta_1, \beta_2 < +\infty$  be such that (1.1) and  $\mathcal{D} \neq 0$  hold. One can check that

$$(2.34) \quad \gamma_{cs} \text{ satisfies (1.1) and } \beta_1 \gamma_{cs} + \beta_2 \gamma_{cs} = 1$$

(see, e.g., the proof of [18, Lemma 2.5]). Using (2.34), we obtain

$$(2.35) \quad \beta_1 \gamma_{cs} + \frac{\beta_2 \gamma_{cs}(d + \alpha - 2)}{d - 1} - 1 = \frac{\beta_2 \gamma_{cs}(\alpha - 1)}{d - 1}.$$

Considering (2.27) with  $\gamma = \gamma_{cs}$  and using the above identity, we obtain

$$(2.36) \quad \frac{\beta_2 \gamma_{cs}(\alpha - 1)}{d - 1} = \frac{sp(d + \alpha - 2) + (d - \alpha)}{(d - 1)(p(d + \alpha) - 2q(d - sp))} (\gamma_{cs} - \gamma_{rad}).$$

Since  $1 < \alpha < d$  and  $\beta_2 > 0$ , it follows from (2.36) that

$$(2.37) \quad \gamma_{rad} < \gamma_{cs} \text{ if } \mathcal{D} > 0$$

and

$$(2.38) \quad \gamma_{cs} < \gamma_{rad} \text{ if } \mathcal{D} < 0.$$

Applying Theorem 1.1 and using Lemma 2.2, we deduce that (1.3) holds if  $\gamma$  satisfies (2.33). Consequently, (1.15) holds if  $\gamma$  satisfies (2.33). Using this fact, we only need to prove Theorem 1.3 for  $\gamma_{rad} < \gamma < \gamma_{cs}$ , when  $\gamma$  varies in Range A and for  $\gamma_{cs} < \gamma < \gamma_{rad}$  when  $\gamma$  varies in Range B.

2.2.1. *Proof of Theorem 1.3 when  $\gamma$  varies in the Range A.* As mentioned in Remark 2.1, one only needs to establish (1.15) for

$$\gamma_{rad} < \gamma < \gamma_{cs},$$

which is assumed from later on in this part.

Let  $\epsilon > 0$  be small enough so that  $p(d + \alpha - 2\epsilon) - 2q(d - sp) > 0$ . This can be done since  $\mathcal{D} > 0$ . Set

$$\gamma_\epsilon := q + \frac{(q(sp - 1) + p)(d - \alpha + 2\epsilon)}{sp(d + \alpha - 2\epsilon - 2) + (d - \alpha + 2\epsilon)}.$$

We claim that (1.15) holds for  $\gamma = \gamma_\epsilon$  for  $\epsilon$  sufficiently small.

We first admit this claim and continue the proof. We have, by (2.23),

$$(2.39) \quad \begin{aligned} \gamma_\epsilon - \gamma_{rad} &= \frac{(q(sp - 1) + p)(d - \alpha + 2\epsilon)}{sp(d + \alpha - 2\epsilon - 2) + (d - \alpha + 2\epsilon)} - \frac{(q(sp - 1) + p)(d - \alpha)}{sp(d + \alpha - 2) + (d - \alpha)} \\ &= \frac{4\epsilon sp(d - 1)(p + q(sp - 1))}{(sp(d + \alpha - 2\epsilon - 2) + (d - \alpha + 2\epsilon))(sp(d + \alpha - 2) + (d - \alpha))}. \end{aligned}$$

We first claim that

$$(2.40) \quad p + (sp - 1)q > 0.$$

Indeed, (2.40) is clear for  $sp \geq 1$ . We now establish (2.40) for  $sp < 1$ . Since  $\mathcal{D} = p(d + \alpha) - 2q(d - sp) > 0$ , we obtain using  $\alpha < d$

$$(2.41) \quad q < \frac{p(d + \alpha)}{2(d - sp)} < \frac{dp}{d - sp}.$$

We have

$$\frac{p}{1 - sp} - \frac{dp}{d - sp} = \frac{p(d - sp - d + dsp)}{(1 - sp)(d - sp)} = \frac{sp^2(d - 1)}{(1 - sp)(d - sp)}.$$

Claim (2.40) then follows in the case  $sp < 1$ .

Thus (2.39) implies  $\gamma_\varepsilon \rightarrow \gamma_{\text{rad}+}$  as  $\varepsilon \rightarrow 0+$ . Since (1.15) holds for  $\gamma = \gamma_{cs}$  by (2.34) and Theorem 1.1, using interpolation (Hölder's inequalities), we derive from the claim that (1.15) holds for  $\gamma_{\text{rad}} < \gamma < \gamma_{cs}$ .

It hence remains to show that (1.15) holds for  $\gamma = \gamma_\varepsilon$  for  $\varepsilon$  sufficiently small. Let  $g \in C_{c,\text{rad}}^1(\mathbb{R}^d)$ . Since  $p(d + \alpha) - 2q(d - sp) \neq 0$  and (1.15) is invariant under scaling, without loss of generality, we can assume that

$$(2.42) \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p}{|x - y|^{d+sp}} dx dy = 1 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x)|^q |g(y)|^q}{|x - y|^{d-\alpha}} dx dy.$$

By (2.1) in Lemma 2.1, it suffices to prove that

$$(2.43) \quad \int_{\mathbb{R}^d} \frac{|g(x)|^q dx}{|x|^{\frac{d-\alpha}{2}+\epsilon}} \leq C,$$

for some constant  $C > 0$  independent of  $g$ . Because of (2.47) with  $R = 1$  in the Lemma 2.5 below, we only need to show that

$$(2.44) \quad \int_{|x|<1} \frac{|g(x)|^q dx}{|x|^{\frac{d-\alpha}{2}+\epsilon}} < C,$$

for some constant  $C > 0$  independent of  $g$ .

Since  $p + (sp - 1)q > 0$  by (2.40), we derive from (2.23) that

$$q < \gamma_{\text{rad}}.$$

This implies, by (2.37), that

$$q < \gamma_{cs}.$$

Since (1.15) holds for  $\gamma = \gamma_{cs}$  by (2.34) and Theorem 1.1, it follows from (2.42) that

$$(2.45) \quad \int_{|x|<1} |g(x)|^{\gamma_{cs}} dx \leq C.$$

Applying Hölder's inequality, we derive that

$$(2.46) \quad \int_{|x|<1} \frac{|g(x)|^q dx}{|x|^{\frac{d-\alpha}{2}+\epsilon}} \leq \left( \int_{|x|<1} |g(x)|^{\gamma_{cs}} dx \right)^{\frac{q}{\gamma_{cs}}} \left( \int_{|x|<1} \frac{dx}{|x|^{\frac{\gamma_{cs}-q}{\gamma_{cs}-q} \left( \frac{d-\alpha}{2} + \epsilon \right)}} \right)^{\frac{\gamma_{cs}-q}{\gamma_{cs}}} \\ \stackrel{(2.45)}{\leq} C \left( \int_{|x|<1} \frac{dx}{|x|^{\frac{\gamma_{cs}-q}{\gamma_{cs}-q} \left( \frac{d-\alpha}{2} + \epsilon \right)}} \right)^{\frac{\gamma_{cs}-q}{\gamma_{cs}}}.$$

Since

$$d - \frac{\gamma_{cs}}{\gamma_{cs} - q} \left( \frac{d - \alpha}{2} + \epsilon \right) = \frac{1}{\gamma_{cs} - q} \left[ \frac{\alpha(p(d + \alpha) - 2q(d - sp))}{2(\alpha + sp)} - \gamma_{cs}\epsilon \right] > 0,$$

for  $\epsilon > 0$  small enough, assertion (2.44) follows for  $\epsilon$  sufficiently small. The proof is complete.  $\square$

The following result due to Ruiz [27] is used in the proof.

**Lemma 2.5.** ([27, Theorem 1.1]) Let  $d \in \mathbb{N}$ ,  $0 < \alpha < d$ ,  $1 \leq q < \infty$ . Then for every  $\epsilon > 0$  and  $R > 0$  there exists  $C = C(d, \alpha, q, \epsilon) > 0$  such that for all  $u \in L^{\frac{2dq}{d+\alpha}}(\mathbb{R}^d)$  we have,

$$(2.47) \quad \int_{|x|>R} \frac{|u(x)|^q}{|x|^{\frac{d-\alpha}{2}+\epsilon}} dx \leq \frac{C}{R^\epsilon} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^q |u(y)|^q}{|x-y|^{d-\alpha}} dx dy \right)^{\frac{1}{2}} \text{ and}$$

$$(2.48) \quad \int_{|x|<R} \frac{|u(x)|^q}{|x|^{\frac{d-\alpha}{2}-\epsilon}} dx \leq CR^\epsilon \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^q |u(y)|^q}{|x-y|^{d-\alpha}} dx dy \right)^{\frac{1}{2}}.$$

2.2.2. *Proof of Theorem 1.3 when  $\gamma$  varies in the Range B.* As mentioned in Remark 2.1, it suffices to prove (1.15) for

$$\gamma_{cs} < \gamma < \gamma_{rad},$$

which will be assumed from later on in this part.

Let  $g \in C_{c,rad}^1(\mathbb{R}^d)$ . Since  $p(d + \alpha) - 2q(d - sp) \neq 0$  and (1.15) is invariant under scaling, without loss of generality we can assume that

$$(2.49) \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p}{|x - y|^{d+sp}} dx dy = 1 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x)|^q |g(y)|^q}{|x - y|^{d-\alpha}} dx dy.$$

We now consider three cases separately:

- Case 1:  $1 < sp < d$ .
- Case 2:  $sp \leq 1$  and  $q(sp - 1) + p > 0$ .
- Case 3:  $sp \leq 1$  and  $q(sp - 1) + p \leq 0$ .

We now proceed the proof.

- Case 1:  $1 < sp < d$ . Set, for  $\epsilon$  sufficiently small,

$$\gamma_\epsilon = q + \frac{(q(sp - 1) + p)(d - \alpha - 2\epsilon)}{sp(d + \alpha + 2\epsilon - 2) + (d - \alpha - 2\epsilon)}.$$

We claim that (1.15) holds for  $\gamma = \gamma_\epsilon$  for  $\epsilon$  sufficiently small.

We first admit this claim and continue the proof. We have, by (2.23),

$$(2.50) \quad \begin{aligned} \gamma_\epsilon - \gamma_{rad} &= \frac{(q(sp - 1) + p)(d - \alpha - 2\epsilon)}{sp(d + \alpha + 2\epsilon - 2) + (d - \alpha - 2\epsilon)} - \frac{(q(sp - 1) + p)(d - \alpha)}{sp(d + \alpha - 2) + (d - \alpha)} \\ &= - \frac{4\epsilon sp(d - 1)(p + q(sp - 1))}{(sp(d + \alpha + 2\epsilon - 2) + (d - \alpha - 2\epsilon))(sp(d + \alpha - 2) + (d - \alpha))}. \end{aligned}$$

It follows that  $\gamma_\epsilon \rightarrow \gamma_{rad}-$  and  $\epsilon \rightarrow 0+$  since  $sp > 1$ . Now, since (1.15) holds with  $\gamma = \gamma_{cs}$  by (2.34), by interpolation, it suffices to prove the claim.

Next, we establish the claim. Applying (2.3) in Lemma 2.1, it suffices to show that

$$(2.51) \quad \int_{\mathbb{R}^d} \frac{|g(x)|^q}{|x|^{\frac{d-\alpha}{2}-\epsilon}} dx \leq C,$$

for some constant  $C > 0$  independent of  $g$  and for  $\epsilon > 0$  small enough. Using (2.48) in Lemma 2.5, one only needs to prove

$$(2.52) \quad \int_{|x|>1} \frac{|g(x)|^q}{|x|^{\frac{d-\alpha}{2}-\epsilon}} dx \leq C.$$

We have, by (2.23),

$$q < \gamma_{\text{rad}}.$$

If  $\gamma_{cs} < q < \gamma_{\text{rad}}$ , using (2.22) and (2.49), we obtain

$$\int_{|x|>1} \frac{|g(x)|^q}{|x|^{\frac{d-\alpha}{2}-\epsilon}} dx \leq C \int_{|x|>1} \frac{dx}{|x|^{(\frac{d-\alpha}{2}-\epsilon)+\frac{q(d-sp)}{p}}} \leq C,$$

for  $\epsilon$  sufficiently small since, by  $\mathcal{D} < 0$ ,

$$\frac{d-\alpha}{2} + \frac{q(d-sp)}{p} > d.$$

If  $q = \gamma_{cs}$ , then applying (1.15) with  $\gamma = \gamma_{cs}$  by (2.34), and using (2.49), we have

$$\int_{|x|>1} \frac{|g(x)|^q}{|x|^{\frac{d-\alpha}{2}-\epsilon}} dx \leq \int_{|x|>1} |g(x)|^{\gamma_{cs}} dx \leq C.$$

If  $q < \gamma_{cs} < \gamma_{\text{rad}}$ , applying (1.15) with  $\gamma = \gamma_{cs}$  by (2.34), and using (2.49), we have

$$(2.53) \quad \int_{|x|>1} \frac{|g(x)|^q}{|x|^{\frac{d-\alpha}{2}-\epsilon}} dx \leq \left( \int_{|x|>1} |g(x)|^{\gamma_{cs}} dx \right)^{\frac{q}{\gamma_{cs}}} \left( \int_{|x|>1} \frac{dx}{|x|^{\frac{\gamma_{cs}-q}{\gamma_{cs}}(\frac{d-\alpha}{2}-\epsilon)}} \right)^{\frac{\gamma_{cs}-q}{\gamma_{cs}}} \\ \leq C \left( \int_{|x|>1} \frac{dx}{|x|^{\frac{\gamma_{cs}-q}{\gamma_{cs}}(\frac{d-\alpha}{2}-\epsilon)}} \right)^{\frac{\gamma_{cs}-q}{\gamma_{cs}}}.$$

Since  $\mathcal{D} = p(d+\alpha) - 2q(d-sp) < 0$ , it follows that

$$d - \frac{\gamma_{cs}}{\gamma_{cs}-q} \left( \frac{d-\alpha}{2} - \epsilon \right) = \frac{1}{\gamma_{cs}-q} \left[ \frac{\alpha(p(d+\alpha) - 2q(d-sp))}{2(\alpha+sp)} + \gamma_{cs}\epsilon \right] < 0,$$

for  $\epsilon > 0$  small enough. It follows from (2.53) that

$$(2.54) \quad \int_{|x|>1} \frac{|g(x)|^q dx}{|x|^{\frac{d-\alpha}{2}-\epsilon}} \leq C.$$

The proof is complete in this case.

- Case 2:  $sp \leq 1$  and  $p + q(sp-1) > 0$ . Set

$$\gamma = \gamma_\epsilon = q + \frac{(q(sp-1) + p)(d-\alpha-2\epsilon)}{sp(d+\alpha+2\epsilon-2) + (d-\alpha-2\epsilon)},$$

for  $\epsilon$  sufficiently small.

From (2.23), we derive that  $\gamma_\epsilon \rightarrow \gamma_{\text{rad}}$  as  $\epsilon \rightarrow 0+$  (see for e.g (2.50)). Since (1.15) holds with  $\gamma = \gamma_{cs}$  by (2.34), by interpolation, it suffices to prove Theorem 1.3 for  $\gamma_\epsilon$  with  $\epsilon$  sufficiently small. Applying (2.3), we only need to establish

$$(2.55) \quad \int_{\mathbb{R}^d} \frac{|g(x)|^q}{|x|^{\frac{d-\alpha}{2}-\epsilon}} dx \leq C,$$

for some constant  $C > 0$  independent of  $g$  and for  $\epsilon > 0$  small enough. Applying (2.48) in Lemma 2.5, we only need to show

$$(2.56) \quad \int_{|x|>1} \frac{|g(x)|^q}{|x|^{\frac{d-\alpha}{2}-\epsilon}} dx \leq C.$$

If  $q < \frac{dp}{d-sp}$ , then  $q < \gamma_{cs}$  given by (2.32) since  $\mathcal{D} < 0$  and  $sp < d$ . Similar to (2.54), we obtain (2.56). If  $\frac{dp}{d-sp} \leq q$  then it follows, since  $p + q(sp - 1) > 0$ , that

$$\frac{1}{p} - s < \frac{1}{q} \leq \frac{1}{p} - \frac{s}{d}.$$

Applying Lemma 2.3 with  $\gamma = q$  and  $R = 1$ , we have

$$(2.57) \quad \int_{|x|>1} |g(x)|^q dx \leq C.$$

Assertion (2.56) now follows from (2.57) by noting that  $\frac{1}{|x|^{\frac{d-\alpha}{2}-\epsilon}} \leq 1$  for  $|x| > 1$ .

The proof of (2.56) is complete.

• Case 3:  $sp \leq 1$  and  $p + q(sp - 1) \leq 0$ . Since,  $p \geq 1$ , so we must have  $sp < 1$  in this case.

We first consider the case  $p + q(sp - 1) = 0$ . Then  $\gamma_{rad} = q$  by (2.23). Applying Lemma 2.3 and using (2.49), we have

$$(2.58) \quad \int_{|x|>1} |g|^q \leq C.$$

On the other hand, by (2.49),

$$(2.59) \quad \int_{|x|<1} |g|^q dx \leq C.$$

From (2.58) and (2.59), we obtain

$$\int_{\mathbb{R}^d} |g(x)|^q dx \leq C.$$

Since (1.15) holds with  $\gamma = \gamma_{cs}$  by (2.34), by interpolation, the conclusion holds for  $\gamma_{cs} < \gamma < \gamma_{rad} = q$ .

We next deal with the case  $p + q(sp - 1) < 0$ . Set

$$(2.60) \quad \gamma_\epsilon = q + \frac{(q(sp - 1) + p)(d - \alpha + 2\epsilon)}{sp(d + \alpha - 2\epsilon - 2) + (d - \alpha + 2\epsilon)},$$

for  $\epsilon$  sufficiently small. Since  $p + q(sp - 1) < 0$ , it follows from (2.39) that  $\gamma_\epsilon \rightarrow \gamma_{rad}$  as  $\epsilon \rightarrow 0+$ . Since (1.15) holds with  $\gamma = \gamma_{cs}$  by (2.34), by interpolation, it suffices to prove Theorem 1.3 for  $\gamma_\epsilon$ , i.e., to prove

$$\int_{\mathbb{R}^d} |g(x)|^{\gamma_\epsilon} dx < C,$$

for  $\epsilon$  sufficiently small.

We have, by (2.49),

$$\int_{|x|<1} |g(x)|^q dx \leq C.$$

Since  $\gamma_\epsilon < \gamma_{rad} \leq q$  thanks to the fact  $p + q(sp - 1) \leq 0$ , it follows from Hölder's inequality that

$$\int_{|x|<1} |g(x)|^{\gamma_\epsilon} dx \leq C.$$



It remains to prove that

$$\int_{|x|>1} |g(x)|^{\gamma_\varepsilon} dx \leq C.$$

Set

$$(2.61) \quad r = \frac{p}{1-sp}.$$

We have, by (2.23),

$$\begin{aligned} \gamma_{\text{rad}} - r &= q + \frac{(q(sp-1) + p)(d-\alpha)}{sp(d+\alpha-2) + (d-\alpha)} - \frac{p}{1-sp} \\ &= (q(sp-1) + p) \left( \frac{d-\alpha}{sp(d+\alpha-2) + (d-\alpha)} - \frac{1}{1-sp} \right) \\ &= -\frac{2(q(sp-1) + p)sp(d-1)}{(sp(d+\alpha-2) + (d-\alpha))(1-sp)}. \end{aligned}$$

Since  $p + q(sp-1) < 0$ , it follows that

$$(2.62) \quad r < \gamma_{\text{rad}} \stackrel{(2.23)}{<} q.$$

For  $\varepsilon > 0$  sufficiently small, set

$$(2.63) \quad \eta = \eta_\varepsilon := \frac{d-\alpha}{2} + \varepsilon$$

and let  $\theta = \theta_\varepsilon \in (0, 1)$  be such that

$$\frac{\theta}{r} + \frac{1-\theta}{q} = \frac{1}{\gamma_\varepsilon}$$

(the existence of  $\theta$  follows from (2.62)). One has

$$(2.64) \quad \theta = \frac{q - \gamma_\varepsilon}{q - r} \frac{r}{\gamma_\varepsilon} \quad \text{and} \quad 1 - \theta = \frac{(\gamma_\varepsilon - r)q}{(q - r)\gamma_\varepsilon}.$$

We have, by Hölder's inequality,

$$(2.65) \quad \begin{aligned} \int_{|x|>1} |g(x)|^{\gamma_\varepsilon} dx &= \int_{|x|>1} \left( |g(x)|^{\theta\gamma_\varepsilon} |x|^{\frac{\eta\gamma_\varepsilon(1-\theta)}{q}} \right) \left( |g(x)|^{(1-\theta)\gamma_\varepsilon} \frac{1}{|x|^{\frac{\eta\gamma_\varepsilon(1-\theta)}{q}}} \right) dx \\ &\leq \left( \int_{|x|>1} |g(x)|^r |x|^{\frac{r\eta(1-\theta)}{\theta q}} dx \right)^{\frac{\gamma_\varepsilon\theta}{r}} \left( \int_{|x|>1} \frac{|g|^q}{|x|^\eta} dx \right)^{\frac{(1-\theta)\gamma_\varepsilon}{q}}, \end{aligned}$$

which yields, by (2.63), (2.47), and (2.49)

$$(2.66) \quad \int_{|x|>1} |g(x)|^{\gamma_\varepsilon} dx \leq C \left( \int_{|x|>1} |g(x)|^r |x|^{\frac{r\eta(1-\theta)}{\theta q}} dx \right)^{\frac{\gamma_\varepsilon\theta}{r}} = C \left( \int_{|x|>1} |g|^r |x|^{-r\beta} dx \right)^{\frac{\gamma_\varepsilon\theta\varepsilon}{r}},$$

by (2.64), where

$$(2.67) \quad \beta = \beta_\varepsilon := -\frac{(\gamma_\varepsilon - r)\eta}{(q - \gamma_\varepsilon)r}$$

We claim that

$$(2.68) \quad \beta = -(d-1)s.$$

Since, by (2.61),

$$q(sp - 1) + p = (q - r)(sp - 1),$$

it follows from (2.60) that

$$(2.69) \quad \gamma_\varepsilon = q + \frac{(q - r)(d - \alpha + 2\varepsilon)(sp - 1)}{sp(d + \alpha - 2\varepsilon - 2) + (d - \alpha + 2\varepsilon)}.$$

This implies

$$\gamma_\varepsilon - r = (q - r) \left( 1 + \frac{(d - \alpha + 2\varepsilon)(sp - 1)}{sp(d + \alpha - 2\varepsilon - 2) + (d - \alpha + 2\varepsilon)} \right),$$

which yields

$$(2.70) \quad \gamma_\varepsilon - r = \frac{2(q - r)sp(d - 1)}{sp(d + \alpha - 2\varepsilon - 2) + (d - \alpha + 2\varepsilon)}.$$

We also have, by (2.69)

$$(2.71) \quad q - \gamma_\varepsilon = -\frac{(q - r)(sp - 1)(d - \alpha + 2\varepsilon)}{sp(d + \alpha - 2\varepsilon - 2) + (d - \alpha + 2\varepsilon)}.$$

Combining (2.61), (2.70), and (2.71) yields

$$\beta_\varepsilon = -\frac{(\gamma_\varepsilon - r)\eta_\varepsilon}{(q - \gamma_\varepsilon)r} = -(d - 1)s.$$

Applying Lemma 2.2 with  $\beta = -(d - 1)s$  and using (2.49) and (2.66), we obtain

$$\int_{|x|>1} |g|^{\gamma_\varepsilon} dx \leq C.$$

The proof is complete. □

2.2.3. *Proof of Theorem 1.3 when  $\gamma$  varies in the Range C.* Let  $g \in C_{c,\text{rad}}^1(\mathbb{R}^d)$ . Since  $p(d + \alpha) - 2q(d - sp) \neq 0$  and (1.15) is invariant under scaling, without loss of generality we can assume that

$$(2.72) \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p}{|x - y|^{d+sp}} dx dy = 1 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x)|^q |g(y)|^q}{|x - y|^{d-\alpha}} dx dy.$$

We first note that  $\gamma = q$  in this case. Applying Lemma 2.3 and using (2.72), we have

$$(2.73) \quad \int_{|x|>1} |g|^q \leq C.$$

On the other hand, by (2.72),

$$(2.74) \quad \int_{|x|<1} |g|^q dx \leq C.$$

The conclusion follows from (2.73) and (2.74). □

## 3. OPTIMALITY OF THE RANGE OF PARAMETERS

In this section, we prove Theorem 1.4 and Theorem 1.5. Let  $\eta \in C_c^\infty(\mathbb{R} \setminus \{0\})$  be a nontrivial, nonnegative function with  $\text{spt } \eta \subset (-1, 1)$ . Define

$$(3.1) \quad g_{\lambda,R,S}(x) := \lambda\eta\left(\frac{|x| - R}{S}\right), \forall R > S > 0 \text{ and } \lambda > 0.$$

Applying [5, Theorem 1], we have

$$(3.2) \quad \|g\|_{\dot{W}^{s,p}(\mathbb{R}^d)} \lesssim \|\nabla g\|_{L^p(\mathbb{R}^d)} + \|g\|_{L^p(\mathbb{R}^d)} \quad \forall g \in C_c^\infty(\mathbb{R}^d).$$

Here and in what follows in this section,  $a \lesssim b$  means that  $a \leq Cb$  for some positive constant  $C$  depending only on the parameters in Theorem 1.4 and Theorem 1.5.

By a scaling argument, we derive from (3.2) that

$$(3.3) \quad \|g\|_{\dot{W}^{s,p}(\mathbb{R}^d)} \lesssim \|g\|_{\dot{W}^{1,p}(\mathbb{R}^d)}^s \|g\|_{L^p(\mathbb{R}^d)}^{1-s}, \quad \forall g \in C_c^\infty(\mathbb{R}^d).$$

Using (3.3) we can estimate

$$(3.4) \quad \|g_{\lambda,R,S}\|_{\dot{W}^{s,p}(\mathbb{R}^d)}^p \lesssim \lambda^p R^{d-1} S^{1-sp}, \text{ for } 0 < s < 1 \text{ and } 1 \leq p < +\infty.$$

On the other hand, one has, see e.g., [2, (5.8)],

$$(3.5) \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g_{\lambda,R,S}(x)|^q |g_{\lambda,R,S}(y)|^q}{|x-y|^{d-\alpha}} dx dy \lesssim \begin{cases} \lambda^{2q} R^{d+\alpha-2} S^2 & \text{if } 1 < \alpha < d, \\ \lambda^{2q} R^{d-1} S^2 \log(R/S) & \text{if } \alpha = 1, \\ \lambda^{2q} R^{d-1} S^{1+\alpha} & \text{if } 0 < \alpha < 1. \end{cases}$$

A straightforward computation gives

$$(3.6) \quad \lambda^\gamma (R-S)^{d-1} S \lesssim \|g_{\lambda,R,S}\|_{L^\gamma(\mathbb{R}^d)}^\gamma.$$

We are now ready to give the proofs of Theorem 1.4 and Theorem 1.5. We will only focus on the case  $0 < s < 1$ . The cases  $s = 1$  and  $s = 0$  are either trivial or follow by the same arguments.

**3.1. Proof of Theorem 1.4.** We establish the conclusion by considering two cases separately:

- *Case 1:*  $\beta_1\gamma + \frac{d+\alpha-2}{d-1}\beta_2\gamma < 1$ .
- *Case 2:*  $\beta_1\gamma + \frac{d+\alpha-2}{d-1}\beta_2\gamma = 1$  and  $q(sp-1) + p \neq 0$ .

*Case 1:*  $\beta_1\gamma + \frac{d+\alpha-2}{d-1}\beta_2\gamma < 1$ . In this case, if (1.15) holds, then we can apply it to  $g = g_{1,R,1}$  for large  $R$ . Therefore, using estimates (3.4), (3.5), (3.6), and the inequality (1.15) we have, for large  $R$ ,

$$(3.7) \quad R^{d-1} \lesssim R^{(d-1)\beta_1\gamma + (d+\alpha-2)\beta_2\gamma}$$

Taking  $R \rightarrow \infty$  in (3.7), we obtain a contradiction.

*Case 2:*  $\beta_1\gamma + \frac{d+\alpha-2}{d-1}\beta_2\gamma = 1$  and  $q(sp-1) + p \neq 0$ .

Since, in addition,  $\gamma$  satisfy (1.1) so we can compute (see for e.g. (2.27))

$$(3.8) \quad \gamma = q + \frac{(q(sp-1) + p)(d-\alpha)}{sp(d+\alpha-2) + (d-\alpha)}.$$

If (1.15) holds, then in particular it holds for

$$(3.9) \quad g = g_{m,R} = \sum_{k=1}^m g_{R^{k\xi_2}, R^k, R^{k\xi_1}}, \quad \forall R > 0 \text{ and } m \in \mathbb{N},$$

where  $\xi_1 \neq 1$  and  $\xi_2 \in \mathbb{R}$  to be chosen subsequently. If  $\xi_1 > 1$ , then we start with a small enough  $0 < R < 1$  and if  $\xi_1 < 1$ , then we start with a large enough  $R \gg 1$  so that the supports of  $g_{R^{k\xi_2}, R^k, R^{k\xi_1}}$  remain disjoint for all  $k \in \mathbb{N}$ . With this choice and with the help of (3.4), (3.5), (3.6), and the inequality (1.15), we derive that

$$\begin{aligned} \sum_{k=1}^m R^{k\gamma\xi_2} \left( R^k - R^{k\xi_1} \right)^{d-1} R^{k\xi_1} \\ \lesssim \left( \sum_{k=1}^m R^{kp\xi_2} R^{k(d-1)} R^{(1-sp)k\xi_1} \right)^{\beta_1\gamma} \left( \sum_{k=1}^m R^{2qk\xi_2} R^{k(d+\alpha-2)} R^{2k\xi_1} \right)^{\beta_2\gamma}. \end{aligned}$$

Simplifying this estimate, we have

$$\begin{aligned} (3.10) \quad \sum_{k=1}^m R^{k(\gamma\xi_2+\xi_1+d-1)} \left( 1 - R^{k(\xi_1-1)} \right)^{d-1} \\ \lesssim \left( \sum_{k=1}^m R^{k(p\xi_2+(1-sp)\xi_1+d-1)} \right)^{\beta_1\gamma} \left( \sum_{k=1}^m R^{k(2q\xi_2+2\xi_1+d+\alpha-2)} \right)^{\beta_2\gamma}. \end{aligned}$$

We now choose  $\xi_1$  and  $\xi_2$  such that

$$(3.11) \quad p\xi_2 + (1-sp)\xi_1 + (d-1) = 0 = 2q\xi_2 + 2\xi_1 + (d+\alpha-2).$$

Since  $q(sp-1) + p \neq 0$ , the exact expressions of  $\xi_1$  and  $\xi_2$  can be computed from (3.11) as follows

$$(3.12) \quad \xi_1 = \frac{2q(d-1) - p(d+\alpha-2)}{2(q(sp-1)+p)} \quad \text{and} \quad \xi_2 = -\frac{sp(d+\alpha-2) + (d-\alpha)}{2(q(sp-1)+p)}.$$

Notice that using (3.8), we can simplify the expression of  $\xi_2$  as follows

$$(3.13) \quad \xi_2 = -\frac{d-\alpha}{2(\gamma-q)}.$$

Making use of (3.13) and (3.11) we calculate

$$\begin{aligned} (3.14) \quad \gamma\xi_2 + \xi_1 + (d-1) &= (\gamma-q)\xi_2 + q\xi_2 + \xi_1 + (d-1) \\ &= -\frac{d-\alpha}{2} - \frac{d+\alpha-2}{2} + (d-1) = 0 \end{aligned}$$

Plugging (3.11) and (3.14) into (3.10) we get

$$(3.15) \quad \sum_{k=1}^m \left( 1 - R^{k(\xi_1-1)} \right)^{d-1} \lesssim m^{\beta_1\gamma+\beta_2\gamma}, \quad \forall m \in \mathbb{N}.$$

Since  $\beta_1\gamma + \frac{d+\alpha-2}{d-1}\beta_2\gamma = 1$  and  $1 < \alpha < d$ , it follows that  $\beta_1\gamma + \beta_2\gamma < 1$ . Taking  $R \rightarrow +\infty$  in (3.15) if  $\xi_1 < 1$  and taking  $R \rightarrow 0+$  in (3.15) if  $\xi_1 > 1$ , we obtain a contradiction.

It remains to show that  $\xi_1 \neq 1$ . Indeed, this follows from the assumption  $p(d+\alpha) - 2q(d-sp) \neq 0$  and (3.12), as  $\xi_1$  can be rewritten as  $\xi_1 = 1 - \frac{p(d+\alpha)-2q(d-sp)}{2(q(sp-1)+p)}$ .

The proof is complete.  $\square$

**3.2. Proof of Theorem 1.5.** We prove the result by contradiction. Assume (1.15). Consider the family of radial functions  $g_{1,R,1}$  for  $R > 1$  (see (3.1) for the definition). Using (3.4), (3.5), and (3.6), and applying (1.15) to  $g_{1,R,1}$ , we obtain

$$(3.16) \quad R^{d-1} \lesssim \begin{cases} R^{(d-1)(\beta_1\gamma+\beta_2\gamma)}, & \text{if } 0 < \alpha < 1, \\ R^{(d-1)(\beta_1\gamma+\beta_2\gamma)} ((\log(R))^{\beta_2\gamma} + 1), & \text{if } \alpha = 1. \end{cases}$$

Taking  $R \rightarrow +\infty$  in (3.16) and noticing  $\beta_1\gamma + \beta_2\gamma < 1$ , we obtain a contradiction. The conclusion follows.  $\square$

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