Group Equivariant Networks Using Morphological Operators

Valentin Penaud–Polge, Santiago Velasco-Forero, Jesus Angulo-Lopez

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Valentin Penaud--Polge, Santiago Velasco-Forero and Jesus Angulo
Center for Mathematical Morphology
Mines Paris, PSL University
Fontainebleau
{valentin.penaud_polge, santiago.velasco, jesus.angulo}@minesparis.psl.eu

ABSTRACT

With the increase of interest upon rotation invariance and equivariance for Convolutional Neural Network (CNN), a fair amount of papers have been published on the subject and the literature keeps increasing. This paper aims to fill the lack of morphological approaches on the matter. We propose a set of group equivariant layers using morphological operators, several model configurations are tested and compared with a convolutional equivalent network. The results show that the proposed morphological networks are capable of classifying rotated images even when trained only with upright samples.

***K*eywords** Mathematical Morphology · Deep Learning · Equivariant Network

1 Introduction

In the intense interest towards deep learning in the computer vision community and the plethora of papers coming with it, one particular contribution remains the cornerstone of most proposed methods: the convolutional layer. Through this contribution in [1], LeCun *et al.* have shown that using translation equivariant transformations offers great improvements to neural networks for image processing and analysis. This powerful property allows to process in the same way (or at least predictively) an object in an image independently of its position. Yet, other types of fluctuation may arise in the pose of an object in an image: orientations or scales, for example, may also change between two images or between two parts of a same image. Equivariance or invariance to certain transformations can also be beneficial to process objects presenting symmetries. A popular method generalizing regular convolution to group convolution have shown interesting capacities [2]. Even though a theory of Group Morphology has been proposed before the deep learning era [3] and the idea of using group equivariant morphological operators in deep learning has been suggested by Angulo [4], there have not been any morphological version

of the *Group Equivariant Convolutional Network* (GECN) [2]. We propose in this paper to combine the theory of Group Morphology with GECN to fill this gap. The paper is organized in the following way. Section 2 recalls definitions and mathematical results of the fields of group action and mathematical morphology. Section 3 reviews the most related work concerning group morphology and group equivariant neural networks. Then section 4 introduces the contribution of the paper: sixteen group equivariant layers based on mathematical morphology operators. Section 5 describes the preliminary results obtained with the proposed layers on the Fashion-MNIST dataset. Finally section 6 concludes the paper and gives future perspectives.

2 Mathematical Preliminaries

2.1 Group Action

This section begins by giving a general definition of the notion of equivariance and in-variance. We provide definitions inspired by the ones in [5] and slightly modified. We will consider that a transformation is a bijective map between a set and itself. In this case, we say that the transformation acts on the set. For any set X, we will denote by $\mathcal{T}(X)$, the set of all transformations acting on X. In the following, $\mathcal{F}(X, \mathbb{R})$ will denote the space of maps from X to $\mathbb{R}$.

Definition 1 (Equivariant and Invariant Mapping to a Transformation) *Given a set X and a transformation $\pi \in \mathcal{T}(X)$ acting on it ($\pi : X \to X$), π can also act on $\mathcal{F}(X, \mathbb{R})$ through an induced transformation $\Pi : \mathcal{F}(X, \mathbb{R}) \to \mathcal{F}(X, \mathbb{R})$ defined as follows :*

$$\forall f \in \mathcal{F}(X, \mathbb{R}), \forall x \in X, \Pi(f)(x) := f\left(\pi^{-1}(x)\right).$$

Given two sets X, Y, a map $\Phi : \mathcal{F}(X, \mathbb{R}) \to \mathcal{F}(Y, \mathbb{R})$, two transformations $\pi \in \mathcal{T}(X)$ and $\psi \in \mathcal{T}(Y)$ acting respectively on X and Y and their induced transformations $\Pi \in \mathcal{T}(\mathcal{F}(X, \mathbb{R}))$ and $\Psi \in \mathcal{T}(\mathcal{F}(Y, \mathbb{R}))$:

- *Φ is said to be (Π, Ψ)-equivariant if and only if*

$$\forall f \in \mathcal{F}(X, \mathbb{R}),\ \Phi(\Pi(f)) = \Psi(\Phi(f)).$$

  When the context is clear and without doubts on the nature of Ψ, we can abusively say that Φ is Π-equivariant.

- *Φ is said to be Π-invariant if and only if it is ($\Pi, id_{\mathcal{F}(Y,\mathbb{R})}$)-equivariant:*

$$\forall f \in \mathcal{F}(X, \mathbb{R}),\ \Phi(\Pi(f)) = \Phi(f).$$

When considering specific kinds of transformations, e.g. rotations or translations, one can sometimes benefit from a group structure if the transformations behave accordingly. The scope of this paper is limited to such situations.

Definition 2 (Group Actions) *Given a set X and a group $(\mathcal{G}, \cdot)$, $\mathcal{G}$ is acting on X if it exists a subgroup $\mathcal{T}_{\mathcal{G}}(X)$ of $(\mathcal{T}(X), \circ)$, where $\circ$ is map composition, and a homomorphic mapping*

$$\begin{array}{c} T : (\mathcal{G}, \cdot) \to (\mathcal{T}_{\mathcal{G}}(X), \circ) \\ g \mapsto T_g \end{array}.$$

Such transformations T_g are called $\mathcal{G}$-actions. It can be noted that if a group $\mathcal{G}$ acts on X, then $\mathcal{G}$ acts on $\mathcal{F}(X, \mathbb{R})$ through the induced transformations from $\mathcal{T}_{\mathcal{G}}(X)$ with the induced $\mathcal{G}$-actions defined as:

$$\forall f \in \mathcal{F}(X, \mathbb{R}), \forall x \in X, \mathbb{T}_g(f)(x) = f\left(T_g^{-1}(x)\right) = f\left(T_{g^{-1}}(x)\right)$$

Definition 3 (Group Equivariant and Group Invariant Mapping) *Given a group $\mathcal{G}$ acting on two sets X and Y with $\mathcal{G}$-actions $\mathcal{T}_{\mathcal{G}}(X) = \{T_g^X \in \mathcal{T}(X) \mid g \in \mathcal{G}\}$ and $\mathcal{T}_{\mathcal{G}}(Y) = \{T_g^Y \in \mathcal{T}(Y) \mid g \in \mathcal{G}\}$. Let us consider the induced $\mathcal{G}$-actions on $\mathcal{F}(X, \mathbb{R})$ and $\mathcal{F}(Y, \mathbb{R})$, denoted as $\mathbb{T}_g^X$ and $\mathbb{T}_g^Y$ for all $g \in \mathcal{G}$. Given a mapping $\Phi : \mathcal{F}(X, \mathbb{R}) \to \mathcal{F}(Y, \mathbb{R})$,*

- *Φ is said to be $\mathcal{G}$-equivariant (or equivariant to the action of $\mathcal{G}$) if for all $g \in \mathcal{G}$, Φ is $(\mathbb{T}_g^X, \mathbb{T}_g^Y)$-equivariant.*
- *Φ is said to be $\mathcal{G}$-invariant (or invariant to the action of $\mathcal{G}$) if for all $g \in \mathcal{G}$, Φ is $\mathbb{T}_g^X$-invariant.*

Definition 4 (Transitivity and Homogeneous Space) *Given a set X and group $\mathcal{G}$, $\mathcal{G}$ is said to* act transitively *on X if for all $x, y \in X$ it exists a group element $g \in \mathcal{G}$ such that $y = T_g(x)$. In this case, X is said to be a homogeneous space of $\mathcal{G}$. The existence of the group element g is not necessarily unique. If a point x_o is chosen to be the origin of the space X, the set of group elements mapping the origin to itself is called the stabilizer of x_o and will be written Σ_{x_o} and by an abuse of notation, Σ will be used.*

Remark 1 *The stabilizer of the origin Σ is a subgroup of $\mathcal{G}$. For every element $y \in X$, if X is a homogeneous space of $\mathcal{G}$, there is at least one $g \in \mathcal{G}$ such that $y = T_g(x_o)$. It can be shown that every other group element mapping x_o to y is an element of the left-coset $g\Sigma = \{gh \in \mathcal{G} \mid h \in \Sigma\}$. The set of all the left-cosets, called the left quotient-space $\mathcal{G}/\Sigma$, is then isomorphic to X.*

Example 1 *Considering $X = \mathbb{R}^2$ and the group of 2-dimensional translations and rotations $\mathcal{G} = \mathbb{R}^2 \rtimes [-\pi, \pi]$. $\mathcal{G}$ acts on X with the following group action:*

$$\forall (\tau, \theta) \in \mathcal{G}, \forall x \in \mathbb{R}^2 \mid T_{(\tau,\theta)}(x) = R_\theta x + \tau,$$

where R_θ is a rotation matrix. The stabilizer $\Sigma = \{(0, \theta) \in \mathcal{G} \mid \theta \in [-\pi, \pi]\}$ when setting $x_o = (0, 0)$.

In practice, to obtain group equivariance, a commonly used strategy is to lift the input image to the desired group, apply an operator or a filter defined on the group and project it back on the original space. For a map $f : X \to \mathbb{R}$, the lifting of f to $\mathcal{G}$ is defined as follow:

$$\forall g \in \mathcal{G},\ f^{X \to \mathcal{G}}(g) := f(T_g(x_o)). \tag{1}$$

One possibility proposed in [5], in the case of discrete groups, for the projection of a map $f : \mathcal{G} \to \mathbb{R}$ is the following one

$$\forall x \in X,\ \exists g_x \in \mathcal{G} \text{ such that } x = T_g(x_o),\ \text{then } f_{\mathcal{G} \to X}(x) = \frac{1}{\#(\Sigma)} \sum_{h \in g_x \Sigma} f(h), \tag{2}$$

where $\#$ gives the cardinality of a set.

2.2 Basis of Mathematical Morphology

This section aims to recall the theoretical foundation of Mathematical Morphology (MM) in order to enhance the self-contained nature of the paper. The subsequent content is strongly inspired by [6] and interested readers are invited to consult this reference for a more detailed overview of MM. Originally defined using set theory to process and analyse binary images, the main idea of MM was to use probes, defined as sets and called *Structuring Elements*, to describe and modify binary shapes. Two operators called *Dilation* and *Erosion* based on the Minkowski addition and subtraction are essential to this end. Given an Euclidean space $\mathcal{E}$ and two subsets X and B of $\mathcal{E}$, where X plays the role of the image/shape to study and B is the structuring element, the dilation and erosion of X using B are defined by the following equations

$$\delta_B(X) := X \oplus B = \cup_{x \in X} B_x = \cup_{b \in B} X_b. \tag{3}$$

$$\varepsilon_B(X) := X \ominus B = \{y \in \mathcal{E} \,|B_y \in X\} = \cap_{b \in B} X_{-b}. \tag{4}$$

For any $B \subset \mathcal{E}$ and any $x \in \mathcal{E}$, the notation B_x denotes the set $\{x + b \in \mathcal{E} \mid b \in B\}$.

MM has then been generalized to other mathematical objects of various natures (e.g. bounded functions on Euclidean spaces [7] or Riemannian manifolds [8], graphs [9], etc) under a common framework of complete lattices defined as follow,

Definition 5 (Complete Lattice) *A* complete lattice *is defined to be a set $\mathcal{L}$ equipped with a partial ordering $\leq$ satisfying the following properties: for any family of elements $(X_i)_{i \in I} \in \mathcal{L}^I$, it exists a supremum $\bigvee_{i \in I} X_i$ and an infimum $\bigwedge_{i \in I} X_i$ such that:*

- $\forall i \in I,\ \bigwedge_{i \in I} X_i \leq X_i \leq \bigvee_{i \in I} X_i$
- $\forall Y \in \mathcal{L}\ (\forall i \in I, X_i \leq Y) \implies \left(\bigvee_{i \in I} X_i \leq Y\right)$
- $\forall Y \in \mathcal{L}\ (\forall i \in I, Y \leq X_i) \implies \left(Y \leq \bigwedge_{i \in I} X_i\right)$

Two important elements arise from this definition: the supremum and the infimum of the entire complete lattice. They will be denoted $\bot_{\mathcal{L}}$ for the infimum of $\mathcal{L}$ and $\top_{\mathcal{L}}$ for the supremum of $\mathcal{L}$.

Remark 2 *Using the reverse partial ordering $\geq$, if $(\mathcal{L}, \leq)$ is a complete lattice, then $(\mathcal{L}, \geq)$ is a complete lattice were the supremum (resp. infimum) of $(\mathcal{L}, \geq)$ is the infimum (resp. supremum) of $(\mathcal{L}, \leq)$. The two lattices are said to be dual lattices. This notion of duality is recurrent in MM.*

Under the framework of complete lattices, erosions and dilations are defined to be the mappings commutating with supremums and infimums.

Definition 6 (Erosion and Dilation) *Given a complete lattice $(\mathcal{L}, \leq)$, a map $\delta : \mathcal{L} \to \mathcal{L}$ is called a dilation if for any family of elements $(X_i)_{i \in I} \in \mathcal{L}^I$, $\delta\left(\bigvee_{i \in I} X_i\right) = \bigvee_{i \in I} \delta(X_i)$. In the same manner, a map $\varepsilon : \mathcal{L} \to \mathcal{L}$ is called an erosion if for any family of elements $(X_i)_{i \in I} \in \mathcal{L}^I$, $\varepsilon\left(\bigwedge_{i \in I} X_i\right) = \bigwedge_{i \in I} \varepsilon(X_i)$.*

Following the notations of [6], the set of maps $\mathcal{L}^{\mathcal{L}}$ between a complete lattice and itself will be denoted $\mathcal{O}$.

Remark 3 *The notions of dilation and erosion are dual and it can be shown (Proposition 2.3 of [6]) that the set of dilations and the set of erosions are both complete lattices with the following partial ordering: Given* $\Psi, \Phi \in \mathcal{O}$

$$\Psi \leq \Phi \iff \forall X \in \mathcal{L},\ \Psi(X) \leq \Phi(X).$$

Definition 7 (Adjunction) *Given two mappings* $\delta, \varepsilon \in \mathcal{O}$*, the pair* (ε, δ) *is said to be an adjunction if*

$$\forall X, Y \in \mathcal{L},\ \delta(X) \leq Y \iff X \leq \varepsilon(Y).$$

Proposition 1 *Proposition 2.5 of [6]* Given $\delta, \varepsilon \in \mathcal{O}$, (ε, δ) being an adjunction implies that δ is a dilation and ε is an erosion.

Definition 8 *For any* $\Psi \in \mathcal{O}$*, the two dual mappings* $\Psi^\bullet$ *and* $\Psi_\bullet$ *can be defined in the following way:* $\forall Y \in \mathcal{L}$

$$\Psi^\bullet(Y) := \bigvee\{Z \in \mathcal{L} \mid \Psi(Z) \leq Y\} \text{ and } \Psi_\bullet(Y) := \bigwedge\{Z \in \mathcal{L} \mid Y \leq \Psi(Z)\}.$$

Remark 4 *Heijmans and Ronse showed in Proposition 2.6 of [6] that for any adjunction* $(\varepsilon, \delta) \in \mathcal{O}^2$*, the following relations between* ε *and* δ *holds:*

$$\varepsilon = \delta^\bullet \text{ and } \delta = \varepsilon_\bullet. \tag{5}$$

This remark goes even further with the following theorem

Theorem 1 (Theorem 2.7 of [6]) *The set of adjunctions forms a dual isomorphism between the complete lattice of erosions and the complete lattice of dilations. Meaning that for every dilation* δ *(resp. erosion* ε*), it exists a unique erosion* ε *(resp. dilation* δ*) such that* (ε, δ) *is an adjunction and thus respects* (5)*.*

The concepts of dilations, erosions and more generally the concept of adjunction can be generalized to mappings between two distinct complete lattices $\mathcal{L}_1$ and $\mathcal{L}_2$. In this case, it is possible to have an adjunction (ε, δ) such that $\delta : \mathcal{L}_1 \to \mathcal{L}_2$ and $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$.

Definition 9 (Sup-Generating family) *Given a complete lattice* $\mathcal{L}$ *and a subset* $l \subset \mathcal{L}$*,* l *is said to be sup-generating if every element* $X \in \mathcal{L}$ *can be associated with a subset of* l*, written* $l(X)$*, defined as* $l(X) = \{x \in l \mid x \leq X\}$ *and satisfying* $X = \bigvee l(X)$*. In other words, every element of the complete lattice can be expressed as the supremum of a collection of elements of* l*.*

When studying gray level images, it comes naturally to consider the image as functions to real numbers, maybe bounded [10], on a Euclidean space. More generally, a gray level image can be seen as a map between a set (or a Euclidean space) $\mathcal{E}$ and a complete lattice $\mathcal{L}$. The space $\mathcal{F}(\mathcal{E}, \mathcal{L})$ of such maps is again a complete lattice where the partial ordering, the infimum and the supremum are defined using the partial ordering, the infimum and the supremum of $\mathcal{L}$: Considering any collection $(f_i)_{i \in I} \in \mathcal{F}(\mathcal{E}, \mathcal{L})^I$

- $\forall f, g \in \mathcal{F}(\mathcal{E}, \mathcal{L}),\ f \leq g \iff \forall x \in \mathcal{E},\ f(x) \leq g(x)$
- $\forall x \in \mathcal{E}, \left(\bigvee_{i \in I} f_i\right)(x) = \bigvee_{i \in I} f_i(x)$ and $\left(\bigwedge_{i \in I} f_i\right)(x) = \bigwedge_{i \in I} f_i(x)$

We will consider two sets $\mathcal{E}_1$ and $\mathcal{E}_2$ and a complete lattice $\mathcal{L}$, the notion of generating family can be useful to describe erosions and dilations on the complete lattices $\mathcal{F}(\mathcal{E}_1, \mathcal{L})$ and $\mathcal{F}(\mathcal{E}_2, \mathcal{L})$ by looking at the dilations (or erosions) of the elements of the generating family. More precisely, we will consider a dilation $\delta : \mathcal{F}(\mathcal{E}_1, \mathcal{L}) \to \mathcal{F}(\mathcal{E}_2, \mathcal{L})$ and for the complete lattice $\mathcal{F}(\mathcal{E}_1, \mathcal{L})$ we will consider the sup-generating family $l_{\mathcal{F}(\mathcal{E}_1,\mathcal{L})} = \{f_{x,t} \in \mathcal{F}(\mathcal{E}_1, \mathcal{L}) \mid x \in \mathcal{E}_1,\ t \in \mathcal{L}\}$ with

$$\forall y \in \mathcal{E}_1,\ f_{x,t}(y) = \begin{cases} t, & \text{if } y = x \\ \bot_{\mathcal{L}}, & \text{if } y \neq x \end{cases},$$

as for any $F \in \mathcal{F}(\mathcal{E}_1, \mathcal{L})$, $F = \bigvee_{x \in \mathcal{E}_1} f_{x,F(x)}$. In a similar way that a sup generating family can be used to describe an element of the complete lattice, a dilation can be described by its effect on the sup generating family. The set of mappings $\delta_{y,x} : \mathcal{L} \to \mathcal{L}$ are defined using the elements of $l_{\mathcal{F}(\mathcal{E}_1,\mathcal{L})}$ and δ:

$$\forall x \in \mathcal{E}_1,\ \forall y \in \mathcal{E}_2,\ \forall t \in \mathcal{L},\ \delta_{y,x}(t) = \delta(f_{x,t})(y)$$

Using the assumption that δ is a dilation, it can be shown that $\delta_{y,x}$ is a dilation for all $x \in \mathcal{E}_1,\ y \in \mathcal{E}_2$. The dilations $\delta_{y,x}$ define the dilation δ which gives the following proposition.

Proposition 2 (Proposition 2.10 of [6]) *Given two sets $\mathcal{E}_1$, $\mathcal{E}_2$ and a complete lattice $\mathcal{L}$. The map $\delta : \mathcal{F}(\mathcal{E}_1, \mathcal{L}) \to \mathcal{F}(\mathcal{E}_2, \mathcal{L})$ is a dilation if and only if for every $x \in \mathcal{E}_1$ and $y \in \mathcal{E}_2$, there exists a dilation $\delta_{y,x} : \mathcal{L} \to \mathcal{L}$ such that for all $F \in \mathcal{F}(\mathcal{E}_1, \mathcal{L})$ and for all $y \in \mathcal{E}_2$, it holds*

$$\delta(F)(y) = \bigvee_{x \in \mathcal{E}_1} \delta_{y,x}(F(x)).$$

The adjoint erosion $\varepsilon : \mathcal{F}(\mathcal{E}_2, \mathcal{L}) \to \mathcal{F}(\mathcal{E}_1, \mathcal{L})$ is such that, for all $F \in \mathcal{F}(\mathcal{E}_2, \mathcal{L})$ and for all $x \in \mathcal{E}_1$:

$$\varepsilon(F)(x) = \bigwedge_{y \in \mathcal{E}_2} \varepsilon_{x,y}(F(y)),$$

with every $\varepsilon_{x,y}$ being the adjoint of $\delta_{y,x}$ for $\mathcal{L}$.

To make the transition to group morphology, Heijmans and Ronse [6] pointed out that by taking $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}^d$, $\mathcal{L} = \mathbb{R}$ and by imposing $\delta_{y,x} = \delta_{y-x,0}$ to respect translation equivariance, then, using a structuring function G, one can choose to consider

$$\forall h \in \mathbb{R}^d, \forall t \in \mathbb{R},\ \delta_{h,0}(t) = t + G(h). \tag{6}$$

3 Related Work

3.1 Group Morphology

Under the name of invariance, equivariance to translation has always been a concern in MM. As MM is usually applied to image analysis and processing using "small" structuring elements running spatially through the entire image of study, it is natural to be concerned with translation equivariance. Some authors extended MM to general groups [3] or gave fairly general results applicable to any group when studying translations equivariance [6]. This subsection focuses on the main results of these two references. The first and intuitive approach was to generalize the Minkowski addition and subtraction. Roerdink [3] proposed such a generalization for groups that are not necessarily Abelian. Given a group

$\mathcal{G}$ and two of its subsets H and G, Roerdink defined the left-equivariant erosion and left-equivariant dilation on the complete lattice $\mathcal{P}(\mathcal{G})$ as follows:

$$\delta_H^L(G) := G \oplus_{\mathcal{G}} H = \cup_{h\in H} Gh = \cup_{g\in G} gH, \tag{7}$$

$$\varepsilon_H^L(G) := G \ominus_{\mathcal{G}}^L H = \cap_{h\in H} Gh^{-1} = \{g \in \mathcal{G} \mid gH \subset G\}. \tag{8}$$

The right-equivariant dilation and erosion are defined in the same manner:

$$\delta_H^R(G) := H \oplus_{\mathcal{G}} G = \cup_{h\in H} hG = \cup_{g\in G} Hg, \tag{9}$$

$$\varepsilon_H^R(G) := G \ominus_{\mathcal{G}}^R H = \cap_{h\in H} h^{-1}G = \{g \in \mathcal{G} \mid Hg \subset G\}. \tag{10}$$

In practice, a binary image is defined on a Euclidean space $\mathcal{E}$. In this case, the considered complete lattice is the set of subsets of $\mathcal{E}$, written $\mathcal{P}(\mathcal{E})$ with the inclusion as partial ordering. If the space $\mathcal{E}$ is a homogeneous space of $\mathcal{G}$, then, Roerdink [3] proposed the following lifting ϑ and projections ϖ, ϖ_Σ in order to apply the erosions and dilations defined by (7), (8) or (9), (10).

$$\forall X \in \mathcal{P}(\mathcal{E}),\ \vartheta(X) := \{g \in \mathcal{G} \mid T_g(x_o) \in X\}, \tag{11}$$

$$\forall G \in \mathcal{G},\ \varpi(G) := \{T_g(x_o) \mid g \in G\} \text{ and } \varpi_\Sigma(G) := \{T_g(x_o) \mid g\Sigma \subset G\}, \tag{12}$$

where x_o is a chosen origin for $\mathcal{E}$. Using these maps, Roerdink characterized the adjunctions on $\mathcal{P}(\mathcal{E})$ using (7), (8). They are called left $\mathcal{G}$-adjunctions and they take the following form:

$$\delta_H^{\mathcal{G}}(G) := \varpi(\vartheta(G) \oplus_{\mathcal{G}} \vartheta(H)) \text{ and } \varepsilon_H^{\mathcal{G}}(G) := \varpi_\Sigma\left(\vartheta(G) \ominus_{\mathcal{G}}^L \vartheta(H)\right). \tag{13}$$

The name left $\mathcal{G}$-adjunction comes from the fact that given such an adjunction $\left(\varepsilon_H^{\mathcal{G}}, \delta_H^{\mathcal{G}}\right)$, the dilation $\delta_H^{\mathcal{G}}$ and the erosion $\varepsilon_H^{\mathcal{G}}$ are left $\mathcal{G}$-equivariant. A similar definition can be given for right $\mathcal{G}$-adjunctions using (9) and (10). When the image of study is not boolean, it was proposed to use a sup-generative family as an intermediate step. The principal requirement is that the group $\mathcal{G}$ acts transitively on the sup-generative family, allowing to use the lifting and projections defined by the equations (11), (12) and therefore the $\mathcal{G}$-adjunction $\left(\varepsilon_H^{\mathcal{G}}, \delta_H^{\mathcal{G}}\right)$.

3.2 Group Equivariant Networks

Conceptually there are two different approaches to the construction of invariant and equivariant models [11]: *symmetrization* based one and the *intrinsic* one. In the first case, one starts with an non-invariant model and symmetrizing it by a group averaging. In the second case, the intrinsic approach consists of imposing prior structural constraints on the model that guarantee its invariance.

In Deep Learning, both approaches can justify practices that are common to induce equivariance and invariances. For instance, a) *data augmentation* techniques advocates to augment the available set of training samples $(x, f(x))$ by new ones of the form $(T_g(x), f(x))$, then loss values of theses realizations are average on batches during training [12]. b) *Invariance by regularization* uses differences of $f(x)$ and many realization of $f(T_g(x))$ are computed to regularize the model during training [12, 13, 14]. Both approaches can be seen as exemplifications of the symmetrization-based approach. On the other hand, the *weight sharing* mechanism specially used in convolutional networks [15], and other approaches as PDE based CNNs [16], Elementary Symmetric Polynomials based CNNs [17], Moving Frame based CNNs [18], Deep Scattering CNNs [19], Steerable CNNs [20] and Group Convolutional networks [2, 5] are manifestation of intrinsic approach. The method proposed in [2] will be recalled in more details as

the contribution of the paper uses a similar approach. The main idea is to generalize the convolution to any compact group. In the usual convolution, a filter runs through the entire input image to locally match its pattern with the image, i.e, the filter is translated to every position in the domain of the image. When considering other transformations forming a different group than the translations, Cohen and Welling proposed to apply all the transformations of the group to the filter. For example, if the considered group is composed of all the translations with also a finite number of rotations discretizing the circle (subgroup of the circle). Then the proposition would be to transform the filter using all the possible translations and all the rotations. In practice, two types of layers are proposed: group lifting layers and group convolutional layers. The first one plays the role of (1) with in addition a learnable filtering part. To be more precise, copies of the input image (or tensor) are created, which correspond to (1) and rotated copies of a learned two-dimensional filter are used to perform regular convolutions. This layer can be summarized in one formula. Given a discrete group $\mathcal{G}$, an input image f composed with N feature maps and a kernel ϕ, the lifting layer performs the following formula

$$\forall g \in \mathcal{G},\ \left(f \star^{\mathbb{Z}^2 \to \mathcal{G}} \phi\right)(g) := \sum_{k=1}^{N} \sum_{x \in \mathbb{Z}^2} f_k(x)\, \mathbb{T}_g(\phi_k)(x), \tag{14}$$

followed by a bias addition and an activation function. The output of this operation is a map whose domain is the group $\mathcal{G}$. The group layers, also followed by a bias addition and an activation function, apply the group convolution in the following way

$$\forall g \in \mathcal{G},\ \left(f \star^{\mathcal{G}} \phi\right)(g) := \sum_{k=1}^{N} \sum_{h \in \mathcal{G}} f_k(h)\, \mathbb{T}_g(\phi_k)(h). \tag{15}$$

4 Proposed Layers

This section aims to define group-equivariant morphological layers. These layers will be named after classical morphological operators due to their evident parallels. Nevertheless, it is important to note that no formal proofs are provided regarding their inherent properties. As an illustration, layers labeled as *dilation* can be characterized by a formula closely resembling that of an established dilation operation. We introduce a total of sixteen layers, with eight of them serving as nonlinear counterparts to the lifting layer outlined in (14). The other half are nonlinear counterparts of the group convolutional layer described by (15). We denote by $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ the extended real number line and by $\mathbb{R}_+^*$ the real strictly positive numbers. The proposed layers are based on two types of adjunction on $\bar{\mathbb{R}}$. Given a group $\mathcal{G}$, each element $h \in \mathcal{G}$ gives rise to an adjunction on $\bar{\mathbb{R}}$. Such an adjunction can be denoted by (e_h, d_h). Following the work of Heijmans in [21], we propose to use the following forms for e_h and d_h

$$d_h^\star(t) = A(h)\, t + G(h) \text{ and } e_h^\star(t) = (t - G(h))\, /A(h), \tag{16}$$

where $A(\cdot) : \mathcal{G} \mapsto \mathbb{R}_+^*$ and $G(\cdot) : \mathcal{G} \mapsto \mathbb{R}$ play the role of structuring functions. An easier alternative (see (6) for $\mathcal{G} = \mathbb{R}^d$) consists of using

$$d_h^\dagger(t) = t + G(h) \text{ and } e_h^\dagger(t) = t - G(h). \tag{17}$$

From these adjunctions on $\bar{\mathbb{R}}$, we define the following operators on $\mathcal{G}$. Given $F : \mathcal{G} \to \bar{\mathbb{R}}$,

$$D(F)(g) := \bigvee_{h \in \mathcal{G}} d_h\left(F\left(gh^{-1}\right)\right) \text{ and } E(F)(g) := \bigwedge_{h \in \mathcal{G}} e_h(F(gh)) \tag{18}$$

where the pair (d_g, e_g) can be (16) or (17). We use (18) to propose nonlinear equivariant layers. The eight *lifting* layers can be described by the following formulas. Given a discrete group $\mathcal{G}$, an input image $f : \mathbb{Z}^2 \to \bar{\mathbb{R}}^N$ composed of N feature maps and N adjunctions $(e_{k,x}, d_{k,x})_{k\in[\![1,N]\!]}$ defined for all $x \in \mathbb{Z}^2$ following either equation (16) or (17), then for all $g \in \mathcal{G}$

$$\mathfrak{e}_\uparrow (f)(g) := \sum_{k=1}^{N} \bigwedge_{x\in\mathbb{Z}^2} e_{k,T_{g^{-1}}(x)} (f_k(x)), \tag{19}$$

$$\mathfrak{d}_\uparrow (f)(g) := \sum_{k=1}^{N} \bigvee_{x\in\mathbb{Z}^2} d_{k,-T_{g^{-1}}(x)} (f_k(x)), \tag{20}$$

$$\mathfrak{o}_\uparrow (f)(g) := \sum_{k=1}^{N} \bigvee_{y\in\mathbb{Z}^2} d_{k,-T_{g^{-1}}(y)} \left(\bigwedge_{x\in\mathbb{Z}^2} e_{k,T_{g^{-1}}(x)-T_{g^{-1}}(y)} (f_k(x)) \right), \tag{21}$$

$$\mathfrak{c}_\uparrow (f)(g) := \sum_{k=1}^{N} \bigwedge_{y\in\mathbb{Z}^2} e_{k,T_{g^{-1}}(y)} \left(\bigvee_{x\in\mathbb{Z}^2} d_{k,-T_{g^{-1}}(x)+T_{g^{-1}}(y)} (f_k(x)) \right), \tag{22}$$

where $\mathfrak{e}$, $\mathfrak{d}$, $\mathfrak{o}$ and $\mathfrak{c}$ stand respectively for erosion, dilation, opening and closing. In the same manner, we define nonlinear counterparts of (15). Given a map $F : \mathcal{G} \to \bar{\mathbb{R}}$:

$$\mathfrak{e}_\mathcal{G} (F)(g) := \sum_{k=1}^{N} \bigwedge_{h\in\mathcal{G}} e_{k,h} (F(gh)) \tag{23}$$

$$\mathfrak{d}_\mathcal{G} (F)(g) := \sum_{k=1}^{N} \bigvee_{h\in\mathcal{G}} d_{k,h} \left(F\left(gh^{-1}\right)\right) \tag{24}$$

$$\mathfrak{o}_\mathcal{G} (F)(g) := \sum_{k=1}^{N} \bigvee_{h\in\mathcal{G}} d_{k,h} \left(\bigwedge_{m\in\mathcal{G}} e_{k,m} \left(F\left(gh^{-1}m\right)\right) \right) \tag{25}$$

$$\mathfrak{c}_\mathcal{G} (F)(g) := \sum_{k=1}^{N} \bigwedge_{h\in\mathcal{G}} e_{k,h} \left(\bigvee_{m\in\mathcal{G}} d_{k,m} \left(F\left(ghm^{-1}\right)\right) \right) \tag{26}$$

These layers are followed by a bias addition and an activation function.

5 Experiments

In practice, we used the group $P4$ defined to be the group of translations and rotations of $\frac{\pi}{2}$ [2]. Several configurations[1] have been tested on a classification task using the Fashion-MNIST dataset. The configurations use either the adjunctions defined by equations (16) or (17) and use different types of layers among those presented in section 4. We use the notation † in superscript to denote the use of the

[1] `https://github.com/Penaud-Polge/Group_Equivariant_Morphological_Layers`

Table 1: Test accuracies in percentage of several configurations of $P4$ equivariant morphological networks depending on the rotations applied to the test dataset.

Networks	Acc_0	$Acc_{\frac{\pi}{6}}$	$Acc_{\frac{\pi}{4}}$	$Acc_{\frac{\pi}{2}}$	Acc_{π}	$Acc_{\frac{3\pi}{2}}$
CNN	**87.13** %	**38.41** %	23.30 %	7.28 %	34.76 %	10.24%
CNN P4	80.16 %	23.08 %	27.17 %	**80.16** %	**80.16** %	**80.16** %
$\mathfrak{c}^{\dagger}_{\mathcal{G}}$ $\mathfrak{c}^{\dagger}_{\uparrow}$	62.75 %	17.68 %	17.26 %	62.73 %	62.73 %	62.73 %
$\mathfrak{d}^{\dagger}_{\mathcal{G}}$ $\mathfrak{d}^{\dagger}_{\uparrow}$	64.90 %	26.63 %	22.46 %	64.90 %	64.90 %	64.90 %
$\mathfrak{e}^{\dagger}_{\mathcal{G}}$ $\mathfrak{e}^{\dagger}_{\uparrow}$	69.83 %	25.54 %	23.31 %	69.83 %	69.83 %	69.83 %
$\mathfrak{o}^{\dagger}_{\mathcal{G}}$ $\mathfrak{o}^{\dagger}_{\uparrow}$	70.23 %	30.44 %	27.70 %	70.23 %	70.23 %	70.23 %
$\mathfrak{c}^{\star}_{\mathcal{G}}$ $\mathfrak{c}^{\star}_{\uparrow}$	62.71 %	22.71 %	19.97 %	62.71 %	62.71 %	62.71 %
$\mathfrak{d}^{\star}_{\mathcal{G}}$ $\mathfrak{d}^{\star}_{\uparrow}$	71.64 %	23.67 %	23.89 %	71.64 %	71.64 %	71.64 %
$\mathfrak{e}^{\star}_{\mathcal{G}}$ $\mathfrak{e}^{\star}_{\uparrow}$	76.21 %	21.96 %	15.76 %	76.21 %	76.21 %	76.21 %
$\mathfrak{o}^{\star}_{\mathcal{G}}$ $\mathfrak{o}^{\star}_{\uparrow}$	57.66 %	34.51 %	**33.60** %	57.66 %	57.66 %	57.66 %

adjunctions of (17) and $\star$ to denote the use of the adjunctions of (16). The type of layer used will be given by their names. Therefore, an architecture denoted "$\mathfrak{d}^{\dagger}_{\mathcal{G}}$ $\mathfrak{d}^{\dagger}_{\uparrow}$" stands for a two layer network where the lifting is given by (20) and the group layer, i.e. the second one, is given by (24) and with adjunctions (e_g, d_g) given by (17). For every configuration, a dense layer is used after a global average pooling to predict the class for each orientation. A max-pooling operation is then applied between the different orientations to obtain invariance to rotations of $\frac{\pi}{2}$. All the structuring functions were of size $(3, 3)$ for the lifting layers and of size $(3, 3, 3)$ for the group layers. For all the configurations, the network has been trained using only original (untransformed) images. On the other hand, rotated copies of the test dataset have been used to determine if the networks were able to classify images having orientations unseen during the training process. An equivalent network using group convolution and an equivalent CNN have been used as references for comparison. The results obtained are presented in table 1. The results show that all networks except the regular CNN were able to generalize to unseen orientations of $P4$. All the networks are less efficient when evaluating at an orientation not contained in the group $P4$. Results also highlight the fact that for dilation and erosion networks, using adjunctions defined by equation (16) offers better performances. Using the opening and closing layers seems to lower the performances. And finally, except $\mathfrak{e}^{\star}_{\mathcal{G}}$ $\mathfrak{e}^{\star}_{\uparrow}$ that tends to have performances closer to the convolutional reference, it seems that morphological layers, on this task, are less efficient. It is important to highlight that CNNs have benefited from extensive and in-depth studies by the research community regarding their appropriate optimization, initialization, and other aspects. This is not the case for morphological counterparts [22, 23]. We hope that the presented results related to the morphological layers will improve with future research on the subject.

6 Conclusion

This paper is a first approach to generalize morphological layers to group morphology. Multiple layers, incorporating morphological operators, have been proposed and tested on the Fashion-MNIST dataset. The networks exhibited almost perfect invariance to the chosen group but offered lower performances

compared to their convolutional counterpart. The morphological operators employed in these layers have not been theoretically described yet; therefore, providing a theoretical foundation is an essential aspect of future work.

Acknowledgments

This work was granted access to the HPC resources of IDRIS under the allocation 2023-[AD011013367R1] made by GENCI.

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