Control design for linear systems with asymmetric input backlash and dead-zone through LMI conditions

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Control design for linear systems with asymmetric input backlash and dead-zone through LMI conditions

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***Abstract*— The interconnection between a linear time invariant system and a nonlinear operator gathering an asymmetric backlash and an asymmetric dead-zone is studied, which are relevant in mechanics and hydraulics. This paper aims to design a controller as a static linear state feedback to ensure Uniform Ultimately Bounded (UUB) property for the closed-loop system. A UUB-Lyapunov-based approach is used to provide sufficient conditions as Linear Matrix Inequalities (LMIs) for this control design and a resulting optimization problem is offered to minimize the UUB set. A numerical illustration presents the efficiency of our contribution and related discussions.**

***Index Terms*— Backlash, dead-zone, stability analysis, piecewise quadratic Lyapunov function, uniform ultimate boundedness.**

I. INTRODUCTION

THIS paper is focused on the control design for systems with asymmetric input backlash and dead-zone in order to ensure stability properties. Many mechanical systems, such as hydraulic cylinder, intrinsically exhibit backlash effect [1]–[3]. This memory-based phenomena is a consequence of gaps between parts of the mechanism. When using a hydraulic cylinder to set in motion a mechanical device, dry friction arises and is specially significant at low speed. Dry friction is typically modelled by the dead-zone operator [4], which can be interpreted by the fact that applying small efforts on the device at zero speed may not be enough to set it in motion, because the dry friction opposes to the effort. In addition, some mechanical systems may not have the same behavior depending the direction of motion. For example, this is the case for simple effect cylinder. Therefore it is crucial for some application to deal with these three non-linear phenomena at the same time, leading to a resulting difficult non-linear operator [5]. The main difficulty lies in the non-differentiable nature (only piecewise differentiable) of the non-linear operator [6].

There is a wealth of literature concerning control design of systems presenting hysteresis phenomena. Among them, we can cite the hybrid system approach [7] where the nonlinear operator is modelled as a hybrid dynamical system; the use of linear matrix inequalities and cone-bounded sector conditions [8]; a modified anti-windup approach [9]; or an inverse based model for hysteresis compensation [10].

We consider in this paper a linear dynamical system with an actuator whose behavior is described by the non-linear operator presented above. The control law is assumed to be a linear state feedback controller where the matrix feedback gain needs to be selected. Because of the fact that the dead-zone makes the system uncontrollable near the origin, we propose here a control design approach that ensures the Uniform Ultimately Boundedness (UUB) property of the closed-loop, which guarantees that the state trajectories converge in finite time to a neighborhood of the origin and remain there forever [11, Section 4.8].

The contribution of this paper is to provide sufficient conditions as Linear Matrix Inequalities (LMIs) to design a static and linear state feedback ensuring the UUB property in closed-loop of the investigated system. In addition, an optimization problem is formalized to minimize the size of the UUB set. We can emphasize the main improvements with respect to the recent literature:

- In [5] and [12], the same nonlinear operator is considered, but only UUB analysis results are provided.
- In [8], the control problem is investigated, but the nonlinear operator consists only in a symmetric backlash without dead-zone.
- The paper [13] copes with backlash associated with dead-zone in the actuator and control design with sliding mode techniques. Our contribution deals with a linear state feedback, continuously time-differentiable.

The paper is organized as follows. Section II formalizes the framework of the paper by describing the structure of the nonlinear system, the characteristic of the nonlinear operator including an asymmetric backlash and a dead-zone and finally the notion of UUB. The investigated problem ends this section. Section III offers the main tools and technical results to build the main result given in Section IV, which contains sufficient conditions to ensure the UUB property and also an optimization problem to minimize the size of the induced UUB set. A numerical illustration is given and discussed in Section V before concluding remarks in Section VI.

Notation: The sets $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the set of real numbers, the set of real non-negative numbers and the set of real positive definite numbers, respectively. For a vector x, $\|x\|$ is its Euclidean norm. For a vector or a matrix, $(\cdot)^\top$ denotes its transpose. For $M \in \mathbb{R}^{n\times n}$, $\mathrm{Tr}(M)$ denotes its trace and $\mathrm{He}(M) = M + M^\top$. For a rectangular matrix, $\mathrm{rk}(\cdot)$ denotes its rank. I_n stands for the identity matrix of dimension n and $0_{n\times m}$ stands for the null matrix of dimensions $n \times m$, with

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$0_n = 0_{n\times n}$. A symmetric and positive definite matrix M is denoted $M \succ 0$. A symmetric and negative semidefinite is denoted $M \preceq 0$. In a block matrix, $\star$ stands for a symmetric block. $\mathrm{diag}(A, B)$ denotes the block diagonal matrix with matrices A and B as diagonal blocks. For $\mathbb{R}^{n\times n} \ni P = P^\top \succ 0$, $\mathcal{E}(P) = \{x \in \mathbb{R}^n, x^\top P x = 1\}$ is the associated unit ellipsoid.

II. PROBLEM FORMULATION

A. Description of the structure of the system

The following continuous-time system, depicted on Figure 1, is investigated in this paper:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1}$$
$$u(t) = \Phi[f](t), \tag{2}$$
$$f(t) = Kx(t). \tag{3}$$

This system is the interconnection between:

- The Linear Time Invariant (LTI) system (1) with a single input $u(t) \in \mathbb{R}$ and its state $x(t) \in \mathbb{R}^n$, characterized by constants matrices A and B of appropriate dimension;
- A one dimension nonlinear operator Φ (properly defined in the following), given by Equation (2). This nonlinear operator relates the signal f, defined by (3) and the input signal u of the LTI system (1);
- A linear and constant state feedback (3), with a gain $K \in \mathbb{R}^{1\times n}$ to be designed.

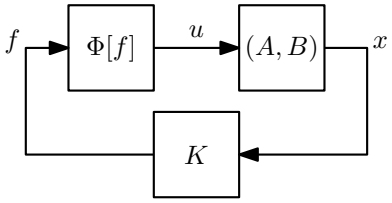

Fig. 1. Description of the considered system (1)–(3).

Remark 1: For a sake of clarity, the input u of LTI (1) and the signal f, given by (3) are one dimension signals. The framework can be easily extended to higher dimension $p > 1$, i.e. $u(t), f(t) \in \mathbb{R}^p$, because the nonlinear operator can be defined componentwise.

B. Description of the nonlinear operator

The nonlinear operator Φ is at the heart of this paper. It combines several simple nonlinear operators into a single isolated one: a backlash [5], [8], an asymmetric dead-zone [4], [14] and finally an asymmetry with respect to the sign of the input. The memory effect implies that Φ is well defined for functions f that are continuous and piecewise-differentiable and is defined by its derivative as follows, for all times t in the open intervals where $\dot{f}(t)$ exists:

$$\dot{\Phi[f](t)} = \begin{cases} l_a\dot{f}(t) \text{ if } \Phi[f](t) \geq 0 \text{ and} \big(\big(\dot{f}(t) \geq 0 \text{ and} \\ \quad \Phi[f](t) = l_a(f(t) - \rho_a - h)\big) \text{ or} \\ \quad \big(\dot{f}(t) \leq 0 \text{ and } \Phi[f](t) = l_a(f(t) - \rho_a)\big)\big); \\ l_b\dot{f}(t) \text{ if } \Phi[f](t) \leq 0 \text{ and} \big(\big(\dot{f}(t) \leq 0 \text{ and} \\ \quad \Phi[f](t) = l_b(f(t) + \rho_b + h)\big) \text{ or} \\ \quad \big(\dot{f}(t) \geq 0 \text{ and } \Phi[f](t) = l_b(f(t) + \rho_b)\big)\big); \\ 0 \quad \text{otherwise.} \end{cases} \tag{4}$$

The parameters of the nonlinearity Φ are $h, l_a, l_b, \rho_a, \rho_b \in \mathbb{R}_{>0}$. They are respectively called the backlash width, the inclination and the threshold when f is positive and negative (see the characteristic of Φ on Figure 2).

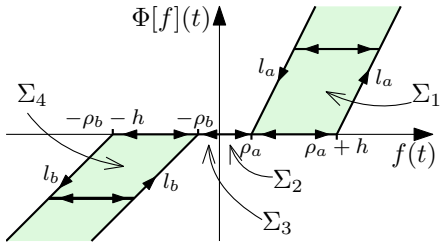

Fig. 2. Characteristic of the nonlinear operator Φ defined by (4).

It is important to note that the function $t \mapsto \Phi[f](t)$ is continuous. It ensures that the state is continuously differentiable due to (1) and that this is also the case of the input f thanks to relation (3). In our framework, $t \mapsto \Phi[f](t)$ is then continuously differentiable.

In order to completely define the nonlinearity, we need to specify its initial condition $\Phi[f](0)$, which is not given by an unequivocal correspondance of the initial condition $f(0)$ but depends also on the history of f. As discussed in [12], the initial condition of the nonlinearity should verify

$$\Phi[f](0) \in \begin{cases} [l_b(f(0) + \rho_b); \min(0, l_b(f(0) + \rho_b + h)], \\ \quad \text{if } f(0) \leq -\rho_b; \\ \{0\}, \quad \text{if } -\rho_b \leq f(0) \leq \rho_a; \\ [\max(0, l_a(f(0) - \rho_a - h)); l_a(f(0) - \rho_a)], \\ \quad \text{if } \rho_a \leq f(0). \end{cases} \tag{5}$$

The condition (5) ensures that the nonlinearity is uniquely defined and is always active (see [12, Lemma 1] and [13]). By active, we mean that the couples $(f(t), \Phi[f](t))$, for all $t \geq 0$, belongs in the union of $[-\rho_b, \rho_a] \times \{0\}$ and the two areas in green on Figure 2. It is noteworthy, due to the relation (3), that the initial condition $\Phi[f](0)$ should be chosen after setting K.

To provide accurate sector conditions in the future developments, let us introduce the four following zones:

$$\Sigma_1 = \{(f, \phi) \in \mathbb{R}^2, \phi \geq 0, \phi + l_a\rho_a \leq l_a f \leq \phi + l_a(\rho_a + h)\},$$
$$\Sigma_2 = \{(f, \phi) \in \mathbb{R}^2, \phi = 0, 0 \leq f \leq \rho_a\},$$
$$\Sigma_3 = \{(f, \phi) \in \mathbb{R}^2, \phi = 0, -\rho_b \leq f \leq 0\},$$
$$\Sigma_4 = \{(f, \phi) \in \mathbb{R}^2, \phi \leq 0, \phi - l_b(\rho_b + h) \leq l_b f \leq \phi - l_b\rho_b\}.$$

In the following, we denote $l_1 \coloneqq l_2 \coloneqq l_a$, $l_3 \coloneqq l_4 \coloneqq l_b$, $\rho_1 \coloneqq \rho_2 \coloneqq \rho_a$ and $\rho_3 \coloneqq \rho_4 \coloneqq \rho_b$ and also with a slight abuse of notation $\Phi(t) = \Phi[f](t)$, $\dot{\Phi}(t) = \dot{\Phi[f](t)}$.

C. Uniform Ultimately Boundedness property

The presence of the dead-zone and the fact that the matrix A is not assumed to be Hurwitz bring about a main concern in the neighborood of the origin, preventing to investigate the stability of the origin. Furthermore, such a nonlinear system may exhibit complex asymptotic behaviors, that may coexist. Among them, we can cite several equilibrium points (possibly an infinite number) and limit cycles. We are focused here on a

weaker notion of stability: Uniform Ultimately Boundedness (UUB), whose definition is recalled here after.

Definition 1 (UUB): [11, Definition 4.6] The trajectory of system (1)–(3) is *uniformly ultimately bounded* with ultimate bound b if there exist positive constants b and c, independent of $t_0 > 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0, such that $\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b,\ \forall t \geq t_0 + T$. □

The framework is now in place to set the studied problem.

Problem 1 (UUB Control Design): Determine a feedback gain K such that the closed-loop system (1)–(3) is UUB, accordingly to Definition 1, and such that the related UUB set, denoted $\mathcal{A}$, is as small as possible.

III. PRELIMINARIES

In this section, we provide the main tools that will be used in the result to answer Problem 1. First of all, the Lyapunov analysis approach [11, Section 4.8], summarized in the next lemma, is used here to ensure the UUB property.

Lemma 1: Consider the system (1)–(3). If there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and three positive definite scalars c_1, c_2, c_3 such that

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2, \tag{6}$$

$$\dot{V}(x) \leq -c_3\|x\|^2, \text{ if } V(x) \geq 1, \tag{7}$$

then the system (1)–(3) is UUB, with the UUB set $\mathcal{A} = \{x \in \mathbb{R}^n, V(x) \leq 1\}$, the UUB bound $\|x\| \leq 1/\sqrt{c_1}$ and the UUB-time T is less that $T^* = (c_2/c_3)\ln(c_2\|x_0\|^2)$. □

The idea to ensure the conditional inequality (7) will be to guarantee a suitable upper bound via S-procedure and Linear Matrix Inequalities (LMIs) [15] and specifying, in each zone, cone-sector bounded conditions related to the nonlinear operator Φ. We will also use a change of variable related to the dual nonlinear operator of Φ, allowing to make appear a Hurwitz matrix in the linear part of the closed-loop dynamics, which will be required in the LMIs. Finally conditional inequality (7) will be ensured under equality constraints translating the link between the variables of the system and their time-derivatives. In accordance with this approach, we present the next preliminaries and tools.

Let us introduce the dual nonlinear operator Ψ related to Φ that takes into account the asymmetry (see a plot of its characteristic in [5, Figure 3]) by:

$$\begin{cases} \Psi(t) \coloneqq \Psi[f](t) \coloneqq \Phi(t) - l_i f(t), \\ \dot{\Psi}(t) \coloneqq \overbrace{\Psi[f](t)} \coloneqq \dot{\Phi}(t) - l_i \dot{f}(t), \end{cases} (f(t), \Phi(t)) \in \Sigma_i, \tag{8}$$

The closed-loop dynamics is thus recast in zone Σ_i:

$$\dot{x}(t) = (A + Bl_iK)x(t) + B\Psi(t),$$

where the term $A + Bl_iK$ appears, describing the system behaviour far from the origin. We denote the extended vector

$$\eta \coloneqq \left(\begin{array}{c|c|c|c} x^\top & \dot{x}^\top & \Psi \quad \dot{\Psi} & f \quad \dot{f} \end{array} \right)^\top \in \mathbb{R}^{2n+5}, \tag{9}$$

which contains all the variables $x(t)$, $\Psi(t)$ and $f(t)$ and their derivatives as independent variables (which will be linked thanks to equality constraints in the sequel). The presence of the constant 1 is made necessary by the existence of constants in the partition in zones.

In the different zones, we can take advantage of geometrical constraints that can be seen on Figure 2. Their detailed proofs are available in [5, Lemmas 1 and 4]:

- In Zones $i \in \{1, 4\}$:

$$\dot{\Phi}\left(\Psi - (-1)^i l_i\left(\rho_i + h/2\right)\right) \leq 0; \tag{10}$$

$$\dot{\Phi}\left(\dot{\Phi} - \alpha_i l_i \dot{f}\right) \leq 0;\ \forall \alpha_i \geq 1; \tag{11}$$

$$\begin{aligned} &\Psi^2 - (-1)^i 2\Psi l_i(\rho_i + h/2) + l_i^2\rho_i(\rho_i + h) \leq 0 \\ &\Leftrightarrow \begin{cases} (-1)^i\Psi \leq l_i(\rho_i + h), \\ l_i\rho_i \leq (-1)^i\Psi. \end{cases} \end{aligned} \tag{12}$$

- In Zones 2 and 3:

$$\Phi = 0; \quad \dot{\Phi} = 0. \tag{13}$$

It is noteworthy that due to Equations (8) : i) Conditions (10)–(12) may be recast as quadratic form in the vector η; ii) Condition (11) is verified for the most restrictive set of nonlinearities when the parameters $\alpha_1 = \alpha_4 = 1$; iii) Conditions (13) are linear constraints with respect to the vector η.

IV. MAIN RESULTS

This section contains the main results of the paper. They consist in two main theorems and one optimization problem:

- Theorem 1 offers (nonlinear) matrix inequalities that are sufficient conditions to solve Lemma 1 by providing a suitable UUB-Lyapunov function $V(x) = x^\top Px$.
- Theorem 2 offers LMIs leading to a solution of the latter nonlinear inequalities.
- Based on these two theorems, the Optimization Problem 1 aims at minimizing the size of the UUB-set $\mathcal{E}(P)$, under the constraints of LMIs in Theorem 2.

The UUB-analysis of the closed-loop dynamics, with a common quadratic UUB-Lyapunov function, for instance in [5], reveals weighted products involving P and K or products between slack variables and K. An adequate technique to avoid this kind of terms consists in the following: derive the inequalities in Lemma 1 in function of the extended vector η, defined by (9), where the components are considered independent of each other; and impose, at the same time, equality constraints on η translating the relations between its components. This approach leads to Theorem 1.

Theorem 1: Let us consider the LTI system (1) and the nonlinear operator (2). Assume that there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a matrix $K \in \mathbb{R}^{1\times n}$, and scalars $\tau_{j,i} > 0$, $(j, i) \in \{0, \cdots, 5\} \times \{1, 4\}$ and $\tau_{j,i} > 0$, $(j, i) \in \{0, 1\} \times \{2, 3\}$, $\alpha_1 \geq 1$, $\alpha_4 \geq 1$, $\tilde{c}_3 > 0$ and $\omega_{1,i}$, $\omega_{2,i}, \omega_{3,i} \in \mathbb{R}$, $(i \in \{2, 3\})$, such that the following inequalities hold: $\forall i \in \{1, \cdots, 4\}$

$$\mathcal{Q}_i(\tau_{0,i} + \tilde{c}_3, P) + \text{He}\left(\chi_i(P)\mathcal{B}_i(I_n, K)\right) \preceq 0, \tag{14}$$

where the miscellaneous notations (18)–(22) are used to enlighten the reading. Under these conditions, $V(x) = x^\top Px$ satisfies the assumptions of Lemma 1, with c_1 and c_2 respectively the minimum and maximum eigenvalues of P and

$c_3 = \tilde{c}_3 c_1$. The conclusion of Lemma 1 applies and the closed-loop system (1)–(3) is UUB. □

Proof: Let us determine that the quadratic function $V(x) = x^T P x$ is a UUB-Lyapunov function candidate verifying Lemma 1. P being positive definite, choosing c_1 and c_2 as the minimum and maximum eigenvalues of P ensures classically the lower and upper bounds (6).

The inequalities (14) have a structure similar to the one appearing in the Finsler's lemma (see [16, Lemma 2]), but involving a semidefinite matrix instead of a definite one. The Finsler's lemma cannot be applied but one of its implication that will be used in the sequel is still valid.

In our framework, the extended vector η does not have independent components, which are related by several equalities that are linear in η. For each zone $i \in \{1, \cdots, 4\}$, we have the dynamics (1), the relation $f = Kx$ and its time-derivative $\dot{f} = K\dot{x}$ and finally for the zones $i \in \{2, 3\}$, we need to take into account equations (13), that are, $\Phi = \Psi + l_i f = 0$ and $\dot{\Phi} = \dot{\Psi} + l_i \dot{f} = 0$. All these conditions are gathered into the following equality constraints $\mathcal{B}_i(I_n, K)\eta = 0$, $\forall i \in \{1, \cdots, 4\}$. Thanks to these equations, pre- and post-multiplying inequalities (14) by $\eta^\top$ and η leads to $\eta^\top \mathcal{Q}_i(\tau_{0,i} + \tilde{c}_3, P)\eta \leq 0$. By developing the latter inequalities, and thanks to the S-procedure related to quadratic constraints of the nonlinear operator described in Section III, we ensure $\dot{V}(x(t)) \leq -\tilde{c}_3 V(x(t)) \leq -\tilde{c}_3 c_1 \|x(t)\|^2$, if $V(x(t)) \geq 1$, which is the second assumption in Lemma 1 and concludes the proof. ■

Remark 2: The first diagonal block of dimension n in Inequality (14) leads to

$$(A + Bl_i K + \frac{\tau_{0,i}}{2} I_n)^\top P + P(A + Bl_i K + \frac{\tau_{0,i}}{2} I_n) \preceq 0_n. \quad (15)$$

This necessary condition means that the pairs (A, Bl_i) should be quadratically stabilized by the same K and related to the same Lyapunov matrix P, with a stability margin $\tau_{0,i}/2$.

The inequalities (14) are still nonlinear in the variables of Theorem 1. Nevertheless, thanks to changes of basis and of variables, that are not conservative, and the use of the Finsler's lemma with particular slack variables, that is conservative, it results in parameterized LMIs in the following theorem.

Theorem 2: Let us consider the LTI system (1) and the nonlinear operator (2). If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a matrix $Y \in \mathbb{R}^{1 \times n}$, and scalars $\tau_{j,i} > 0$, $(j,i) \in \{0, \cdots, 5\} \times \{1, 4\}$ and $\tau_{j,i} > 0$, $(j,i) \in \{0, 1\} \times \{2, 3\}$, $\alpha_1 \geq 1$, $\alpha_4 \geq 1$, $\tilde{c}_3 > 0$ and $\omega_{1,i}$, $\omega_{2,i}$, $\omega_{3,i} \in \mathbb{R}$, $(i \in \{2, 3\})$, such that the following inequalities hold, $\forall i \in \{1, \cdots, 4\}$:

$$\mathcal{Q}_i(\tau_{0,i} + \tilde{c}_3, W) + \mathrm{He}(\chi_i(I_n)\mathcal{B}_i(W, Y)) \preceq 0, \quad (16)$$

by referring to the notations (18)–(22), then with the feedback gain $K = YW^{-1}$ in (3), and $P = W^{-1}$, the system is UUB and $V(x) = x^\top P x$ is solution of Lemma 1. □

Proof: Matrix W being positive definite, it is invertible. Let us denote $P = W^{-1}$ its inverse which is also positive definite. Let us now be focused on the structure of the matrices $\mathcal{Q}_i$, χ_i and $\mathcal{B}_i$ appearing in (16). Thanks to considering the components of η independent of each others, the matrix W appears only in the first $2n \times 2n$ block of $\mathcal{Q}_i(W)$, that is in $\Theta(W)$ defined in (18). We have also the relation, for any scalar $\alpha \in \mathbb{R}$: $\mathrm{diag}(P, P)\Theta(\alpha, W)\mathrm{diag}(P, P) = \Theta(\alpha, P)$. As a consequence of the structure of $\mathcal{Q}_i(W)$, as diagonal per blocks, we have $\mathcal{P}(P)^\top \mathcal{Q}_i(\tau_{0,i} + \tilde{c}_3, W)\mathcal{P}(P) = \mathcal{Q}_i(\tau_{0,i} + \tilde{c}_3, P)$, where $\mathcal{P}(P)$ is defined by $\mathcal{P} : \begin{cases} \mathbb{R}^{n \times n} \to \mathbb{R}^{(2n+5) \times (2n+5)}, \\ W \mapsto \mathrm{diag}(W, W, I_5). \end{cases}$

In $\mathcal{B}_i(W, Y)$, the matrices W and Y appear only in the two first columns and linearly as products of a constant matrix times W or Y. By denoting $K = YP = YW^{-1}$, it yields

$$\mathcal{B}_i(W, Y)\mathcal{P}(P) = \mathcal{B}_i(I_n, K);\ \mathcal{P}(P)^\top \chi_i(I_n) = \chi_i(P). \quad (17)$$

To sum up, by pre- and post-multiplying (16) by $\mathcal{P}(P)^\top$ and $\mathcal{P}(P)$, and denoting $K = YP$ leads to inequalities (14) in Theorem 1. Applying this theorem finishes the proof. ■

Remark 3: Fixing scalars $\tau_{0,i}$, $i \in \{1, \cdots, 4\}$ α_1, α_4 and $\tilde{c}_3$ implies that Inequalities (16) are LMIs, in the other variables cited in Theorem 2, and thus tractable by standard LMIs tools.

Remark 4: The structure of the (Finsler's) auxiliary matrices $\chi_i(P)$ in (14) calls some comments: the two first block lines involve the same matrix P, because the change of variables $K = YP$ appears in several blocks in (14) to linearize these terms (see relation (17)). The reminders of the matrices components are zeros, except a few components, with the aim to compensate the zeros on the diagonal blocks of $\mathcal{Q}_i(P)$, that are at different places depending on the zone i (see (19) and (20)). Two cases occur: i) we impose a constant (1 here) where resulting in a product with the variable Y in $\mathcal{B}_i(W, Y)$; ii) a variable (ω_1, ω_2 and ω_3) where resulting in a product with constants in $\mathcal{B}_i(W, Y)$. Of course, other choices – respecting these rules allowing the developments) – are possible. Constants may be also tuned by the designer on specific systems to facilitate/allow the feasibility of Inequalities (16).

Thanks to Theorem 2, we would like to solve Problem 1. It is necessary to provide an optimization problem minimizing the size of the ellipsoid $\mathcal{E}(P)$. We select the convention that this size is $\mathrm{Tr}(P^{-1}) = \mathrm{Tr}(W)$. Let us fix a priori $\tilde{c}_3 > 0$, α_1 and α_4. The following optimization problem, with $n(n+3)/2 + 22$ variables, provides a solution to Problem 1:

Optimization Problem 1:

$$\min_{\substack{W, Y, \omega_{1,i}, \omega_{2,i}, \omega_{3,i},\ i \in \{2,3\} \\ \tau_{j,i}, (j,i) \in \{0, \cdots, 5\} \times \{1,4\},\ \tau_{j,i}, (j,i) \in \{0,1\} \times \{2,3\}}} \mathrm{Tr}(W),$$

$$\text{under (16),}\ W \succ 0_n,\ \tau_{j,i} > 0. \quad \square$$

V. Illustration and discussions

In order to present the implementation of our contribution and provide discussions related to UUB analysis in the literature, we consider the numerical example coming from [5] and inspired by [8], that is, a double integrator:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$l_a = 1,\ l_b = 1.2,\ \rho_a = 0.1,\ \rho_b = 0.2,\ h = 0.2.$$

We impose also $\alpha_1 = \alpha_4 = 1$ and $\tilde{c}_3 = 10^{-3}$ and the four positive parameters $\tau_{0,i}$, $i \in \{1, \cdots, 4\}$ to be identical. The cost $\mathrm{Tr}(W)$ generated by Optimization Problem 1 depends on these parameters $\tau_{0,i}$, as depicted on Figure 3. For $\tau_{0,i} \geq$

$$\mathcal{Q}_i(\alpha,P)=\text{diag}(\Theta(\alpha,P),Q_i),\qquad \Theta:\mathbb{R}\times\mathbb{R}^{n\times n}\to\mathbb{R}^{2n\times 2n},\ (\alpha,P)\mapsto\begin{pmatrix}\alpha P & P\\ \star & 0_n\end{pmatrix}. \tag{18}$$

$$Q_1=\begin{pmatrix} -\tau_{5,1} & -\tau_{3,1} & 0 & -l_1\tau_{3,1} & \tau_{2,1}-l_1(\rho_1+h/2)\tau_{5,1}\\ \star & -2\tau_{4,1} & 0 & -(2-\alpha_1)l_1\tau_{4,1} & -l_1(\rho_1+h/2)\tau_{3,1}\\ \star & \star & 0 & 0 & \rho_1\tau_{1,1}+l_1\tau_{2,1}\\ \star & \star & \star & -2(1-\alpha_1)l_1^2\tau_{4,1} & -l_1^2(\rho_1+h/2)\tau_{3,1}\\ \star & \star & \star & \star & \begin{pmatrix}-\tau_{0,1}-2\rho_1^2\tau_{1,1}\\ -l_1^2\rho_1(\rho_1+h)\tau_{5,1}\end{pmatrix}\end{pmatrix};\ Q_2=\left(\begin{array}{c|ccc} 0_2 & \multicolumn{3}{c}{0_{2\times 3}}\\ \hline \star & -\tau_{1,2} & 0 & (\rho_2/2)\tau_{1,2}\\ \star & \star & 0 & 0\\ \star & \star & \star & -\tau_{0,2}\end{array}\right); \tag{19}$$

$$Q_3=\left(\begin{array}{c|ccc} 0_2 & \multicolumn{3}{c}{0_{2\times 3}}\\ \hline \star & -\tau_{1,3} & 0 & -(\rho_3/2)\tau_{1,3}\\ \star & \star & 0 & 0\\ \star & \star & \star & -\tau_{0,3}\end{array}\right);\ Q_4=\begin{pmatrix} -\tau_{5,4} & -\tau_{3,4} & 0 & -l_4\tau_{3,4} & -\tau_{2,4}+l_4(\rho_4+h/2)\tau_{5,4}\\ \star & -2\tau_{4,4} & 0 & -(2-\alpha_4)l_4\tau_{4,4} & l_4(\rho_4+h/2)\tau_{3,4}\\ \star & \star & 0 & 0 & -\rho_4\tau_{1,4}-l_4\tau_{2,4}\\ \star & \star & \star & -2(1-\alpha_4)l_4^2\tau_{4,4} & l_4^2(\rho_4+h/2)\tau_{3,4}\\ \star & \star & \star & \star & \begin{pmatrix}-\tau_{0,4}-2\rho_4^2\tau_{1,4}\\ -l_4^2\rho_4(\rho_4+h)\tau_{5,4}\end{pmatrix}\end{pmatrix}; \tag{20}$$

$$\chi_i(P)=\begin{pmatrix} P & 0_{n\times 2} & 0_{n\times 1} & 0_{n\times 1}\\ P & 0_{n\times 2} & 0_{n\times 1} & 0_{n\times 1}\\ 0_{1\times n} & 0_{1\times 2} & \omega_{1,i} & 0\\ 0_{1\times n} & 0_{1\times 2} & 0 & \omega_{2,i}\\ 0_{1\times n} & 0_{1\times 2} & 0 & 0\\ 0_{1\times n} & 0_{1\times 2} & 0 & \omega_{3,i}\\ 0_{1\times n} & 0_{1\times 2} & 0 & 0\end{pmatrix}\in\mathbb{R}^{(2n+5)\times(n+4)},\forall i\in\{2,3\};\chi_i(P)=\begin{pmatrix} P & 0_{n\times 2}\\ P & 0_{n\times 2}\\ 0_{2\times n} & 0_{2\times 2}\\ 0_{2\times n} & I_2\\ 0_{1\times n} & 0_{1\times 2}\end{pmatrix}\in\mathbb{R}^{(2n+5)\times(n+2)},\forall i\in\{1,4\}. \tag{21}$$

$$\mathcal{B}_i(X_1,X_2)=\left\{\begin{array}{l}\begin{pmatrix} AX_1+Bl_iX_2 & -X_1 & B & 0_{n\times 1} & 0_{n\times 1} & 0_{n\times 1} & 0_{n\times 1}\\ X_2 & 0_{1\times n} & 0 & 0 & -1 & 0 & 0\\ 0_{1\times n} & X_2 & 0 & 0 & 0 & -1 & 0\end{pmatrix}\in\mathbb{R}^{(n+2)\times(2n+5)},\forall i\in\{1,4\};\\ \begin{pmatrix} AX_1+Bl_iX_2 & -X_1 & B & 0_{n\times 1} & 0_{n\times 1} & 0_{n\times 1} & 0_{n\times 1}\\ X_2 & 0_{1\times n} & 0 & 0 & -1 & 0 & 0\\ 0_{1\times n} & X_2 & 0 & 0 & 0 & -1 & 0\\ 0_{1\times n} & 0_{1\times n} & 1 & 0 & l_i & 0 & 0\\ 0_{1\times n} & 0_{1\times n} & 0 & 1 & 0 & l_i & 0\end{pmatrix}\in\mathbb{R}^{(n+4)\times(2n+5)},\forall i\in\{2,3\}.\end{array}\right. \tag{22}$$

1.25, the constraints are not feasible. The argument of the minimum of $\text{Tr}(W)$ is for $\tau_{0,i}=0.88$, which leads to $K=\begin{pmatrix}-1.30 & -2.25\end{pmatrix}$ and $P=P_{\text{DES}}=\begin{pmatrix}10.20 & 7.20\\ 7.20 & 8.63\end{pmatrix}$, satisfying $\text{Tr}(P^{-1})=0.52$. The numerical experiments have been performed using MATLAB r2023a, YALMIP v20230622 and SDPT3-4.0. The interval of $\tau_{0,i}$ leading to a feasible solution, depicted on the horizon axis of Figure 3 is obtained thanks to an adequate grid.

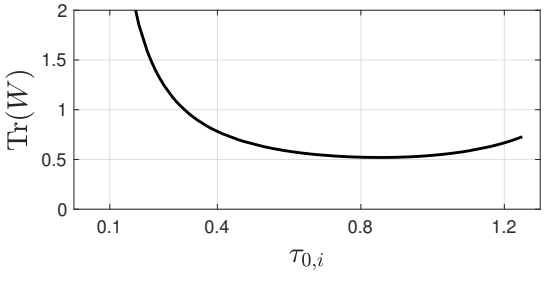

Fig. 3. Size of the induced UUB set, $\text{Tr}(W)$ as a function of $\tau_{0,i}$.

By considering the initial conditions $x_0=\begin{pmatrix}0 & -0.6\end{pmatrix}^\top$ and $\Phi[Kx](0)=1.13$, satisfying the condition (5), we can obtain the state-trajectories drawn on Figure 4, with in addition the value of the UUB Lyapunov function $V(x(t))$. We can observe that the state-trajectories tend to a periodic behavior (see the limit cycle on Figure 6) and that the UUB Lyapunov function decreases faster than exponentially to reach the value 1 and remains below this threshold starting from this time, as expected for the UUB property. We can verify on Figure 5, that the nonlinearity is always active.

As explained previously, the constraints (16) are associated to a choice of χ_i allowing to obtain LMIs. The optimization problem is thus under conservative constraints. When the feedback gain K is generated, it may be beneficial (there is no guarantee nevertheless) to proceed to the analysis of the UUB property for this gain in terms of the size of the UUB set. Here we use the algorithm provided in [5] (with new variables, in particular $\tau_{0,i}=1.68$), leading to a new matrix $P=P_{\text{AN}}=\begin{pmatrix}10.19 & 4.28\\ 4.28 & 13.13\end{pmatrix}$, with $\text{Tr}(P_{\text{AN}}^{-1})=0.20<0.52=\text{Tr}(P_{\text{DES}}^{-1})$. Here, this second-step procedure allows to reduce the UUB ellipsoid, by 59% (for the same gain K). See $\mathcal{E}(P_{\text{DES}})$ and $\mathcal{E}(P_{\text{AN}})$ on Figure 6.

VI. Conclusion

A nonlinear system consisting in the interconnection between a linear time invariant system and a nonlinear operator characterized by an asymmetric backlash and an asymmetric dead-zone has been considered in this paper. The investigation

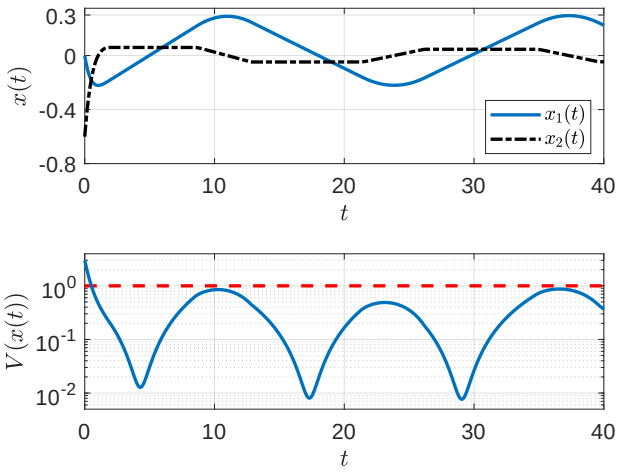

Fig. 4. Time-trajectory of the state $x(t)$ and of UUB-Lyapunov function $V(x(t))$ (with logarithmic scale for y-axis), in function of time t, starting from $x_0 = \begin{pmatrix} 0 & -0.6 \end{pmatrix}^\top$ and $\Phi[Kx](0) = 1.13$.

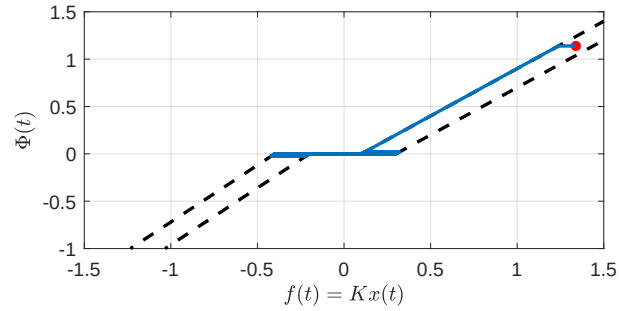

Fig. 5. Characteristic of the nonlinear operator Φ. The starting point corresponds to $(Kx_0, \Phi[Kx](0)) = (1.33, 1.15)$, depicted with a red cross. The trajectory $(Kx(t), \Phi[Kx](t))$ is in solid line. The characteristic of Figure 2 is recalled in back dashes.

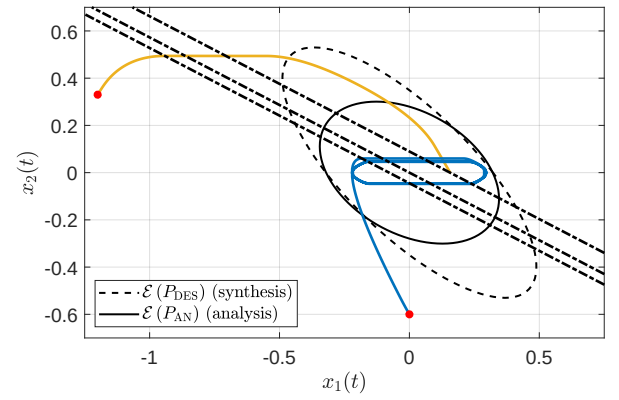

Fig. 6. Phase portrait of the closed-loop system. The lines $\{x \in \mathbb{R}^n, Kx \in \{-\rho_b; 0; \rho_a\}\}$ are depicted with dashed-dot lines. Several trajectories are depicted starting from $x_0 = \begin{pmatrix} -1.2 & 0.33 \end{pmatrix}^\top$ with $\Phi[Kx](0) = 0.57$ (in yellow solid lines, leading to an equilibrium point) and from $x_0 = \begin{pmatrix} 0 & -0.6 \end{pmatrix}^\top$, with $\Phi[Kx](0) = 1.13$ (in blue solid lines, leading to a limit cycle). $\mathcal{E}(P_{\text{DES}})$ and $\mathcal{E}(P_{\text{AN}})$ are in dashed line and solid line.

focused on the design of a linear static state feedback to ensure the Uniform Ultimately Boundedness (UUB) property in the closed-loop and in addition to minimize the size of the UUB-set. A UUB-Lyapunov approach has been used to provide firstly sufficient conditions as nonlinear matrix inequalities and secondly a linearized version of these sufficient conditions. Finally an optimization problem under such constraints has been presented to solve the studied problem. Future work will be focused on alleviating the conservatism of the linearizing techniques to improve the performances and on studying the feasibility of the associated LMIs.

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