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Practical Synchronization of Perturbed Networks of Semi-Passive Systems

Anes Lazri Mohamed Maghenem Elena Panteley Antonio Loría

Abstract—We study practical synchronization for heterogeneous networks of nonlinear systems in the presence of bounded perturbations. Under the assumption that the nodes are state semi-passive and the interconnection graph admits a spanning tree, we establish uniform ultimate boundedness of the solutions and, consequently, practical synchronization. That is, we show that the systems’ trajectories approach each other up to a steady-state error. The magnitude of this steady-state error can be made arbitrarily small by increasing a scalar coupling gain. The results are shown to hold under the assumption that (at least) a single “well-located” node in the network enjoys some robustness properties. Our theoretical results are fairly general in regards to the topology and are illustrated in simulation on a case-study of networked mobile robots.

I. INTRODUCTION

Depending on the context and the domain of study, precise definitions of synchronization may differ. For the purpose of this paper, we simply refer to it as the property by which the solutions of individual systems, after interconnection, match asymptotically. For heterogeneous networks, synchronization is not expected to occur when employing most of the widely-used interconnection protocols, and steady-state errors are unavoidable. This is even more likely to happen when the nodes are affected by external perturbations. In such scenarios, it is more realistic to guarantee practical synchronization [1], [2], [3], [4].

To establish practical synchronization, we use a framework that is based on showing uniform boundedness and uniform ultimate boundedness of the network’s solutions. The latter two properties have been studied in the literature of networked systems. For instance, by appealing to ISS-like conditions for the individual nodes, the so-called mesh-stability property is established in [5], implying boundedness of the network’s solutions. In [1], nonlinear networks, interconnected according to a balanced graph and subject to dynamic uncertainties, are shown to be *semi-passive*, implying ultimate boundedness of the network’s solutions. In [6], when the individual nodes are semi passive, networks interconnected according to a strongly-connected graph, are shown to have ultimately bounded solutions. A similar study of networked semi-passive systems, under undirected graphs, is provided in [7]. The latter two boundedness results have been key to establish synchronization and practical synchronization for homogeneous [8] and for heterogeneous

[4], [9] networks of semi-passive systems, respectively. We recall, roughly speaking, that a semi-passive system defines a passive map away from a compact set around the origin [10], [7]. This property is verified by several physical systems, including robotics systems [1] and some neuronal models such as Hodgkin-Huxley, Morris-Lecar, FitzHugh-Nagumo, and Hindmarsh-Rose models [11].

Most of the works studying boundedness and synchronization for networks of semi-passive systems rely on the assumption that the interconnection graph to be either strongly connected or balanced. Nevertheless, many networks, encountered in control applications [12], physics [13], and opinion-dynamics [14], may not meet these criteria. This explains the interest in considering general directed graphs. Although such a general interconnection scenario has been recently considered in [9], the results therein do not hold in the presence of (even arbitrarily-small) input perturbations.

In this paper, we study practical synchronization for input-disturbed networks of semi-passive systems, interconnected according to a general directed graph admitting a spanning tree. To do so, we first establish ultimate boundedness of the network’s solutions. The value of this result is that it applies to generic directed connected graphs. Furthermore, we handle input perturbations, which can affect the entire network, by requiring a single well-located node to be robust in a specific sense. Without such a robustness property, we are able to find an example where boundedness of the network’s solutions is not satisfied. Later on, we show that ultimate boundedness allows us to guarantee practical synchronization by guaranteeing practical asymptotic stability of the synchronization manifold.

The rest of the paper is organized as follows. The considered class of networks is introduced in Section II. The boundedness and ultimate-boundedness results are in Section III. The practical-synchronization results are in Section IV. Finally, numerical illustrations are provided in Section V.

II. PROBLEM SETTING

Consider n dynamical nonlinear systems modeled by

$$\dot{x}_i = f_i(x_i) + u_i, \quad x_i \in \mathbb{R}, \quad i \in \{1, 2, \dots, n\}, \quad (1)$$

where $f_i : \mathbb{R} \mapsto \mathbb{R}$ is continuous and the input u_i is set to

$$u_i := -\gamma \sum_{j=1}^n a_{ij}(x_i - x_j) + d_i(t) \quad (2)$$

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where $d := [d_1 \ d_2 \ \cdots \ d_n]^\top$ is a bounded input perturbation. It is assumed that there exists $\bar{d} > 0$ such that

$$\sup_{t \geq 0} |d(t)| \leq \bar{d}. \quad (3)$$

Remark 1: In the main content of this paper, it is considered, for notational simplicity and without loss of generality, that $x_i \in \mathbb{R}$. However, all the statements apply for systems such that $x_i \in \mathbb{R}^N$. •

Because we address the problem of synchronization as one of stability, we say that the interconnected network (1)-(2) achieves (global asymptotic) synchronization if the set

$$\mathcal{A} := \{x \in \mathbb{R}^n : x_1 = x_2 = \cdots = x_n\}. \quad (4)$$

is globally asymptotically stable (GAS). However, since (global asymptotic) synchronization is in general not expected to hold for this class of networks, we focus on the following alternative property.

Definition 1 (Global Practical Synchronization): System (1) in closed-loop with (2) achieves global practical synchronization if, for each $\delta > 0$, there exists $\lambda^* > 0$ such that, for each $\lambda \geq \lambda^*$ and $t \mapsto d(t)$ satisfying (3) for some $\bar{d} > 0$, the set $B_\delta(\mathcal{A}) := \{x \in \mathbb{R}^n : |x|_{\mathcal{A}} \leq \delta\}$ is GAS, where $|x|_{\mathcal{A}} := \min\{|x - z| : z \in \mathcal{A}\}$. •

To guarantee global practical synchronization, we rely on three hypotheses, one on the individual-node dynamics (1), one on the network's topology, and one more on a particular node required to have enhanced robustness.

Assumption 1 (State strict semi-passivity): For each $i \in \{1, 2, \dots, n\}$, there exists a continuously differentiable storage function $V_i : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$, a class \mathcal{K}_∞ function $\underline{\alpha}_i$, a constant $\rho_i > 0$, a continuous function $H_i : \mathbb{R} \rightarrow \mathbb{R}$, and a continuous function $\psi_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i), \quad (5)$$

$$\dot{V}_i(x_i) \leq 2u_i x_i - H_i(x_i), \quad \forall x_i \in \mathbb{R}^{n_i} \quad (6)$$

$$H_i(x_i) \geq \psi_i(|x_i|) \quad \forall |x_i| \geq \rho_i. \quad (7)$$

Remark 2: The property described in Assumption 1 is called strict *quasi-passivity* in [15]. The authors of [10] define a similar property without imposing radial unboundedness of the storage function; see also [7]. •

Our second hypothesis concerns the network's topology, which is captured by a directed graph \mathcal{G} to which corresponds the network Laplacian $L \in \mathbb{R}^{n \times n}$ whose entries are

$$[L]_{i,j} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{\substack{\ell=1 \\ \ell \neq i}}^n a_{i\ell}, & i = j, \quad i, j \leq n. \end{cases}$$

Assumption 2: The graph \mathcal{G} is directed and contains a directed spanning tree. •

Under Assumption 2 it is guaranteed that the Laplacian matrix L has exactly one zero eigenvalue $\lambda_1(L) = 0$, and,

$$\operatorname{Re}(\lambda_i(L)) > 0 \quad \forall i \in \{2, 3, \dots, n\},$$

where $\operatorname{Re}(\cdot)$ stems for the real part of (\cdot) . Another very useful and well-established [16] fact is that, under Assumption 2,

\mathcal{G} can be decomposed into a leading strongly-connected subgraph $\{\mathcal{G}_\ell(\mathcal{V}_\ell, \mathcal{E}_\ell)\}$ with no incoming links, and, a subgraph of followers $\{\mathcal{G}_f(\mathcal{V}_f, \mathcal{E}_f)\}$. Note that a single node with no incoming links can also be considered as a leading strongly-connected subgraph. Up to some permutation, the matrix L can be expressed in the lower-block triangular form

$$L = \begin{bmatrix} L_\ell & 0 \\ -A_{\ell f} & M_f \end{bmatrix}, \quad (8)$$

where $L_\ell := D_\ell - A_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ corresponds to the Laplacian matrix of the strongly-connected graph \mathcal{G}_ℓ , the lower-left block $A_{\ell f} \in \mathbb{R}^{n_\ell \times n_f}$, $n_f := n - n_\ell$, is a non-negative matrix, and the lower-right block $M_f \in \mathbb{R}^{n_f \times n_f}$ is a non-singular M -matrix.

Therefore, the network state $x := [x_1 \ \cdots \ x_n]^\top$ and perturbation $d := [d_1 \ \cdots \ d_n]^\top$ may be decomposed into $x := [x_\ell^\top \ x_f^\top]^\top$ and $d := [d_\ell^\top \ d_f^\top]^\top$, where $x_\ell, d_\ell \in \mathbb{R}^{n_\ell}$ contain, respectively, the states and the perturbations of the nodes forming the leading strongly-connected subnetwork, and $x_f, d_f \in \mathbb{R}^{n_f}$ contain, respectively, the states and the perturbations of the remaining nodes, considered to be “followers”. Correspondingly, we define

$$\begin{aligned} f_\ell(x_\ell) &:= [f_{\ell 1}(x_{\ell 1}) \ \cdots \ f_{\ell n_\ell}(x_{\ell n_\ell})]^\top, \\ &:= [f_1(x_1) \ \cdots \ f_{n_\ell}(x_{n_\ell})]^\top, \\ f_f(x_f) &:= [f_{f 1}(x_{f 1}) \ \cdots \ f_{f n_f}(x_{f n_f})]^\top, \\ &:= [f_{n_\ell+1}(x_{n_\ell+1}) \ \cdots \ f_{n_\ell+n_f}(x_{n_\ell+n_f})]^\top. \end{aligned}$$

Consequently, the closed-loop system (1)-(2), which in a compact multi-variable form corresponds to

$$\dot{x} = F(x) - \gamma Lx + d(t), \quad (9)$$

may be re-expressed in the convenient cascaded form

$$\dot{x}_\ell = f_\ell(x_\ell) - \gamma L_\ell x_\ell + d_\ell(t), \quad (10)$$

$$\dot{x}_f = f_f(x_f) + \gamma A_{\ell f} x_\ell - \gamma M_f x_f + d_f(t). \quad (11)$$

Equation (10) defines the dynamics of the leading strongly-connected component, which is a networked system with the underlying strongly-connected graph \mathcal{G}_ℓ . Furthermore, (11) gathers the dynamics of the followers. As a result, many properties for (9) are expected to be achieved using cascades-type arguments applied to (10)-(11).

Remark 3: It is worth stressing that the leader-follower representation in (10)-(11) is always possible under Assumption 2. A similar cascaded structure can also be obtained if, instead of Assumption 2, we assume that the graph \mathcal{G} is only weakly connected. •

Our third hypothesis is a robustness condition on an arbitrary node belonging to the strongly-connected leaders' subgraph \mathcal{G}_ℓ . For a future reference, we label this node with the index $k \in \{1, 2, \dots, n_\ell\}$.

Assumption 3: There exists $k \in \{1, 2, \dots, n_\ell\}$ and a positive constant $c > 0$ such that $\psi_k(|x_k|) \geq c|x_k|$ for all $|x_k| \geq \rho_k$, where (ψ_k, ρ_k) are introduced in Assumption 1. •

If Assumption 3 is not verified, there may be cases in which the solutions are not bounded. In Section V, we give a numerical example that is illustrative of this fact.

In the following section, we make some statements on uniform boundedness and uniform ultimate boundedness for networks of the form (9) satisfying Assumptions 1–2, as well as Assumption 3 for specific values of the constant c introduced therein. These values of c are derived based on the following technical statement.

Lemma 1: The Laplacian matrix $L_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ of a directed and strongly connected graph \mathcal{G}_ℓ admits a vector $v_\ell \in \mathbb{R}^{n_\ell}$ of positive elements $v_{\ell i} > 0$, such that $v_\ell^\top L_\ell = 0$. Moreover, for the diagonal matrix $\Pi_\ell := \text{diag}\{v_{\ell i}\}$, the matrix $Q_\ell := \Pi_\ell L_\ell + L_\ell^\top \Pi_\ell$ is positive semi-definite and its kernel is spanned by $\mathbf{1}_{n_\ell} := [1 \ 1 \ \dots \ 1]^\top \in \mathbb{R}^{n_\ell}$. \square

III. BOUNDEDNESS PROPERTIES

Our first result guarantees global uniform boundedness (GUB) and global uniform ultimate boundedness (GUUB), where uniformity is with respect to the coupling gain γ . For clarity, we recall the definitions of these properties.

Definition 2 (GUB): The solutions $t \mapsto x(t)$ to (9) are globally bounded, uniformly in γ , if, for every $r_o > 0$ and $\gamma_o > 0$, there exists $R = R(r_o, \gamma_o) \geq r_o$ such that, for all $\gamma \geq \gamma_o$ and $t \mapsto d(t)$ satisfying (3), we have

$$|x(0)| \leq r_o \Rightarrow |x(t)| \leq R \quad \forall t \geq 0.$$

Definition 3 (GUUB): The solutions $t \mapsto x(t)$ to (9) are globally ultimately bounded, uniformly in γ , if given $\gamma_o > 0$, there exists $r = r(\gamma_o) > 0$ such that, for all $r_o > 0$, there exists $T = T(r_o, \gamma_o) \geq 0$ such that, for all $\gamma \geq \gamma_o$ and $t \mapsto d(t)$ satisfying (3),

$$|x(0)| \leq r_o \Rightarrow |x(t)| \leq r \quad \forall t \geq T.$$

Then, we have the following result.

Theorem 1 (GUB & GUUB): Consider the network in (9) such that (3) holds for some $\bar{d} > 0$ and such that Assumptions 1–2 hold. Assume further that Assumption 3 holds with

$$c > 2\bar{d}|\Pi_\ell \mathbf{1}_{n_\ell}|/v_{\ell k}, \quad (12)$$

where Π_ℓ and $v_{\ell k}$ are introduced in Lemma 1 and the index k is defined in Assumption 3. Then, the solutions to (9) are GUB and GUUB. \square

Remark 4: It is important to note that both Assumption 3 and Inequality (12) are verified if it is possible to find $k \in \{1, 2, \dots, n_\ell\}$ such that

$$\psi_k(|x_k|) \geq \mathbf{c}(|x_k|)|x_k|, \quad \text{for some } \mathbf{c}(\cdot) \in \mathcal{K}_\infty.$$

Indeed, in this case, one can choose ρ_k in Assumption 1 sufficiently large so that

$$\mathbf{c}(|x_k|) \geq 2\bar{d}|\Pi_\ell \mathbf{1}_{n_\ell}|/v_{\ell k} \quad \forall |x_k| \geq \rho_k.$$

Under such a particular form of ψ_k no global knowledge on the network is required to verify (12) and Assumption 3. \bullet

Sketch of proof: The proof relies on establishing GUB and GUUB, first, for the dynamics of the leading sub-network

(10) and, then, for the follower dynamics (11), considering the latter as a system affected by a disturbance $t \mapsto x_\ell(t)$.

The analysis of (10) uses the Lyapunov-like function

$$W_\ell(x_\ell) := \sum_{i=1}^{n_\ell} v_{\ell i} V_i(x_{\ell i}), \quad (13)$$

where the $v_{\ell i}$ s are positive constants introduced in Lemma 1 and the V_i s are the storage functions provided in Assumption 1. Now, using Lemma 1, one can show the existence of $C > 0$ and $\mu > 0$ such that

$$\dot{W}_\ell(x_\ell) \leq C - \gamma \lambda_2(Q_\ell) |x_\ell|_{\mathcal{A}_\ell}^2 + \mu |x_\ell|_{\mathcal{A}_\ell} \quad \forall x_\ell \in \mathbb{R}^{n_\ell}.$$

As a consequence, given $\gamma_o > 0$ and $\epsilon > 0$, we can find $R_c(\gamma_o, \epsilon) > 0$ such that

$$\dot{W}_\ell(x_\ell) \leq -\epsilon \quad \forall x_\ell \notin \mathcal{C} := \{x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell|_{\mathcal{A}_\ell} \leq R_c\}.$$

Furthermore, using (6) and (7), one can find a constant $\beta_\ell(\gamma_o, \epsilon) > 0$ and a function $\Psi : \mathbb{R}^{n_\ell} \rightarrow \mathbb{R}$ that is continuous and positive on $\mathcal{C} \setminus \mathcal{B}_{\beta_\ell}$, where $\mathcal{B}_{\beta_\ell} := \{x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell| \leq \beta_\ell\}$, such that

$$\dot{W}_\ell(x_\ell) \leq -\Psi(x_\ell) \quad \forall x_\ell \in \mathcal{C} \setminus \mathcal{B}_{\beta_\ell}. \quad (14)$$

Thus, we conclude that

$$\dot{W}_\ell(x_\ell) \leq -\min\{\Psi(x_\ell), \epsilon\} < 0 \quad \forall x_\ell \in \mathbb{R}^{n_\ell} \setminus \mathcal{B}_{\beta_\ell}.$$

The latter is enough to find $\sigma_\ell(\gamma_o) > 0$ such that the set

$$\mathcal{S}_{\sigma_\ell} := \{x_\ell \in \mathbb{R}^{n_\ell} : W_\ell(x_\ell) \leq \sigma_\ell\},$$

which satisfies $\mathcal{B}_{\beta_\ell} \subset \mathcal{S}_{\sigma_\ell}$, is globally attractive and forward invariant. Finally, the first inequality in Assumption 1 allows to deduce a candidate uniform ultimate bound

$$r_\ell := \left[\min_{i \in \{1, \dots, n_\ell\}} \{\underline{\alpha}_{\ell i}\} \right]^{-1} (\sigma_\ell).$$

Next, we compute an upperbound, $T_\ell(r_o, \gamma_o)$, on the time that the solutions to (10) with $\gamma \geq \gamma_o$, and starting from $\mathcal{B}_{r_o} := \{x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell| \leq r_o\}$, take to reach the compact set $\mathcal{B}_{\beta_\ell} \subset \mathcal{S}_{\sigma_\ell}$. For this, we find $\epsilon_o(r_o, \gamma_o) > 0$ such that, along every solution $t \mapsto x_\ell(t)$ to (10) with $x_\ell(0) \in \mathcal{B}_{r_o} \setminus \mathcal{B}_{\beta_\ell}$, we have $\dot{W}_\ell(x_\ell(t)) \leq -\epsilon_o$, up to the earliest time when x_ℓ reaches \mathcal{B}_{β_ℓ} . Thus, we can take $T_\ell := \max\{W_\ell(y) : y \in \mathcal{B}_{r_o}\}/\epsilon_o$.

Finally, to compute the upperbound $R_\ell(r_o, \gamma_o)$ on the solutions to (10), with $\gamma \geq \gamma_o$ and starting from \mathcal{B}_{r_o} , we use the fact that $|x_\ell(t)| \leq r_\ell(\gamma_o)$ for all $t \geq T_\ell(r_o, \gamma_o)$. Hence, it remains to upperbound the norm of those solutions over the interval $[0, T_\ell]$. To that end, we show that on the latter interval

$$W(x_\ell(t)) \leq C + \frac{\theta_1^2 T_\ell(r_o, \gamma_o)}{\gamma_o \lambda_2(Q_\ell)} + \max\{W_\ell(y) : y \in \mathcal{B}_{r_o}\}.$$

Hence, $\forall t \geq 0$, we have

$$|x_\ell(t)| \leq R_\ell := \left[\min_i \{\underline{\alpha}_{\ell i}\} \right]^{-1} \left(C + \frac{\theta_1^2 T_\ell}{\gamma_o \lambda_2(Q_\ell)} + \sigma_o + r_\ell \right).$$

A similar analysis may be carried out for the follower components relying on the boundedness properties established

for the leading sub-network. For that, a key step consists in using the Lyapunov function

$$W_f(x_f) := \sum_{i=1}^{n_f} p_{fi} V_{n_\ell+i}(x_{fi}),$$

where the coefficients $p_{fi} > 0$ are generated based on the M -matrix M_f in (11) via the following statement.

Lemma 2: Let $M_f \in \mathbb{R}^{n_f \times n_f}$ be a non-singular M -matrix. Then, the matrices

$$S_f := R_f M_f + M_f^\top R_f,$$

$$R_f := \text{diag} \left\{ M_f^{-\top} \mathbf{1}_{n_f} \right\} \left(\text{diag} \left\{ M_f^{-1} \mathbf{1}_{n_f} \right\} \right)^{-1}$$

are positive definite. \square

Remark 5: From the sketch of proof of Theorem 1, we can see that $T_\ell(r_o, \gamma_o)$ and $R_\ell(r_o, \gamma_o)$ are continuous with respect to r_o . Even though this is not shown here, the same property holds for the candidates $T(r_o, \gamma_o)$ and $R(r_o, \gamma_o)$. \bullet

IV. PRACTICAL SYNCHRONIZATION

To ensure practical stability of the synchronization manifold \mathcal{A} in (4), we start performing a change of variable, along the lines of [4], [9]. Indeed, after Assumption 2, $\lambda_1(L) = 0$ has multiplicity one and L admits the Jordan-block decomposition:

$$L = V \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix} V^{-1}, \quad (15)$$

where $\Lambda \in \mathbb{R}^{(n-1) \times (n-1)}$ is composed by the Jordan blocks corresponding to the remaining eigenvalues having positive real parts. Moreover, V and V^{-1} correspond to $V = [\mathbf{1}_n \ U]$ and $V^{-1} = \begin{bmatrix} v_\ell^\top \\ U^\dagger \end{bmatrix}$, where $U \in \mathbb{R}^{n \times n-1}$, $v_\ell^\top U = 0$, $v_\ell^\top \mathbf{1}_{n_\ell} = 1$, and $U^\dagger U = I_{n-1}$. Now, using V^{-1} , we define the new coordinates $(x_m, e_v) := (v_\ell^\top x, U^\dagger x)$. In these new coordinates, since $VV^{-1} = \mathbf{1}_n v_\ell^\top + UU^\dagger$, the state vector x can be expressed as

$$x = \mathbf{1}_n x_m + U e_v. \quad (16)$$

Therefore, the network dynamics consists of two interconnected dynamics: the ‘‘average-state’’ dynamics (the dynamics of x_m) and the projected ‘‘synchronization-error’’ dynamics (the dynamics of e_v). Under this transformation, we obtain

$$\dot{x}_m = F_m(d, x_m, e_v) \quad (17a)$$

$$\dot{e}_v = -\gamma \Lambda e_v + G_e(d, x_m, e_v), \quad (17b)$$

$$F_m(d, x_m, e_v) := v_\ell^\top [F(\mathbf{1}_n x_m + U e_v) + d],$$

$$G_e(d, x_m, e_v) := U^\dagger [F(\mathbf{1}_n x_m + U e_v) + d].$$

Note that, in view of (16), $e_v = 0$ is equivalent to $x = \mathbf{1}_n x_m$. Hence, the following equality holds true

$$\mathcal{A} = \{x \in \mathbb{R}^n : e_v = U^\dagger x = 0\}.$$

As a consequence, to analyze global practical synchronization for (9) (according to Definition 1), it is enough to

analyze global practical asymptotic stability (GPAS) of the set $\{(x_m, e_v) : e_v = 0\}$ for (17). More precisely, we establish the following property.

Property 1 (GPAS of $\{(x_m, e_v) : e_v = 0\}$): There exist $\beta_1, \beta_2 \in \mathcal{KL}$ such that, for each $\delta > 0$, there exists $\gamma^*(\delta) > 0$ such that, for each $\gamma \geq \gamma^*$, for each $x(0) \in \mathbb{R}^n$, and for all $t \mapsto d(t)$ verifying (3) for some $\bar{d} > 0$, the solutions to (17) satisfy

$$|e_v(t)|^2 \leq \delta + \beta_1(|e_v(0)|, t) + \frac{\beta_2(|x(0)|, t)}{\gamma} \quad \forall t \geq 0. \quad (18)$$

Theorem 2: Consider the network (9) subject to (3) for some $\bar{d} > 0$ and such that Assumptions 1–2 hold. Assume further that Assumption 3 holds with the constant c therein verifying (12). Then, Property 1 above holds and the network achieves global practical synchronization. \square

Proof: Since the eigenvalues of Λ in (15) have positive real parts, we conclude the existence of a positive-definite matrix $P \in \mathbb{R}^{(n-1) \times (n-1)}$ such that $PA + AP^\top \geq I_{n-1}$. Then, we let $V(e_v) := e_v^\top P e_v$, which satisfies

$$\lambda_m |e_v|^2 \leq V(e_v) \leq \lambda_M |e_v|^2,$$

where $\lambda_m > 0$ and $\lambda_M > 0$ are the minimum and the maximum eigenvalues of P , respectively. The total derivative of V along (17b) satisfies

$$\dot{V}(e_v) \leq -\gamma |e_v|^2 + 2\lambda_M |G_e(d, x_m, e_v)| |e_v|. \quad (19)$$

Now, after Theorem 1, the network solutions are GUB and GUUB. As a result, given $\gamma_o > 0$ and an initial condition $|x(0)| = r_o \geq 0$ and , there exist $T(r_o, \gamma_o)$, $r_a(\gamma_o)$, $R_a(r_o, \gamma_o)$, and $R_e(r_o, \gamma_o) > 0$ such that

$$|e_v(t)| \leq r_e, \quad |x_m(t)| \leq r_a \quad \forall t \geq T, \quad (20)$$

$$|e_v(t)| \leq R_e, \quad |x_m(t)| \leq R_a \quad \forall t \geq 0. \quad (21)$$

Back to (19), and letting

$$\bar{G}(r_o, \gamma_o) := R_e(r_o, \gamma_o) \max_{\substack{|d| \leq \bar{d}, \\ |x_m| \leq R_a(r_o, \gamma_o), \\ |e_v| \leq R_e(r_o, \gamma_o)}} |G_e(d, x_m, e_v)|,$$

we see, in view of the continuity of G_e and Remark 5, that $r_o \mapsto \bar{G}(r_o, \gamma_o)$ is continuous and non-decreasing. Furthermore, for each $\gamma \geq \gamma_o$, we have

$$\dot{V}(e_v(t)) \leq -\frac{\gamma}{\lambda_M} V(e_v(t)) + 2\lambda_M \bar{G}(r_o, \gamma_o).$$

It follows that, on the interval $[0, T]$ and for $\gamma \geq \gamma_o$, we have

$$V(e_v(t)) \leq \lambda_M |e_v(0)|^2 e^{-\frac{\gamma t}{\lambda_M}} + \frac{2\lambda_M^2}{\gamma} \bar{G}(r_o, \gamma_o).$$

Hence, defining $\beta_1(a, t) := \frac{\lambda_M a^2}{\lambda_m} e^{-\frac{\gamma a t}{\lambda_M}}$ and

$$\kappa(a) := \frac{2\lambda_M^2}{\lambda_m} [\bar{G}(a, \gamma_o) - \bar{G}(0, \gamma_o)],$$

we conclude that, for all $\gamma \geq \gamma_o > 0$ and $t \in [0, T]$,

$$|e_v(t)|^2 \leq \beta_1(|e_v(0)|, t) + \frac{\kappa(|x(0)|)}{\gamma} + \frac{2\lambda_M^2 \bar{G}(0)}{\lambda_m \gamma}.$$

As a result, for each $\delta > 0$, there exists $\gamma_1^* \geq \gamma_o$ such that, for each $\gamma \geq \gamma_1^*$, we have

$$|e_v(t)|^2 \leq \beta_1(|e_v(0)|, t) + \frac{\kappa(|x(0)|)}{\gamma} + \delta \quad \forall t \in [0, T]. \quad (22)$$

We now focus on the interval $[T, +\infty)$. After Theorem 1 and in view of (20), we let

$$\hat{G}(\gamma_o) := r_e(\gamma_o) \max_{\substack{|d| \leq \bar{d}, \\ |x_m| \leq r_a(\gamma_o), \\ |e_v| \leq r_e(\gamma_o)}} |G_e(d, x_m, e_v)|,$$

leading, for each $\gamma \geq \gamma_o$, to

$$\dot{V}(e_v(t)) \leq -\frac{\gamma}{\lambda_M} V(e_v(t)) + 2\lambda_M \hat{G}(\gamma_o) \quad \forall t \geq T.$$

Hence, for each $\gamma \geq \gamma_o > 0$, we obtain

$$|e_v(t)|^2 \leq \beta_1(|e_v(T)|, t - T) + \frac{2\lambda_M^2 \hat{G}(\gamma_o)}{\gamma \lambda_m} \quad \forall t \geq T.$$

We conclude that, for each $\delta > 0$, there exists $\gamma_2^* \geq \gamma_o$ such that, for each $\gamma \geq \gamma_2^*$, we have

$$|e_v(t)| \leq \beta_1(|e_v(T)|, t - T) + \delta \quad \forall t \geq T. \quad (23)$$

Finally, using (22) and (23), we conclude that Property 1 is verified with $\gamma^* \geq \sup\{\gamma_1^*, \gamma_2^*\}$ and $\beta_2(a, t) := e^{-\frac{\gamma_o t}{\lambda_M}} \left(\kappa(a) e^{\frac{\gamma_o T(a, \gamma_o)}{\lambda_M}} \right)$. ■

V. NUMERICAL EXAMPLE

To illustrate our theoretical results, we present a numerical example involving a network of $n = 4$ nonholonomic mobile robots, with kinematics given, for each $i \in \{1, 2, 3, 4\}$, by

$$\dot{r}_{x_i} = v_i \cos(\theta_i), \quad \dot{r}_{y_i} = v_i \sin(\theta_i), \quad \dot{\theta}_i = \omega_i,$$

where (r_{x_i}, r_{y_i}) is the position of the center of the i th robot and θ_i is its orientation. The inputs v_i and ω_i are, respectively, the linear and angular velocities of the i th robot.

To render each system strictly semi-passive, we apply a preliminary control law and focus on the dynamics of the position of a point x_i located at a distance $a > 0$ off the axis joining the wheels. To that end, we let

$$x_i := \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = \begin{bmatrix} r_{x_i} + a \cos(\theta_i) \\ r_{y_i} + a \sin(\theta_i) \end{bmatrix},$$

$$\varphi_{1i} := \frac{x_{1i}(x_{1i}^2 - \lambda_i)}{x_{1i}^4 + 1}, \quad \varphi_{2i} := \frac{x_{1i}(x_{1i}^2 - \lambda_i)}{x_{1i}^4 + 1}$$

Then, we define the preliminary control inputs

$$v_i := -(\varphi_{1i} - u_{1i}) \cos(\theta_i) - (\varphi_{2i} - u_{2i}) \sin(\theta_i)$$

$$\omega_i := -\frac{1}{a} \left[-(\varphi_{1i} - u_{1i}) \sin(\theta_i) - (\varphi_{2i} - u_{2i}) \cos(\theta_i) \right],$$

where $\lambda_i > 0$. We obtain

$$\dot{x}_{1i} = -\varphi_{1i} + u_{1i}, \quad \dot{x}_{2i} = -\varphi_{2i} + u_{2i}. \quad (24)$$

The systems in (24) define semi passive maps $u_{1i} \mapsto x_{1i}$ and $u_{2i} \mapsto x_{2i}$ with respective storage functions $V(x_{1i}) = x_{1i}^2$ and $V(x_{2i}) = x_{2i}^2$, so Assumption 1 holds for the two

networks with states $x_1 := [x_{11} \ x_{12} \ x_{13} \ x_{14}]^\top$ and $x_2 := [x_{21} \ x_{22} \ x_{23} \ x_{24}]^\top$, separately.

Next, we assume that every input pair (u_{1i}, u_{2i}) is affected by a disturbance d_i and is given by

$$u_{1i} := -\gamma \sum_{j=1}^4 a_{ij}(x_{1i} - x_{1j}) + \nu_{1i} + d_i(t)$$

$$u_{2i} := -\gamma \sum_{j=1}^4 a_{ij}(x_{2i} - x_{2j}) + \nu_{2i} + d_i(t),$$

where $\gamma > 0$ is the coupling gain and $a_{ij} \in \{0, 1\}$ are the entries of the Laplacian matrix

$$L := \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

whose graph verifies Assumption 2, and ν_{1i} and ν_{2i} are additional design inputs. First, we carried out some simulation tests with $\nu_{1i} = \nu_{2i} = 0$ for all $i \in \{1, 2, 3, 4\}$, $\lambda := [0.5 \ 0.3 \ 1.2 \ 0.8]^\top$, and $d_i(t) := \arctan(t)/\pi$. In Figure 1 are shown the network's state trajectories, which grow unboundedly as $t \rightarrow \infty$.

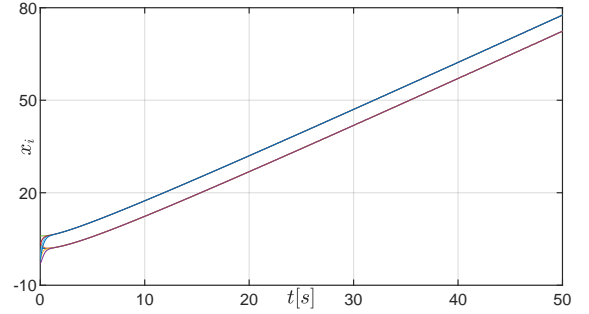


Fig. 1. Trajectories of $x_i = [x_{1i}, x_{2i}]$, $i \in \{1, 2, 3, 4\}$, for $\gamma = 1$.

Then, we added a robust node to the network, with state $x_c \in \mathbb{R}^2$ and dynamics

$$\dot{x}_{1c} = -x_{1c}^3 + 3x_{1c} + u_{1c},$$

$$\dot{x}_{2c} = -x_{2c}^3 + 3x_{2c} + u_{2c}$$

This control node is interconnected to the node $i = 1$, that is, we set $u_{1c} := -\gamma(x_{1c} - x_{11})$ and $u_{2c} := -\gamma(x_{2c} - x_{21})$. Hence, Assumption 2 holds for the new graph and, in fact, the first and control nodes form a leading strongly-connected component with the latter—see Figure 2. Furthermore, we set

$$\nu_{1i} = \nu_{2i} = 0, \quad \forall i \in \{2, 3, 4\}$$

$$\nu_{11} = -\gamma(x_{11} - x_{1c}), \quad \nu_{21} = -\gamma(x_{21} - x_{2c}),$$

Assumption 1 also holds for the network dynamics governing $[x_{1c} \ x_{11} \ x_{12} \ x_{13} \ x_{14}]^\top$ and the one governing $[x_{2c} \ x_{21} \ x_{22} \ x_{23} \ x_{24}]^\top$, separately. Finally, note that Assumption 3 is verified too, for $k \equiv c$. Indeed, using the storage functions $V(x_{1c}) := x_{1c}^2$, we conclude that (5) holds

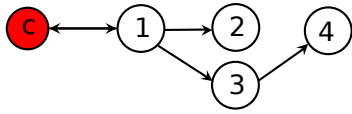


Fig. 2. The graph representation after the red robust node is included. The leading strongly-connected component is composed of nodes $\{1, c\}$.

for $H_c(x_{1c}) = \psi_c(|x_{1c}|) = x_{1c}^2(x_{1c}^2 - 3)$. Note that, when $|x_{1c}| \geq \rho_k = \rho_c = 3$, then $\psi_{1c}(|x_{1c}|) = x_{1c}^2(x_{1c}^2 - 3) > c|x_{1c}|$ for $c = 2$. The same properties hold for x_{2c} . As a result, condition (12) in Theorem 1 holds. Figure 3 shows the trajectories of the new network, which remain bounded and approach a common point. Furthermore, the error vector e_v becomes less and less significant as the coupling gain increases.

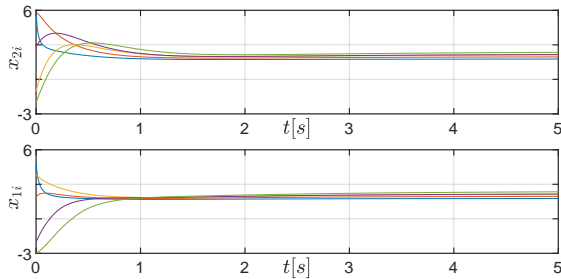


Fig. 3. Trajectories of the new network for $\gamma = 5$.

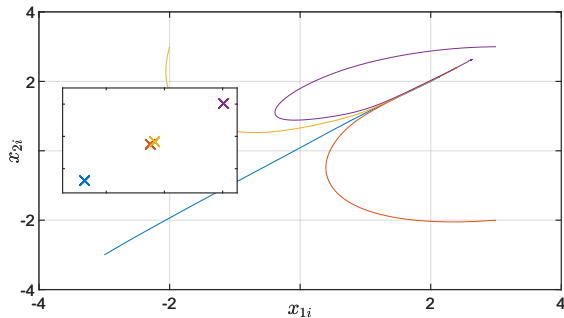


Fig. 4. Phase portrait of the new network trajectories for $\gamma = 5$.

VI. CONCLUSION

This paper studied heterogeneous networks in the presence of input perturbations. We proposed a framework to guarantee GUUB and GUB. These conclusions were key to ensure global practical synchronization. The presented results show that it is possible to handle bounded input perturbations, that can affect the entire network, by making only a single agent robust enough. These results will be extended to more general weakly-connected graphs, and a fully distributed design of γ will be investigated to achieve any desired practical-synchronization bound δ .

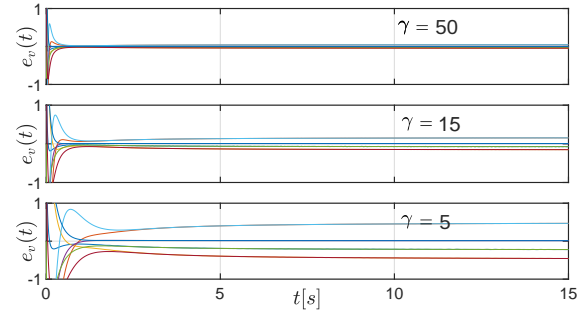


Fig. 5. The behavior of the synchronization errors. In the bottom figure $\gamma = 5$, in the middle figure $\gamma = 15$, and in the top figure $\gamma = 50$.

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