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# Practical Synchronization of Perturbed Networks of Semi-Passive Systems

Anes Lazri   Mohamed Maghenem   Elena Panteley   Antonio Loría

**Abstract**—This paper studies heterogeneous networks of nonlinear systems in the presence of bounded perturbations. We proposed a framework based on semi-passivity of the nodes, interconnected according to a static consensus protocol, to guarantee global uniform ultimate boundedness as well as global uniform ultimate boundedness. These conclusions allow us to ensure global practical synchronization when the coupling gain is allowed to be large. Our main contribution shows that network-wide input perturbations, that can affect the entire network, can be handled provided that a single, well-located, node possesses the right robustness properties. The obtained results are illustrated in simulation on a network of mobile robots.

## I. INTRODUCTION

During the last decades, nonlinear interconnected systems have been extensively studied in various fields of research, including biology [1], sociology [2], and power engineering [3]. Many properties attract this interest, such as synchronization, consensus, and clustering, to name but a few. The trajectories of interconnected systems depend on several factors: the type of interconnection, the coupling gain, and the properties of the interconnected systems. Regarding the properties of the various interconnected systems, the theory of passivity and dissipativity is one of the most considered aspects for studying the synchronization of interconnected systems, for example in [4], [5].

The main interest of this paper is to present conditions for uniform ultimate boundedness of perturbed semi-passive systems interconnected over directed networks. Such systems define a passive map outside a given compact around the origin [6], [4]. Similar properties can be found in the literature, such as quasipassivity [7], set-passivity [8], and almost passivity [9]. Semi-passivity is a property of several physical and biological systems such that conductance-based neuronal model (Hodgkin-Huxley, Morris-Lecar, FitzHugh-Nagumo, and Hindmarsh-Rose) [1], robotics [10]. This property is essential for proving various other network characteristics: asymptotic stability of the origin and almost global orbital asymptotic stability —see [11]. In addition, in [12], such property is essential to find conditions under which the synchronization set is uniformly globally asymptotically stable or globally asymptotically practically stable. Note that perturbations are not considered in the previously cited works. Furthermore, one of the first works to deal with

the ultimate boundedness of interconnected semi-passive systems for undirected networks is [4], and the result was then used to analyze the global asymptotic stability in [13]. In numerous practical scenarios, however, achieving asymptotic stability is conservative due to unmodeled dynamics, measurement noise, and disturbances. Consequently, the primary properties often established for such systems are ultimate boundedness, indicating that errors persist within a certain vicinity of the origin over a prolonged period and practical synchronization. This explains why several papers analyze ultimate boundedness of semi-passive systems. In [14], the authors study ultimate boundedness for strongly connected networked systems. In addition to this work, the authors of [10] aim to synchronize interconnected robotic systems with dynamic uncertainties. They propose a control law that makes the network in closed-loop semi-passive and then show that it guarantees ultimate boundedness. The central assumption on the interconnection graph is that it is balanced.

Note that all these works consider the interconnection graph to be either undirected or directed with certain constraints on connectivity: strongly connected or balanced. Nevertheless, many networks may not meet this criterion, for example, in control applications [15], in physics [16], or in opinion analysis [17], which explains the interest in directed networks with a spanning tree. Moreover, in several practical cases, interconnection can be disrupted. Consequently, it is necessary to study the behavior of this type of networks, considering a perturbation in the interconnection term. In this paper, we analyze the ultimate boundedness of the trajectories of a network of semi-passive systems in the presence of a perturbation, presenting the conditions for which this property is verified. The same conditions will subsequently lead to the deduction of the practical asymptotic stability of the synchronization error.

Regarding the structure of the paper, the following section presents the formulation of the problem and sets the general framework for the work that is carried out. Sections IV and V are devoted to the main results, illustrated with a numerical example in Section VI. Finally, the last section presents the conclusions and prospects for the future.

## II. NOTATIONS

For  $x \in \mathbb{R}^n$ ,  $x^\top$  denotes its transpose,  $|x|$  denotes its Euclidean norm, and,  $\text{diag}\{x\} \in \mathbb{R}^{n \times n}$  denotes the diagonal matrix whose  $i$ th diagonal element is the  $i$ th element of  $x$ . For a set  $K \subset \mathbb{R}^n$ ,  $|x|_K := \min\{|x - y| : y \in K\}$  denotes the distance of  $x$  to the set  $K$ . For a symmetric matrix  $Q \in$

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$\mathbb{R}^{n \times n}$ ,  $\lambda_i(Q)$  denotes the  $i$ th smallest eigenvalue of  $Q$ . A class  $\mathcal{K}$  function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is continuous, increasing, and,  $\alpha(0) = 0$ . A class  $\mathcal{K}^\infty$  function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  satisfies  $\alpha \in \mathcal{K}$  and  $\alpha(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . A class  $\mathcal{L}$  function  $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is continuous, non-increasing, and, tends to zero as its argument tends to infinity. A class  $\mathcal{KL}$  function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  satisfies  $\beta(\cdot, t) \in \mathcal{K}$  for any fixed  $t \geq 0$ , and,  $\beta(s, \cdot) \in \mathcal{L}$  for any fixed  $s \geq 0$ . Furthermore  $\alpha^-$  denotes the inverse function of  $\alpha$ . We denote by  $\mathbf{1}_n := [1 \ 1 \ \dots \ 1]^\top$  the vector of  $n$  entries equal to 1.

### III. PROBLEM FORMULATION

Consider a group of  $n$  dynamical nonlinear systems modeled by<sup>1</sup>

$$\dot{x}_i = f_i(x_i) + u_i, \quad x_i \in \mathbb{R}, \quad i \in \{1, 2, \dots, n\}, \quad (1)$$

where  $f_i : \mathbb{R} \mapsto \mathbb{R}$  is continuous for all  $i \in \{1, 2, \dots, n\}$  and  $u_i$  is set to

$$u_i := -\gamma \sum_{j=1}^n a_{ij}(x_i - x_j) + d_i(t) \quad \forall i \in \{1, 2, \dots, n\}, \quad (2)$$

where  $d := [d_1 \ d_2 \ \dots \ d_n]^\top$  is a bounded input perturbation, for which, there exists  $\bar{d} > 0$  such that

$$\sup_{t \geq 0} |d(t)| \leq \bar{d}. \quad (3)$$

In the case that the systems are heterogeneous (i.e. the  $f_i$ s are not necessarily equal) and  $d_1 = d_2 = \dots = d_n = 0$ , asymptotic synchronization cannot be guaranteed, and only ‘‘practical’’ synchronization is usually ensured [18]-[19]. Considering this, this paper studies practical synchronization for perturbed networks by imposing robustness properties on only a *single* agent.

A standing hypothesis is that each unit possesses certain physical properties reminiscent of energy dissipation. More precisely, we assume the following.

*Assumption 1 (State strict semi-passivity):* For each  $i \in \{1, 2, \dots, n\}$ , the input-output map  $u_i \mapsto x_i$  defined by the dynamics (1) is state strict semipassive [4]. Furthermore, there exists a continuously differentiable storage function  $V_i : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ , a class  $\mathcal{K}^\infty$  function  $\underline{\alpha}_i$ , a constant  $\rho_i > 0$ , a continuous function  $H_i : \mathbb{R} \rightarrow \mathbb{R}$ , and a continuous function  $\psi_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ , such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i), \quad \dot{V}_i(x_i) \leq 2u_i x_i - H_i(x_i), \quad (4)$$

and  $H_i(x_i) \geq \psi_i(|x_i|)$  for all  $|x_i| \geq \rho_i$ . •

*Remark 1:* This property is called strict *quasipassivity* in [7]. The authors of [6] define a similar property without imposing radial unboundedness of the storage function; see also [4]. Uniform ultimate boundedness of the trajectories of networks of semi-passive systems when  $d(t) = 0$  is demonstrated in [20]. •

<sup>1</sup>To simplify the notation, we consider each  $x_i$  to be scalar. The results apply even when the  $x_i$ s are vectors of the same dimension, as illustrated in the simulation example.

Now, defining  $x := [x_1 \ \dots \ x_n]^\top$ , and  $F(x) := [f_1(x_1) \ f_2(x_2) \ \dots \ f_n(x_n)]^\top$ , we may write the system in a compact form as

$$\dot{x} = F(x) - \gamma Lx + d(t), \quad (5)$$

where the entries of the Laplacian  $L$  are given by

$$[L]_{i,j} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{\substack{\ell=1 \\ \ell \neq i}}^n a_{i\ell}, & i = j, \quad i, j \leq n. \end{cases}$$

Note that (5) corresponds to the dynamics of a networked system with an underlying topology represented by a graph  $\mathcal{G}$ , for which, we assume the following.

*Assumption 2:* The graph  $\mathcal{G}$  is directed and contains a spanning tree. •

Assumption 2 is fundamental because it guarantees that the Laplacian matrix  $L$  has exactly one zero eigenvalue  $\lambda_1(L) = 0$ , and,  $\lambda_i(L) > 0$  for all  $i \in \{2, 3, \dots, n\}$ .

Under Assumption 2,  $\mathcal{G}$  can be decomposed into a leading strongly connected subgraph  $\{\mathcal{G}_\ell(\mathcal{V}_\ell, \mathcal{E}_\ell)\}$  with no incoming links, and, a sub-graph of followers  $\{\mathcal{G}_f(\mathcal{V}_f, \mathcal{E}_f)\}$ . Note that a single node with no incoming links can also be considered as a leading strongly connected subgraph. Up to some permutation, the matrix  $L$  can be expressed in the lower-block triangular form

$$L = \begin{bmatrix} L_\ell & 0 \\ -A_{lf} & M_f \end{bmatrix}, \quad (6)$$

where  $L_\ell := D_\ell - A_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$  corresponds to the Laplacian matrix of the strongly connected graph  $\mathcal{G}_\ell$ , the lower-left block  $A_{lf} \in \mathbb{R}^{n_f \times n_\ell}$ ,  $n_f := n - n_\ell$ , is a non-negative matrix, and the lower-right block  $M_f \in \mathbb{R}^{n_f \times n_f}$  is a non-singular  $M$ -matrix. Therefore, the state  $x$  may be decomposed into  $x := [x_\ell^\top \ x_f^\top]^\top$  and system (5) takes the cascaded form

$$\dot{x}_\ell = f_\ell(x_\ell) - \gamma L_\ell x_\ell + d_\ell(t), \quad (7a)$$

$$\dot{x}_f = f_f(x_f) + \gamma A_{lf} x_\ell - \gamma M_f x_f + d_f(t), \quad (7b)$$

$$f_\ell(x_\ell) := [f_1(x_{\ell 1}) \ \dots \ f_{n_\ell}(x_{\ell n_\ell})]^\top,$$

$$f_f(x_f) := [f_{n_\ell+1}(x_{f1}) \ \dots \ f_{n+n_f}(x_{fn_f})]^\top.$$

Equation (7a) governs the dynamics of the *leading strongly-connected* component, which is a networked system with the underlying strongly connected graph  $\mathcal{G}_\ell$ , and (7b) corresponds to the dynamics of the *follower* component.

At this stage, we assume a certain robustness property for a single agent of index  $k \in \{1, 2, \dots, n_\ell\}$ . We will show that, when such a robust agent takes part of  $\mathcal{G}_\ell$ , it allows the network to handle perturbations affecting any other agents.

*Assumption 3:* There exists  $k \in \{1, 2, \dots, n_\ell\}$  and two positive constants  $c_1$  and  $c_2$  such that

$$\psi_k(|x_k|) \geq c_1 |x_k| - c_2, \quad \forall |x_k| \geq \rho_k,$$

where  $(\psi_k, \rho_k)$  are introduced in Assumption 1. •

Under Assumptions 1-2 and when Assumption 3 holds for a specific choice of  $(c_1, c_2)$ , the solutions to (5) can be proven to be globally uniformly bounded (GUB) and globally ultimately uniformly bounded (GUUB).

*Definition 1 (GUB):* The solutions  $t \mapsto x(t)$  to (5) are globally bounded, uniformly in  $\gamma$ , if, for every  $r_o > 0$  and  $\gamma_o > 0$ , there exists  $R = R(r_o, \gamma_o) \geq r_o$  such that, for all  $\gamma \geq \gamma_o$  and  $t \mapsto d(t)$  satisfying (3),

$$|x(0)| \leq r_o \Rightarrow |x(t)| \leq R \quad \forall t \geq 0.$$

*Definition 2 (GUUB):* The solutions  $t \mapsto x(t)$  to (5) are globally ultimately bounded, uniformly in  $\gamma$ , if given  $\gamma_o > 0$ , there exists  $r = r(\gamma_o) > 0$  such that, for all  $r_o > 0$ , there exists  $T = T(r_o, \gamma_o) \geq 0$  such that, for all  $\gamma \geq \gamma_o$  and  $t \mapsto d(t)$  satisfying (3),

$$|x(0)| \leq r_o \Rightarrow |x(t)| \leq r \quad \forall t \geq T.$$

In addition to guaranteeing GUB and GUUB of the trajectories of (5), Assumptions 1-3 will allow us to conclude global practical synchronization, which corresponds to global asymptotic stability (GAS) of a neighborhood of the set

$$\mathcal{A} := \{x \in \mathbb{R}^n : x_1 = x_2 = \dots = x_n\}.$$

*Definition 3 (Global Practical Synchronization):* System (5) achieves global practical synchronization if, for each  $\delta > 0$ , there exists  $\lambda^* > 0$  such that, for each  $\lambda \geq \lambda^*$  and  $t \mapsto d(t)$  satisfying (3), the set  $B_\delta(\mathcal{A}) := \{x \in \mathbb{R}^n : |x|_{\mathcal{A}} \leq \delta\}$  is GAS.

#### IV. MAIN RESULT 1: BOUNDEDNESS PROPERTIES

The main result of this section establishes GUB and GUUB of the trajectories to (5) (see Theorem 1). Based on such a result, in the next section, we prove global practical synchronization for (5) (see Theorem 2).

The following Lemma is key to find the right conditions that  $(c_1, c_2)$  in Assumption 3 need to verify.

*Lemma 1:* Let  $L_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$  be the Laplacian matrix of a directed and strongly connected graph. Let  $v_\ell := [v_1 \ v_2 \ \dots \ v_{n_\ell}]^\top \in \mathbb{R}^{n_\ell}$  such that  $v_\ell^\top L_\ell = 0$ . Then,  $v_i > 0$  for all  $i \in \{1, 2, \dots, n_\ell\}$ ,  $\text{Ker}(V_\ell L_\ell + L_\ell^\top V_\ell) = \text{Span}\{\mathbf{1}_{n_\ell}\}$ , and  $Q := V_\ell L_\ell + L_\ell^\top V_\ell$  is positive semi-definite, where  $V_\ell := \text{diag}\{v_\ell\}$ .  $\square$

*Theorem 1 (GUB & GUUB):* Consider system (5) such that Assumptions 1 and 2 and (3) hold. Then, the solutions to (5) are GUB and GUUB according to Definitions 1 and 2, respectively, provided that Assumption 3 holds with

$$c_1 > \frac{2\bar{d}|V_\ell \mathbf{1}_{n_\ell}|}{v_k}. \quad (8)$$

$\square$

Due to space constraints, a sketch of a proof is provided in the appendix, the full proof will appear elsewhere.

According to Theorem 1, for each  $\gamma \geq \gamma_o > 0$ , the solutions to (5) starting from a ball  $B_{r_o}$  of radius  $r_o > 0$  remain bounded, i.e., they remain in a bigger ball  $B_R$  of radius  $R(r_o, \gamma_o) > 0$ . Furthermore, those trajectories reach and remain in a ball  $B_r$  of radius  $r(\gamma_o) > 0$ , which does not depend on  $r_o$ , after a time  $T(r_o, \gamma_o) > 0$  which can depend on both  $r_o$  and  $\gamma_o$ . This shows, in particular, that the presence of input perturbations  $t \mapsto d(t)$  satisfying (3) cannot render the trajectories to (5) unbounded, thanks to Assumption 3 establishing the existence of a well-located robust node. In particular, we show that imposing an appropriate robustness property on a single agent, belonging to the leading strongly-connected component of a network whose graph contains a spanning tree, is enough to handle a class of perturbations that can affect the entire network.

*Remark 2:* The proof of Theorem 1 is constructive and explicit upper bounds of  $(r(r_o), T(r_o, \gamma_o))$ , and  $R(r_o, \gamma_o)$  can be deduced from the proof. Furthermore, we can always choose  $R(r_o, \gamma_o)$  so that  $r_o \mapsto R(r_o, \gamma_o)$  is continuous and non-decreasing.  $\bullet$

#### V. MAIN RESULT 2: PRACTICAL SYNCHRONIZATION

To analyse practical synchronization, we rely on an equivalent representation of the network dynamics (5), following [12]. That is, after Assumption 2, we conclude that  $\lambda_1(L) = 0$  has multiplicity one, and  $L$  admits the Jordan-block decomposition

$$L = V \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix} V^{-1},$$

where  $\Lambda \in \mathbb{R}^{(n-1) \times (n-1)}$  is composed by the Jordan blocks corresponding to the  $n$  positive eigenvalues. Moreover,  $V$  and  $V^{-1}$  correspond to

$$V = [\mathbf{1}_n \quad U], \quad V^{-1} = \begin{bmatrix} v_l^\top \\ U^\dagger \end{bmatrix},$$

where  $U \in \mathbb{R}^{n \times n-1}$ , and

$$v_l^\top U = 0, \quad U^\dagger U = I_{n-1}.$$

Now, using  $V^{-1}$ , we define the new coordinates

$$\begin{bmatrix} x_m \\ e_v \end{bmatrix} := \begin{bmatrix} v_l^\top x \\ U^\dagger x \end{bmatrix}.$$

In these new coordinates, since  $VV^{-1} = \mathbf{1}_n v_l^\top + UU^\dagger$ , the state vector  $x$  can be expressed as

$$x = \mathbf{1}_n x_m + U e_v. \quad (9)$$

Therefore, the network dynamics consists of two interconnected dynamics: the ‘‘average-state’’ dynamics (the dynamics of  $x_m$ ) and the projected ‘‘synchronization-error’’ dynamics (the dynamics of  $e_v$ ). Under this transformation, we obtain

$$\dot{x}_m = F_m(d, x_m, e_v) \quad (10a)$$

$$\dot{e}_v = -\gamma \Lambda e_v + G_e(d, x_m, e_v), \quad (10b)$$

where

$$\begin{aligned} F_m(d, x_m, e_v) &= v_l^\top [F(\mathbf{1}_n x_m + U e_v) + d], \\ G_e(d, x_m, e_v) &= U^\dagger [F(\mathbf{1}_n x_m + U e_v) + d]. \end{aligned}$$

In view of (9), when  $e_v = 0$ , we conclude that  $x = \mathbf{1}_n x_m$ , thus,

$$\mathcal{A} = \{x \in \mathbb{R}^n : e_v = U^\dagger x = 0\}.$$

Consequently, to analyse global practical synchronization for (5) (i.e., GAS of  $B_\delta(\mathcal{A})$ ), it is enough to analyze global practical asymptotic stability (GPAS) of the origin  $\{e_v \in \mathbb{R}^{n-1} : e_v = 0\}$  for (10b). More precisely, based on [21], we establish the following property.

*Property 1 (GPAS of  $\{e_v = 0\}$ ):* For each  $\delta > 0$  and  $R_m > 0$ , there exists  $\lambda^* > 0$  and  $\beta \in \mathcal{KL}$  such that, for each  $\lambda \geq \lambda^*$ , the solutions to (10b) satisfy

$$|e_v(t)| \leq \delta + \beta(|e_v(0)|, t) \quad \forall t \geq 0, \quad \forall e_v(0) \in \mathbb{R}^{n-1},$$

for any  $t \mapsto x_m(t)$  solution to (10a) with  $|x_m(0)| \leq R_m$ , and for any  $t \mapsto d(t)$  verifying (3). •

*Remark 3:* In view of the GUB and GUUB properties established in Theorem 1, when  $x \in B_r$ , then

$$\begin{aligned} |e_v| &= |U^\dagger x| \leq |U^\dagger| r =: r_e, \\ |x_m| &= |v_\ell^\top x| \leq |v_\ell| r =: r_a. \end{aligned}$$

Similarly, when  $x \in B_R$ , we conclude that

$$\begin{aligned} |e_v| &= |U^\dagger x| \leq |U^\dagger| R =: R_e, \\ |x_m| &= |v_\ell^\top x| \leq |v_\ell| R =: R_a. \end{aligned} \quad (11)$$

As a result, we conclude that

$$|e_v(t)| \leq r_e(\gamma_o), \quad |x_m(t)| \leq r_a(\gamma_o) \quad \forall t \geq T(r_o, \gamma_o), \quad (12)$$

$$|e_v(t)| \leq R_e(r_o, \gamma_o), \quad |x_m(t)| \leq R_a(r_o, \gamma_o) \quad \forall t \geq 0. \quad (13)$$

The inequalities in (12)-(13) are key to verify Property 1, and thus to establish global practical synchronization of the network (5).

*Theorem 2:* Consider the network (5) such that Assumptions 1, 2 and (3) hold. Assume further that Assumption 3 holds for  $(c_1, c_2)$  as in Theorem 1. Then the network achieves global practical synchronization. □

*Proof:* To prove global practical synchronization of (5), we verify Property 1. Since the eigenvalues of  $\Lambda$  have positive real parts, we conclude that the origin for  $\dot{e}_v = -\Lambda e_v$  is globally exponentially stable. As a result, there exists a positive definite matrix  $P \in \mathbb{R}^{(n-1) \times (n-1)}$  such that

$$P\Lambda + \Lambda P^\top \geq I_{n-1}.$$

Let  $V(e_v) := e_v^\top P e_v$  and note that

$$\lambda_m |e_v|^2 \leq V(e_v) \leq \lambda_M |e_v|^2,$$

where  $\lambda_m$  and  $\lambda_M$  are, respectively, the minimum and the maximum eigenvalues of  $P$ .

The total derivative of  $V$  along (10b) satisfies

$$\dot{V}(e_v) \leq -\gamma |e_v|^2 + 2\lambda_M |G_e| |e_v|. \quad (14)$$

Now, we let a solution  $(x_m, e_v)$  to (10a)-(10b) such that  $|x_m(0)| \leq R_m$ . According to Theorem 1 and in view of Remark 2, (11), and (13), we can write

$$\begin{aligned} |e_v(t)| &\leq \bar{R}_e(|e_v(0)|) := R_e(R_m + |e_v(0)|, \gamma_o) \quad \forall t \geq 0, \\ |x_m(t)| &\leq \bar{R}_a(|e_v(0)|) := R_a(R_m + |e_v(0)|, \gamma_o) \quad \forall t \geq 0. \end{aligned}$$

As a result, we let

$$\bar{G}(|e_v(0)|) := \bar{R}_e(|e_v(0)|) \sup \left\{ |G_e| : \begin{array}{l} |d| \leq \bar{d} \\ |x_m| \leq \bar{R}_a(|e_v(0)|) \\ |e_v| \leq \bar{R}_e(|e_v(0)|) \end{array} \right\}.$$

Note that  $\bar{G}$  is continuous and non-decreasing with respect to its argument. As a result, we obtain

$$\begin{aligned} \dot{V}(e_v) &\leq -\gamma |e_v|^2 + 2\lambda_M \bar{G}(|e_v(0)|) \\ &\leq -\frac{\gamma}{\lambda_M} V(e_v) + 2\lambda_M \bar{G}(|e_v(0)|). \end{aligned}$$

As a consequence, on the interval  $[0, T]$ , we have

$$\begin{aligned} V(e_v(t)) &\leq \exp\left(-\frac{\gamma t}{\lambda_M}\right) V(e_v(0)) \\ &\quad + \left(1 - \exp\left(-\frac{\gamma t}{\lambda_M}\right)\right) \frac{2\lambda_M^2 \bar{G}(|e_v(0)|)}{\gamma} \\ &\leq \lambda_M |e_v(0)|^2 + \frac{2\lambda_M^2}{\gamma} \bar{G}(|e_v(0)|) \\ &= \lambda_M |e_v(0)|^2 + \frac{2\lambda_M^2}{\gamma} [\bar{G}(|e_v(0)|) - \bar{G}(0)] + 2 \frac{\lambda_M^2 \bar{G}(0)}{\gamma}. \end{aligned}$$

Hence, for  $\gamma \geq \gamma_o > 0$ , we have

$$\begin{aligned} V(e_v(t)) &\leq \lambda_M |e_v(0)|^2 + \frac{2\lambda_M^2}{\gamma_o} [\bar{G}(|e_v(0)|) - \bar{G}(0)] \\ &\quad + 2 \frac{\lambda_M^2 \bar{G}(0)}{\gamma}. \end{aligned}$$

At this point, by letting

$$\kappa(a) := \lambda_M a^2 + \frac{2\lambda_M^2}{\gamma_o} [\bar{G}(a) - \bar{G}(0)],$$

we conclude that, for each  $\delta > 0$ , there exists  $\gamma_1^* \geq \gamma_o$  such that, for each  $\gamma \geq \gamma_1^*$ , we have

$$|e_v(t)| \leq \kappa(|e_v(0)|) + \delta \quad \forall t \in [0, T]. \quad (15)$$

We now focus on the interval  $[T, +\infty)$ . After Theorem 1 and in view of (12), by letting

$$\hat{G} := r_e \sup \{|G_e| : |d| \leq \bar{d}, |x_m| \leq r_a, |e_v| \leq r_e\},$$

we conclude that

$$\dot{V}(e_v(t)) \leq -\frac{\gamma}{\lambda_M} V(e_v(t)) + 2\lambda_M \hat{G} \quad \forall t \geq T.$$

The latter implies that

$$\begin{aligned} V(e_v(t)) &\leq \exp\left(-\frac{\gamma(t-T)}{\lambda_M}\right) V(e_v(T)) \\ &\quad + \left(1 - \exp\left(-\frac{\gamma(t-T)}{\lambda_M}\right)\right) \frac{2\lambda_M^2 \hat{G}}{\gamma} \\ &\leq \exp\left(-\frac{\gamma(t-T)}{\lambda_M}\right) V(e_v(T)) + \frac{2\lambda_M^2 \hat{G}}{\gamma}. \end{aligned}$$

Hence, for  $\gamma \geq \gamma_o > 0$ , we obtain

$$V(e_v(t)) \leq \exp^{-\frac{\gamma_o(t-T)}{\lambda_M}} V(e_v(T)) + \frac{2\lambda_M^2 \hat{G}}{\gamma}.$$

At this point, by letting

$$\hat{\beta}(a, t) := \exp^{-\frac{\gamma_o t}{\lambda_M}} \lambda_M a^2 \leq \exp^{-\frac{\gamma_o t}{\lambda_M}} \kappa(a),$$

we conclude that, for each  $\delta > 0$ , there exists  $\gamma_2^* \geq \gamma_o$  such that, for each  $\gamma \geq \gamma_2^*$ , we have

$$|e_v(t)| \leq \hat{\beta}(|e_v(T)|, t - T) + \delta \quad \forall t \geq T. \quad (16)$$

As a result, Property 1 is verified for  $\gamma^* \geq \sup\{\gamma_1^*, \gamma_2^*\}$  and for

$$\beta(a, t) := \exp^{-\frac{\gamma_o t}{\lambda_M}} \kappa(a) \exp^{T-t}.$$

## VI. NUMERICAL EXAMPLE

To illustrate our theoretical results, we present a numerical example involving a network of  $n = 4$  nonholonomic mobile robots, with kinematics given by

$$\begin{aligned} \dot{r}_{x_i} &= v_i \cos(\theta_i) \\ \dot{r}_{y_i} &= v_i \sin(\theta_i) \\ \dot{\theta}_i &= \omega_i \end{aligned} \quad i \in \{1, 2, \dots, n\},$$

where  $(r_{x_i}, r_{y_i})$  is the position of the center of the  $i$ th robot and  $\theta_i$  is its orientation. The inputs  $v_i$  and  $\omega_i$  are, respectively, the linear and angular velocities of the  $i$ th robot.

To render each system strictly semi-passive, we apply a preliminary control law and focus on the dynamics of the position of a point  $x_i$  located at a distance  $a > 0$  off the axis joining the wheels. In other words, let

$$x_i := \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = \begin{bmatrix} r_{x_i} + a \cos(\theta_i) \\ r_{y_i} + a \sin(\theta_i) \end{bmatrix}.$$

Then, we define the preliminary control as,

$$\begin{aligned} v_i &:= -\left( \frac{x_{1i}(x_{1i}^2 - \lambda_i)}{x_{1i}^4 + 1} - u_{1i} \right) \cos(\theta_i) \\ &\quad - \left( \frac{x_{2i}(x_{2i}^2 - \lambda_i)}{x_{2i}^4 + 1} - u_{2i} \right) \sin(\theta_i) \end{aligned}$$

and

$$\begin{aligned} \omega_i &:= \frac{1}{a} \left[ -\left( \frac{x_{1i}(x_{1i}^2 - \lambda_i)}{x_{1i}^4 + 1} - u_{1i} \right) \sin(\theta_i) \right. \\ &\quad \left. - \left( \frac{x_{2i}(x_{2i}^2 - \lambda_i)}{x_{2i}^4 + 1} - u_{2i} \right) \cos(\theta_i) \right], \end{aligned}$$

where  $\lambda_i > 0$ . Then,

$$\begin{aligned} \dot{x}_{1i} &= -\frac{x_{1i}(x_{1i}^2 - \lambda_i)}{x_{1i}^4 + 1} + u_{1i} \quad \forall i \in \{1, 2, 3, 4\} \\ \dot{x}_{2i} &= -\frac{x_{2i}(x_{2i}^2 - \lambda_i)}{x_{2i}^4 + 1} + u_{2i} \quad \forall i \in \{1, 2, 3, 4\}. \end{aligned} \quad (17)$$

As a result, the dynamics governing every  $x_{1i}$  can be shown to be strict semi-passive using the quadratic storage function  $V(x_{1i}) = x_{1i}^2$ . The same property holds for the dynamics governing the  $x_{2i}$ s. Hence, Assumption 1 is verified for network dynamics governing  $x_1 := [x_{11} \ x_{12} \ x_{13} \ x_{14}]^\top$  and for the dynamics  $x_2 := [x_{21} \ x_{22} \ x_{23} \ x_{24}]^\top$ , separately.

Next, we assume that the control inputs  $(u_{1i}, u_{2i})$  are affected by disturbances and are given by

$$\begin{aligned} u_{1i} &:= -\gamma \sum_{j=1}^4 a_{ij}(x_{1i} - x_{1j}) + \nu_{1i} + d_i(t) \\ u_{2i} &:= -\gamma \sum_{j=1}^4 a_{ij}(x_{2i} - x_{2j}) + \nu_{2i} + d_i(t), \end{aligned}$$

where  $\gamma > 0$  is the coupling gain and  $a_{ij} \geq 0$  are the entries of the laplacian matrix

$$L := \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

whose graph verifies Assumption 2.

That is, the terms  $-\gamma \sum_{j=1}^4 a_{ij}(x_{1i} - x_{1j}) + \nu_{1i}$  and  $-\gamma \sum_{j=1}^4 a_{ij}(x_{2i} - x_{2j}) + \nu_{2i}$  correspond to the control law, which is affected by the disturbance  $d_i(t)$ . For the simulations, we set

$$\lambda := [0.5 \ 0.3 \ 1.2 \ 0.8]^\top, \quad d_i(t) := \frac{\arctan(t)}{\pi}.$$

When  $\nu_{1i} = \nu_{2i} = 0$  for all  $i \in \{1, 2, 3, 4\}$ , the network trajectories are given in Figure 1. Clearly, the trajectories

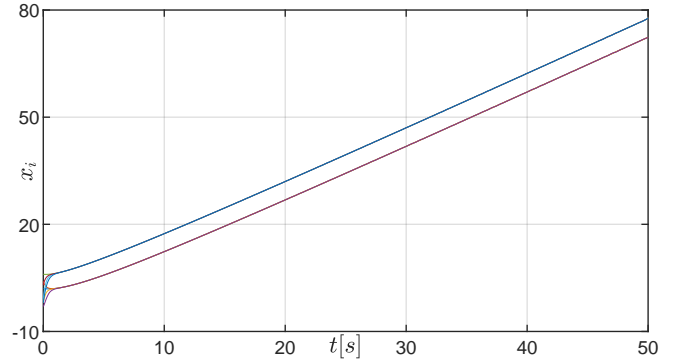


Fig. 1. Trajectories of  $x_i = [x_{1i}, x_{2i}]$ ,  $i \in \{1, 2, 3, 4\}$ , for  $\gamma = 1$ .

diverge as  $t \rightarrow \infty$ . Hence, we do not guarantee GUB nor GUUB.

Next, we propose to include a robust node to the network whose state is  $x_c$ , by interconnecting it with node  $i = 1$ , so that they form the leading strongly-connected component of the new network; see Figure 2. More precisely, we let

$$\begin{aligned} \nu_{1i} &= \nu_{2i} = 0, \quad \forall i \in \{2, 3, 4\} \\ \nu_{11} &= -\gamma(x_{11} - x_{1c}), \quad \nu_{21} = -\gamma(x_{21} - x_{2c}), \end{aligned}$$

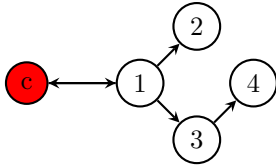


Fig. 2. The graph representing the new-network interconnections, the red node is the added robust node. The leading strongly-connected component is composed of nodes  $\{1, c\}$ .

and we assume that the behavior of  $x_c$  is governed by

$$\dot{x}_{1c} = -x_{1c}^3 + 3x_{1c} + u_{1c}, \quad \dot{x}_{2c} = -x_{2c}^3 + 3x_{2c} + u_{2c}. \quad (18)$$

For the new network (17)-(18), Assumption 1 is verified for the network dynamics governing  $[x_{1c} \ x_{11} \ x_{12} \ x_{13} \ x_{14}]^\top$  and for the dynamics  $[x_{2c} \ x_{21} \ x_{22} \ x_{23} \ x_{24}]^\top$  separately. To see this, we can simply use the storage function  $V(x) := x^2$ . Now, to verify Assumption 2 for the new graph, we let

$$u_{1c} := -\gamma(x_{1c} - x_{11}), \quad u_{2c} := -\gamma(x_{2c} - x_{21}).$$

Next, we can show that Assumption 3 is verified for  $k = c$ . Indeed, using the storage functions  $V(x_{1c}) := x_{1c}^2$ , we conclude that (4) holds for  $H_c(x_{1c}) = \psi_c(|x_{1c}|) = x_{1c}^2(x_{1c}^2 - 3)$ . Note that, when  $|x_{1c}| \geq \rho_k = \rho_c = 3$ , then  $\psi_{1c}(|x_{1c}|) = x_{1c}^2(x_{1c}^2 - 3) > c_1|x_{1c}|$  for  $c_1 = 2$  and  $c_2 = 0$ . The exact same properties hold for  $x_{2c}$ . As a result, condition (8) in Theorem 1 holds for  $\gamma_o = 1$ .

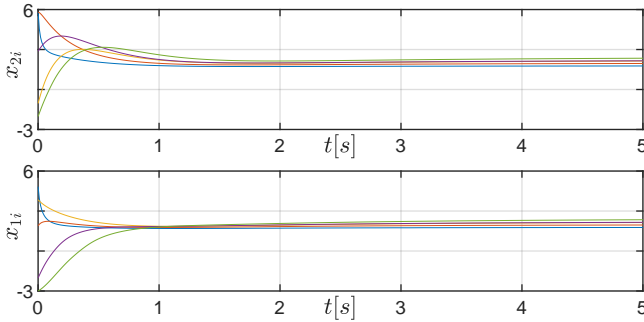


Fig. 3. Trajectories of the new network for  $\gamma = 5$ .

In Figure 3, the trajectories of the new network are shown to remain bounded. Furthermore, it can be seen that after a certain time, the trajectories approach a common point on the plane. Consequently, a practical synchronisation is achieved. That is, the error vector  $e_v$  becomes less and less significant as the coupling gain increases.

## VII. CONCLUSION

This paper studied networks of heterogeneous nonlinear systems in the presence of bounded input perturbations. We proposed a framework to guarantee GUUB and GUB of the solutions. These conclusions are key to ensure global practical synchronization when the coupling gain  $\gamma$  can

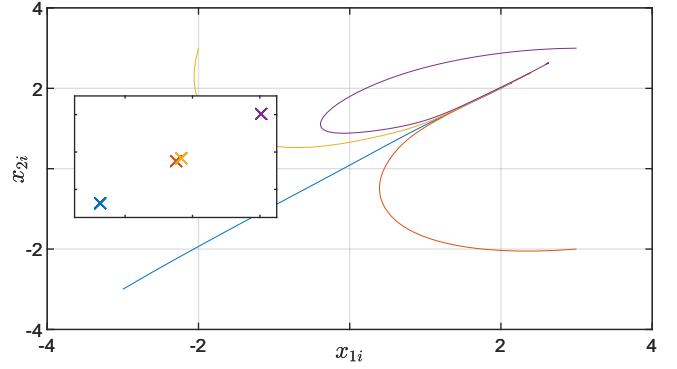


Fig. 4. Phase portrait of the new network trajectories for  $\gamma = 5$ .

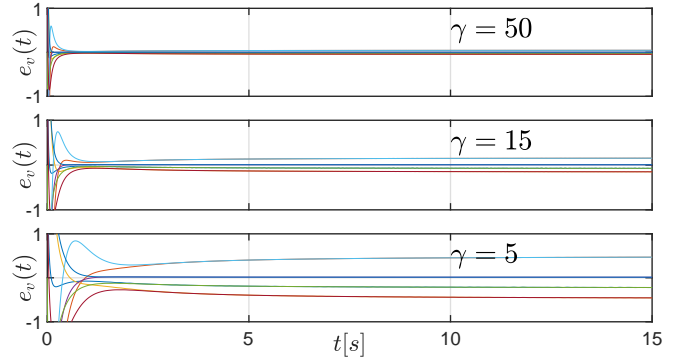


Fig. 5. The behavior of the synchronization errors. In the bottom figure  $\gamma = 5$ , in the middle figure  $\gamma = 15$ , and in the top figure  $\gamma = 50$ .

be tuned. The presented results show that it is possible to handle bounded input perturbations, that can affect the entire network, by making only a single agent robust in a certain sense. These results will be extended to other forms of disturbances, not necessarily affecting the input. Further extensions to systems with nonlinear interconnections will also be considered in future works.

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## APPENDIX

### SKETCH OF PROOF OF THEOREM 1

**GUUB:** First, we show GUUB for the leading dynamics (7a). After Assumption 1, there exists a storage function  $V_i$  such that, along the trajectories of (1), we have

$$\begin{aligned} \dot{V}_i(x_{\ell i}) &\leq 2u_i^\top x_{\ell i} - H_i(x_{\ell i}), \\ H_i(x_{\ell i}) &\geq \psi_i(|x_{\ell i}|) \quad \forall |x_{\ell i}| \geq \rho_i. \end{aligned} \quad (19)$$

Now, from (19), we obtain, for all  $x_\ell \in \mathbb{R}^{n_\ell}$ ,

$$\dot{W}(x_\ell) \leq 2 \sum_{i=1}^{n_\ell} v_i u_i^\top x_{\ell i} - \sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}). \quad (20)$$

The first term on the right-hand side of (20) satisfies

$$\sum_{i=1}^{n_\ell} v_i u_i^\top x_{\ell i} = -\gamma x_\ell^\top L_\ell^\top V_\ell x_\ell + \sum_{i=1}^{n_\ell} v_i d_{\ell i}(t) x_{\ell i}. \quad (21)$$

It follows that

$$\begin{aligned} \dot{W}(x_\ell) &\leq -\sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}) - \gamma x_\ell^\top [L_\ell^\top V_\ell + V_\ell L_\ell] x_\ell \\ &\quad + 2 \sum_{i=1}^{n_\ell} v_i d_{\ell i}(t) x_{\ell i} \\ &\leq -\sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}) - \gamma x_\ell^\top Q x_\ell + 2 \sum_{i=1}^{n_\ell} v_i d_{\ell i}(t) x_{\ell i}. \end{aligned} \quad (22)$$

Furthermore, using Lemma 1, we can show that

$$-x_\ell^\top Q x_\ell \leq -\lambda_2(Q) |x_\ell|_{\mathcal{A}_\ell}^2,$$

where  $|x_\ell|_{\mathcal{A}_\ell}$  denotes the distance of  $x_\ell$  to the set  $\mathcal{A}_\ell := \{x_{\ell 1} = x_{\ell 2} = \dots = x_{\ell n_\ell}\}$  and  $\lambda_2(Q)$  is the second smallest eigenvalue of  $Q$ . Next, for a given  $k \leq n_\ell$ , note that

$$\begin{aligned} 2 \sum_{i=1}^{n_\ell} v_i d_{\ell i}(t) x_{\ell i} &= 2d_\ell(t)^\top V_\ell x_\ell \\ &= 2d_\ell(t)^\top V_\ell (x_\ell - 1_{n_\ell} x_k) + 2d_\ell(t)^\top V_\ell 1_{n_\ell} x_k. \end{aligned}$$

Which we can show that it yields to

$$\begin{aligned} \dot{W}(x_\ell) &\leq -\sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}) - \gamma \lambda_2(Q) |x_\ell|_{\mathcal{A}_\ell}^2 + \theta_2 |x_k| \\ &\quad + \theta_1 |x_\ell|_{\mathcal{A}_\ell}, \end{aligned}$$

where

$$\theta_1 := 2(1 + \sqrt{n_\ell}) \bar{d} |V_\ell|$$

and

$$\theta_2 := 2\bar{d} |V_\ell 1_{n_\ell}|.$$

Next, we let node  $k$  be the one used in Assumption 3. Hence, when  $|x_k| < \rho_k$ , using the constant

$$C_1 := \sum_{i=1}^{n_\ell} \max_{|x_{\ell i}| \leq \rho_i} \{-v_i H_i(x_{\ell i})\} + \theta_2 \rho_k > 0$$

and (22), we obtain

$$\dot{W}(x_\ell) \leq C_1 - \gamma \lambda_2(Q) |x_\ell|_{\mathcal{A}_\ell}^2 + \theta_1 |x_\ell|_{\mathcal{A}_\ell}. \quad (23)$$

Secondly, when  $|x_k| \geq \rho_k$ ,

$$H_k(x_k) \geq \psi(|x_k|) \geq c_1 |x_k| - c_2.$$

For  $c_1$  satisfying the first inequality in (8), and

$$C_2 := c_2 + \sum_{i=1}^{n_\ell} \max_{|x_{\ell i}| \leq \rho_i} \{-v_i H_i(x_{\ell i})\},$$

we obtain

$$\dot{W}(x_\ell) \leq C_2 - \gamma \lambda_2(Q) |x_\ell|_{\mathcal{A}_\ell}^2 + \theta_1 |x_\ell|_{\mathcal{A}_\ell}. \quad (24)$$

Hence, for all  $x_\ell \in \mathbb{R}^{n_\ell}$ ,

$$\dot{W}(x_\ell) \leq C - \gamma \lambda_2(Q) |x_\ell|_{\mathcal{A}_\ell}^2 + \theta_1 |x_\ell|_{\mathcal{A}_\ell}, \quad (25)$$



where  $C := \max\{C_1, C_2\}$ . In turn, given  $\gamma_o > 0$  and  $\epsilon > 0$ , for all  $\gamma \geq \gamma_o$ , we show that

$$\dot{W}(x_\ell) \leq -\epsilon \quad \forall x_\ell \notin \mathcal{C}, \quad (26)$$

where  $\mathcal{C} := \{x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell|_{\mathcal{A}_\ell} \leq R_e\}$ , and

$$R_e := \sqrt{\frac{\theta_1 + \sqrt{\theta_1^2 + 4\gamma_o\lambda_2(Q)}(C + \epsilon)}{2\gamma_o\lambda_2(Q)}}.$$

From (26), we can show that  $x_\ell$  converges to the set  $\mathcal{C}$  in finite time. On the latter set, when  $|x_\ell|$  is larger than a certain constant, then necessarily every agent must satisfy  $|x_{\ell i}| > \rho_{\ell i}$ . Hence,  $H_i(x_{\ell i}) \geq \psi_i(|x_{\ell i}|)$  for all  $i \in \{1, 2, \dots, n_\ell\}$ , and thus the  $C$  in (26) becomes negative. Hence, we prove that  $\dot{W}$  is negative when ever  $|x_\ell|$  is larger than a certain constant, which allows us to conclude GUUB.

We now establish GUUB for the non-leading component governed by (7b). After Lemma 2, we have that

$$S := PM_f + M_f^\top P, \quad P := \text{diag} \left\{ M_f^{-\top} \mathbf{1}_n \right\} \text{diag} \left\{ M_f^{-1} \mathbf{1}_n \right\}^{-1}$$

are symmetric and positive definite. Let  $p_i$  be the  $i$ th diagonal element of  $P$  and  $Z(x_f) := \sum_{i=1}^{n_f} p_i v_i x_{f_i}^2$ . Along the trajectories of (7b), we obtain

$$\begin{aligned} \dot{Z}(x_f) &\leq - \sum_{i=1}^{n_f} p_i H_{f,i}(x_{f_i}) - \gamma x_f^\top [PM_f + M_f^\top P] x_f \\ &\quad + 2\gamma x_f^\top [PA_{\ell f}] x_\ell + 2d_f(t)^\top P x_f. \end{aligned} \quad (27)$$

On the one hand, we already established the existence of  $r_\ell > 0$  and  $T_\ell > 0$  such that  $|x_\ell(t)| < r_\ell$  for all  $t \geq T_\ell$ . On the other, for all  $|x_\ell| \leq r_\ell$ , we have

$$\dot{Z}(x_f) \leq H_f - \gamma \lambda_1(S) |x_f|^2 + 2 \left( \gamma \bar{p} r_\ell + \bar{d} |P| \right) |x_f|, \quad (28)$$

where  $\bar{p} := |PA_{\ell f}|$  and

$$H_f := \sum_{i=1}^{n_f} \max_{|x_{f_i}| \leq \rho_i} \{-v_i H_i(x_{f_i})\}.$$

Letting  $\theta_3 := 2(\gamma \bar{p} r_\ell + \bar{d} |P|)$  and given  $\epsilon > 0$ , we see that, for all  $\gamma \geq \gamma_o$  and for all  $x_f$  and  $x_\ell$  such that  $|x_\ell| \leq r_\ell$  and

$$|x_f| > \beta_1 := \sqrt{\frac{\theta_3 + \sqrt{\theta_3^2 + 4\gamma\lambda_1(S)}(H_f + \epsilon)}{2\gamma\lambda_1(S)}},$$

we conclude that  $\dot{Z}(x_f) \leq -\epsilon$ . Hence, GUUB follows.

**GUB:** Since GUUB holds, then given  $r_o > 0$  and  $\gamma_o > 0$ , for all  $\gamma \geq \gamma_o$ ,

$$|x_\ell(0)| \leq r_o \implies |x_\ell(t)| \leq r_\ell(\gamma_o) \quad \forall t \geq T_\ell(r_o, \gamma_o).$$

Furthermore, integrating (25), on the interval  $[0, T_\ell(r_o, \gamma_o)]$  we obtain,

$$W(x_\ell(t)) \leq CT_\ell + \theta_1 \int_0^{T_\ell} |x_\ell(t)|_{\mathcal{A}_\ell} dt + W(x_\ell(0)).$$

Hence, if we let  $\sigma_\ell := \max\{W(y) : |y| \leq r_o\}$ , for all  $t \geq 0$ , it follows that  $|x_\ell(t)| \leq \bar{R}_\ell$ , where

$$\bar{R}_\ell := \left[ \min_i \{\underline{\alpha}_i\} \right]^{-1} \left( \sigma_\ell + CT_\ell + \theta_1 \int_0^{T_\ell} |x_\ell(t)|_{\mathcal{A}_\ell} dt + r_\ell \right).$$

As a result  $|x_\ell(t)| \leq \bar{R}_\ell$  for all  $t \geq 0$ .

Next, for the solutions to (7b), for any  $\gamma > \gamma_o$  and  $|x_f(0)| \leq r_o$ , we know that  $|x_f(t)| \leq r_f$  for all  $t \geq T_f(\gamma_o, r_o)$ . Also, from the previous proof we have

$$\dot{Z}(x_f) \leq -\gamma \lambda_1(S) |x_f|^2 + 2 \left( \gamma \bar{p} \bar{R}_\ell + |2d_f(t)^\top P| \right) |x_f| + H_f.$$

As a result, when  $|x_f| \geq d_f := \sqrt{\frac{\theta_4 + \sqrt{\theta_4^2 + 4\gamma\lambda_1(S)}H_f}{2\gamma\lambda_1(S)}}$ , where  $\theta_4 := 2(\gamma |PA_{\ell f}| \bar{R}_\ell + \bar{d} |P|)$ , then  $\dot{Z}(x_f) \leq 0$ . Hence,

$$\begin{aligned} Z(x_f(t)) &\leq \max\{\sigma_{f_o}, \sigma_f\} \quad \forall t \geq 0, \\ \sigma_f &:= \max\{Z(x_f) : |x_f| \leq d_f\}, \\ \sigma_{f_o} &:= \max\{Z(x_f) : |x_f| \leq r_o\}. \end{aligned}$$

In turn, for each  $t \geq 0$ , we have

$$|x_f(t)| \leq R_f := \left[ \left( \sum_{i=1}^{n_f} p_i \right) \min_i \{\underline{\alpha}_i\} \right]^{-1} \left( \max\{\sigma_{f_o}, \sigma_f\} \right).$$

#### PROOF OF LEMMA 1

After [22, Theorem C.3.], the matrix  $L_\ell$  is irreducible if and only if the corresponding graph is strongly connected. Furthermore, since the off-diagonal elements of  $L_\ell$  are nonpositive and  $L_\ell \mathbf{1}_{n_\ell} = 0$ , we conclude that  $L_\ell$  is a singular M-matrix. Hence, using [23, Theorem 4.31], we conclude that the only left eigenvector of  $L_\ell$  corresponding to the null eigenvalue has strictly positive entries and the matrix  $V_\ell L_\ell + L_\ell^\top V_\ell$  is positive semi-definite. Finally, to show that  $\mathbf{1}_{n_\ell}$  spans the kernel of  $V_\ell L_\ell + L_\ell^\top V_\ell$ , we start noticing that  $(V_\ell L_\ell + L_\ell^\top V_\ell) \mathbf{1}_{n_\ell} = 0$ . Furthermore, each off-diagonal element of  $V_\ell L_\ell + L_\ell^\top V_\ell$  is given by  $a_{ij} v_i + a_{ji} v_j$ ,  $(i, j) \in \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ . As a result,  $V_\ell L_\ell + L_\ell^\top V_\ell$  is the Laplacian matrix associated with a bi-directional graph that is as connected as the graph of  $L_\ell$  disregarding the direction of the interconnections. Since the graph of  $L_\ell$  is already a strongly connected graph, we conclude that  $\text{rank}(V_\ell L_\ell + L_\ell^\top V_\ell) = n_\ell - 1$ .

The following result is inspired from [23, Section 4.3.5].

**Lemma 2:** Let  $M \in \mathbb{R}^{n \times n}$  be a non-singular M-matrix. Then, the matrices

$$\begin{aligned} S &:= RM + M^\top R, \\ R &:= \text{diag} \{ M^{-\top} \mathbf{1}_n \} \left( \text{diag} \{ M^{-1} \mathbf{1}_n \} \right)^{-1} \end{aligned}$$

are positive definite.  $\square$