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# Collapsing-ring blowup solutions for the nonlinear heat equation

June 12, 2024

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## Abstract

In this paper, we construct a singular standing ring solution of the nonlinear heat in the radial case. We give rigorous proof for the existence of a ring blow-up solution in finite time. This result was predicted formally by Baruch, Fibich and Gavish [BFG10]. We also prove the stability of these dynamics among radially symmetric solutions.

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## 1 Introduction

In this paper, we consider the following nonlinear heat equation,

$$(1.1) \quad \begin{cases} \partial_t u &= \Delta u + |u|^{p-1}u, \\ u(., 0) &= u_0 \in L^\infty(\mathbb{R}), \end{cases}$$

where  $u(t) : x \in \mathbb{R}^N \rightarrow u(x, t) \in \mathbb{R}$  and  $p > 1$ . Equation (1.1) is considered as a model for many physical situations such as heat transfer, combustion theory, thermal explosion, etc. (see more in Kapila [Kap80], Kassoy and Poland [KP80], Bebernes and Eberly [BE89]).

The local Cauchy problem for equation (1.1) can be solved within  $L^\infty(\mathbb{R})$ . Furthermore, it can be shown that the solution  $u(t)$  exists either in the interval  $[0, +\infty)$  or within  $[0, T)$  where  $T < +\infty$ . In the latter case,  $u$  undergoes a finite-time blow-up, indicating that

$$\lim_{t \rightarrow T} \|u(t)\|_{L^\infty} = \infty.$$

Despite the extensive research conducted on these equations over the past four decades, it is crucial to acknowledge that no single review can comprehensively cover all aspects. In this context, our attention is directed towards the development of solutions displaying a distinct blow-up behavior. Consequently, our references will be limited to prior work within this scope. Interested readers may refer to [QS07] for a comprehensive sight of research on equation (1.1).

The pioneering work by Giga and Kohn [GK85] and [GMS04] yielded the first insight into the asymptotics of blow-up. They established that, up to changing  $u$  by  $-u$ , for each  $K > 0$ , the following holds:

$$(1.2) \quad \lim_{t \rightarrow T} \sup_{|x| \leq K\sqrt{T-t}} \left| (T-t)^{\frac{1}{p-1}} u(x, t) - \kappa \right| = 0, \text{ with } \kappa = (p-1)^{-\frac{1}{p-1}}.$$

Based on a numerical analysis conducted by Berger and Kohn [BK88], it was hypothesized that if the decay pattern is non-exponential, the solution  $u$  to equation (1.1) would converge towards a specific universal profile, denoted as  $f(z)$ . An extensive literature is devoted to the blow-up profile for NLH equation see Vélazquez [Vel92], [Vel93a], [Vel93b], and Zaag [Zaa02a], [Zaa02b] for partial results. In one space dimension, given  $a$  a blow-up point, these are the situations:

1.  $(T-t)^{\frac{1}{p-1}} u(x, t) \equiv \kappa$ .

2. either

$$(1.3) \quad \sup_{|x-a| \leq K\sqrt{(T-t)\log(T-t)}} \left| (T-t)^{\frac{1}{p-1}} u(x, t) - f\left(\frac{x-a}{\sqrt{(T-t)\log(T-t)}}\right) \right| \rightarrow 0,$$

3. or for some  $m \in \mathbb{N}$ ,  $m \geq 2$ , and  $C_m > 0$

$$(1.4) \quad \sup_{|x-a| \leq K(T-t)^{1/2m}} \left| (T-t)^{\frac{1}{p-1}} u(x, t) - f_m\left(\frac{C_m(x-a)}{(T-t)^{1/2m}}\right) \right| \rightarrow 0,$$

as  $t \rightarrow T$ , for any  $K > 0$ , where

$$(1.5) \quad \begin{aligned} f(z) &= (p-1 + b_0 z^2)^{-\frac{1}{p-1}}, \text{ with } b_0 = \frac{(p-1)^2}{4p} \\ f_m(z) &= (p-1 + b|z|^{2m})^{-\frac{1}{p-1}}, \text{ with } b > 0. \end{aligned}$$

In the higher dimensional case, we would like to mention works by Herrero and Vélazquez [HV92a] on the asymptotic behavior of the blow-up solution to equation (1.1) and [Vel93a] on the classification of such behavior. Following these behaviors, there are also interesting examples. One may refer to Nguyen and Zaag [NZ18], for constructed solutions showing a refinement of behavior (1.3) in 2 dimensions. In the supercritical case, we have an example of a single-point blow-up with a degenerate profile given by Merle, Raphaël, and Szeftel [MRS20]. More recently, Merle and Zaag [MZ22] have provided an example of a blow-up solution with a completely new blow-up profile, which is cross-shaped.

Going back to the one-dimensional case, Bricmont and Kupiainen [BK88] construct a solution for (1.1) behaves just like (1.3). Recently, Duong et al. [DNZ23b] revisited the construction of *flat* profile given by  $f_m$  in (1.5) using modulation theory. The methods used in [BK88] were enhanced after by Merle and Zaag [MZ97] by using a more geometrical approach. In general speaking, regarding the linearized equation, Merle and Zaag cope with this problem in two steps:

1. Reduce the problem into a finite-dimensional one.
2. Solve the finite-dimensional problem with a topological shooting argument.

This powerful method is then applied to many other different fields. Interested readers are invited to see [MZ08] and [DNZ23a] for an application in the complex Ginzburg-Landau equation and a more direct way to accomplish the first step in [MZ97]. Besides, Dávila, Del Pino, and Wei [DDPW20] applied this method to deal with the formation of the singularities for harmonic map flow. Among the numerous results, the blow-up behavior of the solution to the non-linear Schrodinger equation remarked by Raphael [Rap06] on a sphere arouses our interest.

Nevertheless, Baruch, Fibich, and Gavish have presented numerical evidence indicating that the nonlinear heat equation (1.1) also allows for singular standing-ring solutions in cases of radially symmetric solutions [BFG10]. We hence like to study (1.1) in radially symmetric settings. The existence and stability of solution blowing up on a sphere were already studied for the  $L^2$ -supercritical nonlinear Schrödinger equation by Raphaël [Rap06], this result was extended to a higher dimension in the work of Raphaël and Szeftel [RS09]. We cite also the work of Collot et al. [CGMN23] where the authors proved the existence and stability of a collapsing-ring blow-up solution to the Keller-Segel system in 3 dimensions and higher.

In this paper, we prove the existence and stability of radially symmetric blow-up along the unit sphere of  $\mathbb{R}^n$  for the non-linear heat equation (1.1). Whence proposes our main theorem as follows:

**Theorem 1.1.** *(Existence of a singular standing solution for equation (1.1) with Prescribed Profile). There exists  $T > 0$  such that equation (1.1) has a solution  $u(x, t)$  in  $\mathbb{R}^n \times [0, T)$ , with radial symmetry such that:*

1. *the solution  $u$  blows up in finite time  $T$  on the sphere of radius  $r_{max}$ ;*

2. there holds that for all  $R > 0$ ,

$$\sup_{\Lambda_R} \left| (T-t)^{\frac{1}{p-1}} u(|x|, t) - f \left( \frac{|x| - r_{max}}{\sqrt{(T-t)|\log(T-t)|}} \right) \right| \rightarrow 0 \text{ as } t \rightarrow T, \quad (7)$$

where  $\Lambda_R := \left\{ ||x| - r_{max}| \leq R\sqrt{(T-t)|\log(T-t)|} \right\}$ ,

$$f(z) = (p-1 + bz^2)^{-\frac{1}{p-1}} \text{ and } b = \frac{(p-1)^2}{4p}.$$

3. for all  $r > 0$ ,  $r \neq r_{max}$ ,  $u(r, t) \rightarrow u(r, T)$  as  $t \rightarrow T$ , with  $u(r, T) \sim u^*(r - r_{max})$  as  $r \rightarrow r_{max}$ , where

$$(1.6) \quad u^*(r) \sim \left[ \frac{b}{2} \frac{r^2}{|\log r|} \right]^{-\frac{1}{p-1}} \text{ as } r \rightarrow 0.$$

Using the ideas from [MZ97] and [Zaa98], we are able to interpret the two-dimensional variable in terms of blow-up point and blow-up time. This results in the stability of profile (1.6) in Theorem 1.1 .

**Proposition 1.2** (Stability of the singular standing solution). *Denote by  $\hat{u}$  the solution constructed in Theorem 1.1 which blowup at the sphere of radius  $\hat{r}$  and note by  $T_{\hat{u}}$  its blow-up time. Then, there exists  $\varepsilon_0 > 0$  such that for any radially symmetric initial data  $u_0 \in H$ , satisfying  $\|u_0 - \hat{u}(\cdot, 0)\|_{L^\infty} \leq \varepsilon_0$ , the solution of (1.1), with initial data  $u_0$  blows up at finite time  $T_{u_0}$  at only one collapsing ring with radius  $r_0$  in  $\mathbb{R}^n$  . Moreover, the function  $u(|x|, \cdot)$  satisfies the same estimates as  $u$  with  $T_{\hat{u}}$  replaced by  $T_{u_0}$ . Furthermore, it follows that*

$$T_{u_0} \rightarrow T_{\hat{u}}, \quad r_0 \rightarrow \hat{r} \quad \text{as } u_0 \rightarrow \hat{u}(0).$$

**Remark 1.3.** *To prove Theorem 1.1 we project the linearized partial differential equation on the eigenfunctions  $h_m$  given by (2.19). This is technically different from the work of [MNZ16], [MZ97] and [BKL94], where the authors use the integral equation. We will follow the two steps proposed in [MZ97] but in a more straight way. Indeed, we have an additional problem coming from the fact that the equation in radial coordinates presents a singularity in zero. To solve this problem we will use ideas from [MNZ16] and [Rap06].*

**Remark 1.4.** *In this paper, we are focused on the radial dynamics of the circle which reduces to the one-dimensional dynamic. We will give the proof in dimension 2, but it can be extended to a higher dimension with no difficulties.*

**Remark 1.5.** *Note that Herrero and Velázquez showed the genericity of the behavior given by (1.3), in [HV92b] and [HV92c] dedicated to the one dimensional case, and in a non published document in higher space dimensions. In Proposition 1.2, we focused on the radially symmetric perturbations. While, under non-radial perturbations, due to the genericity of the profile, stability of the blow-up profile broke down.*

This paper is organized as follows: in Section 2, we will give the formulation of our problem. Then, in Section 3 we give the proof of Theorem 1.1 without technical details and solve the finite dimension problem. Finally, in Section 4 we conclude by giving the proofs of propositions cited in Section 3.

## 2 Formulation of the problem

For simplicity, we give the proof in dimension  $n = 2$ . Inspired by the numerical results [BFG10], we consider the radially symmetric solution  $u(r, t) = u(|x|, t)$ , then we write equation (1.1) in radial coordinates as follows

$$(2.1) \quad \partial_t u = \partial_r^2 u + \frac{d-1}{r} \partial_r u + |u|^{p-1} u.$$

In general, the two terms  $\partial_r^2 u$  and  $\frac{\partial_r u}{r}$  forming the Laplacian certainly scale the same way. Heuristically, if we assume that the singularity formation of a priori takes place exclusively around the circle  $r \sim 1$ , then on this circle, the term  $\frac{\partial_r u}{r}$  scales below  $\partial_r^2 u$ , and thus the singular part of the equation is governed by the one-dimension nonlinear heat equation

$$(2.2) \quad \partial_t u = \partial_r^2 u + |u|^{p-1} u.$$

for which the rigorous construction of blow-up solution is very well known, see [MZ97], while the existence of the term: " $\frac{d-1}{r} \partial_r u$ " which has a singularity at " $\{(t, x) | x = 0\}$ ", prevents us from the estimations in a neighborhood of the origin. We naturally think of separating the space into two parts: the regular part and the blow-up part. The first part contains the origin, where the solution is supposed to be regular, while the other is away from the origin and the solution is expected to be explosive.

We introduce the following smooth nonnegative cut-off functions:

$$(2.3) \quad \chi = \begin{cases} 0 & 0 \leq \xi \leq \frac{3}{8}, \\ 1 & \xi \geq \frac{3}{4}, \end{cases}$$

and

$$(2.4) \quad \bar{\chi} = \begin{cases} 0 & \xi \geq \frac{1}{4}, \\ 1 & 0 \leq \xi \leq \frac{1}{8}. \end{cases}$$

**In regular region**, we define  $\bar{u}(x, t) = \bar{\chi} \left( \frac{|x|}{\varepsilon_0} \right) u(x, t)$ , for  $x \in \mathbb{R}^2$ , where  $u(x, t)$  is assumed to satisfy the following:

$$u_t = \Delta u + |u|^{p-1} u.$$

Then for all  $x \in \mathbb{R}$ ,  $\bar{u}$  satisfies the following equation:

$$(2.5) \quad \partial_t \bar{u} = \Delta \bar{u} + |u|^{p-1} \bar{u} - 2\nabla \bar{\chi} \nabla u - \Delta \bar{\chi} u$$

$\bar{u}$  will be controlled using classical parabolic estimates.

**In the blow-up region:** First, we note that by an invariable scaling, we can take  $r_{max} = 1$ . In the following, we consider the equation in radial coordinates given by (2.1).

Let us introduce  $U(r, t) = u(|x|, t)$  with  $r = |x|$ , then  $U$  satisfies the following equation:

$$(2.6) \quad \partial_t U = \partial_r^2 U + \frac{d-1}{r} \partial_r U + |U|^{p-1} U.$$

If we consider the following self-similar transformation:

$$(2.7) \quad W(y, s) = (T-t)^{\frac{1}{p-1}} U(r, t) \text{ with } y = \frac{r-1}{\sqrt{T-t}}, \quad s = -\log(T-t),$$

then  $W$  satisfies:

$$(2.8) \quad \partial_s W = \partial_y^2 W - \frac{1}{2} y \partial_y W + e^{-s/2} \frac{d-1}{ye^{-s/2} + 1} \partial_y W - \frac{W(y, s)}{p-1} + |W|^{p-1} W,$$

With  $y \in [-e^{s/2}, +\infty)$  and  $s \in [-\log T, +\infty)$ . If we set  $w = W \cdot \mathcal{X}(\frac{ye^{-s/2}+1}{\varepsilon_0})$ , where  $\mathcal{X}$  is defined by (2.3), then  $w$  satisfies:

$$(2.9) \quad \partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{1}{p-1} w - |w|^{p-1} w + e^{-s/2} \frac{d-1}{ye^{-s/2} + 1} \partial_y w + F(y, s),$$

where  $F(y, s)$  is defined as following:

$$(2.10) \quad F(y, s) = \begin{cases} W \partial_s \mathcal{X} - 2 \partial_y \mathcal{X} \partial_y W - W \partial_y^2 \mathcal{X} \\ + \frac{1}{2} y W \partial_y \mathcal{X} - \frac{d-1}{y+e^{s/2}} W \partial_y \mathcal{X} + |W|^{p-1} W (\mathcal{X} - \mathcal{X}^p), & \text{if } y \geq -\frac{3}{4} e^{-s/2} \\ 0, & \text{otherwise.} \end{cases}$$

In the ring  $\{r = 1\}$ , we introduce the perturbation  $q$  defined by

$$w = \varphi + q,$$

with

$$(2.11) \quad \varphi = f\left(\frac{y}{\sqrt{s}}\right) + \frac{\kappa}{2ps},$$

where

$$(2.12) \quad f(z) = (p-1 + bz^2)^{-\frac{1}{p-1}}, \quad \kappa = (p-1)^{-\frac{1}{p-1}}, \quad \text{and } b = \frac{(p-1)^2}{4p}.$$

The problem is then reduced to constructing a function  $q$  satisfying

$$\lim_{s \rightarrow \infty} \sup_{y \in [-e^{-\frac{s}{2}}, +\infty)} |q(y, s)| = 0.$$

The equation for  $q$  is as follows:

$$(2.13) \quad \partial_s q = (\mathcal{L} + V)q + H(y, s) + \partial_y G(y, s) + R(y, s) + B(y, s) + N(y, s)$$

where

$$(2.14) \quad \mathcal{L} = \partial_y^2 - \frac{1}{2} y \partial_y + 1, \quad V = p\varphi^{p-1} - \frac{p}{p-1},$$

$$(2.15) \quad B(y, s) = |\varphi + q|^{p-1}|\varphi + q| - \varphi^p - p\varphi^{p-1}q,$$

and

$$(2.16) \quad \begin{aligned} R(y, s) &= \partial_y^2 \varphi - \frac{1}{2}y\partial_y \varphi - \frac{1}{p-1}\varphi + \varphi^p - \partial_s \varphi, \\ H(y, s) &= W(\partial_y^2 \mathcal{X} + \partial_s \mathcal{X} + \frac{1}{2}y\partial_y \mathcal{X}\partial_y \mathcal{X}) + |W|^{p-1}W(\mathcal{X} - \mathcal{X}^p), \\ G(y, s) &= -2\partial_y \mathcal{X}W, \\ N(y, s) &= \frac{d-1}{y + e^{s/2}}W\partial_y \mathcal{X}. \end{aligned}$$

The control of  $q$  near the collapsing ring  $\{r = 1\}$  obeys two facts:

- **Localization**

Looking at the expression provided in (2.11), we note that the variable  $z = \frac{y}{\sqrt{s}}$  plays a fundamental role. Consequently, we will analyze the behavior of  $q$  separately when  $|z| > 2K$ , namely the outer region, and when  $|z| \leq 2K$ , namely the inner region, with the specific value of  $K > 0$  to be chosen later sufficiently large.

Let us consider the cut-off function  $\mathcal{X}_0 \in C_0^\infty([0, +\infty))$ , such that  $\mathcal{X}_0(\xi) = 1$  for  $\xi < 1$  and  $\mathcal{X}_0(\xi) = 0$  for  $\xi > 2$  and introduce

$$(2.17) \quad \mathcal{X}_c(y, s) = \mathcal{X}_0\left(\frac{|y|}{2K\sqrt{s}}\right) \text{ where } K > 0 \text{ is chosen large enough,}$$

and we introduce:

$$(2.18) \quad q_e = q(1 - \mathcal{X}_c).$$

- **Spectral properties of the linear operator  $\mathcal{L}$**

The operator  $\mathcal{L}$  is self adjoint on  $\mathcal{D}(\mathcal{L}) \subset L^2(\mathbb{R}, d\mu)$  with

$$d\mu(y) = \frac{e^{-\frac{y^2}{4}}}{(4\pi)^{1/2}} dy.$$

The spectrum of  $\mathcal{L}$  is

$$\text{spec}(\mathcal{L}) = \{1 - \frac{m}{2} | m \in \mathbb{N}\}.$$

All the eigenvalues are simple and the corresponding eigenfunctions are derived from Hermite polynomials:

$$(2.19) \quad h_m(y) = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{n!(m-2n)!} (-1)^n y^{m-2n}.$$

$h_m$  satisfies

$$\int_{\mathbb{R}} h_m h_n d\mu = 2^n n! \delta_{nm}.$$



Thanks to the above spectral properties, we can define the following projections:

•**Decomposition of  $q$**

For the sake of controlling  $q$  in the region  $|y| < \sqrt{s}$ , we will expand the unknown function  $q$  (and not  $q\mathcal{X}_c$ ) concerning the Hermite polynomial.

$$(2.20) \quad P_m(f) = f_m = \frac{\int_{\mathbb{R}} f h_m d\mu}{\left(\int h_m^2 d\mu\right)^{1/2}} \text{ for } m \in \{0, 1, 2\},$$

$$(2.21) \quad P_-(f) = f_- = \sum_{m \geq 3} P_m(f).$$

Then we study

$$(2.22) \quad q(y, s) = \sum_{m=0}^2 q_m(s) h_m(y) + q_-(y, s).$$

### 3 The existence assuming some technical results

This section is devoted to the proof of Theorem 1.1 and as mentioned before, we only give the proof in  $\mathbb{R}^2$ . We proceed in four steps, each of them making a separate subsection.

- In the first subsection, we define the bootstrap regime and translate our goal of making  $q(s)$  go to 0 in terms of belonging to  $\mathcal{S}$ .
- In the second subsection, we give an initial data family for equation (1.1), such that the initial datum is trapped in the shrinking set.
- In the third subsection, using spectral properties of the linearized operator in the blow-up region and parabolic regularity in the regular region, we reduce our goal from the control of  $u \in \mathcal{S}$  to the control of the two first component of  $q$  ( $q_0$  and  $q_1$ ).
- We end this section by solving the finite-dimensional problem using the shooting lemma and conclude the proof of Theorem (1.1).

#### 3.1 Bootstrap regime

In this part, we introduce the following shrinking set

**Definition 3.1.** For  $A, K_0, \varepsilon_0 > 0, 0 < \eta_0 \leq 1, T > 0$ , we define for all  $t \in [0, T)$

$$\mathcal{S}(t) = \mathcal{S}[A, K_0, \varepsilon_0, \eta_0](t),$$

the set of all functions  $u \in L^\infty(\mathbb{R})$  satisfying:

- (i) *Estimates in  $\mathcal{R}_1$* : we consider  $\mathcal{V}(s) = \mathcal{V}[K_0, A](s)$  (where  $s = -\log(T - t)$ ), the set of all functions  $r \in L^\infty(\mathbb{R})$  such that

$$\begin{aligned} |r_m(s)| &\leq As^{-2} \quad (m = 0, 1), \\ |r_2(s)| &\leq A^2 s^{-2} \log(s), \\ |r_-(y, s)| &\leq As^{-2}(1 + |y|^3), \\ |r_e(y, s)| &\leq As^{-1/2}, \end{aligned}$$

where

$$\begin{aligned} r_e(y, s) &= (1 - \chi_c(y, s))r(y, s), \\ r_-(s) &= P_-(r), \end{aligned}$$

for  $m \in \mathbb{N}$ , where  $r_m(y, s)$  and  $P_-$  are defined in (2.20) and (2.21).

- (ii) *Estimates in  $\mathcal{R}_2$* : For all  $0 \leq |x| \leq \frac{\varepsilon_0}{4}$ ,  $|u(x, t)| \leq \eta_0$ .

This definition yields the following priori estimates on the functions in  $\mathcal{V}(s)$ .

**Proposition 3.2.** *For any  $s > 1$ , let  $r$  be in the shrinking set  $\mathcal{V}(s)$  defined in Definition 3.1. Then the following estimates hold.*

1.  $\|r\|_{L^\infty(\mathbb{R})} \leq C(K) \frac{A^2}{\sqrt{s}}$ ,
2. for all  $y \in \mathbb{R}$ ,  $|r(y)| \leq CA \frac{\log s}{s^2} (1 + |y|^3)$

*Proof.* The proof is the same as Proposition 4.1 in [MZ08], we hence omit it here.  $\square$

## 3.2 Preparation of initial data

In this part, we aim to give a suitable family of initial data for our problem. Let us consider  $(d_0, d_1) \in \mathbb{R}^2$ ,  $T > 0$ , we consider initial data for the equation (1.1) defined for all  $x \in \mathbb{R}^2$  by:

$$(3.1) \quad u_0(x, d_0, d_1) = T^{-\frac{1}{p-1}} \left\{ \varphi(y, s_0) \mathcal{X}\left(\frac{ye^{-s_0/2}}{\varepsilon_0}\right) + \frac{A}{s_0^2} (d_0 + d_1 y) \mathcal{X}_c \right\},$$

where  $s_0 = -\log T$ ,  $y = \frac{|x|-1}{\sqrt{T}}$ ,  $\mathcal{X}$  is defined in (2.3) and  $\mathcal{X}_c$  is given by (2.17).

**Lemma 3.3.** *[Decomposition of Initial Data in Different Components] There exists  $K_0 > 0$  such that for  $\varepsilon_0 > 0$ ,  $A \geq 1$ , there exists  $s_0(K_0, \varepsilon_0, A) \geq e$  such that :*

1. *There exists a rectangle*

$$(3.2) \quad \mathcal{D}_{K_0, \varepsilon_0, A, T} = \mathcal{D}_T \subset [-2, 2]^2,$$

*such that the mapping  $(d_0, d_1) \rightarrow (q_0(s_0), q_1(s_0))$  is linear and one-to-one from  $\mathcal{D}_T$  onto  $[-\frac{A}{s_0^2}, \frac{A}{s_0^2}]^2$  and maps the boundary  $\partial\mathcal{D}_T$  into the boundary  $\partial[-\frac{A}{s_0^2}, \frac{A}{s_0^2}]^2$ . Moreover, it is of degree one on the boundary.*

2. For all  $(d_0, d_1) \in \mathcal{D}_T$ , we have:

$$(3.3) \quad \begin{aligned} |q_2(s_0)| &\leq CAe^{-s_0}, \quad |q_-(y, s_0)| \leq \frac{c}{s_0^2}(1 + |y|^3), \\ \text{and } q_e(y, s_0) &= 0, \quad |d_0| + |d_1| \leq 1. \end{aligned}$$

3. For all  $(d_0, d_1) \in \mathcal{D}_T$  and  $|x| \leq \varepsilon_0/4$ , we have  $u(x, d_0, d_1) = 0$ .

*Proof.* The proof is purely technical and follows as the analogous step in [MNZ16] and [MZ97], for that reason we refer the reader to Lemma 3.5 page 156 and Lemma 3.9 page 160 in [MZ97].  $\square$

### 3.3 Reduction to a finite-dimensional problem

In this part, we show that the control of the infinite problem is reduced to a finite-dimensional one. Since the definition of the bootstrap  $\mathcal{S}(s)$  shows two different types of estimates, in the regions  $\mathcal{R}_1$  and  $\mathcal{R}_2$ , accordingly, we need two different approaches to handle those estimates:

- In  $\mathcal{R}_1$ , we work in similarity variables (2.7), in particular we crucially use the projection of equation (2.13) with respect to the decomposition given in (2.22).
- In  $\mathcal{R}_2$ , we directly work in the variables  $u(x, t)$ , using standard parabolic estimates. For more details see subsection 4.2.

In the following, we restrict ourselves to the blow-up region. It is sufficient to prove there exists a unique global solution  $q$  on  $[s_0, +\infty)$  for some  $s_0$  large enough such that

$$q(s) \in \mathcal{V}(s), \quad \forall s \geq s_0.$$

In particular, we show that the control of the infinite problem is reduced to a finite-dimensional one. To obtain this key result, we first claim the following priori estimates. We should emphasize that the parameters  $K, A, T$  and  $s_0$  in the following lemmas are allowed to vary from one to one. When proving Proposition 3.4, we will prove that the conclusions of all lemmas are simultaneously valid for values of  $K, A, T$  and  $s_0$  as described in the proposition.

**Proposition 3.4** (A priori estimates). *There exists  $A \geq 1$  and  $s_0 \geq 0$  such that for all  $s \geq s_0$  if  $q(s) \in \mathcal{V}(s)$  is true, then the following holds:*

1. (ODE satisfied by the expanding models) For  $m = 0$ , or 1, we have

$$(3.4) \quad \left| q'_m - \left(1 - \frac{m}{2}\right) q_m \right| \leq \frac{AC}{s^2}.$$

2. (Control of null and negative modes)

$$\begin{aligned} |q_2(s)| &\leq \left(\frac{\tau}{s}\right)^2 q_2(\tau) + CA^2 s^{-2} \log(s/\tau) \\ \left\| \frac{q_-(s)}{1 + |y|^3} \right\|_{L^\infty} &\leq e^{-\frac{3}{4}(s-\tau)} \left\| \frac{q_-(\tau)}{1 + |y|^3} \right\|_{L^\infty} + \frac{CA^2}{\cdot} s^2 \end{aligned}$$

3. (Control of outer part  $q_e$ )

$$\|q_e(s)\|_{L^\infty} \leq e^{-\frac{(s-\tau)}{2(p-1)}} \|q_e(\tau)\|_{L^\infty} + C \frac{A^2}{\sqrt{\tau}} (1 + s - \tau).$$

The idea of the proof of Proposition 3.4 is to project (2.13) according to the decomposition (2.22). The computations become too long, so we postpone the proof of Proposition 3.4. That is why the whole section 4.1 is devoted to the proof of Proposition 3.4.

Consequently, we have the following result

**Proposition 3.5** (Control of  $q(s)$  in  $\mathcal{V}(s)$  by  $(q_0(s), q_1(s))$ ). *There exists  $A > 1$  such that there exists  $T(A) \in (0, 1/e)$  such that the following holds: If  $q$  is a solution of (21)–(51) with initial data at  $s = s_0 = -\log T$  given by (45) with  $(d_0, d_1) \in D_T$ , and  $q(s) \in \mathcal{S}(s)$  for all  $s \in [s_0, s_1]$  with  $q(s_1) \in \partial\mathcal{S}(s_1)$  for some  $s_1 > s_0$ , then:*

(i)  $(q_0(s_1), q_1(s_1)) \in \partial[-\frac{A}{s_1^2}, \frac{A}{s_1^2}]^2$ .

(ii) (Transverse crossing) *There exist  $m \in \{0, 1\}$  and  $\omega \in \{-1, 1\}$  such that*

$$\omega q_m(s_1) = \frac{A}{s_1^2} \text{ and } \omega \frac{d}{ds} q_m(s_1) > 0.$$

**Remark 3.6.** *In (ii) of Proposition 3.5, we show that the solution  $q(s)$  crosses the boundary  $\partial\mathcal{V}(s)$  at  $s_1$ , with positive speed, in other words, that all points on  $\partial\mathcal{V}(s_1)$  are strict exit points. The construction is essentially an adaptation of Wazewski's principle (see [Con78], chapter II and the references given there).*

*Proof of Proposition 3.5.* : Assuming Proposition 3.4, we argue as in the proof of Proposition 4.5, page 1632 from [MZ08]. By choosing proper  $A$  and  $T$ , we can use the conclusions of Proposition 4.6.

To prove (i), we notice that from Definition 3.1 and the fact that  $q_0(s) = 0$ , it is enough to show that for all  $s \in [s_0, s_1]$ ,

$$(3.5) \quad \begin{aligned} \|q_e\|_{L^\infty(\mathbb{R})} &\leq \frac{A}{2\sqrt{s}}, \\ \|q_-(y)\|_{L^\infty(\mathbb{R})} &\leq \frac{A(1+|y|)^3}{2s^2}, \\ |q_2| &\leq \frac{A^2}{2s^2}. \end{aligned}$$

Define  $\sigma = \log A$  and take  $s_0 \geq \sigma$  (that is,  $T = e^{-\sigma} = 1/A$ ) so that for all  $\tau \geq s_0$  and  $s \in [\tau, \tau + \sigma]$ , we have

$$\tau \leq s \leq \tau + \sigma \leq \tau + s_0 \leq 2\tau \implies \frac{1}{2} \leq \frac{\tau}{s} \leq \frac{s}{\tau}.$$

We consider two cases in the proof.

**Case 1:**  $s \leq s_0 + \sigma$ . Note that (54) holds with  $\tau = s_0$ . Using (ii) of Proposition 3.4 and estimate (ii) of Proposition 4.2 on the initial data  $q(\cdot, s_0)$ , we write

$$\begin{aligned} \|q_2(s)\| &\leq CA^2 e^{-\gamma s/2} + \frac{CA^2}{s^2}, \\ \left\| \frac{q_-(s)}{1+|y|^3} \right\|_{L^\infty} &\leq C \frac{A}{(s/2)^3} + C \frac{A^2}{s^2}, \\ \|q_e(s)\|_{L^\infty} &\leq CA^3 (s/2)^{-1/2} (1 + \log A). \end{aligned}$$

Thus, for sufficiently large  $A$  and  $s_0$  we see that (3.5) holds.

**Case 2:**  $s > s_0 + \sigma$ . Let  $\tau = s - \sigma > s_0$ , by Proposition 3.4 and using the fact that  $q(\tau) \in \mathcal{V}(\tau)$ , we write

$$\begin{aligned} \|q_2(s)\| &\leq A^2 (s/2)^{-2} \log(s) + \frac{CA^2}{2} s^{-2} \log(s), \\ \left\| \frac{q_-(s)}{1+|y|^3} \right\|_{L^\infty} &\leq e^{-\frac{3}{4}\sigma} \frac{A}{(s/2)^2} + C \frac{A^2}{s^2}, \\ \|q_e(s)\|_{L^\infty} &\leq e^{-\frac{\sigma}{2(p-1)}} \frac{A}{(s/2)^{1/2}} + \frac{CA^2}{(s/2)^{1/2}} (1 + \sigma). \end{aligned}$$

Thus in this case, we see clearly that there exists sufficiently large  $A$  and  $s_0$  such that conditions in (3.5) are satisfied.

Conclusion of (i): We select  $A$ , and  $s_0$  large enough so that (3.5) are verified. Then the fact that  $q(s_1) \in \partial\mathcal{V}(s_1)$  together with the definition of  $\mathcal{V}(s)$  shows that (i) of 3.5 is true. From (i) in 3.5 we deduce (ii) as follows:

Firstly from (i), there is  $(m, \omega) \in \{0, 1\} \times \{-1, 1\}$  such that  $q_m(s_1) = \omega \frac{A}{s_1^2}$  using 1 of 3.4 we see that

$$(3.6) \quad \omega q'_m(s_1) \geq \left(1 - \frac{m}{2}\right) \omega q_m(s_1) - \frac{C}{s_1^2} \geq \frac{(1 - m/2)A - C}{s_1^2}.$$

Taking  $A$  large enough concludes the proof of Proposition 3.5.  $\square$

### 3.4 Control of the solution in the bootstrap regime and proof of Theorem 1.1

We prove Theorem 1.1 using the previous results. We proceed in two parts:

#### Part 1: Solution of the finite-dimensional problem

Let  $A$ , and  $T(= e^{-s_0})$  be chosen that Proposition 3.5 and Proposition 3.4 valid, We will find the parameters  $(d_0, d_1) \in \mathcal{D}_T$  defined in (3.2) and advance by assuming that for all  $(d_0, d_1) \in \mathcal{D}_T$ , there exists  $s_*(d_0, d_1) \geq -\log T$  such that  $q_{d_0, d_1}(s) \in \mathcal{V}(s)$  for all  $s \in [-\log T, s_*]$  and  $q_{d_0, d_1}(s_*) \in \partial\mathcal{V}(s_*)$ . From (i) of Proposition 3.5, we see that  $(\tilde{q}_0(s_*), \tilde{q}_1(s_*)) \in \partial[-\frac{A}{s_*^2}, \frac{A}{s_*^2}]^2$  and the following function is well defined:

$$(3.7) \quad \begin{aligned} \Phi : \mathcal{D}_T &\rightarrow \partial[-1, 1] \\ (d_0, d_1) &\rightarrow \frac{s_*^2}{A} (\tilde{q}_0, \tilde{q}_1)_{d_0, d_1}(s_*). \end{aligned}$$

This function is continuous by (ii) of Proposition 3.5. If we manage to show that  $\Phi$  is of degree 1 on the boundary, then we have a contradiction from the degree theory. We now focus on proving that.

Using the fact that  $q(-\log T) = \psi_{d_0, d_1}$ , we see that when  $(d_0, d_1)$  is on the boundary of the quadrilateral  $\mathcal{D}_T$ ,  $(\tilde{q}_0, \tilde{q}_1)(-\log T) \in \partial[-A(\log T)^{-2}, A(\log T)^{-2}]^2$  and  $q(-\log T) \in \mathcal{V}_A(-\log T)$  with strict inequalities for the other components. Applying the transverse crossing property of Proposition 3.5, we see that  $q(s)$  leaves  $\mathcal{V}(s)$  at  $s = -\log T$ , hence  $s_*(d_0, d_1) = -\log T$ . Using (3.7), we see that the restriction of  $\Phi$  to the boundary is of degree 1. A contradiction then follows. Thus, there exists a value  $(d_0, d_1) \in \mathcal{D}_T$  such that for all  $s \geq -\log T$ ,  $q_{d_0, d_1}(s) \in \mathcal{V}(s)$ .

**Part 2: Proof of Theorem 1**

Consider the solution constructed in Part 1, such that  $q(s) \in \mathcal{V}(s)$ . Then by Definition 3.1, we see that

$$\forall y \in \mathbb{R}, \quad \forall s \geq -\log T, \quad |q(y, s)| \leq \frac{CA^2}{\sqrt{s}}.$$

By definitions (2.7) (2.11), we see that

$$\forall s \geq -\log T, \quad \forall |x| \geq \frac{\varepsilon_0}{4}, \quad \left| W(y, s) - f\left(\frac{y}{\sqrt{s}}\right) \right| \leq \frac{CA^2}{\sqrt{s}} + \frac{C}{s}.$$

By definition (2.7) of  $W$ , we see that  $\forall t \in [0, T)$ ,  $\forall |x| \geq \frac{\varepsilon_0}{4}$ ,

$$\left| (T-t)^{1/(p-1)}u(r, t) - f\left(\frac{r-r_{max}}{\sqrt{(T-t)\log(T-t)}}\right) \right| \leq \frac{C(A)}{\sqrt{|\log(T-t)|}}.$$

(i) If  $r = r_{max}$ , then we see from above that  $|u(0, t)| \sim \kappa(T-t)^{-1/(p-1)}$  as  $t \rightarrow T$ . Hence  $u$  blows up at time  $T$  at  $r = r_{max}$ .

It remains to prove that any  $r_0 \neq r_{max}$  is not a blow-up point. Since we know from item (ii) in Definition 3.1, that if  $r_0 \leq \frac{\varepsilon_0}{4}$ , and  $0 \leq t \leq T$ ,  $|u(\theta, t)| \leq \eta_0$ , it follows that  $r_0$  is not a blow-up point, provided  $r \leq \frac{\varepsilon_0}{4}$ .

Now, if  $r_0 \geq \frac{\varepsilon_0}{4}$ , the following result from Giga and Kohn [13] allows us to conclude.

**Proposition 3.7** (Giga and Kohn). *For all  $C_0 > 0$ , there is  $\eta_0 > 0$  such that if  $v(\xi, \tau)$  solves*

$$|v_\tau - \Delta v| \leq C_0(1 + |v|^p),$$

and satisfies

$$|v(\xi, \tau)| \leq \eta_0(T-t)^{-1/(p-1)},$$

for all  $(\xi, \tau) \in B(a, R) \times [T-R^2, T)$  for some  $a \in \mathbb{R}$  and  $R > 0$ , then  $v$  does not blow up at  $(a, T)$ .

**Proof.** See Theorem 2.1 page 850 in [GK85].  $\square$

Since  $r \geq \frac{\varepsilon_0}{4}$ , the estimate

$$\left| (T-t)^{1/(p-1)}u(r, t) - f\left(\frac{r-r_{max}}{\sqrt{(T-t)\log(T-t)}}\right) \right| \leq \frac{C(A)}{\sqrt{|\log(T-t)|}}$$

together with Proposition 3.7 concludes that  $r_0$  is not a blow-up point.

## 4 Reduction to a finite-dimensional problem

Since the definition of the shrinking set  $\mathcal{S}$ , given by Definition 3.1, shows two different types of estimates, in the blow-up region and regular region, accordingly, we need two different approaches to handle those estimates:

- In the blow-up region, we work in similarity variables (2.7), in particular, we crucially use the projection of equation (2.13) with respect to the decomposition given in (2.22).
- In the regular region, we directly work in the variables  $u(x, t)$ , using standard parabolic estimates.

### 4.1 Estimates in the blow-up region

#### Proof of Proposition 3.4

In this section, we prove Proposition 3.4. More precisely, we project the linearized equation (2.13) on the Hermite polynomials to get the equations satisfied by the different coordinates of the decomposition (2.22).

In the following, we will find the main contribution in the projections  $P_i$  (for  $0 \leq i \leq 2$ ) and  $P_-$  of the different terms appearing in equation (2.13). More precisely, the proof will be carried out in 2 parts,

- In the first subsection, we write equations satisfied by  $q_j$ , for  $0 \leq j \leq 2$ , and  $q_-$ . Then, we prove (1) and (2) of Proposition 4.5.
- In the second subsection, we first derive from equation (2.13) the equation satisfied by  $q_e$  and prove the last identity in (3) of Proposition 3.4.

#### Part 1: Proof of items (1) and (2) from Proposition 3.4

##### First term $\partial_s q$

Let  $P_i$  and  $P_-$  defined as in (2.20) and (2.21), then the following holds:

$$(4.1) \quad \begin{aligned} P_i(\partial_s q) &= \partial_s q_i \text{ with } i \in \{0, 1, 2\}, \\ P_-(\partial_s q) &= \partial_s q_-. \end{aligned}$$

##### Second term $\mathcal{L}q$

By the definition of  $h_i$  given by (2.19), we easily obtain the projection of  $\mathcal{L}q$  as follows

**Lemma 4.1.** *Let  $P_i$  and  $P_-$  defined as in (2.20) and (2.21), then the following holds:*

$$(4.2) \quad \begin{aligned} P_i(\mathcal{L}q) &= \left(1 - \frac{i}{2}\right) q_i, \text{ for } i = 0, 1, 2. \\ P_-(\mathcal{L}q) &= \mathcal{L}q_-. \end{aligned}$$

##### Third term $Vq$

**Lemma 4.2.** For all  $A > 0$ , there exists an  $s_0 \geq 0$  such that for all  $s \geq s_0$ , if  $q(s) \in \mathcal{V}(s)$ , the following estimations holds:

$$(4.3) \quad \begin{aligned} P_i(Vq) &\leq ACs^{-2} \quad i = 0 \text{ or } 1, \\ |P_2(Vq) + 2s^{-1}q_2| &\leq 2As^{-3}, \\ P_-(Vq) &\leq CA^2s^{-3}\log(s)(1 + |y|^3). \end{aligned}$$

*Proof.* Let us recall that  $V = p\varphi^{p-1} - \frac{p}{p-1}$ , then using Taylor expansion for  $\{|y| \leq \sqrt{s}\}$ , we obtain:

$$(4.4) \quad \begin{aligned} V &= p \left( (p-1 + b\frac{y^2}{s})^{-1/(p-1)} + \frac{\kappa}{2ps} \right)^{p-1} - \frac{p}{p-1}, \\ &= \frac{1}{2s} - \frac{bpy^2}{s(p-1)^2} + O(\frac{y^4}{s^2}). \end{aligned}$$

Using the fact that  $q \in \mathcal{V}(s)$ ,  $q_0$  and  $q_1$  are then controlled by  $s^{-2}$ . Therefore,

$$(4.5) \quad \begin{aligned} |P_0(Vq)| &= \left| \int_{\mathbb{R}} Vqh_0 d\mu \right|, \\ &\leq \left| \int_{|y| \leq \sqrt{s}} C\frac{y^2}{s} (\sum_{i=0}^2 q_i h_i + q_-) h_0 d\mu \right| + \left| \int_{|y| > \sqrt{s}} C\frac{y^2}{s} (\sum_{i=0}^2 q_i h_i + q_-) h_0 d\mu \right|, \\ &\leq C \left| \int_{|y| \leq \sqrt{s}} (\sum_{i=0}^2 q_i h_i + q_-) h_0 d\mu \right| + \left| \int_{|y| > \sqrt{s}} C\frac{y^2}{s} (\sum_{i=0}^2 q_i h_i + q_-) h_0 \frac{e^{-\frac{y^2}{4}}}{(4\pi)^{1/2}} dy \right|, \\ &\leq C|q_0| + \left| \int_{|y| > \sqrt{s}} C\frac{y^2}{s} (\sum_{i=0}^2 q_i h_i + q_-) h_0 \frac{e^{-\frac{y^2}{4}}}{(4\pi)^{1/2}} dy \right|. \end{aligned}$$

Notice that  $q(s)$  is in  $\mathcal{V}(s)$ , then by Definition 3.1 and Proposition 3.2. We obtain that:

$$(4.6) \quad |P_0(Vq)| \leq \frac{AC}{s^2} + Ce^{-\frac{s}{8}} \leq \frac{AC}{s^2}.$$

This is exactly the desired result for  $P_0(Vq)$ . The proof for  $P_1(Vq) \leq e^{-\frac{\kappa s^2}{2}}$  is parallel to above, we hence omit it. Using (4.4) and argue as above, we obtain:

$$(4.7) \quad \begin{aligned} |P_2(Vq) + 2s^{-1}q_2| &\leq \left| \int_{\mathbb{R}} -\frac{bp}{(p-1)^2} (s^{-1})(y^2) (\sum_{i=0}^2 q_i h_i + q_-) h_2 d\mu + 2s^{-1}q_2 \right| \\ &\leq ACs^{-3}. \end{aligned}$$

The above implies that

$$(4.8) \quad P_-(Vq) = \left| Vq - \sum_{i=0}^2 P_i(Vq) \right| \leq CA^2s^{-3}\log(s)(1 + |y|^3).$$

□

#### Fourth term $R(y, s)$

**Lemma 4.3** (Estimates for term  $R$ ). For  $i \leq 1$

$$(4.9) \quad |P_i(R)| \leq Cs^{-2},$$

$$(4.10) \quad |P_2(R)| \leq Cs^{-3},$$



and we have also:

$$(4.11) \quad |P_-(R)| \leq Cs^{-2}(1 + |y|^3)$$

*Proof.* To give the estimates on  $R$ , we first compute each term in  $R$  for  $|y| < \sqrt{s}$  with Taylor expansion. To have a better vision of this, we remind the readers that

$$R(y, s) = \partial_y^2 \varphi - \frac{1}{2} y \partial_y \varphi - \frac{1}{p-1} \varphi + \varphi^p - \partial_s \varphi,$$

$$\text{with } \varphi(y, s) = \left[ (p-1 + b \frac{y^2}{s})^{-\frac{1}{p-1}} + \frac{a}{s} \right].$$

We note that for  $|y| < \sqrt{s}$ ,  $f$  is bounded. By Taylor expansion, we obtain the following

$$(4.12) \quad \begin{aligned} R(y, s) &= \left( a - \frac{wb\kappa}{(p-1)^2} \right) \frac{1}{s} + O\left(\frac{1}{s^2}\right) \\ &+ \left( -\frac{abp}{(p-1)^2} + \frac{b\kappa}{(p-1)^2} \left( \frac{6bp}{(p-1)^2} - 1 \right) \right) \frac{y^2}{s^2} + O\left(\frac{y^6}{s^3}\right). \end{aligned}$$

From the above Taylor expansion, one can easily see that:

$$(4.13) \quad P_0(R) = \left( a - \frac{2b\kappa}{(p-1)^2} \right) \frac{1}{s} + O\left(\frac{1}{s^2}\right),$$

and

$$(4.14) \quad P_2(R) = 2 \left( -\frac{abp}{(p-1)^2} + \frac{b\kappa}{(p-1)^2} \left( \frac{6bp}{(p-1)^2} - 1 \right) \right) \frac{1}{s^2} + O\left(\frac{1}{s^3}\right).$$

Together with our choice of  $a, b$ :

$$(4.15) \quad a = \frac{(p-1)^{-\frac{1}{p-1}}}{2p}, \quad b = \frac{(p-1)^2}{4p},$$

we therefore obtain  $P_0(R) = O\left(\frac{1}{s^2}\right)$  and  $P_2(R) = O\left(\frac{1}{s^3}\right)$ . The estimation for  $P_1(R)$  can be argued as in Corollary 5.13 and Lemma 5.18 in [MZ08], we hence omit here.

Using (2.21), (4.12), (4.13) and (4.14), we obtain the following:

$$|P_-(R)| \leq \left| R - \sum_{i=0}^2 P_i(R) \right| \leq Cs^{-2}(1 + |y|^3).$$

□

**Fifth term  $B$ :**

For the quadratic term  $B$ , we first remind the readers of the following Lemma:

**Lemma 4.4.** *For all  $A > 0$ , there exists  $s_0 \geq 0$  such that for all  $\tau \geq s_0$ , if  $q(\tau) \in \mathcal{V}(\tau)$ , then*

$$(4.16) \quad |\chi_c(y, \tau) B(q(y, \tau))| \leq C|q|^2,$$

and

$$(4.17) \quad |B(q)| \leq C|q|^{\bar{p}},$$

where  $\bar{p} = \min(p, 2)$ .

*Proof.* This Lemma was argued in Lemma 3.6 of [MZ97], interested readers are invited to read the proof in [MZ97].  $\square$

Then we are enabled to claim the following lemma:

**Lemma 4.5.** *There exists  $s_0 \geq 0$  such that if  $q(s) \in \mathcal{V}(s)$  for  $s > s_0$ , then  $B$  verifies:*

$$(4.18) \quad \begin{aligned} P_i(B) &\leq CA^2s^{-3}, \quad i \in \{0, 1, 2\} \\ P_-(B) &\leq CA s^{-2}(1 + |y|^3). \end{aligned}$$

*Proof.* We argue as in the proof of Lemma 5.10 and Lemma 5.17 in [MZ08].  $\square$

**Sixth term  $H$ :**

**Lemma 4.6.** *The following estimations holds:*

$$(4.19) \quad \begin{aligned} P_i(H) &\leq Ce^{-s/2} \quad i = 0, 1 \text{ or } 2, \\ P_-(H) &\leq Ce^{-s/2}(1 + |y|^3), \end{aligned}$$

*Proof.* We argue it as in the proof of Lemma 3.9 from [MNZ16].  $\square$

**Seventh term  $\partial_y G$ :**

**Lemma 4.7.** *For  $\partial_y G$  we have the following estimations:*

$$(4.20) \quad \begin{aligned} P_i(\partial_y G) &\leq Ce^{-s/2} \quad i = 0, 1 \text{ or } 2, \\ P_-(\partial_y G) &\leq Ce^{-s/2}(1 + |y|^3), \end{aligned}$$

*Proof.* This can be done with integration by part, interested readers are invited to see the proof of Lemma 5.19 in [MNZ16].  $\square$

**Eighth term  $N$ :**

**Lemma 4.8** (projection of the last term:  $N$ ).

$$(4.21) \quad |P_i(N)| \leq Ce^{-s} \text{ where } i = 0, 1 \text{ or } 2.$$

and

$$(4.22) \quad |P_-(N)| \leq e^{-s/2}(1 + |y|^3).$$

*Proof.* Let us first recall that

$$N(y, s) = \frac{d-1}{y + e^{s/2}} W \partial_y \mathcal{X},$$

where  $W$  is defined by (2.8) and  $\mathcal{X}$  is the cut-off function defined in (2.3).

We will now give the estimation on the terms  $P_i(N)$ ,  $i \in \{0, 1, 2\}$ . From Lemma A.1, we have that for  $|y| \geq e^{\frac{s}{2}}(\frac{3}{8}\varepsilon_0 - 1)$ ,  $\|W(s)\|_{L^\infty} \leq \kappa + 2$ . By the definition (2.3) of  $\mathcal{X}$  we easily have that

$$(4.23) \quad |\partial_y \mathcal{X}| \leq e^{-s/2} \frac{C}{\varepsilon_0} \mathbb{I}_{\{(\frac{3}{8}\varepsilon_0 - 1)e^{s/2} \leq y \leq (\frac{3}{4}\varepsilon_0 - 1)e^{s/2}\}}.$$

Using the estimations above, we get

$$\begin{aligned}
(4.24) \quad |P_0(N)| &\leq e^{-s/2} \int_{\mathbb{R}} \left| \frac{d-1}{ye^{-s/2}+1} W \partial_y \mathcal{X} \right| \frac{e^{-|y|^2/4}}{(4\pi)^{1/2}} dy, \\
&\leq Ce^{-s} \int_{\{y \geq (\frac{3}{8}\varepsilon_0 - 1)e^{s/2}\}} \left| \frac{d-1}{ye^{-s/2}+1} \right| \frac{e^{-|y|^2/4}}{(4\pi)^{1/2}} dy, \\
&\leq Ce^{-s} \int_{\{y \geq (\frac{3}{8}\varepsilon_0 - 1)e^{s/2}\}} \frac{e^{-|y|^2/4}}{(4\pi)^{1/2}} dy \leq Ce^{-s}.
\end{aligned}$$

Arguing in a similar fashion, we obtain the desired estimation for  $P_1(N)$  and  $P_2(N)$ .

We conclude the proof with the estimation of  $P_-(N)$ . Using Lemma A.1, (4.21) and (4.23), we obtain,

$$(4.25) \quad |P_-(N)| = |N - \sum_{i=0}^2 P_i(N)h_i| \leq e^{-s/2}(1 + |y|^3).$$

□

### Proof of Proposition 3.4

*Proof of item (1) and (2) of Proposition 3.4*

Using Lemma 4.1, Lemma 4.2, Lemma 4.3, Lemma 4.5, Lemma 4.6, Lemma 4.7 and Lemma 4.8 and arguing as the proof of Proposition 4.6 in [MZ08], we can easily obtain

$$|q'_0(s) - q_0(s)| \leq \frac{AC}{s^2} \text{ and } |q'_1(s) - \frac{1}{2}q_1(s)| \leq \frac{AC}{s^2},$$

this conclude (1) from Proposition 3.4.

The case (2) is more delicate. From Lemma 4.1, Lemma 4.2, Lemma 4.3, Lemma 4.5, Lemma 4.6, Lemma 4.7 and Lemma 4.8, we obtain:

$$(4.26) \quad \left| q'_2(s) + \frac{2}{s}q_2(s) \right| \leq C \frac{A^2}{s^3}.$$

Integrating this inequality between  $\tau$  and  $s$  gives the desired estimates on  $q_2$ ,

$$(4.27) \quad |q_2(s)| \leq \left(\frac{\tau}{s}\right)^2 q_2\tau + C \frac{A^2}{s^2} \log(s/\tau).$$

For  $q_-$ , we can use the properties of the semi-group generated by  $\mathcal{L}$ , and obtain that for all  $s \in [\tau, s_1]$ ,

$$\begin{aligned}
q_-(s) &= e^{(s-\tau)\mathcal{L}} q_-(\tau) \\
&\quad + \int_{\tau}^s e^{(s-s')\mathcal{L}} P_-(Vq + H(y, s) + \partial_y G(y, s) + R(y, s) + B(y, s) + N(y, s)) ds'.
\end{aligned}$$

Arguing as Lemma A.2 in [MZ08] gives us:

$$\begin{aligned} \left\| \frac{q_-(s)}{1+|y|^3} \right\|_{L^\infty} &= e^{-\frac{3}{2}(s-\tau)} \left\| \frac{q_-(\tau)}{1+|y|^3} \right\|_{L^\infty} \\ &+ \int_\tau^s e^{-\frac{3}{2}(s-s')} \left\| \frac{P_-(Vq + H(y, s) + \partial_y G(y, s) + R(y, s') + B(y, s') + N(y, s'))}{1+|y|^3} \right\|_{L^\infty} ds'. \end{aligned}$$

Assuming that  $q(s') \in V_A(s')$ , the estimations Lemma 4.1, Lemma 4.2, Lemma 4.3, Lemma 4.5, Lemma 4.6, Lemma 4.7 and Lemma 4.8 implies the following

$$\begin{aligned} \left\| \frac{q_-(s)}{1+|y|^3} \right\|_{L^\infty} &= e^{-\frac{3}{2}(s-\tau)} \left\| \frac{q_-(\tau)}{1+|y|^3} \right\|_{L^\infty} \\ &+ \int_\tau^s e^{-\frac{3}{2}(s-s')} \left[ \frac{A^2}{s'^3} \log(s') + \frac{C}{s'^2} + \frac{CA}{s'^2} + Ce^{-\frac{s'}{2}} \right] ds'. \end{aligned}$$

Using Gronwall's Lemma we deduce that:

$$\begin{aligned} e^{\frac{3}{2}s} \left\| \frac{q_-(s)}{1+|y|^3} \right\|_{L^\infty} &= e^{-\frac{3}{4}(s-\tau)} e^{\frac{3}{2}\tau} \left\| \frac{q_-(\tau)}{1+|y|^3} \right\|_{L^\infty} \\ &+ e^{\frac{3}{2}s} 2^{\frac{5}{2}} \left[ \frac{A^2}{s^3} \log(s) + \frac{C}{s^2} + \frac{CA}{s^2} + Ce^{-\frac{s}{2}} \right]. \end{aligned}$$

This concludes the estimation on  $P_-(q)$ .

### Part 2: The outer region $q_e$ : proof of item (3) from Proposition 3.4

Here, we conclude the proof of Proposition 3.4 by demonstrating the final inequality about  $q_e$ . As  $q(s) \in \mathcal{V}(s)$  for all  $s \in [\tau, s_1]$ , it follows that

$$\|q(s)\|_{L^\infty(|y| < 2K\sqrt{s})} \leq \frac{CA^2}{\sqrt{s}}.$$

We note that terms  $H$ ,  $\partial_y G$  and  $N$  defined in (2.16) are compactly supported in  $[(\frac{3}{8}\varepsilon_0 - 1)e^{s/2}, (\frac{3}{4}\varepsilon_0 - 1)e^{s/2}]$ . Then, we derive the equation satisfied by  $q_e$  is (4.28)

$$\begin{aligned} \partial_s q_e &= (\mathcal{L} + V)q_e - q(s) \left( \partial_s \mathcal{X}_c + \Delta \mathcal{X}_c + \frac{1}{2} y \nabla \mathcal{X}_c \right) \\ &+ (R(y, s) + B(y, s) + H(y, s) + \partial_y G(y, s) + N(y, s)) (1 - \mathcal{X}_c) + 2 \operatorname{div}(q(s) \nabla \mathcal{X}_c). \end{aligned}$$

Writing this equation in its integral form and using the maximum principle satisfied by  $e^{\tau \mathcal{L}}$ , we deduce that:

$$\begin{aligned} \|q_e\|_{L^\infty} &\leq e^{-\frac{s-\tau}{p-1}} \|q_e(\tau)\|_{L^\infty}, \\ &+ \int_\tau^s e^{-\frac{s-s'}{p-1}} \|((1 - \mathcal{X}_c)(R + B + H + \partial_y G + N))\|_{L^\infty} ds', \\ (4.29) \quad &+ \int_\tau^s e^{-\frac{s-s'}{p-1}} \|q(s') \left( \partial_s \mathcal{X}_c + \Delta \mathcal{X}_c + \frac{1}{2} y \nabla \mathcal{X}_c \right)\|_{L^\infty} ds', \\ &+ \int_s^\tau e^{-\frac{s-s'}{p-1}} \left( \frac{1}{\sqrt{1 - e^{-(s-s')}}} \right) \|q(s') \nabla \mathcal{X}_c\|_{L^\infty} ds'. \end{aligned}$$

Notice that with Lemma 4.3, Lemma 4.5, Proposition 3.2 and (2.17), by arguing as Section 5.3 in [MZ08], we obtain the following bound:

$$(4.30) \quad \begin{aligned} \|q(s') (\partial_s \mathcal{X}_c + \Delta \mathcal{X}_c + \frac{1}{2} y \nabla \mathcal{X}_c)\|_{L^\infty} &\leq C \frac{A^2}{\sqrt{s'}}, \\ \|q(s') \nabla \mathcal{X}_c\|_{L^\infty} &\leq C \frac{A^{\frac{3}{2}}}{s'}, \\ \|(1 - \mathcal{X}_c) R(y, s')\|_{L^\infty} &\leq \frac{C}{s'}, \\ \|(1 - \mathcal{X}_c) B(y, s')\|_{L^\infty} &\leq \frac{1}{2(p-1)} \|q_e\|_{L^\infty}. \end{aligned}$$

Then with Lemma A.1 and (4.23) we gain the following

$$(4.31) \quad (1 - \mathcal{X}_c)(H(y, s') + \partial_y G(y, s') + N(y, s')) \leq C e^{-\frac{s'}{2}} \leq \frac{C}{s'}$$

By choosing  $K$  large enough such that estimations (4.30) are verified, and we write:

$$(4.32) \quad \begin{aligned} \|q_e\|_{L^\infty} &\leq e^{-\frac{s-\tau}{p-1}} \|q_e(\tau)\|_{L^\infty}, \\ &+ \int_\tau^s e^{-\frac{s-s'}{p-1}} \left( \frac{1}{2(p-1)} \|q_e(s')\|_{L^\infty} + \frac{CA^2}{\sqrt{s'}} + \frac{A^2}{s'} \frac{1}{\sqrt{1-e^{-(s-s')}}} \right) ds', \end{aligned}$$

We then conclude with Gronwall's inequality:

$$\|q_e(s)\|_{L^\infty} \leq e^{-\frac{(s-\tau)}{2(p-1)}} \|q_e(\tau)\|_{L^\infty} + C \frac{A^2}{\sqrt{\tau}} (1 + s - \tau).$$

## 4.2 Estimates in the regular region

Our goal here is to show that:

$$(4.33) \quad |x| \leq \frac{\varepsilon_0}{4}, \text{ then we have } u(x, t^*) \leq \frac{\eta}{2}.$$

This is shown in 3 steps:

- In the first step, we improve the bounds on the solution  $u(x, t)$  in the intermediate region.
- In the second step, we use parabolic regularity to obtain an estimation of the solution in the region  $\mathcal{R}_2$ .
- Finally, we use the two steps above to get (4.33)

### Step 1: Improved estimates in the intermediate region

Here, we refine the estimates on the solution in the following intermediate region:

$$(4.34) \quad \frac{\varepsilon_0}{8} \leq |x| \leq K \sqrt{(T-t) \log(T-t)}.$$

By Lemma A.1, we have

$$(4.35) \quad \forall t \in [0, t^*], \text{ and } \forall x \in \mathbb{R}^n, |u(t)| \leq C(T-t)^{-\frac{1}{p-1}},$$

valid in particular in the intermediate region given by (4.34). This bound is unsatisfactory since it goes to infinity as  $t \rightarrow T$ . In order to refine it, given a  $x$  small enough in norm  $|x|$ , we use this bound when  $t = t_0(x)$  defined by

$$(4.36) \quad |x| = K_0 \sqrt{(T - t_0(x)) |\log(T - t_0(x))|},$$

to see that the solution is, in fact, flat at that time. Then, advancing the PDE (2.1), we see that the solution remains flat for later times. More precisely, we claim the following:

**Lemma 4.9** (Flatness of the Solution in the Intermediate Region in (4.34)). *There exists  $\zeta_0 > 0$  such that for all  $K_0 > 0$ ,  $\varepsilon_0 > 0$ ,  $A \geq 1$ , there exists  $s_{0,9}(K_0, \varepsilon_0, A)$  such that if  $s_0 \geq s_{0,9}$  and  $0 < \eta_0 \leq 1$ , then,  $\forall t_0(x) \leq t \leq t^*$ ,*

$$(4.37) \quad \left| \frac{u(x, t)}{u^*(x)} - \frac{U_K(x)}{U_K(1)} \right| \leq \frac{C}{|\log |x||^{\zeta_0}},$$

where  $u^*$  is defined in and

$$(4.38) \quad U_K(\tau) = \kappa \left( (1 - \tau) + \frac{(p-1)K^2}{4p} \right)^{-\frac{1}{p-1}}.$$

In particular,  $|u(x, t)| \leq 2|u^*(\theta)|$ .

*Proof.* Proof see in [MNZ16] P.316 Lemma 3.12.  $\square$

**Step 2: A parabolic estimate in regular region:**

Recall from the definition on  $\mathcal{V}$ , that:

$$\forall x \in \mathbb{R} \text{ such that } 0 \leq |x| \leq \frac{\varepsilon_0}{4}, u(x, t) \leq \eta_0.$$

Using parabolic estimation on the solution, for  $u(x, t)$  in region  $\mathcal{R}^2$ , we claim the following:

**Proposition 4.10.** *For all  $\varepsilon > 0$ ,  $\varepsilon_0 > 0$ ,  $\sigma_1 \geq 0$ , there exists  $T \geq 0$  such that for all  $\bar{t} \leq T$ , if  $u$  is a solution of*

$$\partial_t u = \Delta u + |u|^{p-1}u \quad \text{for all } x \in [0, \varepsilon_0/4], t \in [0, \bar{t}],$$

which satisfies:

$$(i) \text{ For } |x| \in [\frac{\varepsilon_0}{8}, \frac{\varepsilon_0}{4}], |u(x, t)| \leq \sigma_1.$$

$$(ii) \text{ For } 0 \leq |x| \leq \frac{\varepsilon_0}{8}, u(x, 0) = 0.$$

Then, for all  $t$  in  $[0, \bar{t}]$ , for all  $|x| \leq \frac{\varepsilon_0}{4}$ ,  $|u(x, t)| \leq \varepsilon$ .

*Proof.* Consider  $\bar{u}$ , recalled here, after a trivial chain rule to transform the  $\partial_x u$  term:

$$\forall t \in [0, \bar{t}], \quad \forall \theta \in \mathbb{R}, \quad \partial_t \bar{u} = \Delta \bar{u} + |\bar{u}|^{p-1} \bar{u} - 2\nabla(\bar{\chi}' u) + \bar{\chi}'' u.$$

Therefore, since  $\bar{u}(x, 0) \equiv 0$ , we write

$$\|\bar{u}(t)\|_{L^\infty} \leq \int_0^t S(t-t') \left( |u|^{p-1} I_{|x| \leq \frac{\varepsilon_0}{4}} \bar{u} - 2\nabla \left( \bar{\chi}' u I_{|x| \leq \frac{\varepsilon_0}{4}} \right) \right) + \bar{\chi}'' u(t') I_{|x| \leq \frac{\varepsilon_0}{4}} dt'.$$

where  $S(t)$  is the heat kernel. Since  $\bar{\chi}'$  and  $\bar{\chi}''$  are supported by  $\{\frac{\varepsilon_0}{8} \leq |x| \leq \frac{\varepsilon_0}{4}\}$  and satisfy  $|\bar{\chi}'| \leq \frac{C}{\varepsilon_0}$ ,  $|\bar{\chi}''| \leq \frac{C}{\varepsilon_0^2}$  and using parabolic regularity, we write

$$\|\bar{u}(t)\|_{L^\infty} \leq \sigma_1^{p-1} \int_0^t \|\bar{u}(t')\| dt' + C\sigma_1 \frac{1}{\varepsilon_0} \int_0^t \frac{1}{\sqrt{t-t'}} dt' + C\sigma_1 \frac{1}{\varepsilon_0^2} \int_0^t dt'.$$

If  $\bar{t} < 1$ , by Gronwall's estimate, this implies that

$$\|\bar{u}(t)\|_{L^\infty} \leq Ce^{\sigma_1^{p-1}} \left( \frac{\sigma_1}{\varepsilon_0} \sqrt{\bar{t}} + \frac{\sigma_1}{\varepsilon_0^2} \bar{t} \right).$$

Taking  $\bar{t}$  small enough, we can obtain  $\forall t \in [0, \bar{t}]$ ,  $\|\bar{u}(t)\|_{L^\infty} \leq \varepsilon$ . □

### Step 3: Proof of the improvement in Definition 3.1

Here, we use Step 1 and Step 2 to prove (4.33), for a suitable choice of parameters.

Let us consider  $K > 0$ , and  $\delta_0(K) > 0$  defined in Lemma 4.9. Then, we consider  $\varepsilon_0 \leq 2\delta_0$ ,  $0 < \eta_0 \leq 1$  defined in Lemma 4.9 and Proposition 4.10;  $A \geq 1$ ,  $s_0$  sufficiently large such that conditions in (3.5) and Lemma 4.9 and Proposition 4.10 holds.

Applying Lemma 4.9, we see that for all  $|x| \leq \delta_0$ ,  $A \geq 1$ , for all  $t \in [0, t^*]$ ,  $|u(x, t)| \leq 2|u^*(x)|$ .

In particular, for all  $\delta_0 \leq \frac{\varepsilon_0}{8} \leq |x| \leq \frac{\varepsilon_0}{4}$ , for all  $t \in [0, t^*]$ ,  $|u(|x|, t)| \leq 2|u^*(\frac{\varepsilon_0}{8})|$ .

Using item (iii) of Lemma 3.3, we see that for all  $|\theta| \leq \frac{\varepsilon_0}{8}$ ,  $u(|x|, 0) = 0$ .

Therefore Proposition 4.10 applies with  $\varepsilon = \frac{\eta_0}{2}$  and  $\sigma_1 = 2u^*(\frac{\varepsilon_0}{4})$ , and we see that for all  $|x| \leq \frac{\varepsilon_0}{4}$ , for all  $t \in [0, t^*]$ ,  $|u(|x|, t)| \leq \frac{\eta_0}{2}$  and estimate (4.33) holds.

## A Appendix

**Lemma A.1.** *For all  $K_0 > 0$ ,  $\varepsilon_0 > 0$ ,  $A \geq 1$ , there exists  $s_0$  such that if  $s \geq s_0$ ,  $0 < \eta_0 \leq 1$ , and we assume that  $u(t) \in \mathcal{S}(t)$  defined in Definition 3.1, where  $t = T - e^{-s}$ , then we have:*

$$\|W(\cdot, s)\|_{L^\infty} \leq \kappa + 2.$$

*Proof.* For  $W$  we can see that:

- If  $|y| \geq \varepsilon_0 e^{s/2} (\frac{3}{4}\varepsilon_0 - 1)$ , then  $W(y, s) = w(y, s) = \varphi(y, s) + q(y, s)$ . Since  $|\varphi|_{L^\infty} \leq \kappa + 1$  from (2.11), using (ii), we see that  $|W|_{L^\infty} \leq \kappa + 2$  for  $s$  large enough, which is for  $T$  small enough.
- If  $|y| < \varepsilon_0 e^{s/2} (\frac{3}{4}\varepsilon_0 - 1)$ , then  $W(y, s) = e^{-s(p-1)} u(\theta e^{-s/2}, t)$  with  $|\theta| \geq \varepsilon_0^2/2$ . By (ii) of Definition 3.1, we see that  $|W(y, s)| \leq \eta_0 e^{-s(p-1)} \leq \eta_0 T^{1/(p-1)} \leq 1$  if  $\eta_0 \leq 1$  and  $T \leq 1$ .

□

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