Elliptic Units Above Fields With Exactly One Complex Place

Pierre Morain

To cite this version:

Pierre Morain. Elliptic Units Above Fields With Exactly One Complex Place. 2024. hal-04607848

HAL Id: hal-04607848
https://hal.science/hal-04607848
Preprint submitted on 11 Jun 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution 4.0 International License
Elliptic units above fields with exactly one complex place

Pierre L. L. Morain

Abstract:
In this work we explore the construction of abelian extensions of fields with exactly one complex place using multivariate analytic functions in the spirit of Hilbert’s 12th problem. To this end we study the special values of the multiple elliptic Gamma functions introduced in the early 2000s by Nishizawa following the work of Felder and Varchenko on the elliptic Gamma function of Ruijsenaars. We construct geometric variants of these functions enjoying transformation properties under an action of $\text{SL}_d(\mathbb{Z})$ for $d \geq 2$. The evaluation of these functions at points of a degree $d$ field $K$ with exactly one complex place following the scheme of a recent article by Bergeron, Charollois and García [1] seems to produce algebraic numbers. More precisely, we conjecture that such infinite products yield algebraic units in abelian extensions of $K$ related to conjectural Stark units and we provide numerical evidence to support this conjecture for cubic, quartic and quintic fields.

Acknowledgments: This work is part of an on-going PhD work and the author would like to thank his advisors Pierre Charollois and Antonin Guilloux for their guidance and their helpful comments on this work.

1Sorbonne Université, IMJ-PRG, CNRS, Ouragan Team, INRIA, Paris, France. This PhD work is funded by the École polytechnique, Palaiseau, France.
1 Introduction

In explicit class field theory the main task is to provide a way to compute the abelian extensions of a number field $K$. The celebrated theorem of Kronecker and Weber shows that the exponential function plays a key role in the description of the abelian extensions of $\mathbb{Q}$. Since then, the theory of Complex Multiplication has provided explicit constructions to build abelian extensions of imaginary quadratic fields. There are many candidates for the function which plays the role of the exponential function here, including Klein’s $j$-invariant, Weber’s functions or Jacobi’s $\theta$ functions, which are all related to modular forms for $SL_2(\mathbb{Z})$. Elliptic units are obtained by evaluating ratios of $\theta$ functions at points of an imaginary quadratic field. For instance, define

$$\theta(z, \tau) = \prod_{m \geq 0} (1 - \exp(2i\pi((m + 1)\tau - z)))(1 - \exp(2i\pi(m\tau + z)))$$
for \( z \in \mathbb{C} \) and \( \tau \in \mathbb{H} = \{ w \in \mathbb{C}, \Im(w) > 0 \} \). Then we may compute

\[
\theta \left( \frac{7}{91}, \frac{10 + e^{2i\pi/3}}{91} \right)^7 \theta \left( \frac{7}{13}, \frac{10 + e^{2i\pi/3}}{13} \right)^{-1} \approx 0.0196475... - i \cdot 0.6399892...
\]

which is a unit outside an ideal above 13 in the field \( \mathbb{Q}[x]/(x^4 + 3x^3 + 32x^2 + 13) \), an extension of \( \mathbb{Q}(e^{2i\pi/3}) \) ramified only above a prime ideal of norm 13. For a more complete presentation on the subject, we refer to [17].

Finding general analytic functions which play the role of the exponential function for other number fields constitutes Hilbert’s 12th problem. The situation becomes more complicated outside of the two cases above due to the presence of an infinite set of units in \( \mathbb{K} \). Other efforts have been made to compute elements in class fields using analytic functions, especially in the case of real quadratic fields. In [20], Stark analyses regulators in abelian extensions and studies how they can be factored. This led him to refine Dirichlet’s class number formula and to conjecture the existence of special units in class fields which should yield the exponential of the values of \( L \)-functions at \( s = 0 \) as a combination of their valuations at infinite places. To relate this to Hilbert’s 12th problem, we are interested in finding a way to compute these units directly, instead of computing absolute values. It was later shown that there are two simpler cases in the Stark conjectures (see for instance [6]). The first case deals with totally real fields and it was solved using \( p \)-adic methods by Dasgupta and Kakde in [7]. In this article we focus on the second case where \( \mathbb{K} \) is an “almost totally real field” (ATR), meaning that \( \mathbb{K} \) has exactly one complex place. Note that imaginary quadratic fields are ATR in this sense, so that we may hope to find an analogue to the construction of elliptic units above ATR fields. To this end, Bergeron, Charollois and García [1] used the elliptic Gamma function from mathematical physics of Ruijsenaars [18] studied by Felder and Varchenko [10] to build conjectural elliptic units above complex cubic fields and related their construction to the conjectural Stark units. For instance, if \( z \) is the root of the polynomial \( x^3 - 2 \) in the upper half-plane \( \mathbb{H} \) we may compute with high precision (more than 1000 digits) the quotient

\[
\frac{\Gamma \left( \frac{-1}{3}, \frac{-z^2 + z - 3}{15}, \frac{-z + 8}{15} \right)^5}{\Gamma \left( \frac{-5}{3}, \frac{-z^2 + z - 3}{3}, \frac{-z + 8}{3} \right)} \approx -0.3362599... - i \cdot -0.3832104...
\]

(see section 2.1.1 for the definition of the elliptic Gamma function). It is close to a root of the polynomial \( x^6 + 3x^5 + 6x^4 + 5x^3 + 6x^2 + 3x + 1 \) which is the minimal polynomial over \( \mathbb{Q} \) of a unit in an abelian extension of \( \mathbb{Q}(z) \). We treat this example in section 5.1.2 below.
This work proposes a generalisation beyond complex cubic fields of the construction carried out in [1] using multiple elliptic Gamma functions (denoted $G_r$ for $r \geq 0$) which were introduced by Nishizawa in [13] and studied by Narukawa in [12] (see section 2.2). In order to achieve this, we first construct geometric variants of the $G_r$ functions of Nishizawa, upgrading the construction of the geometric elliptic Gamma function in [9] and we show that these functions enjoy nice properties under the action of $\text{SL}_{r+2}(\mathbb{Z})$. In section 3, we show how to evaluate these functions at points in an ATR field $K$ of degree $d = r + 2$ to obtain conjectural algebraic units in abelian extensions of $K$. This is expressed in conjecture 10 which generalises the conjecture in [1] to higher degree ATR fields. In section 4, under some assumptions, we show how to choose the evaluation points for the geometric $G_r$ functions to efficiently obtain conjectural algebraic units. We also describe the algorithmic approach to obtaining numerical evidence to support conjecture 10, and in section 5, we provide many examples for ATR cubic, quartic and quintic fields.

2 Multiple elliptic Gamma functions and their geometric variants

In this section, we recall the properties of the elliptic Gamma function and its geometric variants. Then we will present the multiple elliptic Gamma functions of Nishizawa and construct geometric variants of these functions, upgrading the construction of the geometric elliptic Gamma functions in [9].

2.1 The geometric elliptic Gamma functions

2.1.1 The elliptic Gamma function

The elliptic Gamma function was first introduced by Ruijsenaars [18] and was studied later at length by Felder and Varchenko [10] who made the connexion with the above $\theta$ function. For $z \in \mathbb{C}$ and $\tau, \sigma \in \mathbb{H}$ put

$$\Gamma(z, \tau, \sigma) = \prod_{m,n \geq 0} \left( \frac{1 - \exp(2i\pi((m + 1)\tau + (n + 1)\sigma - z))}{1 - \exp(2i\pi(m\tau + n\sigma + z))} \right)$$

This function converges absolutely for $z \in \mathbb{C}, z \notin \mathbb{Z} + \mathbb{Z}_{\leq 0}\tau + \mathbb{Z}_{\leq 0}\sigma$ and it is meromorphic in this domain. It is worth noting that this function has a nice
expression as the exponential of an infinite sum involving sinuses:

\[ \Gamma(z, \tau, \sigma) = \exp \left( \sum_{j \geq 1} \frac{1}{2ij} \frac{\sin(\pi j(2z - \tau - \sigma))}{\sin(\pi j \tau) \sin(\pi j \sigma)} \right) \]  

The range of parameters \( \tau, \sigma \) can thus be extended to \( \tau, \sigma \in \mathbb{C} - \mathbb{R} \) by putting:

\[ \Gamma(z, -\tau, \sigma) = \frac{1}{\Gamma(z + \tau, \tau, \sigma)}, \quad \Gamma(z, \tau, -\sigma) = \frac{1}{\Gamma(z + \sigma, \tau, \sigma)} \]

In [10], Theorems 3.1 and 4.1, Felder and Varchenko proved the following properties for the elliptic Gamma function related to the \( \theta \) function of Complex Multiplication. If \( \tau, \sigma \in \mathbb{C} - \mathbb{R} \) then

\[ \Gamma(z + 1, \tau, \sigma) = \Gamma(z, \tau + 1, \sigma) = \Gamma(z, \tau, \sigma + 1) = \Gamma(z, \tau, \sigma) \]

\[ \Gamma(z + \tau, \tau, \sigma) = \theta(z, \sigma) \Gamma(z, \tau, \sigma) \]

\[ \Gamma(z + \tau + \sigma, \tau, \sigma) = \Gamma(-z, \tau, \sigma)^{-1} \]

Finally, if \( \sigma/\tau \not\in \mathbb{R} \) then

\[ \Gamma(z, \tau, \sigma)^{-1} \Gamma \left( \frac{z}{\tau}, \frac{-1}{\tau}, \frac{\sigma}{\tau} \right) \Gamma \left( \frac{z - \tau}{\sigma}, \frac{-\tau}{\sigma}, \frac{-1}{\sigma} \right)^{-1} = \exp(i \pi P_1(z, \tau, \sigma)) \]  

where

\[ P_1(z, \tau, \sigma) = \frac{z^3}{3 \tau \sigma} - \frac{\tau + \sigma - 1}{2 \tau \sigma} z^2 + \frac{\tau^2 + \sigma^2 + 3 \tau \sigma - 3 \tau - 3 \sigma + 1}{6 \tau \sigma} z \]

\[ + \frac{1}{12} (\tau + \sigma - 1) \left( \frac{1}{\tau} + \frac{1}{\sigma} - 1 \right) \]

These properties are transformation properties for the elliptic Gamma function under the action of \( \text{SL}_3(\mathbb{Z}) \). This will be more apparent for the geometric elliptic Gamma functions which we introduce now.

**2.1.2 Geometric variants of the elliptic Gamma function**

In [9], Felder, Henriques, Rossi and Zhu have built elliptic Gamma functions associated to pairs of primitive vectors in a free \( \mathbb{Z} \)-module of rank 3. We briefly review the construction which we will adapt to Nishizawa’s \( G_r \) functions. Consider two linearly independent primitive vectors \( a, b \) in \( \Lambda \) a free \( \mathbb{Z} \)-module of rank 3. Fix a basis \( B \) (and therefore an orientation) of \( \Lambda \) and write \( \det_B(a, b, \cdot) = s \gamma \) for a unique integer \( s > 0 \) and a unique linear form
\(\gamma \in L = \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})\). The basis \(B\) identifies \(\Lambda \simeq \mathbb{Z}^3\) which fixes an action of \(\text{SL}_3(\mathbb{Z})\) on \(\Lambda\). Now write

\[C(a, b) = \{\delta \in L, \delta(a) > 0, \delta(b) \leq 0\}\]

The geometric elliptic \(\Gamma_{a,b}\) function associated to the pair \((a, b)\) is defined as follows:

\[
\Gamma_{a,b}(w, x, \Lambda) = \frac{\prod_{\delta \in C(a,b)/\mathbb{Z}/\gamma} (1 - \exp(2i\pi(\delta(x) - w)/\gamma(x)))}{\prod_{\delta \in C(b,a)/\mathbb{Z}/\gamma} (1 - \exp(2i\pi(\delta(x) - w)/\gamma(x)))}
\]

where \(w \in \mathbb{C}\) and \(x \in \text{Hom}_\mathbb{Z}(L, \mathbb{C})\) and where we understand \(\delta(x) = x(\delta)\) by biduality. The study of the convergence properties of this function was carried out in [9] and is better understood with an alternative definition of this function. Consider \(\alpha, \beta \in L\) such that \(\beta(a) = \alpha(b) = 0, \alpha(a) > 0, \text{ and } \beta(b) > 0\). We call such a pair a positive dual family to \((a, b)\). We define

\[F(\alpha, \beta) = \{\delta \in L, 0 \leq \delta(a) < \alpha(a), 0 \leq \delta(b) < \beta(b)\}\]

Then in [9] the authors prove that the following alternative definition holds:

\[
\Gamma_{a,b}(w, x, \Lambda) = \prod_{\delta \in F(\alpha, \beta)/\mathbb{Z}/\gamma} \Gamma\left(\frac{w + \delta(x)}{\gamma(x)}, \frac{\alpha(x)}{\gamma(x)}, \frac{\beta(x)}{\gamma(x)}\right)
\]  

(3)

The right-hand side is a finite product of elliptic Gamma functions and is independent of the choice for \(\alpha, \beta\). As a function of \(x\) it converges outside the hyperplane \(\gamma(x) = 0\) whenever \(\alpha(x)/\gamma(x)\) and \(\beta(x)/\gamma(x)\) are not real. As a function of \(w\), it converges outside the discrete set of poles of every elliptic Gamma function involved. Note that the conditions \(\alpha(x)/\gamma(x), \beta(x)/\gamma(x) \in \mathbb{R}\) describe a \(\mathbb{R}\)-hyperplane of the \(\mathbb{R}\)-vector space \(\text{Hom}_\mathbb{Z}(L, \mathbb{C})\), so that the geometric elliptic Gamma function is well-defined on a dense open set of \(\mathbb{C} \times \text{Hom}_\mathbb{Z}(L, \mathbb{C})\) endowed with the finite dimensional \(\mathbb{C}\)-vector space topology.

**Remark:** Formula (3) could have been taken as the definition of the geometric elliptic Gamma function associated to the pair \((a, b)\). This will be done below (see Proposition 4) when considering \(G_r\) functions associated to \(r + 1\) vectors in a free \(\mathbb{Z}\)-module of rank \(r + 2\). It will remain to check that this definition is independent of the choice for \(\alpha, \beta\), and this can be done by direct computation in reverting the proof of formula (3) carried out in [9].

On the computational side, the geometric function \(\Gamma_{a,b}\) is here computed as a product of \(#|F(\alpha, \beta)/\mathbb{Z}/\gamma|\) ordinary elliptic Gamma functions. A standard choice gives \(\alpha(a) = \beta(b) = s = #|F/\mathbb{Z}/\gamma|\). This choice is given by considering a third vector \(c\) such that \(\text{det}_B(a, b, c) = s\) and writing the transpose of the comatrix of \((a, b, c)\) as \((\alpha, \beta, s\gamma)^T\). The properties of the elliptic
Gamma function translate into nice properties for the geometric $\Gamma_{a,b}$ functions under the action of $\text{SL}_3(\mathbb{Z})$ as shown in [9]. Namely, in the domains where every term converges (which are dense open sets), we have the following inversion relation:

$$\Gamma_{a,b}(w, x, \Lambda)\Gamma_{b,a}(w, x, \Lambda) = 1$$

We also have for $a, b, c$ linearly independent:

$$\Gamma_{a,b}(w, x, \Lambda)\Gamma_{b,c}(w, x, \Lambda)\Gamma_{c,a}(w, x, \Lambda) = \exp(i\pi P_{a,b,c}(w, x, \Lambda)) \quad (4)$$

where $P_{a,b,c}$ is a polynomial in $w$ of degree 3. Furthermore, we have the equivariance relations: for all $g \in \text{SL}_3(\mathbb{Z})$,

$$\Gamma_{ga,gb}(w, gx, \Lambda) = \Gamma_{a,b}(w, x, \Lambda)$$

$$P_{ga,gb,gc}(w, gx, \Lambda) = P_{a,b,c}(w, x, \Lambda)$$

The polynomial $P_{a,b,c}$ will be presented below alongside other Bernoulli polynomials serving the same purpose for higher degree (see definition 5 in section 2.2.3). It is worth noting that the 3-term relation (4) can be understood as a 1-cocycle relation for $\text{SL}_3(\mathbb{Z})$. Consider the cocycle $\psi_a(g) = \Gamma_{a,ga}$ for a primitive vector $a \in \Lambda$ and $g \in \text{SL}_3(\mathbb{Z})$. Then we may rewrite formula (4) for fixed $w, x$ as

$$\Gamma_{a,g_1,a,g_1g_2,a} = \Gamma_{g_1g_2,a,a} = \exp(i\pi P_{a,g_1,a,g_1g_2,a})$$

which yields the coboundary relation:

$$d\psi_a(g_1, g_2) := \psi_a(g_1)(g_1 \cdot \psi_a(g_2))\psi_a(g_1g_2)^{-1} = \exp(i\pi P_{a,g_1,a,g_1g_2,a}) \quad (5)$$

2.2 Multiple elliptic Gamma functions

Here we recall the definition and properties of Nishizawa’s $G_r$ functions which generalise the $\theta$ and elliptic Gamma functions above, and we construct their geometric variants $G_{r,2}$ upgrading the construction in [9] to higher degrees.

2.2.1 Nishizawa’s $G_r$ functions

In [13], Nishizawa introduced the hierarchy of $G_r$ functions that generalise both the $\theta$ and elliptic Gamma functions. Later, Narukawa proved the modularity property for the $G_r$ functions by relating them to multiple sine
functions. To simplify the notations, we will write as in [13], with a clear separation for base and exponential variables:

$$\tau = (\tau_j)_{0 \leq j \leq r} \in \mathbb{C}^{r+1} ; \ q = (q_j)_{0 \leq j \leq r} = (\exp(2i\pi \tau_j))_{0 \leq j \leq r}$$

$$|\tau| = \sum_{j=0}^{r} \tau_j ; \ |q| = \prod_{j=0}^{r} q_j$$

$$\tau^{-}(j) = (\tau_n)_{0 \leq n \neq j \leq r} ; \ q^{-}(j) = (q_n)_{0 \leq n \neq j \leq r}$$

$$\tau[j] = (\tau_0, \ldots, \tau_{j-1}, -\tau_j, \tau_{j+1}, \ldots, \tau_r) ; \ q[j] = (q_0, \ldots, q_{j-1}, q_j^{-1}, q_{j+1}, \ldots, q_r)$$

Moreover, for a \((r+1)\)-uple \(m = (m_0, \ldots, m_r) \in \mathbb{Z}^{r+1}\), we write:

$$m + 1 = (m_j + 1)_{0 \leq j \leq r} , \ m \cdot \tau = (m_j \tau_j)_{0 \leq j \leq r} , \ q^m = (q_j^m)_{0 \leq j \leq r}$$

$$\prod_{m \geq 0} = \prod_{m_0 \geq 0} \cdots \prod_{m_r \geq 0}$$

Let \(r \geq 0\) and \(\tau = (\tau_0, \ldots, \tau_r) \in \mathbb{H}^{r+1}\). For \(z \in \mathbb{C} , \ (z \notin \mathbb{Z} + \sum_{j=0}^{r} \mathbb{Z} \tau_j\) if \(r\) is odd), put \(x = \exp(2i\pi z)\) and define as in [13]:

$$G_r(z, \tau) = \prod_{m \geq 0} \left(1 - |q^m| x^{-1}\right) \left(1 - |q^m| x\right)^{(-1)^r}$$

Note that \(G_0\) and \(G_1\) are respectively the \(\theta\) function and the elliptic Gamma function. It is once again worth noting that the \(G_r\) functions have nice expressions as the exponential of infinite sums involving sinuses (see [[13], Proposition 3.6]), namely:

$$G_r(z, \tau) = \begin{cases} 
\exp \left( \sum_{j \geq 1} \frac{1}{(2i\tau_j)^2} \prod_{k=0}^{r} \sin(\pi j(2z-|\tau|)) \right) & \text{if } r \text{ is odd} \\
\exp \left( \sum_{j \geq 1} \frac{1}{(2i\tau_j)^2} \prod_{k=0}^{r} \cos(\pi j(2z-|\tau|)) \right) & \text{if } r \text{ is even}
\end{cases} \quad (6)$$

This expression allows us to extend the range of parameters to \(\tau \in (\mathbb{C} - \mathbb{R})^{r+1}\) by putting:

$$G_r(z, \tau[j]) = G_r(z + \tau_j, \tau)^{-1} \quad (7)$$

The \(G_r\) functions satisfy modular properties similar to that of the elliptic Gamma function, as proven by Nishizawa in [13] and later by Narukawa in [12]. First, the \(G_r\) functions are 1-periodic in each of their arguments and they satisfy:

$$G_r(z + |\tau|, \tau) = G_r(-z, \tau)^{(-1)^r}$$
$$G_r(-z, -\tau) = G_r(z, \tau)^{-1} \quad (8)$$
Furthermore, if \( r \geq 1 \), the \( G_r \) functions are almost periodic in \( z \) with periods \( \tau_j \) for \( 0 \leq j \leq r \) with a correction factor involving a lower degree function:

\[
G_r(z + \tau_j, \bar{z}) = G_{r-1}(z, \bar{z}^{-}(j))G_r(z, \bar{z})
\]

A consequence of these properties which we will use is the following formula for complex conjugation:

\[
G_r(\bar{z}, \bar{\tau}_0, \ldots, \bar{\tau}_r) = G_r(z, \tau_0, \ldots, \tau_r)^{(-1)^r}
\]

### 2.2.2 Bernoulli polynomials and Narukawa’s theorem

The last missing property is crucial for arithmetic purposes and was later proved by Narukawa. To state Narukawa’s theorem, we first need to introduce the multiple Bernoulli polynomials. Let \( \omega = (\omega_1, \ldots, \omega_d) \in (\mathbb{C} - \{0\})^d \) with \( \omega_j \neq 0 \) for all \( 1 \leq j \leq d \). We define the multiple Bernoulli polynomials \( B_{d,n}^*(z, \omega) \) with the following generating function:

\[
e^{zt} \prod_{j=1}^{d} \frac{\omega_j t}{e^{\omega_j t} - 1} = \sum_{n \geq 0} B_{d,n}^*(z, \omega) \frac{t^n}{n!} = \left( \sum_{m \geq 0} \frac{z^m t^m}{m!} \right) \prod_{1 \leq j \leq d} \left( \sum_{k_j \geq 0} B_{k_j} \omega_j \frac{k_j t^{k_j}}{k_j!} \right)
\]

where the \( B_{k_j} \) are the usual Bernoulli numbers. The classic Bernoulli polynomials are obtained with the choice \( d = 1, \omega_1 = 1 \), and the Bernoulli numbers are obtained for \( z = 0 \). These polynomials obey many relations which we sum up below and which can easily be obtained from the properties of the generating function. For \( \omega \in (\mathbb{C} - \{0\})^d \), \( B_{d,n}^*(z, \omega) \) is a degree \( n \) homogeneous polynomial in \( d + 1 \) variables, which is symmetric in the \( d \) variables of \( \omega \). Moreover, the coefficients of \( B_{d,n}^*(z, \omega) \) are rational numbers. In [12], Narukawa used the rescaled version \( B_{d,n}(z, \omega) = (\prod_{j=1}^{d} \omega_j^{-1}) B_{d,n}^*(z, \omega) \) and proved the following properties:

\[
B_{d,n}(\alpha z, \alpha \omega) = \alpha^{n-d} B_{d,n}(z, \omega), \forall \alpha \in \mathbb{C} - \{0\} \\
B_{d,n}(z + |\omega|, \omega) = (-1)^n B_{d,n}(-z, \omega) \\
B_{d,n}(z + \omega_j, \omega) = B_{d,n}(z, \omega) + nB_{d-1,n-1}(z, \omega^{-}(j)) \\
B_{d,n}(z + \omega_j, \omega) = -B_{d,n}(z, \omega[j]) \\
\frac{\partial}{\partial z} B_{d,n}(z, \omega) = nB_{d,n-1}(z, \omega)
\]

Narukawa’s modularity theorem (see [12], Theorem 7) involves the polynomials \( B_{d,d}^* \) or rather the rescaled versions \( B_{d,d} \). For instance, we have

\[
(\omega_1 \omega_2) B_{2,2}(z, \omega_1, \omega_2) = B_{2,2}^*(z, \omega_1, \omega_2) = z^2 - z(\omega_1 + \omega_2) + \frac{\omega_1^2 + \omega_2^2 + 3\omega_1 \omega_2}{6}
\]
Then, Narukawa’s theorem can be stated as follows. Let $r \geq 0$. Take $\omega \in (\mathbb{C} - \{0\})^{r+2}$ and suppose that $\omega_j/\omega_n \in \mathbb{C} - \mathbb{R}$ for all $j \neq n$. Then for $z \in \mathbb{C}$ outside the discrete set of poles of the left-hand side

$$\prod_{j=0}^{r+1} G_r \left( \frac{z}{\omega_j}, \left( \frac{\omega_n}{\omega_j} \right)_{n \neq j} \right) = \exp \left( \frac{-2i\pi}{(r+2)!} B_{r+2,r+2}(z, \omega) \right) \quad (10)$$

### 2.2.3 Geometric $G_{r,a}$ functions

In this section we upgrade the construction of section 2.1.2 and define geometric variants of the $G_r$ functions. We also prove (Theorem 6) the modular property of the geometric variants and their equivariance property under the action of $\text{SL}_{r+2}(\mathbb{Z})$. In the literature one may find other variants of these functions, such as the $G_r$ functions associated to cones by Winding in [23]. Although our construction inspired by [9] differs from Winding’s construction, it would be interesting to understand the links between the two variants. As mentioned, we will use the alternative definition of 2.1.2 to start and prove that this definition is indeed valid. Throughout this section, we consider a free $\mathbb{Z}$-module $\Lambda$ of rank $r+2$ with an orientation form given by the determinant in a fixed basis. Let us start by defining positive dual families which will be used in the rest of this work.

**Definition 1:** Let $(a_0, \ldots, a_m)$ be a family of $m+1$ independent primitive vectors in a lattice $\Lambda$. We call $(a_0, \ldots, a_m) \in L = \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$ a positive dual family to $a = (a_0, \ldots, a_m)$ if for all $0 \leq j \leq m$ the following holds:

$$\alpha_j(a_j) > 0, \quad \alpha_j(a_k) = 0, \quad \forall k \neq j$$

The important cases in this work will be those where $m = r$ and $m = r + 1$ in the lattice $\Lambda$ of rank $r+2$. The following lemmas show that in these cases two positive dual families to the same family $a$ are closely related.

**Lemma 2:** Let $a = (a_0, \ldots, a_r)$ be a family of $r + 1$ linearly independent primitive vectors in the oriented lattice $\Lambda$ of rank $r+2$. Take two positive dual families $(a_0, \ldots, a_r), (a'_0, \ldots, a'_r)$ to $a$ in $L = \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$. For $0 \leq j \leq r$ write $s_j = \alpha_j(a_j)$ and $s'_j = \alpha'_j(a_j)$. Then, there are rational numbers $r_j$ such that $\alpha_j = (s_j/s'_j)\alpha'_j + r_j \det(a_0, \ldots, a_r, \cdot)$ for all $0 \leq j \leq r$.

**Proof:**
Consider $\gamma_j = s'_j \alpha_j - s_j \alpha'_j$. Then for all $0 \leq k \leq r$, we get $\gamma_j(a_k) = 0$. This means that either $\gamma_j = 0$ and then $r_j = 0$ or the $\mathbb{Q}$-linear forms $\gamma_j$ and
det($a_0, \ldots, a_r, \cdot$) defined on the $\mathbb{Q}$-vector space $\Lambda \otimes \mathbb{Q}$ share the same kernel, and therefore, they must be linearly dependent.

\[ \square \]

**Lemma 3:** A family $a = (a_0, \ldots, a_{r+1})$ of $r+2$ linearly independent primitive vectors in a lattice $\Lambda$ of rank $r+2$ has exactly one positive dual family $\alpha$ containing only primitive vectors. We call this family the primitive positive dual family to $a$.

**Proof:**
Consider two such families $\alpha, \alpha'$ and write again $s_j = \alpha_j(a_j), s'_j = \alpha'_j(a_j)$. Consider once again $\gamma_j = s'_j \alpha_j - s_j \alpha'_j$ such that for all $0 \leq k \leq r+1$, $\gamma_j(a_k) = 0$. The family $(a_0, \ldots, a_{r+1})$ is a basis of the $\mathbb{Q}$-vector space $\Lambda \otimes \mathbb{Q}$, so $\gamma_j = 0$ for all $0 \leq j \leq r+1$. This means that $\alpha_j = s_j \alpha'_j/s'_j$ and we write $s_j n_j + s'_j n'_j = d_j$ where $d_j = \gcd(s_j, s'_j)$. Then $n_j \alpha_j + n'_j \alpha'_j = d_j \alpha'_j/s'_j \in L$. Because $\alpha'_j$ is a primitive vector in $L$ by assumption, we must have $d_j = s'_j$, and then because $\alpha_j = s_j \alpha'_j/d_j$ is primitive in $L$ we must also have $d_j = s_j$. The conditions $\alpha_j(a_j) = s_j, \alpha_j(a_k) = 0$ for $k \neq j$ describe exactly one linear form on the $\mathbb{Q}$-vector space $\Lambda \otimes \mathbb{Q}$. To compute $\alpha$ we compute the comatrix of $a$ in a suitable basis of $\Lambda$ and we rescale the linear forms to obtain primitive elements in $L$.

\[ \square \]

Now we may define the geometric variants of the $G_r$ functions using lemma 2:

**Proposition 4:** Let $a = (a_0, \ldots, a_r)$ be a family of $r+1$ linearly independent primitive vectors in the oriented lattice $\Lambda$ of rank $r+2$. There is a unique $s \in \mathbb{Z}_{>0}$ and a unique primitive element $\gamma \in L = \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ such that $\forall c \in \Lambda, \det(a, c) = s \gamma(c)$. For any choice of positive dual family $(\alpha_0, \ldots, \alpha_r)$ to $a$ in $L$ the function

$$ G^a_{r,2}(w, x, \Lambda) = \prod_{\delta \in F(a)/\mathbb{Z}\gamma} G_r \left( \frac{w + \delta(x)}{\gamma(x)}, \frac{1}{\gamma(x)} \alpha(x) \right) \quad (11) $$

$$ F(a) = \{ \delta \in L, \forall 0 \leq j \leq r, 0 \leq \delta(a_j) < \alpha_j(a_j) \} $$

is well defined for $(w, x)$ in a dense open set of the $\mathbb{C}$-vector space $\mathbb{C} \times \text{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \simeq \mathbb{C} \times \mathbb{C}^{r+2}$ endowed with the finite dimensional $\mathbb{C}$-vector space topology. Furthermore, it is independent of the choice for $\alpha$.

**Proof:**
The set $\text{Hom}_{\mathbb{Z}}(L, \mathbb{C})$ is a $\mathbb{R}$-vector space of dimension $2r + 4$. The condition $\gamma(x) = 0$ describes a subspace of dimension $2r + 2$ and the condition $\alpha_j(x)/\gamma(x) \in \mathbb{R}$ describes a subspace of dimension $2r + 3$. The function $G^a_{r,2}(w, x, \Lambda)$ is therefore well defined for $x$ outside a finite union of
\( \mathbb{R} \)-hyperplanes in \( \text{Hom}_Z(L, \mathbb{C}) \), i.e. in a dense open set. The right-hand side in (11) is a finite product of meromorphic functions in \( w \) and it has a discrete set of poles.

Let us now prove that this definition is indeed independent of the choice for the positive dual family \( \alpha \). Take \( x \in \text{Hom}_Z(L, \mathbb{C}) \) such that \( \gamma(x) \neq 0 \) and \( \alpha_j(x)/\gamma(x) \notin \mathbb{R} \) for all \( 0 \leq j \leq r \). Put for \( 0 \leq j \leq r \), \( d_j = \pm 1 \) such that \( d_j\alpha_j(x)/\gamma(x) \in \mathbb{H} \). Put also \( D = \sum_{j=0}^{r}(d_j - 1)/2 \). Then using the inversion relation (7) we get:

\[
G_{r,a}^\alpha(w, x, \Lambda)^{-1}\rho = \prod_{\delta \in F(\omega)/\mathbb{Z}_\gamma} G_r \left( \frac{w + \delta(x)}{\gamma(x)} + \sum_{j=0}^{r} \frac{d_j - 1}{2} \frac{\alpha_j(x)}{\gamma(x)} \left( \frac{1}{\gamma(x)} d_j\alpha_j(x) \right)_{0 \leq j \leq r} \right)
\]

\[
G_{r,a}^\alpha(w, x, \Lambda)^{-1}\rho = \prod_{\delta \in F(\omega)/\mathbb{Z}_\gamma} \prod_{m \geq 0} \left[ 1 - e^{-2\pi i \left( \sum_{j=0}^{r} \frac{(d_j m_j + (1 + d_j)/2)\alpha_j(x) - w - \delta(x)}{\gamma(x)} \right)} \right]^{-1}\rho \left( \frac{d_j m_j + (1 - d_j)/2\alpha_j(x) + w + \delta(x)}{\gamma(x)} \right)
\]

Now write \( C^\pm(\alpha, x) \) for the set of \( \delta' \in L \) satisfying for all \( 0 \leq j \leq r \):

\[
\begin{cases}
\delta'(a_j) > 0, & \text{if } d_j = \pm 1 \\
\delta'(a_j) \leq 0, & \text{if } d_j = \mp 1
\end{cases}
\]

Consider \( \delta' \in C^+(\alpha, x) \). This means that if \( d_j = 1 \) then \( \delta'(a_j) > 0 \) and we can perform Euclidian division by \( \alpha_j(a_j) \) so that there exists a unique integer \( m_j > 0 \) such that \( 0 \leq (m_j\alpha_j - \delta')(a_j) < \alpha_j(a_j) \). On the contrary, if \( d_j = -1 \) then \( \delta'(a_j) \leq 0 \) and there exists a unique integer \( m_j \geq 0 \) such that \( 0 \leq (-m_j\alpha_j - \delta')(a_j) < \alpha_j(a_j) \). Then for all \( 0 \leq k \leq r \):

\[
0 \leq \left( \sum_{j=0}^{r} d_j m_j \alpha_j - \delta' \right)(a_k) < \alpha_k(a_k)
\]

This shows that \( C^+(\alpha, x) \) can be written as a disjoint union

\[
C^+(\alpha, x) = \bigcup_{\delta \in F(\omega)/\mathbb{Z}_\gamma} \bigcup_{m \geq 0} \left\{ -\delta + \sum_{j=0}^{r} (d_j m_j + (1 + d_j)/2\alpha_j) \right\} + \mathbb{Z}_\gamma
\]

and a similar statement holds for \( C^-(\alpha, x) \) so that \( G_{r,a}^\alpha(w, x, \Lambda)^{-1}\rho \) is equal to

\[
\prod_{\delta' \in C^+(\alpha, x)/\mathbb{Z}_\gamma} \left( 1 - e^{2\pi i \left( \frac{\delta'(x) - w}{\gamma(x)} \right)} \right) \prod_{\delta' \in C^-(\alpha, x)/\mathbb{Z}_\gamma} \left( 1 - e^{-2\pi i \left( \frac{\delta'(x) - w}{\gamma(x)} \right)} \right)
\]

12
Then, we only need to show that the sets $C^\pm(\alpha, x)$ are independent of the choice for $\alpha$. Consider another positive dual family $\alpha'$ to $\alpha$. Write $\alpha_j(a_j) = s_j > 0$ and $\alpha'_j(a_j) = s'_j > 0$. Then from lemma 2 there is a rational number $r_j$ such that

$$\alpha'_j = \frac{s'_j}{s_j} \alpha_j + r_j \gamma$$

For any $x \in \text{Hom}_\mathbb{Z}(L, \mathbb{C})$ we get:

$$d_j \frac{\alpha'_j(x)}{\gamma(x)} = d_j r_j + \frac{d_j s'_j}{s_j} \frac{\alpha_j(x)}{\gamma(x)} \in \mathbb{H}$$

This shows that the signs $d_j$ (and therefore also $D$) are independent of the choice for $\alpha$. By construction we get $C^\pm(\alpha, x) = C^\pm(\alpha', x)$ and the definition of the geometric $G_{r,a}^\alpha$ function is independent of the choice for $\alpha$.

From now on we denote by $G_{r,a} := G_{r,a}^\alpha$ the geometric $G_r$ function associated to $\alpha$ for any suitable choice of $\alpha$. A choice that is interesting for us is defined by the comatrix or rather by the rescaled comatrix. Let $c$ be a vector such that $\gamma(c) = 1$. Then the elementary divisors of the concatenation $(\alpha, c)$ can be written as $[A_r, A_{r-1}, \ldots, A_1, 1, 1]$ with $A_i | A_{i+1}$ so that a possible choice for $\alpha$ is given by:

$$\alpha = A_r I_{r+2}$$

To express the transformation properties of the function $G_{r,a}$ under the action of $\text{SL}_{r+2}(\mathbb{Z})$ we need to introduce the family of geometric Bernoulli polynomials which encompass $P_{a,b,c}$. In what follows, we want to discuss an analogue of formula (4) for $G_{r,a}$, and we must consider families of $r+2$ vectors. Recall that we have defined the polynomials $B^*_{r+2,a_0,\ldots,a_{r+1}}(w, x, \Lambda)$ in section 2.2.2.

**Definition 5:** Let $w \in \mathbb{C}$ and $x \in \text{Hom}_\mathbb{Z}(L, \mathbb{C}) \simeq \mathbb{C}^{r+2}$. Let $a_0, \ldots, a_r, a_{r+1}$ be a family of $r + 2$ linearly independent primitive vectors in the oriented lattice $\Lambda$. Let $\epsilon$ be the sign of $(-1)^{r+1}\det(a_0, \ldots, a_{r+1})$ and put

$$B^*_{r+2,a_0,\ldots,a_{r+1}}(w, x, \Lambda) = \frac{-2\epsilon}{(r+2)!} \sum_{\delta \in F} B^*_{r+2,r+2}(w + \delta(x), a_0(x), \ldots, a_{r+1}(x))$$

where $a_0, \ldots, a_{r+1}$ is the unique primitive positive dual family to $(a_0, \ldots, a_{r+1})$ in $L = \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$ (see lemma 3) and

$$F = F(\alpha_0, \ldots, \alpha_{r+1}) = \{\delta \in L, 0 \leq \delta(a_j) < \alpha_j(a_j), \forall 0 \leq j \leq r + 1\}$$
is a finite set.

These geometric Bernoulli “polynomials” have the form

$$B^*_{r+2,a_0,...,a_{r+1}}(w, x, \Lambda) = \sum_{l+k_0+...+k_{r+1}=r+2} c_{l,k_0,...,k_{r+1}} w^l \prod_{j=0}^{r+1} \alpha_j(x)^{k_j}$$

where the sum is taken over non-negative integers and the coefficients $c_{l,k_0,...,k_{r+1}}$ are rational numbers which explicitly depend on the set $F = F(\alpha_0, \ldots, \alpha_{r+1})$.

The modularity property for the geometric $G_{r,a}$ functions will involve the rescaled versions of the geometric Bernoulli polynomials

$$B_{r+2,a_0,...,a_{r+1}}(w, x, \Lambda) = \left(\prod_{j=0}^{r+1} \alpha_j(x)\right)^{-1} B^*_{r+2,a_0,...,a_{r+1}}(w, x, \Lambda)$$

For $r = 1$ the polynomial $B_{3,a,b,c}$ may be identified with the polynomial $P_{a,b,c}$ appearing in formula (4). We can now generalise formula (4).

**Theorem 6:**

1. [Modular property] For a family $a_0, \ldots, a_{r+1}$ of $r+2$ linearly independent primitive vectors in $\Lambda$:

$$\prod_{j=0}^{r+1} G_{r,(a_k)_{k\neq j}}(w, x, \Lambda)(-1)^j = \exp(i\pi B_{r+2,a_0,...,a_{r+1}}(w, x, \Lambda)) \quad (13)$$

2. [Equivariance relations] For a family $a_0, \ldots, a_r$ of $r+1$ linearly independent primitive vectors in $\Lambda$, for $w \in \mathbb{C}$ and $x \in \text{Hom}_\mathbb{Z}(L, \mathbb{C})$, for all $g \in \text{SL}_{r+2}(\mathbb{Z})$:

$$G_{r,g}(w, gx, \Lambda) = G_{r,g}(w, x, \Lambda)$$

$$B_{r+2,g,a_0,...,a_{r+1}}(w, gx, \Lambda) = B_{r+2,a_0,...,a_{r+1}}(w, x, \Lambda)$$

**Proof:**

1. The modular relation for the geometric $G_{r,a}$ functions is a consequence of the modular property (10) for the ordinary $G_r$ function. Let $\alpha$ be the primitive positive dual family to $a$ in $L$ as given in lemma 3. Write $\epsilon$ for the sign of $\det_B(a_0, \ldots, a_{r+1}).(-1)^{r+1}$. Then for all $0 \leq j \leq r+1$ we have

$$\det_B(a_0, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{r+1}, \cdot) = s_j.\epsilon.(-1)^j \alpha_j$$

14
so that $\gamma_j = \epsilon.(-1)^j\alpha_j$ and

$$G_{r,(a_k)_{k\neq j}}(w,x,\Lambda) = \prod_{g \in F_j/\mathbb{Z}\alpha_j} G_r\left( \frac{w + \delta(x)}{\alpha_j(x)}, \left( \frac{\alpha_k(x)}{\alpha_j(x)} \right)_{k \neq j} \right)^{(-1)^j \epsilon}$$

where

$$F_j = \{ \delta \in \mathbb{L}, 0 \leq \delta(a_k) < \alpha_k(a_k), \forall 0 \leq k \neq j \leq r + 1 \}$$

Then

$$\prod_{j=0}^{r+1} G_{r,(a_k)_{k\neq j}}(w,x,\Lambda)^{(-1)^j} = \prod_{j=0}^{r+1} \prod_{\delta \in F_j/\mathbb{Z}\alpha_j} G_r\left( \frac{w + \delta(x)}{\alpha_j(x)}, \left( \frac{\alpha_k(x)}{\alpha_j(x)} \right)_{k \neq j} \right)^{\epsilon}$$

Put $\mathcal{F} = F(\alpha_0, \ldots, \alpha_{r+1}) = \{ \delta \in \mathbb{L}, 0 \leq \delta(a_j) < \alpha_j(a_j), \forall 0 \leq j \leq r + 1 \}$ so that we can write uniformly $\mathcal{F} \simeq F_j/\mathbb{Z}\alpha_j$ for all $0 \leq j \leq r + 1$. Then, using Narukawa’s theorem (10) for each $\delta$ in the finite set $\mathcal{F}$, yields:

$$\prod_{j=0}^{r+1} G_{r,(a_k)_{k\neq j}}(w,x,\Lambda)^{(-1)^j} = \prod_{\delta \in \mathcal{F}} \exp\left( \frac{-2i\pi \epsilon}{(r+2)!} B_{r+2,r+2}(w + \delta(x), \alpha(x)) \right)$$

The identification of the right-hand side gives the conclusion:

$$\prod_{j=0}^{r+1} G_{r,(a_k)_{k\neq j}}(w,x,\Lambda)^{(-1)^j} = \exp(i\pi B_{r+2,a_0,\ldots,a_{r+1}}(w,x,\Lambda))$$

2. Consider $g \in \text{SL}_{r+2}(\mathbb{Z})$. In the construction of $G_{r,g\alpha}$ or $B_{r,a_0,\ldots,a_{r+1}}$ replacing $\alpha$ with $g\alpha$ replaces $\alpha$ with $\alpha g^{-1}$, $gamma$ with $\gamma g^{-1}$ and $F(\alpha)$ with $F(\alpha g^{-1})$ but $\epsilon$ is left unchanged. Therefore we may write:

$$G_{r,g\alpha}(w,gx,\Lambda) = \prod_{\delta \in F(\alpha g^{-1})/\mathbb{Z}\gamma} G_r\left( \frac{w + \delta(gx)}{\gamma^{-1}(gx)} \right)$$

Then, identifying $F(\alpha g^{-1}) = F(\alpha)g^{-1}$ and putting $\delta' = \delta g$ gives

$$G_{r,g\alpha}(w,gx,\Lambda) = \prod_{\delta' \in F(\alpha)/\mathbb{Z}\gamma} G_r\left( \frac{w + \delta'(x)}{\gamma^{-1}(x)} \right)$$

$$G_{r,g\alpha}(w,gx,\Lambda) = G_{r,a}(w,x,\Lambda)$$

As for $B_{r+2,a_0,\ldots,a_{r+1}}$, write $\mathcal{F} = F(\alpha_0, \ldots, \alpha_{r+1})$:

$$B_{r+2,a_0,\ldots,a_{r+1}}(w,gx,\Lambda) = \frac{-2\epsilon}{(r+2)!} \sum_{\delta \in \mathcal{F}_{g^{-1}}} B_{r+2,r+2}(w + \delta(gx), \alpha(g^{-1}gx))$$

15
Put once again $\delta' = \delta g$, which gives:

$$B_{r+2, g_0, \ldots, g_{r+1}}(w, gx, \Lambda) = \frac{-2\epsilon}{(r + 2)!} \sum_{\delta' \in F(a_0, \ldots, a_{r+1})} B_{r+2, r+2}(w + \delta(x), \Omega(x))$$

Identify the right-hand side to conclude that

$$B_{r+2, g_0, \ldots, g_{r+1}}(w, gx, \Lambda) = B_{r+2, a_0, \ldots, a_{r+1}}(w, x, \Lambda)$$

We also add that the geometric $G_{r,a}$ functions behave nicely under permutation of vectors, namely for all permutation $\sigma \in S_{r+1}$, $G_{r,\sigma(a)} = G_{r,a}^{\epsilon(\sigma)}$ where $\epsilon(\sigma)$ is the sign of the permutation $\sigma$. As in the cubic case, the relation (13) can be restated and best understood as a multiplicative $r$-cocycle relation for $SL_{r+2}(\mathbb{Z})$. For $g_1, \ldots, g_l \in SL_{r+2}(\mathbb{Z})$ put

$$[g_1| \ldots | g_l] = \left( g_1, \ldots, \prod_{j=1}^{k} g_j, \ldots, \prod_{j=1}^{l} g_j \right)$$

Then we may write $\psi_a(g_1, \ldots, g_r) = G_{r,a,g_1,a,\ldots,\left(\prod_{j=1}^{l} g_j\right)a} = G_{r,|g_1| \ldots |g_r|,a}$ for a primitive vector $a \in \Lambda$ and $g_1, \ldots, g_r \in SL_{r+2}(\mathbb{Z})$. The coboundary of $\psi_a$ is given by the modular property (13):

$$d\psi_a(g_1, \ldots, g_{r+1}) = \exp(i\pi B_{r+2,|g_1| \ldots |g_{r+1}|,a})$$

where 1 stands for the identity matrix $I_{r+2}$. This property is essential to understand the type of evaluation of the $G_r$ functions we carry out in section 3.1. Indeed, in [19] Sczech uses Eisenstein $(d - 1)$-cocycles for $SL_d(\mathbb{Z})$ to express the values of $L$-functions of totally real fields of degree $d$ at negative integers. In the case of a totally real field $K$ of degree $d$, fixing a basis of $O_K$, the unit group $O_K^\times$ may be embedded inside $SL_d(\mathbb{Z})$ as an abelian group of rank $d - 1$. In the case of ATR fields which we are concerned with in this article, we have here obtained a $(d - 2)$-cocycle $\psi_a$ (up to the $\exp(i\pi B_{r+2,|g_1| \ldots |g_{r+1}|,a})$ term) for $SL_d(\mathbb{Z})$ which is compatible with the unit group of an ATR field of degree $d$ having rank $d - 2$.

### 2.3 Modified $G_r$ functions and smoothings

Following an observation by Zudilin for $r = 2, 3$ (see for instance [14]), we modify the $G_r$ functions with an exponential prefactor to mimic the correction factors one may find attached to modular functions to ensure better
arithmetic properties, as for the Dedekind $\eta$ function:

$$\eta(\tau) = q^{1/24} \prod_{m \geq 1} (1 - q^m)$$

We will now derive the exponential prefactor from Narukawa’s modularity theorem:

$$\prod_{j=0}^{r+1} G_r \left( \frac{z}{\omega_j}, \left( \frac{\omega_n}{\omega_j} \right)_{n \neq j} \right) = \exp \left( \frac{-2i\pi}{(r + 2)!} B_{r+2,r+2}(z, \omega) \right)$$

The rescaled Bernoulli polynomial may be written explicitly using the generating function:

$$e^{zt} \prod_{j=0}^{r+1} \omega_j = \left( \sum_{n \geq 0} (zt)^n / n! \right) \prod_{j=0}^{r+1} \left( \sum_{k_j \geq 0} (\omega_j t)^{k_j} / (k_j)! \right) B_{k_j}$$

where the $B_{k_j}$ are the usual Bernoulli numbers. This gives:

$$B_{r+2,r+2}^{*,(z, \omega)} = B_{r+2,r+2}(z, \omega) = \sum_{l+k_0+\cdots+k_{r+1}=r+2} \frac{z^l}{l!} \prod_{j=0}^{r+1} \frac{\omega_j^{k_j-1}}{k_j!} B_{k_j}$$

where the sum ranges over non-negative integers. Specialising to the case where $z = \sum_{j=0}^{r+1} a_j \omega_j$ with rational coefficients $a_j$ gives:

$$B_{r+2,r+2}(\sum_{j=0}^{r+1} a_j \omega_j, \omega) = (r + 2)! \sum_{|j|+|k|=r+2} \prod_{j=0}^{r+1} \frac{(a_j \omega_j)^{k_j}}{l_j!} \frac{\omega_j^{k_j-1}}{k_j!} B_{k_j}$$

Therefore we define

$$D_{r+2,m}(a, \omega) = \sum_{|j|+|k|=r+2} (r + 2)! \sum_{\#\{j, l_j = k_j = 0\}} \prod_{j=0}^{r+1} \frac{(a_j \omega_j)^{l_j}}{l_j!} \frac{\omega_j^{k_j-1}}{k_j!} B_{k_j}$$

Then $B_{r+2,r+2}$ naturally decomposes as

$$B_{r+2,r+2}(\sum_{j=0}^{r+1} a_j \omega_j, \omega) = (r + 2)! \prod_{j=0}^{r+1} \left( a_j - \frac{1}{2} \right) + \sum_{m=0}^{r+1} D_{r+2,m}(a, \omega)$$

Thus we define

$$H_r \left( \sum_{j=0}^{r+1} a_j \omega_j \left( \frac{\omega_j}{\omega_m} \right)_{j \neq m} \right) = e^{\frac{2i\pi}{r+2}} D_{r+2,m}(2, \omega) G_r \left( \sum_{j=0}^{r+1} a_j \omega_j \left( \frac{\omega_j}{\omega_m} \right)_{j \neq m} \right)$$

17
This way, Narukawa’s modularity theorem becomes a multiplicative cocycle relation for $H_r$ with values in $\exp(2i\pi \mathbb{Q})$:

$$
\prod_{m=0}^{r+1} H_r \left( \sum_{j=0}^{r+1} a_j \frac{\omega_j}{\omega_m} \left( \frac{\omega_j}{\omega_m} \right)_{j\neq m} \right) = \exp \left( 2i\pi \prod_{j=0}^{r+1} \left( a_j - \frac{1}{2} \right) \right)
$$

In what follows we will smooth out $G_r$ functions, and this will kill the factor $D_{r+2,m}$ because it is homogeneous of degree 1 in $\omega_0, \ldots, \omega_{m-1}, \omega_m, \ldots, \omega_{r+1}$:

$$
D_{r+2,m}(a, N\omega_0, \ldots, N\omega_{m-1}, \omega_m, N\omega_{m+1} \ldots \omega_{r+1} = ND_{r+2,m}(a, \omega)
$$

This means that for any positive integer $N$, the level $N$ smoothings of $G_r$ and $H_r$ functions coincide:

$$
\frac{H_r \left( \sum_{j=0}^{r+1} a_j \frac{\omega_j}{\omega_m}, \left( \frac{\omega_j}{\omega_m} \right)_{j\neq m} \right)^N}{H_r \left( \sum_{j=0}^{r+1} a_j \frac{N\omega_j}{\omega_m}, \left( \frac{N\omega_j}{\omega_m} \right)_{j\neq m} \right)} = \frac{G_r \left( \sum_{j=0}^{r+1} a_j \frac{\omega_j}{\omega_m}, \left( \frac{\omega_j}{\omega_m} \right)_{j\neq m} \right)^N}{G_r \left( \sum_{j=0}^{r+1} a_j \frac{N\omega_j}{\omega_m}, \left( \frac{N\omega_j}{\omega_m} \right)_{j\neq m} \right)}
$$

3 An algebraicity conjecture on special values of $G_r$ functions

3.1 Arithmetic $G_r$ functions

In their recent article [1], Bergeron, Charollois and García describe special values of the elliptic Gamma function in complex cubic fields which they relate to partial zeta values. They also conjecture that these special values are units lying in specific abelian extensions. We shall present a generalisation of this construction for fields of degree $d \geq 2$ with exactly one pair of complex embeddings yielding special values of $G_{d-2}$ functions.

Consider a degree $d \geq 2$ ATR field $\mathbb{K}$ and an integral ideal $\mathfrak{f} \neq \mathcal{O}_\mathbb{K}$. We write $q\mathbb{Z} = \mathbb{Z} \cap \mathfrak{f}$. Fix an integral ideal $\mathfrak{b}$ coprime to $\mathfrak{f}$ and put $L = \mathfrak{f}^{-1} \mathfrak{b}$. Choose $B_L$ a $\mathbb{Z}$-basis of the $\mathbb{Z}$-module $L$ and fix an integral ideal $\mathfrak{a}$ coprime to $\mathfrak{f} \mathfrak{b}$ such that $\mathfrak{a}^{-1}L/L$ is a cyclic group. We call such an ideal a smoothing ideal and we write $N = N(\mathfrak{a})$ for the norm of $\mathfrak{a}$. We consider specific vectors in $L$ which we will use to evaluate geometric $G_r$ functions.

**Definition 7:** A vector $h \in L = \mathfrak{f}^{-1} \mathfrak{b}$ is called admissible for the data $\mathfrak{f}, \mathfrak{b}, \mathfrak{a}$ if $h/q \equiv 1 \mod L$ and $h/N$ generates the cyclic group $\mathfrak{a}^{-1}L/L$, where $q\mathbb{Z} = \mathbb{Z} \cap \mathfrak{f}$ and $N = N(\mathfrak{a})$ is the norm of the ideal $\mathfrak{a}$.
In this section we fix \( h \in L \) an admissible vector. The unit group
\[ \mathcal{O}_f^{+} = \{ \varepsilon \in \mathcal{O}_K^\times, \varepsilon \equiv 1 \mod f, \sigma(\varepsilon) > 0, \text{for all real embedding } \sigma \text{ of } K \} \]
is a free \( \mathbb{Z} \)-module of rank \( r = d - 2 \) acting on \( L \), and we fix for now \( u_1, \ldots, u_r \) a set of fundamental units for \( \mathcal{O}_f^{+} \) such that the linear form \( \det_{B_L}(h, u_1h, \ldots, u_rh, \cdot) \) is non degenerate. This means that it should be non-zero and that if \( K \) contains a totally real subfield an additional condition must be satisfied (see \((\ast)\) in section 4.1). Denote by \( a \) the unique primitive element in \( \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \) satisfying \( na = \det_{B_L}(h, u_1h, \ldots, u_rh, \cdot) \) for a positive integer \( n \).

**Definition 8:** Let \( u_1, \ldots, u_r \) be a set of fundamental units for \( \mathcal{O}_f^{+} \) as above. Let \( h \) be an admissible vector in \( f^{b-1} \) and \( a \) be the linear form described above. We define for the complex embedding \( \sigma_C \) of \( K \) and the choice of sign \( \pm \) the arithmetic \( G_r \) function:
\[
G_{r,f,b,a}^+(u_1, \ldots, u_r; h, \sigma_C) = \frac{G_{r,(a,au_1,\ldots,au_r)}(\frac{h}{q}, \sigma_C, \text{Hom}_\mathbb{Z}(f^{b-1}, \mathbb{Z}))^N}{G_{r,(a,au_1,\ldots,au_r)}(\frac{h}{q}, \sigma_C, \text{Hom}_\mathbb{Z}(f^{ba-1}, \mathbb{Z}))}
\]
\[
G_{r,f,b,a}^-(u_1, \ldots, u_r; h, \sigma_C) = \frac{G_{r,(-a,-au_1,\ldots,-au_r)}(\frac{h}{q}, \sigma_C, \text{Hom}_\mathbb{Z}(f^{b-1}, \mathbb{Z}))^N}{G_{r,(-a,-au_1,\ldots,-au_r)}(\frac{h}{q}, \sigma_C, \text{Hom}_\mathbb{Z}(f^{ba-1}, \mathbb{Z}))}
\]
where in both cases the right-hand side is a quotient of two geometric \( G_r \) functions as defined in Proposition 4.

Note that this level \( N \) smoothing is a product of level \( N \) smoothings of ordinary \( G_r \) functions which coincide with the level \( N \) smoothings of \( H_r \) functions as defined in section 2.3. On another note, we only have two complex embeddings \( \sigma_C, \overline{\sigma_C} \) for the ATR field \( K \) and using formula (9) we get
\[
G_{r,f,b,a}^\pm(u_1, \ldots, u_r; h, \sigma_C) = G_{r,f,b,a}^\pm(u_1, \ldots, u_r; h, \overline{\sigma_C})^{-1)^r}
\]
This means that we only need to treat the case of one of the complex embeddings. Throughout the rest of this work, we will consider fields with a specific defining polynomial equation and we fix the complex embedding \( \sigma_C \) corresponding to the root of the polynomial lying in the upper half-plane. We shall then drop the \( \sigma_C \) in the writing. In the special cases \( r = 0, r = 1 \) we will identify
\[
\theta_{f,b,a}^\pm(h) = G_{0,f,b,a}^\pm(0; h)
\]
\[
\Gamma_{f,b,a}^\pm(\varepsilon; h) = G_{1,f,b,a}^\pm(\varepsilon; h)
\]
In [1] the authors use modular symbols (see the cocycle relation (5)) for \( \text{SL}_3(\mathbb{Z}) \) built from smoothed Eisenstein series to establish a link between this special value and the partial zeta values associated to the conductor \( f \) and the ideal class \( [b] \) of the integral ideal \( b \) in the narrow ideal class group \( Cl^{+}(f) \). Namely, the partial \( \zeta \) function associated to this class is

\[
\zeta_f([b], s) = \sum_{I \in [b]} \mathcal{N}(I)^{-s}
\]

when \( \Re(s) > 1 \) and where the sum ranges over all integral ideals in the class \( [b] \). This function can be extended into a meromorphic function over \( \mathbb{C} \) with a pole at \( s = 1 \). In the specific case where \( \mathbb{K} \) is a complex cubic field, this partial \( \zeta \) function vanishes at the origin. In [1] the authors have proven that the modulus of \( \Gamma_{f,b,a}^{\pm}(\varepsilon; h) \) is independent of the choices for \( \varepsilon \) and \( h \) up to signs and there is a sign \( \delta = +1 \) or \(-1\) such that

\[
\mathcal{N}(a) \zeta'_f([b], 0) - \zeta'_f([ab], 0) = \delta \log |\Gamma_{f,b,a}^{\pm}(\varepsilon; h)|^2
\]

They also conjectured that \( \Gamma_{f,b,a}^{\pm}(\varepsilon; h) \) is a unit in the associated class field \( \mathbb{K}^{+}(f) \) related to the conjectural Stark unit. We believe that a combination of special values of arithmetic \( Gr \) functions for \( r \geq 2 \) will be linked to the partial \( \zeta \) functions of an ATR field of degree \( d = r + 2 \) using modular symbols (see the cocycle relation (14)) for \( \text{SL}_d(\mathbb{Z}) \) related to some smoothed Eisenstein series. In view of this cohomological interpretation and following the work of Sczech in [19] on the Eisenstein cocycles for totally real fields, we consider the following combination of arithmetic \( Gr \) functions:

**Definition 9:** Let \( (\varepsilon_1, \ldots, \varepsilon_r) \) be a set of fundamental units for \( \mathcal{O}_r^{+} \). Let \( h = (h_\rho) \) be a set of admissible vectors for \( f, b, a \) where \( \rho \) ranges over the permutations in \( S_r \), the symmetric group on \( r \) elements. For any choice of orientations for the \( Gr \) functions \( \mu = (\mu_\rho), \nu = (\nu_\rho) \) we define:

\[
I_{r,f,b,a}(\varepsilon_1, \ldots, \varepsilon_r; h, \mu, \nu) = \prod_{\rho \in S_r} G_{r,f,b,a}^{\mu_\rho}(\varepsilon_{\rho(1)}, \ldots, \varepsilon_{\rho(r)}; h_\rho)^{\nu_\rho}
\]

where again

\[
[\varepsilon_1|\ldots|\varepsilon_r] = \left( \varepsilon_1, \ldots, \prod_{j=1}^{k} \varepsilon_j, \ldots, \prod_{j=1}^{r} \varepsilon_j \right)
\]

and the orientations \( \mu_\rho, \nu_\rho \in \{-1, +1\} \). This is well-defined if for all permutation \( \rho \), the family \( (1, \varepsilon_{\rho(1)}, \ldots, \prod_{j=1}^{r} \varepsilon_{\rho(j)}) \) is a family of linearly independent vectors in \( \mathbb{K} \).
We will now present a conjecture on special values of these arithmetic $G_r$ functions for specific choices of $(h_\rho)_{\rho \in \mathfrak{G}_s}$ which we will describe in the next section.

### 3.2 Formulation of the conjecture

One aim of this article is to generalise the following conjecture of Bergeron, Charollois and García [1] on the special values of the elliptic Gamma function evaluated at points in complex cubic fields.

**Conjecture (Bergeron, Charollois and García):**

Suppose that $\mathbb{K}$ is a complex cubic field and $f \neq \mathcal{O}_\mathbb{K}$ is such that $\mathbb{K}^+(f)$ is totally complex. Take $b, a$ as before and further ask that $(N(a), 6) = 1$. Then for any admissible vector $h \in L = fb^{-1}$:

- The number $\Gamma^+_f, b, a(\varepsilon, h)$ is independent of the choice of $h$ up to orientations and is the image in $\mathbb{C}$ of a unit $u_{L,a} \in \mathbb{K}^+(f)$ under a complex embedding $\sigma'_C$ extending $\sigma_C$.

- Any complex embedding of $\mathbb{K}^+(f)$ above the real place of $\mathbb{K}$ sends $u_{L,a}$ to the unit circle.

- For $c \in I(f)$, the reciprocity law is explicitly given by $u_{L,a}^{\sigma_c} = u_{Lc^{-1}, a}$ where $\sigma_c$ is the corresponding element in $\text{Gal}(\mathbb{K}^+(f)/\mathbb{K})$.

Consider the following example for the field $\mathbb{K} = \mathbb{Q}(z)$ where $z = e^{2\pi i / 3}10^{1/3}$ is the root of the polynomial $x^3 - 10$ in the upper half-plane, $f^3 = (5)$, $b = (1)$ and $a$ is the unique degree one prime above 11 in $\mathbb{K}$. The fundamental unit is $\varepsilon = (2z^2 - z - 7)/3$. Then, for $h = -(35z^2 + 20z + 35)/3$ we compute

$$\Gamma^+_f, b, a(\varepsilon, h) = \frac{\Gamma \left( \frac{11}{5}, \frac{-776}{45}, \frac{1751}{45} \right)^{11}}{\Gamma \left( \frac{11}{5}, \frac{-776}{45}, \frac{1751}{45} \right)} \approx -27.5333588... - i \cdot 32.7146180...$$

with 1000 digits of precision and find it to be close to a root of the polynomial

$$x^{12} + 57x^{11} + 1956x^{10} + 4640x^9 + 35415x^8 - 109818x^7 + 150139x^6 - 109818x^5 + 35415x^4 + 4640x^3 + 1956x^2 + 57x + 1$$

which defines $\mathbb{K}^+(f)$. This example is developed in section 5.1.3 below.

In the cubic case, this conjecture has been tested numerically on hundreds of examples and we want to generalise this to ATR fields of degree $d \geq 4$ in some specific cases supported by numerical evidence. In section 4, we will describe a specific set of conditions under which we formulate our conjecture.
Conjecture 10: Suppose that $K$ is an ATR field of degree $d = r + 2$. Suppose $f \neq O_K$ is such that $K^+(f)$ is totally complex. Take $b, a$ as above. Suppose that there exists a system of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ satisfying the conditions $(\ast)$ and $(\ast\ast)$ associated to $f, b, a$ described in section 4.3. For all set of vectors $h$ described in section 4.3, Proposition 22, there exists a choice of orientations $\mu, \nu$ such that

$$N\zeta_f([b], 0) - \zeta_f([ab], 0) = \log |I_{r,f,b,a}(\varepsilon_1, \ldots, \varepsilon_r; h, \mu, \nu)|^2$$

In addition, if $r \geq 1$, there exists a positive integer $m(r, N)$ such that the value $I_{r,f,b,a}(\varepsilon_1, \ldots, \varepsilon_r; h, \mu, \nu)^{m(r, N)}$ is the image in $C$ of an algebraic unit $u_{L,a,h}$ in the class field $K^+(f)$ under a complex embedding $\sigma'_C$ extending $\sigma_C$. Moreover any embedding of $K^+(f)$ above a real embedding of $K$ sends $u_{L,a,h}$ to the unit circle. Furthermore, $m(r, N)$ may be taken such that it is only divisible by prime numbers dividing $N$ and smaller than a bound $M(r)$.

Remarks:

1. The case $r = 0$ in the conjecture is already known as a consequence of the theory of complex multiplication and we refer to [17] for a presentation on the subject. Note that in this case the value

$$\theta_{f,b,a}^+(h)^{m(0, N)}$$

is not necessarily a unit but rather a $q$-unit where once again $qZ = f \cap Z$.

2. This conjecture is formulated with restrictions, but it has been successfully tested in many other cases outside of the conditions $(\ast), (\ast\ast)$. It is our aim to complete the formulation of this conjecture outside of these restrictions and to show that the unit $u_{L,a,h}$ is indeed independent of the choice of $h$ as long as $h$ is “admissible” in a way not yet fully understood. Once completed, it should be part of the conjecture that the explicit reciprocity law is given by $u_{L,a}^{\sigma'} = u_{L,a}^{\sigma - 1}$.

3. The values of $m(r, N)$ in the cubic and quartic cases have been tested on dozens of examples. In the context of Proposition 22, the norm $N$ of the smoothing ideal $a$ is a prime number. The table below sums up our observations on the expected values for $m(r, N)$ in the cubic and quartic cases as well as the known values in the quadratic case.

<table>
<thead>
<tr>
<th>$N$</th>
<th>quartic $m(0, N)$</th>
<th>cubic $m(1, N)$</th>
<th>quartic $m(2, N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>7 ≤</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
This shows that there is a specific behavior for “small” smoothings and
that most of the time \( m(r, N) = 1 \).

We give here an example supporting the conjecture with a “small” smooth-
ing: take the quartic field \( \mathbb{K} = \mathbb{Q}(z) \) where \( z \) is the root of the polynomial
\( x^4 - 6x^3 - x^2 - 3x + 1 \) in the upper half-plane, and take \( f \) the unique degree
one prime above 2 in \( \mathbb{K} \). We choose \( a \) the degree one prime above 5 in \( \mathbb{K} \).
Then we may compute the two quotients

\[
\begin{align*}
v_1 &= \frac{G_2 \left( \frac{1}{2}, \frac{5z^3 - 29z^2 - 15z - 81}{70}, \frac{6z^3 - 39z^2 + 10z + 5}{70}, \frac{-2z^3 + 13z^2 + z + 24}{70} \right)}{G_2 \left( \frac{5}{2}, \frac{5z^3 - 29z^2 - 15z - 81}{14}, \frac{6z^3 - 39z^2 + 10z + 5}{14}, \frac{-2z^3 + 13z^2 + z + 24}{14} \right)}^{-5}, \\
v_2 &= \frac{G_2 \left( -\frac{1}{2}, \frac{-2z^3 + 13z^2 - z + 24}{70}, \frac{-5z^3 + 29z^2 + 15z + 81}{70}, \frac{2z^3 - 13z^2 - 6z - 101}{70} \right)}{G_2 \left( -\frac{5}{2}, \frac{-2z^3 + 13z^2 - z + 24}{14}, \frac{-5z^3 + 29z^2 + 15z + 81}{14}, \frac{2z^3 - 13z^2 - 6z - 101}{14} \right)}^{5}.
\end{align*}
\]

The values obtained are respectively \( v_1 \approx -2.0167576... - i \cdot 5.8008598... \) and
\( v_2 \approx -0.4159958... + i \cdot 0.0018434... \). Their product \( v_1v_2 \approx 0.8496565... - i \cdot 2.4094157... \) is not an algebraic integer in \( \mathbb{K}^+(f) \). Yet the fifth power
\( (v_1v_2)^5 \approx 108.0070738... - i \cdot 13.4979021... \) of this product coincides up to
at least 1000 digits with a root of the polynomial \( x^8 - 215x^7 + 11629x^6 +
11941x^5 + 3913x^4 + 11941x^3 + 11629x^2 - 215x + 1 \) which defines \( \mathbb{K}^+(f) \).
The constant term of this polynomial is 1, so the roots of this polynomial are
units inside \( \mathbb{K}^+(f) \).

The way our conjecture was formulated is also inspired by the rank one
abelian Stark conjecture in the ATR case:

**Rank one abelian Stark conjecture (ATR case, see [6]):**
Write \( e_f \) for the number of roots of unity in \( \mathbb{K}^+(f) \). There is a unit \( u_{Stark} \) in
\( \mathbb{K}^+(f) \) such that

- for all class \([b]\) in \( Cl^+(f) \) and corresponding \( \sigma_b \in \text{Gal}(\mathbb{K}^+(f)/\mathbb{K}) \),
  \[
  \zeta_f^i([b], 0) = -\frac{1}{e_f} \log |u_{Stark}^b|.
  \]
- every complex embedding of \( \mathbb{K}^+(f) \) above a real embedding of \( \mathbb{K} \) sends
  \( u_{Stark} \) to the unit circle.
- \( \mathbb{K}^+(f)(u_{Stark}^{1/e_f}) \) is an abelian extension of \( \mathbb{K} \).

There are ways to make this statement a bit more precise so that the unit
\( u_{Stark} \) is unique and hopefully the unit \( u_{L,a,h} \) will be related to the Stark unit
by the formula:

\[
u_{Stark}^{N - \sigma_a} = u_{L,a,h}\]
Moreover, our conjecture relates to Hilbert’s twelfth problem, allowing for a construction of certain class fields using specific multivariate analytic functions.

4 Computing the arithmetic $G_r$ functions

In order to understand how the choice of vectors $(h_\rho)_{\rho \in \mathfrak{S}}$, should be made, we must carry out an explicit description of the geometric aspects of the construction of the arithmetic $G_{r,f,b,a}$ functions. Thus, we will identify key quantities in the construction, namely the parameters $t$ and $l$ defined in section 4.1 and then show how we can control these quantities. We then try to minimize the quantities that have the greatest impact on the computation time and describe an efficient algorithm to compute the arithmetic products $I_{r,f,b,a}(\varepsilon_1, \ldots, \varepsilon_r; h, \mu, \nu)$ given in definition 9.

4.1 Geometric setup

Let $K$ be an ATR field of degree $d = r + 2$ and let $f$, $b$, $a$ be given as in section 3.1. Fix a basis $B$ of the $\mathbb{Q}$-vector space $K$. Any basis $B'$ of the $\mathbb{Q}$-vector space $K$ is called a positive basis if $\det B(B') > 0$. We say that we have fixed a global orientation of $K$, which is not to be confused with the orientations $\mu$ and $\nu$ associated to the product in definition 9. Put $L = \mathbb{F}^{-1}$. Fix a system of fundamental units $(\varepsilon_1, \ldots, \varepsilon_r)$ of $O_K^+, \times f, b, a$. Write $(u_1, \ldots u_r) = \left[ \varepsilon_\rho^0 | \cdots | \varepsilon_\rho^r \right]$ for any permutation $\rho \in \mathfrak{S}_r$. For convenience, we will always write $u_0 = 1$. Choose $h \in L$ an admissible vector (see definition 7). Put $mh' = h$ where $m$ is a positive integer and $h'$ is primitive in $L$. This means that for all non-zero integer $n \in \mathbb{Z}$, $h'/n \in L$ implies that $n = \pm 1$.

We may choose a positive basis $B_L = (e_0 = h', e_1, \ldots, e_{r+1})$ of $L$ such that $u_i h' = \sum_{j=0} e_{ij} e_j$ where the coefficients $c_{ij0}$ are integers and $c_{i00} > 0$. This means in particular that $\det_B(B_L) > 0$. To evaluate geometric $G_r$ functions we must fix an orientation form on the lattice $L$ and we consider the orientation form $\det_{B_L}$. In the basis $B_L$ we identify the units $u_i$ with the matrix $(c_{ijk})_{i,k} \in \text{SL}_d(\mathbb{Z})$ such that $u_i e_k = \sum_{j=0} c_{ijk} e_j$. Fix the dual basis $C = (f_0, \ldots, f_{r+1})$ of $\Lambda = \text{Hom}_\mathbb{Z}(L, \mathbb{Z})$ such that $f_i(e_j) = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker symbol. In this basis the linear form $a$ defined in (3.1) is exactly $f_{r+1}$. For $1 \leq i \leq r$, the composition of $a$ with multiplication by $u_i$ is written $au_i = \sum_{k=0} c_{i(r+1)k} f_k$. Then there is a unique primitive vector $\gamma \in L$ such that $\det_C(a, au_1, \ldots, au_r, \cdot) = s^r = \pm sh'$ with $s$ a positive integer.

Next, we identify the family $a = (a = au_0, au_1, \ldots, au_r)$ with the matrix obtained by concatenation of the coefficients of the linear forms $au_i$ in the
basis \( C \):

\[
\begin{pmatrix}
au_0 \\
au_1 \\
\vdots \\
au_j \\
\vdots \\
au_r
\end{pmatrix} = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 1 \\
0 & c_1(r+1) & \cdots & c_1(r+1)k & \cdots & c_1(r+1)(r+1) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & c_j(r+1) & \cdots & c_j(r+1)k & \cdots & c_j(r+1)(r+1) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & c_r(r+1) & \cdots & c_r(r+1)k & \cdots & c_r(r+1)(r+1)
\end{pmatrix}
\]

This matrix has a first column filled with zeroes and the submatrix \( A \) is a square matrix of size \( r + 1 \). The elementary divisors of \( A \) associated to its Smith normal form are the positive integers \([A_r, \ldots, A_1, A_0 = 1]\) such that \( A_i | A_{i+1} \). Then we define

\[
\lambda(a_1, \ldots, a_r; h) = \lambda = \prod_{i=1}^r c_{ii0}
\]

\[
s(a_1, \ldots, a_r; h) = s = \prod_{i=1}^r A_i
\]

\[
t(a_1, \ldots, a_r; h) = t = A_r
\]

so that

\[
\lambda a(u_1, \ldots, u_r; h) = \lambda a = \det_{B_L}(h', u_1h', \ldots, u_rh', \cdot)
\]

and \( \det_C(a, au_1, \ldots, au_r, \cdot) = \pm sh' \). Recalling the construction in Proposition 4, we must now consider positive dual families \( \alpha = (\alpha_0, \ldots, \alpha_r) \subset L \) to \( a \). Recall that this means that for all \( 0 \leq j \leq r \):

\[
au_j(\alpha_j) > 0, \quad au_k(\alpha_j) = 0, \quad \forall k \neq j
\]

with once again the convention that \( u_0 = 1 \). Consider the matrix \( B = (\prod_{i=1}^r A_i^{-1})\text{com}(A)^T \) which corresponds to the rescaled comatrix in formula (12). Then the product \( A \cdot B = tI_{r+1} \). Write \( B = (b_{ij})_{1 \leq i,j \leq r+1} \). Then a possible choice for \( \alpha \) is given by \( \alpha_j = \sum_{i=1}^{r+1} b_{ij}e_i \in L \). This family satisfies for all \( 0 \leq j \leq r \):

\[
au_j(\alpha_j) = t, \quad au_k(\alpha_j) = 0, \quad \forall k \neq j
\]

We write this as the multiplication of two matrices \( a \cdot \alpha = tI_{r+1} \). We say that this choice of \( \alpha \) is a uniform positive dual family to \( a \) because the value \( au_j(\alpha_j) = t \) is independent of \( j \). We argue that this choice is minimal amongst the uniform positive dual families to \( a \) in \( L \), as explained in the following lemma.
Lemma 11: Let $B' \in M_{r+1}(\mathbb{Z})$ be a square matrix of size $r + 1$ such that $\mathbf{A} \cdot B' = nI_{r+1}$. Then $t$ divides $n$. Consequently, any uniform positive dual family $\alpha'$ to $\alpha$ satisfies $\alpha' \cdot B' = nI_{r+1}$ where $t$ divides $n$. Furthermore, the minimal uniform positive dual family $\alpha$ to $\alpha$ described above satisfies the following property: for all prime $p$ there exists an index $0 \leq j \leq r$ such that $\forall m \in \mathbb{Z}$, $\alpha_j - m\gamma \notin pL$.

Proof:
Consider the matrices $U, V \in \text{GL}_{r+1}(\mathbb{Z})$ such that $UAV$ is the diagonal matrix in Smith normal form $[A_r = t, \ldots, A_1, 1]$. Then

$$nI_{r+1} = (UAV) \cdot (V^{-1}B'U^{-1}) = \begin{pmatrix} t & 0 & \cdots & 0 \\ 0 & A_{r-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix} V^{-1}B'U^{-1}$$

This shows that $t|n$. Consider now a uniform positive dual family $\alpha' = (\alpha'_0, \ldots, \alpha'_r)$ to $\alpha$ in $L$ such that $\alpha \cdot \alpha' = nI_{r+1}$. Write the coordinates of the $\alpha'_j$ in the basis $B_L$ as $\alpha'_j = \sum_{i=0}^{r+1} b_{ij}e_i$. Then the matrix $B' = (b'_{ij})_{1 \leq i,j \leq r+1}$ satisfies $\mathbf{A} \cdot B' = nI_{r+1}$ and $t$ divides $n$.

Consider now the minimal choice $\alpha$ given above associated to the matrix $\mathbf{B} = (\prod_{i=1}^r A_i^{-1}) \text{com}(\mathbf{A})^T$ such that $\alpha \cdot \alpha = tI_{r+1}$. The elementary divisors of $\mathbf{B}$ are given by the positive integers $[A_r, A_r/A_1, \ldots, A_r/A_{r-1}, 1]$. The last elementary divisor is the $gcd$ of the coordinates of the $\alpha_j$ in the basis $B_L$. Therefore, for each prime $p$ there exists an integer $j$ and a coefficient $b_{ij}$ not divisible by $p$. For any $m \in \mathbb{Z}$, we get $\alpha_j - m\gamma = \pm me_m + \sum_{i=1}^{r+1} b_{ij}e_i \notin pL$.

From now on we will always fix $\alpha = \alpha(u_1, \ldots, u_r; h)$ the minimal uniform positive dual family $\alpha$ to $\alpha$ given above. Define $M = M(u_1, \ldots, u_r; h) = \mathbb{Z}h' \oplus (\oplus_{j=0}^r \mathbb{Z}\alpha_j)$ in $L$ associated to this family. This sublattice has index $\det(\mathbf{B}) = t^{r+1}/s$ in $L$. Recalling the construction in section (3.1) and formula (11) for the geometric $G_{r\mathbb{Z}}$ function we may identify $F(\alpha)/\mathbb{Z}\gamma \simeq L/M$ so that the geometric $G_{r\mathbb{Z}}$ function is a product of $t^{r+1}/s$ ordinary elliptic $G_r$ functions. In the worst case scenario $t = s$ and $G_{r\mathbb{Z}}$ is a product of $t^r$ ordinary elliptic $G_r$ functions.

Let us now analyse $\alpha$ more closely. It follows from the definition of the linear form $a$ in formula (15) and lemma 11 that for $i \neq j$, $\alpha_i$ must satisfy

$$\det_{B_L}(u_0h', u_1h', \ldots, u_rh', u_j\alpha_i) = 0$$

with once again the convention that $u_0 = 1$, so that $\alpha_i/h' = \sum_{k=0}^r d_{ijk}u_j^{-1}u_k$ with rational coefficients $d_{ijk}$. This quotient depends rather loosely upon $h'$.
and this leads us to investigate the case \( h = 1 \) (which is not an admissible vector). In what follows, we will write with a \( \sim \) the counterpart of the quantities defined in the construction above for \( h = 1 \) and \( \mathcal{O}_K \) instead of the lattice \( L \). There is a positive basis \( B_K = [1, \tilde{e}_1, \ldots, \tilde{e}_{r+1}] \) of \( \mathcal{O}_K \) such that \( u_i = \sum_{j=0}^r \tilde{c}_{ij} \tilde{e}_j \) with \( \tilde{c}_{ii} > 0 \). Consider

\[
\tilde{\lambda} \tilde{a} = \left( \prod_{i=1}^r \tilde{c}_{i0} \right) \tilde{a} = \det_{B_K}(1, u_1, \ldots, u_r, \cdot) \tag{17}
\]

and put \( \tilde{a} = (\tilde{a}, \tilde{a} u_1, \ldots, \tilde{a} u_r) \). Associate to \( \tilde{a} \) a square matrix \( \tilde{A} \) of size \( r + 1 \) as above for \( a \). Write as above the elementary divisors of this matrix \( \tilde{A} \) associated to \( \tilde{a} \) as \( \tilde{A}_0, \ldots, \tilde{A}_r, \tilde{A}_r \) such that \( \tilde{A}_i | \tilde{A}_{i+1} \). Then put \( \tilde{s} = \prod_{i=1}^r \tilde{A}_i, \tilde{t} = \tilde{A}_r \) and construct as above \( \tilde{A} \) associated to the square matrix \( B = (\prod_{i=1}^r \tilde{A}_i^{-1}) \text{com}(\tilde{A})^T \) such that \( \tilde{a} \cdot \tilde{A} = \tilde{t} I_{r+1} \). Recall that this means that for all \( 0 \leq j \leq r \):

\[
\tilde{a} u_j(\tilde{A}_j) = \tilde{t}, \quad \tilde{a} u_k(\tilde{A}_j) = 0, \forall k \neq j \tag{18}
\]

In the case \( r = 1 \), \( \tilde{\lambda} \) is the content of the smallest ring \( \mathcal{O} \) containing the unit \( \varepsilon \) in the sense of \([2]\), Definition 14] and \( \tilde{t} \) measures how far the \( \mathbb{Z} \)-module \( \mathbb{Z} + \mathbb{Z} \varepsilon \) is from being a ring. Therefore, we will call \( \tilde{\lambda} \) the content of the unit system \((u_1, \ldots, u_r)\) and \( \tilde{t} \) the overflow of this unit system. We insist that these quantities only depend on the unit system \((u_1, \ldots, u_r)\). We relate this general construction to the construction for a particular admissible vector \( h \) in \( L \) as follows:

**Lemma 12:** The family \( \varepsilon \tilde{a} h' = (\varepsilon \tilde{a} h'_0, \ldots, \varepsilon \tilde{a} h'_r) \) is a uniform positive dual family to \( a \) in \( L \) for some sign \( \varepsilon = \pm 1 \). Therefore there exists a positive integer \( l \) such that \( a \cdot (\varepsilon \tilde{a} h') = l I_{r+1} \). The value of \( l \) is given by the following equality which holds in \( \mathbb{Z} \)

\[
\varepsilon \lambda lt = \frac{\mathcal{N}(h')}{\mathcal{N}(L)} \tilde{\lambda} \tilde{t} \tag{19}
\]

**Proof :**

We use formulae (15) relating \( a \) to the form \( \det_{B_K} \) and (17) relating \( \tilde{a} \) to the form \( \det_{B_L} \) to evaluate \( \lambda u_k \tilde{A}_j h' \).

\[
\lambda u_k \tilde{A}_j h' = \det_{B_L}(h', u_1 h', \ldots, u_r h', u_k \tilde{A}_j h')
\]

Using standard linear algebra and the definition of the norm of the fractional
ideal $L$ we may rewrite this equality using $\det_{B_k}$ instead of $\det_{B_L}$ as

$$\lambda au_k \tilde{\alpha}_j h' = \det_{B_k}(B_k \det_{B_k}(h', u_1 h', \ldots, u_k h', u_k \tilde{\alpha}_j h'))$$

$$\lambda au_k \tilde{\alpha}_j h' = \frac{1}{N(L)} \det_{B_k}(h', u_1 h', \ldots, u_k h', u_k \tilde{\alpha}_j h')$$

Then, using the definition of the norm $N(h')$ of $h'$ we get

$$\lambda au_k \tilde{\alpha}_j h' = \frac{N(h')}{N(L)} \det_{B_k}(1, u_1, \ldots, u_k \tilde{\alpha}_j)$$

We now use formula (18) to conclude that for $k \neq j$, $\lambda au_k \tilde{\alpha}_j h' = 0$ and for $k = j$:

$$\lambda au_j \tilde{\alpha}_j h' = \frac{N(h')}{N(L)} \lambda \tilde{t} \in \mathbb{Z}$$

$$au_j \tilde{\alpha}_j h' = \frac{N(h')}{N(L)} \lambda \tilde{t}$$

The lattice $L$ is a fractional ideal so that $\tilde{\alpha} h' \subset L$. The linear forms $au_k$ take integral values on $L$ so the number $n = \frac{N(h')}{N(L)} \lambda \tilde{t}$ is a non-zero integer. Write $\epsilon = \pm 1$ for the sign of $n$. Then $\epsilon \tilde{\alpha} h'$ is a uniform positive dual family to $\tilde{\alpha}$ in $L$ such that $\tilde{\alpha} h' = |n| l_{r+1}$. Lemma 11 gives $n = \epsilon l t$ for a positive integer $l$ and

$$\epsilon \lambda \tilde{t} = \lambda n = \lambda au_j \tilde{\alpha}_j h' = \frac{N(h')}{N(L)} \lambda \tilde{t}$$

We now give a precise statement on the relation between $\alpha$ and $\tilde{\alpha}$.

**Proposition 13:** Let $h$ be an admissible vector in $L$ and consider a unit system $u_1, \ldots, u_r$ for $\mathcal{O}_L^\times$. Let $\alpha$ and $\tilde{\alpha}$ be as defined in (16) and (18). Write $\epsilon$ for the sign in formula (19). Then for all $0 \leq j \leq r$, there exists an integer $m_j \in \mathbb{Z}$ such that $\epsilon \alpha_j = \tilde{\alpha}_j h' + m_j h'$.

**Proof:**

Using lemma 2 for $\tilde{\alpha}$ and the two positive dual families $\alpha$, $\epsilon \tilde{\alpha} h'$ to $\tilde{\alpha}$ in $L$ we get $\epsilon \alpha_j = \tilde{\alpha}_j h' + m_j h'/n_j$ with $m_j/n_j$ a rational number in irreducible form. Write $p_j m_j + q_j n_j = 1$. Then

$$p_j (\epsilon \alpha_j - \tilde{\alpha}_j h') + q_j h' = h'/n_j \in L$$

28
Because \( h' \) is a primitive vector in \( L \), we conclude that \( n_j = 1 \).

This proposition shows that the choice of \( h \) only affects the value of \( l \) and the values of the \( m_j \) and not the main content of \( \alpha / h' \). Indeed, the special value \( G_{r,f,b,a}^{\pm}(u_1, \ldots, u_r; h) \) we aim to compute is a product of \( t^{r+1}/s \) ordinary elliptic \( G_r \) functions with parameters \( \alpha_0/h', \ldots, \alpha_r/h' \), which we may write as \( \epsilon(\tilde{\alpha}_0 + m_0)/l, \ldots, \epsilon(\tilde{\alpha}_r + m_r)/l \). For instance in the cubic case, \( \Gamma_{f,b,a}^{\pm} \) is a product of \( t \) ordinary elliptic Gamma functions with parameters \( \tau = \pm \varepsilon + n_0/\lambda, \sigma = \pm \varepsilon^{-1} + n_1/\lambda = \sigma_0/\lambda \)

for some integers \( n_0, n_1 \). Therefore, the integer \( l \) can be thought of as a level and will be called the level of the value \( G_{r,f,b,a}^{\pm}(u_1, \ldots, u_r; h) \).

### 4.2 Computing the \( G_r \) functions

#### 4.2.1 Computing the ordinary elliptic \( G_r \) functions

To compute the \( G_r \) functions we make use of [[13], Proposition 3.6] which we write as

\[
G_r(z, \tau) = \exp \left( -\sum_{j \geq 1} \frac{1}{j} \left| q^{j} \frac{x^{-j}}{x^{j}} \right| \prod_{k=0}^{r} (1 - q^{k}) \right) \tag{20}
\]

This formula is only valid for \( \tau \in \mathbb{H}^{r+1} \) and \( 0 < \Im(z) < \sum_{k=0}^{r} \Im(\tau_k) \). We call this domain the center strip. We now only need to make sure that we can use the properties of the \( G_r \) functions to reach this domain. The first step in doing so, starting from \( \tau \in (\mathbb{C} - \mathbb{R})^{r+1} \) is to use the property

\[
G_r(z, \tau_0, \ldots, \tau_{k-1}, -\tau_k, \tau_{k+1}, \ldots, \tau_r) = G_r(z + \tau_k, \bar{\tau})^{-1}
\]

to bring back all arguments in the upper half-plane. Then, to bring back \( \Im(z) \) in the center strip, use recursively the property

\[
G_r(z + \tau_k, \bar{\tau}) = G_{r-1}(z, \tau^{-}[k])G_r(z, \bar{\tau})
\]

Doing so will require the computation of lower degree functions, that is until we reach the case \( r = 0 \). Lastly, to compute \( G_0 = \theta \) one may use Jacobi’s triple product formula and compute directly \( \theta \) using this fast-converging sum:

\[
\theta(z, \tau) = \frac{q^{1/24}}{\eta(\tau)} \sum_{n \in \mathbb{Z}} x^n (-1)^n q^{n(n-1)/2}
\]
where \( \eta(\tau) \) is the Dedekind \( \eta \) function. Let us say a word about the complexity of such computations. Fix \( \tau_0, \ldots, \tau_r \) in the upper half-plane and suppose we want to compute the level \( l \) function \( G_r(z, \tau/l) \) with precision \( \delta > 0 \). Then the number of terms we need to compute in the sum (20) is \( O(l \log(\delta)) \) where the constant explicitly depends on \( \tau \) and \( z \).

### 4.2.2 Computing the arithmetic \( G^\pm_{r,f,b,a}(u_1, \ldots, u_r; h) \) functions

We analyse the definition 8 of \( G^+_r(u_1, \ldots, u_r; h) \) using the geometric setup in section 4.1. For ease of presentation we treat the case of \( G^+_r(u_1, \ldots, u_r; h) \) and we insist that the computation for the other choice of sign is almost identical. Recall definition 8 for the arithmetic function:

\[
G^+_{r,f,b,a}(u_1, \ldots, u_r; h, \sigma_C) = \frac{G_{r,(a,au_1, \ldots, au_r)}(\frac{h}{q}, \sigma_C, \text{Hom}_{\mathbb{Z}}(fb^{-1}, \mathbb{Z}))^N}{G_{r,(a,au_1, \ldots, au_r)}(\frac{a}{q}, \sigma_C, \text{Hom}_{\mathbb{Z}}(fa^{-1}, \mathbb{Z}))}
\]

We drop once again the \( \sigma_C \) and we write using formula (11)

\[
G^+_{r,f,b,a}(u_1, \ldots, u_r; h) = \prod_{\delta \in F(\overline{\alpha})/Z\gamma} \frac{G_r \left( \frac{h+\delta}{q\gamma}, \frac{\alpha_0}{\gamma}, \ldots, \frac{\alpha_r}{\gamma} \right)^N}{G_r \left( \frac{Nh+N\delta}{q\gamma}, \frac{N\alpha_0}{\gamma}, \ldots, \frac{N\alpha_r}{\gamma} \right)}
\]

where \( \alpha \) and \( \gamma \) are defined in section 4.1. We use once again the fact that \( F(\overline{\alpha})/Z\gamma \simeq L/M \) where \( M = M(u_1, \ldots, u_r; h) \) is defined in section 4.1 to write

\[
G^+_{r,f,b,a}(u_1, \ldots, u_r; h) = \prod_{\delta \in L/M} \frac{G_r \left( \frac{h+\delta}{q\gamma}, \frac{\alpha_0}{\gamma}, \ldots, \frac{\alpha_r}{\gamma} \right)^N}{G_r \left( \frac{Nh+N\delta}{q\gamma}, \frac{N\alpha_0}{\gamma}, \ldots, \frac{N\alpha_r}{\gamma} \right)}
\]

\[
G^+_{r,f,b,a}(u_1, \ldots, u_r; h) = \prod_{\delta \in L/M} \frac{G_r \left( \frac{h+\delta}{q\gamma}, \epsilon^\gamma \frac{\alpha_0}{l}, \ldots, \epsilon^\gamma \frac{\alpha_r}{l} \right)^N}{G_r \left( \frac{Nh+N\delta}{q\gamma}, \epsilon^N \frac{\alpha_0}{l}, \ldots, \epsilon^N \frac{\alpha_r}{l} \right)}
\]

where the last expression is obtained using Proposition 13. This special value is a product of \( t^{r+1}/s = \#|L/M| \) ordinary elliptic \( G_r \) functions. In the worst case scenario we get \( t = s \) and \( t^{r+1}/s = t^r \). This means that the computation time of this value is \( O(l \log(\delta)t^r) \) with a constant that depends on \( u_1, \ldots, u_r \). The parameters \( l \) and \( t \) which depend on \( h \) for fixed \( u_1, \ldots, u_r \) are therefore crucial in our computations. For instance, if we want to compute this special value with a precision of 1000 digits (i.e. \( |\log(\delta)| = 1000 \)) where \( r = 2, l = 15 \) and \( t = 100 \), then the computation requires the computation of \( 150 \times 10^6 \) terms appearing in the sum (20). With a personal computer it is generally
an overwhelming computation. In the next section, we describe how we can choose the vector $h$ to obtain efficient computations. Namely, because the parameter $t$ has the biggest impact on computation time, we will show that under some assumptions we may choose a vector $h$ such that $t = 1$ at the cost of a moderate elevation of the level $l$.

4.3 The choice of vectors $(h_\rho)_{\rho \in \mathfrak{S}_r}$

In view of the above estimation on the computation time for the special value $\mathcal{G}_{r,\hat{f},b,d}(\varepsilon_{\rho(1)}, \ldots, \varepsilon_{\rho(r)}; h_\rho)$ for a permutation $\rho \in \mathfrak{S}_r$, we must be careful in choosing the vector $h_\rho$ to perform efficient computations. Also, because we want to build a product of these values for $\rho \in \mathfrak{S}_r$, we must find a way to unify our choices for each permutation $\rho$. To this extent, we will fix the unit system $\varepsilon_1, \ldots, \varepsilon_r$ and the permutation $\rho$. Write once again $(u_1, \ldots, u_r) = [\varepsilon_{\rho(1)}, \ldots, \varepsilon_{\rho(r)}]$. Our aim is to reduce at all costs the value of $t = t(u_1, \ldots, u_r; h)$ by choosing $h$ carefully. Formula (19)

$$\pm \lambda t = \frac{\mathcal{N}(h')}{\mathcal{N}(L)} \tilde{\lambda}$$

will reveal useful to understand how to minimize the value of $t$. The unit system $(u_1, \ldots, u_r)$ is fixed and we therefore have to deal with the fixed constants $\tilde{\lambda}, \tilde{t}$ defined in section 4.1. In what follows, a goal will be to understand how the prime factors of $\tilde{\lambda}$ and $\tilde{t}$ transfer to the left-hand side of this equation. From our computations it seems that the value of $\tilde{\lambda}$ is a hard limit for the value of $t$, which means that for all admissible vector $h$, $\tilde{\lambda}$ divides $t$. We also believe that this value of $\tilde{\lambda}$ is exactly the minimal achievable value for $t$. We are far from being able to prove this statement, but it will be proven in the case $\tilde{\lambda} = 1$ under the conditions $(\ast)$ and $(\ast\ast)$ below. The rest of this section contains the study of the $p$-adic valuations of the quantities appearing in formula (19) first in the cases where $p$ divides $\tilde{t}$ and then in the case where $p$ does not divide $\tilde{t}$. Under some conditions we show that in the first case the $p$-part of $h$ (as in definition 14 below) is best chosen as specific ideals which we call target ideals (see definition 18 and lemma 19 below). In the second case, it will be shown how a vector $h$ may have a non-trivial $p$-part while not affecting the value of $t$ using helper ideals (see lemma 21 below). In the rest of this section, we will discuss a uniform choice for the vectors $h_\rho, \rho \in \mathfrak{S}_r$ under the following conditions, referred to as the $(\ast)$ conditions: suppose that $\mathcal{N}(f) = q = \ell^k$ is a power of a prime number $\ell$ and that the choice of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ for $\mathcal{O}_l^{+,\infty}$ is such that for all $\rho \in \mathfrak{S}_r$:

- $\tilde{\lambda}_\rho = 1$
\[ \tilde{t}_\rho \text{ and } q \mathcal{N}(b) \text{ are relatively coprime.} \quad (*) \]

\[ \tilde{s}_\rho = \tilde{t}_\rho \neq 0 \]

- for all \( 0 \leq j \leq r \), \( \tilde{\alpha}_{\rho,j} \) is not a totally real element of \( \mathbb{K} \)

where the subscript \( \rho \) indicates that the quantities are related to the unit system \( [\varepsilon_{\rho(1)}|\ldots|\varepsilon_{\rho(r)}] \). The last condition is relevant only if \( \mathbb{K} \) contains a totally real subfield and it is there to ensure that we compute \( G_r \) functions with parameters outside of \( \mathbb{R} \). Note that the value of \( \tilde{t}_\rho \) does not depend on \( b \) or \( a \) which means that in the second line, we only require \( \tilde{t}_\rho \) to be coprime to \( q \) and we decide to choose \( b \) and \( a \) conveniently afterwards, so that this part is not restrictive. We will review the other assumptions when we discuss (**) below.

### 4.3.1 Target ideals

While choosing the vector \( h = h_\rho \), we must be very careful of the value of the overflow \( \tilde{t} = \tilde{t}_\rho \). Indeed, for most admissible vectors \( h \), the constant \( \tilde{t} \) depending only on \( u_1, \ldots, u_r \) will divide \( t = t(u_1, \ldots, u_r; h) \) and the computations will require \( t^r \) evaluations of ordinary \( G_r \) functions, which can become quickly overwhelming. We now introduce the notion of target ideal to understand how to choose \( h \) such that \( t = 1 \). Explicitly, we explain that there is a tradeoff with an increase of the level \( l \) as we diminish the value of \( t \). In the rest of this section we will make extensive use of the following definitions:

**Definition 14:** Let \( x \in \mathbb{K} \) and \( I \) be an ideal such that \( \mathcal{N}(I) \) is a power of the prime number \( p \). We say that the \( p \)-part of \( x \in \mathbb{K} \) is the ideal \( I \) if for all prime ideal \( \mathfrak{P} \) above \( p \), \( v_\mathfrak{P}(x) = v_\mathfrak{P}(I) \).

**Definition 15:** Let \( I \) be a non zero integral ideal in \( \mathcal{O}_\mathbb{K} \) such that \( I \cap \mathbb{Z} = n\mathbb{Z} \). Then, for all \( \alpha \in \mathcal{O}_\mathbb{K} \) congruent to an integer modulo \( I \) we write \( \text{rmd} (\alpha, I) \) for the unique integer \( 0 \leq \text{rmd} (\alpha, I) < |n| \) such that \( \alpha - \text{rmd} (\alpha, I) \in I \).

The number \( \text{rmd} (\alpha, I) \) is naturally called the remainder of \( \alpha \) mod \( I \). Note that if \( I \) is a product of degree one primes above different prime numbers, then all elements \( \alpha \in \mathcal{O}_\mathbb{K} \) are congruent to integers mod \( I \).

**Definition 16:** Let \( I \) be a non zero integral ideal in \( \mathcal{O}_\mathbb{K} \). Then, there is a unique primitive ideal \( I^c \) and a unique integer \( n > 0 \) such that \( II^c = n\mathcal{O}_\mathbb{K} \). We call \( I^c \) the complementary ideal of \( I \).
Before we define the notion of target ideal, we prove a lemma which we will need to control the $p$-adic valuation of the integer $\lambda$ in the construction of section 4.1.

**Lemma 17:** Take $h = mh'$ an admissible element in $L$. Suppose that there is a linear combination

$$
\left( u_j + \sum_{i=0}^{j-1} n_i u_i \right) h' \in p^k L
$$

where $j \geq 1$ and the coefficients $n_i$ are integers. Then $p^k$ divides the integer $c_{jj0}$ and therefore $p^k$ divides the integer $\lambda$.

**Proof:**
Recall that the basis $B_L = (e_0, \ldots, e_{r+1})$ defined at the beginning of section 4.1 is such that $e_0 = h'$ and for all $1 \leq i \leq r$, $u_i h' = \sum_{k=0}^{i} c_{ik0} e_k$ where the $c_{ik0}$ are integers and $c_{i0} > 0$. This means that

$$
\left( u_j + \sum_{i=0}^{j-1} n_i u_i \right) h' = c_{jj0} e_j + \left( \sum_{k=0}^{j-1} c_{jk0} e_k \right) + \sum_{i=0}^{j-1} \left( \sum_{k=0}^{i} n_i c_{ik0} e_k \right) \in p^k L
$$

so the coordinate of this element on the vector $e_j$ is exactly $c_{jj0}$ and $p^k$ divides $c_{jj0}$. Recalling that $\lambda = \prod_{j=1}^{r} c_{jj0}$, we get $p^k$ divides $\lambda$. \hfill \Box

We now define the notion of target ideal to deal with the $p$-adic valuation of the overflow $\tilde{t}$.

**Definition 18:** Let $I$ be a non zero primitive integral ideal of $\mathcal{O}_K$ such that $I I^c = p^{k_0} \mathcal{O}_K$. We say that $I$ is a target ideal associated to the unit system $u_0 = 1, u_1, \ldots, u_r$ if the two following conditions are satisfied:

- there is a linear combination $u_r + \sum_{j=0}^{r-1} n_j u_j \in I^c$ with $n_j \in \mathbb{Z}$
- for all $0 \leq j \leq r$, $\tilde{\alpha}_j$ is congruent to an integer $n'_j$ modulo $I^c$

Note that if the first condition is satisfied or if the second condition is satisfied for some $j$ when $I$ is a degree one prime then necessarily $p | \tilde{t}$. Indeed, in the first case, $\tilde{\alpha}_r - \text{rmd}(\tilde{\alpha}_r, I) \in I$ so that $(u_r + \sum_{j=0}^{r-1} n_j u_j)(\tilde{\alpha}_r - \text{rmd}(\tilde{\alpha}_r, I)) \in p\mathcal{O}_K$ and

$$
\tilde{t} = \tilde{a} u_r \tilde{\alpha}_r = \tilde{a} \left( u_r + \sum_{j=0}^{r-1} n_j u_j \right) (\tilde{\alpha}_r - \text{rmd}(\tilde{\alpha}_r, I)) \in p\mathbb{Z}
$$
In the second case, \( u_j - \text{rmd}(u_j, I) \in I \), so that \((u_j - \text{rmd}(u_j, I))(\tilde{\alpha}_j - n'_j) \in p\mathcal{O}_K \) and \( \tilde{a}(u_j - \text{rmd}(u_j, I))(\tilde{\alpha}_j - n'_j) = \tilde{a}u_j\tilde{\alpha}_j = \tilde{t} \in p\mathbb{Z} \). The importance of target ideals is expressed by the following lemma.

**Lemma 19:** Suppose (\( * \)). Let \( p \) be a prime number such that \( v_p(\tilde{t}) > 0 \) and suppose that \( I \) is a target ideal with \( II^c = p^{k_0}\mathcal{O}_K \) where \( k_0 \geq v_p(\tilde{t}) \) for the unit system \( u_0, \ldots, u_r \). Suppose further that \( \mathcal{N}(I) = p^{2k_0 - v_p(\tilde{t})} \). Take \( h \) an admissible vector with \( p \)-part exactly \( I \). Then \( v_p(\lambda) = v_p(l) = k_0 \) and \( v_p(t) = 0 \).

**Proof:**

The ideal \( I \) is a target ideal therefore there are integers \( n_0, \ldots, n_{r-1} \) such that \( (u_r + \sum_{i=0}^{r-1} n_i u_j)h' \in I^cI = p^{k_0}\mathcal{O}_K \). Using lemma 17 we obtain that \( p^{k_0} \) divides \( c_{r+1} \) and so \( p^{k_0} \) divides \( \lambda \). On the other hand, from lemma 11 there is an index \( j \) such that \( p \) does not divide \( \alpha_j + mh' \) for all \( m \in \mathbb{Z} \) and from Proposition 13 there is an integer \( m_j \) such that \( e \alpha_j = \tilde{\alpha}_j h' + m_j h' \) for some sign \( \epsilon = \pm 1 \). The ideal \( I^c \) is a target ideal, therefore there is an integer \( n_j' \) such that \( \tilde{\alpha}_j - n_j' \in I^c \). This implies that \( e \alpha_j = m_j h' = (\tilde{\alpha}_j - n_j')h' \in p^{k_0}\mathcal{O}_K \). Suppose for now that \( p \) does not divide \( l \). Then \( \exists m \in \mathbb{Z} \) such that \( e l m - m_j - n_j' \in p\mathbb{Z} \). Therefore \( e l (\alpha_j - m h') = (\tilde{\alpha}_j - n_j')h' + (m_j + n_j' - e l m)h' \in p\mathcal{O}_K \) but this is a contradiction. Therefore \( p \) must divide the integer \( l \). We then prove by induction that \( p^2, \ldots, p^{k_0} \) divide \( l \). Indeed, if \( p^k | l \) with \( k < k_0 \) then \( e \alpha_j = (m_j + n_j')h' \in p^{k_0}\mathcal{O}_K \) which means that \( m_j + n_j' \in p^k\mathbb{Z} \). Suppose that \( l \notin p^{k+1}\mathbb{Z} \). Then there is an integer \( m \in \mathbb{Z} \) such that \( e l m - m_j - n_j' \in p^{k+1}\mathbb{Z} \) and then \( e l (\alpha_j - m h') \in p^{k+1}\mathcal{O}_K \) which yields again a contradiction. Now, using formula (19) one finds

\[
\begin{align*}
v_p(\lambda) + v_p(l) + v_p(t) &= v_p(\mathcal{N}(h')) - v_p(\mathcal{N}(L)) + v_p(\tilde{\lambda}) + v_p(\tilde{t}) \\
v_p(\lambda) + v_p(l) + v_p(t) &= (2k_0 - v_p(\tilde{t}))/0 + 0 + v_p(\tilde{t}), \quad v_p(\lambda) \geq k_0, \quad v_p(l) \geq k_0
\end{align*}
\]

And this leads to \( v_p(\lambda) = k_0 \), \( v_p(l) = k_0 \) and \( v_p(t) = 0 \). Note once again the most important case where \( I \) is a degree one prime and \( v_p(\tilde{t}) = 1 \). Then we have \( v_p(\lambda) = 1 \), \( v_p(l) = 1 \) and \( v_p(t) = 0 \).

\( \square \)

This allows us to strengthen the conditions under which we might understand special values of arithmetic \( G_r \) functions. We add to the conditions (\( * \)) the following conditions (\( ** \)): for all \( \rho \in \mathcal{S}_r \), for all \( p \)\( \big| \rho \big| \), there exists a target ideal \( I \) such that \( II^c = p^{k_0}\mathcal{O}_K \) and \( \mathcal{N}(I) = p^{2k_0 - v_p(\rho)} \) for the unit system \( [\varepsilon_{\rho(1)}] \cdots [\varepsilon_{\rho(r)}] \). We discuss briefly what we expect from the conditions (\( * \)) and (\( ** \)). In regards to our computations for now, the most restrictive conditions would be the content condition \( \tilde{\lambda}_p = 1 \), as well as the conditions
\( \mathcal{N}(f) = q = \ell^k \) and \( \tilde{t}_p \) coprime to \( q \) as together they rule out many possible ideals \( \mathfrak{f} \). The condition \( s_\rho = \tilde{t}_p \) is expected to be true quite often. Finally, we expect the condition (**) to be true generally under the conditions (*).

Note that there may be more than one target ideal for each prime number \( p \) dividing the overflow \( \tilde{t}_\rho \).

### 4.3.2 Helper ideals

Target ideals helped us understand how to choose the \( p \)-part of \( h \) when \( p \mid \tilde{t} \).

It is not clear however how to choose an admissible vector with this specific \( p \)-part. Especially when the class number of \( K \) is not 1, it is most likely that the \( p \)-part of an admissible vector \( h \) for some \( p \) not dividing \( \tilde{t} \) will be non-trivial, in which case it must be carefully chosen. To understand the best case scenario, we introduce the notion of helper ideals.

**Definition 20:** A primitive ideal \( I \) is a helper ideal if its complementary ideal \( I^c \) is a product of degree one primes with \( \mathcal{N}(I^c) \) squarefree.

These ideals are interesting because they do not impact the value of \( t \) as explained by the following lemma.

**Lemma 21:** Let \( p \) be a prime number not dividing \( \tilde{t}qN(\mathfrak{b}) \). Suppose that \( h = mh' \) is an admissible vector generating an ideal whose \( p \)-part is a helper ideal dividing \( p\mathcal{O}_K \). This means that there is a prime ideal \( \mathfrak{p} \) of norm \( p \) such that \( h' \mathfrak{p} \subseteq pL \) and \( h \not\in pL \). Then under the assumptions (*), \( v_p(l) = 1 \) and \( v_p(t) = 0 \).

**Proof:**

We use once again formula (19). Under the assumptions (*), we have \( \tilde{\lambda} = 1 \). Here, by assumption we also have \( v_p(\tilde{t}) = 0 \), \( v_p(\mathcal{N}(h')/\mathcal{N}(L)) = d - 1 \). Yet, because \( u - rmd(u_j, \mathfrak{p}) \in \mathfrak{p} \) for all \( 1 \leq j \leq r \) one gets \( (u_j - rmd(u_j, \mathfrak{p}))h' \in pL \). Using lemma 17 successively for all indices \( 1 \leq j \leq r \) with integers \( n_0 = -rmd(u_j, \mathfrak{p}) \) and \( n_i = 0 \) for \( 1 \leq i \leq j - 1 \) we get that \( p \) divides \( c_{j0} \).

Therefore, \( v_p(\lambda) \geq r = d - 2 \) and using formula (19) we get \( v_p(lt) \leq 1 \).

We now want to prove that \( v_p(l) = 1 \) so that \( v_p(t) = 0 \). This is essentially the same proof as for target ideals. There exists \( 0 \leq j \leq r \) such that \( p \) does not divide \( \alpha_j + mh' \) for all \( m \in \mathbb{Z} \) (in fact this is true for all \( j \) here). There are an integer \( k_j \) and a sign \( \epsilon = \pm 1 \) such that \( \epsilon l\alpha_j = (\tilde{\alpha}_j - rmd(\tilde{\alpha}_j, \mathfrak{p}))h' + k_jh' \).

Suppose that \( p \) does not divide \( l \). Then there is an integer \( m \) such that \( 
\epsilon lm + k_j \in p\mathbb{Z} \) and \( \epsilon(l(\alpha_j + mh')) = (\tilde{\alpha}_j - rmd(\tilde{\alpha}_j, \mathfrak{p}))h' + (\epsilon lm + k_j)h' \in pL \).

This is once again a contradiction and we must conclude that \( p \) divides \( l \) and \( p \) does not divide \( t \).

\( \square \)
4.3.3 Using target and helper ideals to choose \((h_\rho)_{\rho \in \mathcal{S}_r}\)

We now use both target ideals and helper ideals to describe how we choose the vectors \(h_\rho\) in conjecture 10.

**Proposition 22:** Suppose that \(\varepsilon_1, \ldots, \varepsilon_r\) is a set of fundamental units for \(O_{\mathfrak{f}}^+\) satisfying (*) and (**) for all \(\rho \in \mathcal{S}_r\), write \(p_{\rho,j}\) the prime numbers dividing \(\tilde{t}_\rho\) for \(1 \leq j \leq j_\rho\) and \(I_{\rho,j}\) for any corresponding target ideal in (**).

Put \(D = q \prod_{\rho} \prod_{j=1}^{j_\rho} p_{\rho,j}\). Choose for the smoothing ideal \(a\) a degree one prime ideal of norm \(N\) coprime to \(D\). Choose a representative integral ideal \(b\) for a class in \(Cl^+(f)\) coprime to \(DN\). Then there is an ideal \(I_\rho\) which is either \(O_K\) or a helper ideal prime to \(DN\) such that the ideal

\[
q^{N_{ab}} I_\rho \prod_{j=1}^{j_\rho} I_{\rho,j}
\]

is principal, with an admissible generator \(h_\rho\). For all orientations \(\mu, \nu\), the value \(I_{r,f,b,a}(\varepsilon_1, \ldots, \varepsilon_r; h_\mu, \mu, \nu)\) is a product of \(r!\) smoothed ordinary elliptic \(G_r\) functions.

**Proof:**

We want to pick an admissible generator of the ideal \(q^{N_{ab}} \prod_{j=1}^{j_\rho} I_{\rho,j}\) but this ideal may not be principal nor possess an admissible generator. We use helper ideals to deal with those as follows. By Cebotarev’s density theorem (see [5], Theorem 8.17), there are infinitely many degree one prime ideals in each class of \(Cl^+(f)\) and therefore infinitely many helper ideals in each class of \(Cl^+(f)\). Therefore, there is an ideal \(I_\rho\) which is either \(O_K\) or a helper ideal prime to \(DN\) such that the ideal

\[
N_{ab}^{I_\rho} \prod_{j=1}^{j_\rho} I_{\rho,j}
\]

lies in the trivial class of \(Cl^+(f)\). In particular, this ideal has a generator \(g_\rho\) congruent to 1 mod \(f\) and we put \(h_\rho = qg_\rho\). Then \(h_\rho\) is admissible because \(h_\rho/q \equiv 1\) mod \(L\) and \(h_\rho/N \in a^{-1}L - L\) is a generator of the cyclic group \(a^{-1}L/L\) of prime order \(N\).

Write \(N(I_{\rho,j}) = p_{\rho,j}^{k_{\rho,j}}w_{\rho,j}\) where \(w_{\rho,j} = v_{p_{\rho,j}}(l_\rho)\). Write also \(n_{\rho}^{d-1} = N(I_\rho)\). Then, using lemma 19 we get \(v_{p_{\rho,j}}(l_\rho) = k_{\rho,j}\) and \(v_{p_{\rho,j}}(l_\rho) = 0\) for all \(\rho, j\). Using lemma 21 we get \(n_{\rho}^{d-1}\) and \(n_{\rho}^{d-1} \cdot t_\rho = 1\). Now we have to understand what happens for the \(\ell\)-adic valuations in formula (19) where \(N(f) = q = \ell^k\). In this setting, we get \(v_{\ell}(\lambda_\rho) = rk = (d-2)k\), \(v_{\ell}(l_\rho) = k\), and
\( v_t(\mathcal{N}(h'_\rho)/\mathcal{N}(L)) = kd - k \) such that using formula (19) we get \( v_t(t_\rho) = 0 \). The value of \( \mathcal{N}(b) \) is completely invisible in formula (19) because

\[ \mathcal{N}(h'_\rho)/\mathcal{N}(L) = q^{d-1}N^{d-1}\prod_{j=1}^{j_\rho} p_{p,j}^{2w_{p,j}-w_{p,j}}n_{p}^{\rho d-1} \]

Lastly, \( N/\mathfrak{a} \) is in fact a helper ideal for which we have \( v_N(l_\rho) = 1 \) and \( v_N(t_\rho) = 0 \). We use formula (19) and we obtain the values

\[ l_\rho = qNn_{\rho} \prod_{j=1}^{j_\rho} p_{p,j}^{k_{p,j}} \]

for the levels in the computations of the values \( G_{r,f,b,a}^\pm(\varepsilon_{1,\ldots,\varepsilon_r}; h_\rho, \mu, \nu) \) as well as \( t_\rho = 1 \), which implies that the value \( I_{r,f,b,a}(\varepsilon_1, \ldots, \varepsilon_r; h_\rho, \mu, \nu) \) is a product of \( r! \) smoothed ordinary elliptic \( G_r \) functions.

**Remark:** Our choice of vectors \( h_\rho \) are made to satisfy two important conditions. The main focus is to produce values for \( I_{r,f,b,a}(\varepsilon_1, \ldots, \varepsilon_r; h_\rho, \mu, \nu) \) with arithmetical properties. It is plainly not true that all such products of \( G_r \) functions yield algebraic integers. A key point in computing interesting \( I_{r,f,b,a}(\varepsilon_1, \ldots, \varepsilon_r; h_\rho, \mu, \nu) \) is to make sure that there is some uniformisation on the choice of \( h_\rho \) for \( \rho \in \Gamma_r \). We believe that this uniformisation property has to do with the lattices \( M_\rho \) defined in section 4.1. The condition \( t_\rho = 1 \) for all \( \rho \) guarantees that \( M_\rho = L \) for all \( \rho \), which can be seen as the strongest form of uniformisation possible. On the other hand, this condition is useful to perform quick computations.

Before giving numerical evidence to support our conjecture, we explain how proposition 22 is used to compute the arithmetic \( G_r \) functions.

1. Fix \( \mathbb{K} \) and \( \mathfrak{f} \neq \mathcal{O}_\mathbb{K} \). Compute fundamental units for \( \mathcal{O}_\mathbb{K}^\times \) (as given by Pari/GP for example). Use these units to compute a set of fundamental units for \( \mathcal{O}_\mathfrak{f}^{\mathfrak{f}} \).

2. Compute the values \( \tilde{t}_\rho \) and \( \lambda_\rho \) for all \( \rho \). Check conditions (\() and (**) and compute target ideals. Note that when \( \tilde{t}_\rho = 1 \) there are no target ideals for the permutation \( \rho \) and the empty product \( \prod_{j=1}^{j_\rho} I_{p,j} \) is equal to the trivial ideal \( \mathcal{O}_\mathbb{K} \). If the conditions (\() and (**) are not satisfied, try small base changes for the units.

3. Choose \( a \) and \( b \) as in proposition 22, with the smallest possible norms.
4. Use Proposition 22 to construct the admissible generators $h_\rho$. One way to find $I_\rho$ in Proposition 22 is to try $I_\rho = \mathcal{O}_K$ first, and if unsuccessful, try successively the available helper ideals with increasing norms until successful.

5. Compute $h'_\rho$, $a_\rho$, $\alpha_\rho$ and $G_{r,f,b,a}^{\mu_\rho}(\varepsilon_\rho(1), \ldots, \varepsilon_\rho(r); h_\rho)$ using formula (20) as well as formula (21) for the desired orientation $\mu_\rho$.

6. Compute the product

$$I_{r,f,b,a}(\varepsilon_1, \ldots, \varepsilon_r; h_\rho, \mu_\rho, \nu_\rho) = \prod_{\rho \in \mathfrak{S}_r} G_{r,f,b,a}^{\mu_\rho}(\varepsilon_\rho(1), \ldots, \varepsilon_\rho(r); h_\rho)^{\nu_\rho}$$

for the desired orientations $\mu$ and $\nu$.

5 Numerical evidence to support the conjecture

In this section, we provide numerical examples to support our conjecture. They may be computed with high precision in a low amount of time. In what follows, we will give computation times for 1000 digits precision on a personal computer. Computations were carried out using number fields found in the LMFDB database [11] as well as the computer algebra system PARI/GP [21], making extensive use of algebraic number theory tools it provides. To understand the result of the computations, we use the commands lindep to test if the value obtained from our computations is close to the expected value for $N\zeta'([b], 0) - \zeta'([ab], 0)$ and algdep to test if the value obtained is close to some algebraic integer in an abelian extension of $K$. Note that if $[K^+(f) : K] = m$ and $[K : \mathbb{Q}] = d$ then we may compute using our method $2m$ roots of a polynomial defining a subfield of $K^+(f)$ over $\mathbb{Q}$ out of a maximal amount of $dm$. This is achieved by varying the class $[b]$ and using both complex embeddings $\sigma_{\mathbb{C}}$ and $\overline{\sigma_{\mathbb{C}}}$. This way, we may also compute all the roots of a polynomial in $K[x]$ which defines a subextension of $K^+(f)/K$ and we can thus compute the elementary symmetric polynomials in these roots to recover the polynomial. In what follows, we perform computations for $b = (1)$ because most of the work on target ideals is independent of the choice for the ideal $b$. In this section, we will name prime ideals above a prime $p$ as $\mathfrak{P}_p, \mathfrak{P}'_p, \mathfrak{P}''_p, \ldots$ in their order of appearance using the commands idealprimedec or idealfactor in Pari/GP version 2.15.4.
5.1 Cubic examples

Here we present six examples in the cubic case. The first example showcases the work on target ideal with $\tilde{t} = 131$ in the spirit of $(\ast)$ and $(\ast\ast)$. The second example is one of the simplest ones, with $\tilde{t} = 1$ and class number one. The third example is a case where $q = 5$. The fourth and fifth examples showcase the work on helper ideals when the class number gets larger, for $q = 3$ and $q = 7$. The sixth example shows what can be done outside $(\ast)$, $(\ast\ast)$ in the cubic case.

5.1.1 Example 1

We first discuss in full detail a cubic example. Let $z$ be the complex root of the polynomial $x^3 - 13$ lying in the upper half-plane. Then $\mathbb{K}_1 = \mathbb{Q}(z)$ has class number 3. We choose $\mathfrak{f}^3 = (3)$. The unit group $\mathcal{O}_{\mathbb{K}_1}^+, \times$ is generated by $\varepsilon = 2z^2 - 3z - 4$. We compute $\tilde{\lambda} = 1$, $\tilde{s} = \tilde{t} = 131$. Then, we search for target ideals above the prime number 131 and we find the unique degree one prime $\mathfrak{p}_{131}$ above 131. Choose $b = (1)$ and $a$ the unique degree one prime above 5 in $\mathbb{K}_1$. Then the ideal $qN\mathfrak{p}_{131}/a$ has an admissible generator $h = -21z^2 - 42z - 114$ so that we don’t need to find a helper ideal. The corresponding level is $l = 3 \cdot 5 \cdot 131 = 1965$. The value

$$\Gamma_{i,b,a}^\pm (\varepsilon, h)^{-1} = \frac{\Gamma \left( \frac{1}{3}, \frac{-\varepsilon - 1 - 5348}{393} \right) \Gamma \left( \frac{-2}{3}, \frac{-\varepsilon - 467}{393} \right)^{-5}}{\Gamma \left( \frac{-5}{3}, \frac{-\varepsilon - 1 - 5348}{393} \right) \Gamma \left( \frac{-4}{3}, \frac{-\varepsilon - 467}{393} \right)^{-5}} \approx -0.0660917... + i \cdot 0.0932299...$$

is close to a root of the palindromic polynomial $x^{18} + 384x^{17} + 2310x^{16} - 10646490x^{15} + 1596241353x^{14} + 18608357181x^{13} + 156933809421x^{12} + 21509825680x^{11} + 38140736538x^{10} + 338205493469x^9 + \ldots$ which defines $\mathbb{K}_1^+(\mathfrak{f})$. The computation time for 1000 digits is 11 seconds. Take another admissible vector $h = 3z^2 + 6z + 12$. The valuation of $h$ at any prime ideal dividing 131 is 0. This is one of the simplest admissible vectors we could find. Using this vector $h$ we have to compute $t = 131$ ordinary elliptic Gamma functions and the computation time is 30 seconds for 1000 digits. In the cubic case, because the computation time for the precision $\delta$ is $O(l|\log(\delta)|t)$, reducing the value of $t$ by raising the level $l$ may not result in a massive time gain. In the cases $r \geq 2$, the gain will be more substantial.

5.1.2 Example 2

We now discuss one of the simplest cubic examples which was presented in the introduction. Let $z$ be the complex root of the polynomial $x^3 - 2$ in the
upper half-plane. Then $K_2 = \mathbb{Q}(z)$ has class number 1. We choose \( f^3 = (3) \). Then $\varepsilon = z - 1$ is a generator for $O_f^{-1,\varepsilon}$ such that $\lambda = \bar{t} = 1$. The ideal $qN/\mathfrak{a}$ has an admissible generator $h = 9z^2 - 3z - 9$. The corresponding level is $l = 15$. The value

$$\Gamma_{f,b,a}^+(\varepsilon, h)^{-1} = \frac{\Gamma\left(\frac{-1}{3}, \frac{\varepsilon-1+2}{15}, \frac{-\varepsilon+7}{15}\right)}{\Gamma\left(\frac{-5}{3}, \frac{\varepsilon-1+2}{9}, \frac{-\varepsilon+7}{3}\right)} \approx -1.2937005... + i \cdot 1.4743341...$$

is close to a root of the polynomial $x^6 + 3x^5 + 6x^4 + 5x^3 + 6x^2 + 3x + 1$ which defines $K_2^+(f)$. The computation time for 1000 digits is 1 second.

### Example 3

Here we explain how we have obtained the example in section 3.2 with $q = 5$. Let $z = e^{2\pi i/3}10^{1/3}$ be the root of the polynomial $x^3 - 10$ in the upper half-plane. Then $K_3 = \mathbb{Q}(z)$ has class number 1. We choose $f^3 = (5)$. Then $\varepsilon = (2z^2 - z - 7)/3$ is a generator for $O_f^{-1,\varepsilon}$ such that $\lambda = 1$ and $\bar{t} = 9$. We may choose $b = (1)$ and $a$ the unique degree one prime above 11 in $K_3$. We find the target ideal $\mathfrak{p}_3^2$ above 3 where $\mathfrak{p}_3\mathfrak{p}_3^2 = (3)$. Then, $h = -(35z^2 + 20z + 35)/3$ is an admissible generator for the ideal $qN\mathfrak{p}_3^2/\mathfrak{a}$. The value

$$\Gamma_{f,b,a}^+(\varepsilon, h) = \frac{\Gamma\left(\frac{1}{5}, \frac{-\varepsilon+1+1751}{495}, \frac{-\varepsilon+776}{495}\right)}{\Gamma\left(\frac{11}{5}, \frac{-\varepsilon+1+1751}{45}, \frac{-\varepsilon+776}{45}\right)} \approx -27.5333588... - i \cdot 32.7146180...$$

is close to a root of the polynomial $x^{12} + 57x^{11} + 1956x^{10} + 4640x^9 + 35415x^8 - 199818x^7 + 150139x^6 - 109818x^5 + 35415x^4 + 4640x^3 + 1956x^2 + 57x + 1$ which defines $K_3^+(f)$. The computation time for 1000 digits is 7 seconds.

### Example 4

Let $z$ be the complex root of the polynomial $x^3 - 65$ in the upper half-plane. Then $K_4 = \mathbb{Q}(z)$ has class number 18. This means that most ideals won’t be principal ideals and we will need to use helper ideals to build an admissible vector $h$. We choose $f^3 = (3)$. Then $\varepsilon = z - 4$ is a generator for $O_f^{-1,\varepsilon}$ such that $\lambda = \bar{t} = 1$. We choose $b = (1), b = \mathfrak{p}_5, \ldots$ representatives for the 36 classes in $Cl^+(f)$, and $a$ the unique degree one prime above 5 in $K_4$. The ideals $qN/(ab)$ are unfortunately not principal in general so we use helper ideals. In the case $b = (1)$ we find the helper ideal $\mathfrak{p}_{157}\mathfrak{p}_{157}'$, and in the case $b = \mathfrak{p}_5$ we find the helper ideal $\mathfrak{p}_{101}$. The corresponding admissible generators we found for $qN\mathfrak{p}_{101}/\mathfrak{a}$ and $qN\mathfrak{p}_{157}\mathfrak{p}_{157}'/(a\mathfrak{p}_5)$ are $h_{b=(1)} = 3z^2 + 600z - 2460$
and $h_{b=q_{59}} = (-222z^2 + 4710z - 15375)/59$ respectively. The corresponding levels will be $l_{(1)} = 3 \cdot 5 \cdot 157$, $l_{q_{59}} = 3 \cdot 5 \cdot 101$. Then we compute

$$
\Gamma_{f,(1),a}^{+}(\varepsilon, h)^{-1} = \frac{\Gamma \left( -\frac{1}{3}, -\varepsilon^{-1} + 781, -\varepsilon - 32774 \right)}{\Gamma \left( -\frac{5}{3}, -\varepsilon^{-1} + 781, -\varepsilon - 32774 \right)^{-1}} \approx -1.6691052... + i \cdot 5.7493283...
$$

$$
\Gamma_{f,q_{59},a}^{+}(\varepsilon, h)^{-1} = \frac{\Gamma \left( \frac{1}{3}, -\varepsilon^{-1} + 676, -\varepsilon - 26954 \right)}{\Gamma \left( \frac{5}{3}, -\varepsilon^{-1} + 676, -\varepsilon - 26954 \right)^{-1}} \approx 0.0344135... - i \cdot 0.0123218...
$$

and the remaining 34 out of 36 values $I_b = \Gamma_{f,b,a}(\varepsilon, h, \sigma_C)$ attached to the 36 classes in $Cl^+(f)$. We may then compute the polynomial

$$
\prod_{b \in Cl^+(f)} (X - I_b) = X^{36} - (u + vz + wz^2)X^{35} + \cdots \in \mathcal{O}_K[X]
$$

where $u = 93377174024326, v = 769211619985, w = -5967373310133$. This palindromic polynomial defines a relative equation of the class field $K_f^{+}(f)$ above $K_4$ and we identify the rest of its coefficients in $\mathcal{O}_K$. The computation time for 1000 digits for each of the 36 computations is 1 minute and 20 seconds on average.

5.1.5 Example 5

Here we give an example with $q = 7$. Let $z$ be the complex root of the polynomial $x^3 - 14$ in the upper half-plane. Then $K_5 = \mathbb{Q}(z)$ has class number 3. We choose $f^5 = (7)$. Then $\varepsilon = -z^2 + 2z + 1$ is a generator for $O_f^{rel,x}$ such that $\tilde{\lambda} = 1$, $\tilde{t} = 2 \cdot 11$. We choose $b = (1)$ and $a$ the unique degree one prime above 5 in $K_5$. Computing target ideals gives $\mathfrak{P}_2$ and $\mathfrak{P}_{11}$ the degree one primes above 2 and 11. The ideal $q \mathfrak{P}_2 \mathfrak{P}_{11} / (ab)$ is unfortunately not principal so we need to look for a helper ideal. One of the helper ideals with smallest norm we find is $\mathfrak{P}_{53}$, the degree two prime above 53 in $K_5$. The associated admissible generator is $h = 392z^2 + 1043z + 2212$. The corresponding level will be $l = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 53$. The value

$$
\Gamma_{f,b,a}^{+}(\varepsilon, h)^{-1} = \frac{\Gamma \left( -\frac{1}{7}, -\varepsilon^{-1} - 445073, -\varepsilon - 17177 \right)}{\Gamma \left( -\frac{5}{7}, -\varepsilon^{-1} - 445073, -\varepsilon - 17177 \right)^{-1}} \approx -0.1700923... + i \cdot 3.8609499...
$$

is close to a root of the palindromic polynomial $x^{54} - 4167x^{53} + 7931535x^{52} - 259219286x^{51}...$ which defines $K_f^{+}(f)$ over $\mathbb{Q}$. This polynomial has very large coefficients, and we could alternatively compute the remaining 17 out of 18
values associated to the 18 classes in $\text{Cl}^+(f)$ to identify a relative polynomial in $\mathcal{O}_K[x]$ instead, as we did in the previous example. The computation time for the above value and for 1000 digits of precision is 6 minutes and 50 seconds, because the value of the level $l$ is quite high.

5.1.6 Example 6

This example falls outside of the conditions (* and **) because we will have $\tilde{\lambda} = 2$. Yet, it shows that these conditions are a set of $p$-adic conditions which can be satisfied for most $p$. In other words, the unique prime number for which (*) and (**) are not true in the present case is 2. Because of this, we can try to reduce the number of computations outside of the prime number 2.

Let $z$ be the complex root of the polynomial $x^3 - 5$ in the upper half-plane. Then $K_6 = \mathbb{Q}(z)$ has class number 1. We choose $f_\gamma = (3)$. Then $\varepsilon = 2z^2 - 4z + 1$ is a generator for $\mathcal{O}_f^{+,*}$ such that $\tilde{\lambda} = 2$ and $\tilde{t} = 2 \times 13$. We choose $b = 1$ and $a$ the unique degree one prime above 5 in $K_6$. We find $\mathfrak{p}_{13}$ a target ideal above 13 and we find an admissible generator $h = 6z^2 + 15$ for the ideal $qN\mathfrak{p}_{13}/a$. The corresponding level will be $l = 3 \cdot 5 \cdot 13$. Then $\Gamma_{f,b,a}^+(\varepsilon, h)^{-1}$ is a product of two ordinary smoothed elliptic Gamma functions with parameters

$$\tau = \frac{-\varepsilon^{-1} - 119}{2 \times 195}, \quad \sigma = \frac{-\varepsilon - 59}{2 \times 195}$$

More precisely, put $\delta = -2z^2 - z - 5$. Then the set $L/M$ in formula (21) is equal to $\{0, \delta\}$ and $\delta/q\gamma = (-4z^2 - 5z - 55)/195$. The product

$$\frac{\Gamma\left(\frac{1}{3}, -\varepsilon^{-1} - \frac{119}{390}, -\varepsilon - \frac{59}{390}\right)^5 \Gamma\left(\frac{1}{3} + \frac{-4z^2 - 5z - 55}{195}, -\varepsilon^{-1} - \frac{119}{390}, -\varepsilon - \frac{59}{390}\right)^5}{\Gamma\left(\frac{5}{3}, -\varepsilon^{-1} - \frac{119}{78}, -\varepsilon - \frac{59}{78}\right)^{-1} \cdot \Gamma\left(\frac{5}{3} + \frac{-4z^2 - 5z - 55}{39}, -\varepsilon^{-1} - \frac{119}{78}, -\varepsilon - \frac{59}{78}\right)^{-1}}$$

is close to the root $\approx 9.8439696... + i \cdot 5.1060682...$ of the palindromic polynomial $x^6 - 21x^5 + 150x^4 - 185x^3 + 150x^2 - 21x + 1$ which defines $K_6^+(f)$. The computation time for 1000 digits is 11 seconds. Note that in the present case, we could have chosen a different target ideal, namely $\mathfrak{P}_{13}^2 \mathfrak{P}_{13}'$ where $\mathfrak{P}_{13}'$ is one of the other primes above 13, but in this case the level $l$ would be 169 × 195 instead of 195.

5.2 Quartic examples

We now present four examples in the quartic case. The first example is one of the simplest cases, for which $t_1 = t_2 = 1$. The second example
showcases the work on target ideals with $\tilde{t}_1 = 7, \tilde{t}_2 = 7129$. The third example shows that we can sometimes hope to find better units to gain time in the computations. The fourth example is a case where the quartic field $\mathbb{K}$ contains a real quadratic field and the set of units falls just outside (*), (**). In all examples, we use the ordering $\{Id, (21)\}$ of $\mathfrak{S}_2$ and when we say that we “choose” fundamental units for $\mathcal{O}_f^{+,-}$, we let Pari/GP compute a set of fundamental units for $\mathcal{O}_K^X$, which we use to compute a set of fundamental units for $\mathcal{O}_f^{+,-}$. Most of the time Pari/GP does a great job in finding small fundamental units, and the few operations required to compute $\mathcal{O}_f^{+,-}$ keep these units relatively small.

5.2.1 Example 7

We first discuss one of the simplest quartic examples mentioned in section 3.2. Let $z$ be the complex root of the polynomial $x^4 - 6x^3 - x^2 - 3x + 1$ lying in the upper half-plane. Then $\mathbb{K}_7 = \mathbb{Q}(z)$ has class number 1. We choose $f = \mathfrak{P}_2$ the degree one prime above 2. We choose the fundamental units

$$
\varepsilon_1 = \frac{-2z^3 + 13z^2 - z + 3}{7}, \quad \varepsilon_2 = \frac{-5z^3 + 29z^2 + 15z + 18}{7}
$$

for $\mathcal{O}_f^{+,-}$. We compute $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1, \tilde{t}_1 = \tilde{t}_2 = 1$. We may choose $b = (1)$ and $a$ the unique degree one prime above 13 in $\mathbb{K}_7$. The ideal $qN/a$ has an admissible generator $h_1 = h_2 = (-170z^3 + 972z^2 + 468z + 640)/7$ so that we don’t have to look for helper ideals. The corresponding levels will be $l_1 = l_2 = 2 \cdot 13$. Computations show that the correct orientations should be $\mu = [1, -1]$ and $\nu = [-1, 1]$. Let us write the parameters

$$
\tau = \varepsilon_2 + 11, \quad \tau' = 20 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1 \varepsilon_2}
$$

$$
\sigma = -7 + \frac{1}{\varepsilon_2}, \quad \sigma' = -\varepsilon_2 - 11
$$

$$
\rho = -\varepsilon_1 + 23, \quad \rho' = 4\varepsilon_1 - 85 + \frac{1}{\varepsilon_2}
$$

Then

$$
\frac{G_2 \left( \frac{1}{2}, \frac{\tau}{20}, \frac{\sigma}{20}, \frac{\rho}{20} \right)^{-13}}{G_2 \left( \frac{13}{2}, \frac{\tau'}{20}, \frac{\sigma'}{20}, \frac{\rho'}{20} \right)^{-1}} \times \frac{G_2 \left( \frac{-1}{2}, \frac{\tau'}{20}, \frac{\sigma'}{20}, \frac{\rho'}{20} \right)^{13}}{G_2 \left( \frac{-13}{2}, \frac{\tau'}{20}, \frac{\sigma'}{20}, \frac{\rho'}{20} \right)^{13}}
$$

is close to the root $\approx 4.1210208... - i \cdot 5.0617720...$ of the polynomial $x^8 - 7x^7 + 33x^6 + 49x^5 + 17x^4 + 49x^3 + 33x^2 - 7x + 1$ which defines $\mathbb{K}_7^+(f)$. The
computation time for 1000 digits is 6 seconds. We compare this to a simple choice for the admissible vectors \( h_1, h_2 \) given by Pari/GP. Namely for \( h_1 = h_2 = (86z^3 - 496z^2 - 286z - 206)/7 \) we compute \( t_1 = t_2 = 31 \) and \( l = 26 \). The computation time for this pair and for 1000 digits is 2 minutes, which is roughly 20 times longer than for the previous pair.

5.2.2 Example 8

We now discuss in full detail a quartic example with specific work on target ideals. Let \( z \) be the complex root of the polynomial \( x^4 - 19x^3 + 18x^2 + 8x + 1 \) lying in the upper half-plane. Then \( K_8 = \mathbb{Q}(z) \) has class number 1. We choose \( f = P_3 \) such that \((3) = P_2^3P_3\). We choose the fundamental units

\[
\varepsilon_1 = \frac{19z^3 - 366z^2 + 438z + 44}{9}, \quad \varepsilon_2 = -z^3 + 19z^2 - 18z - 8
\]

for \( \mathcal{O}_f \). We compute \( \tilde{\lambda}_1 = \tilde{\lambda}_2 = 1, \tilde{t}_1 = 7, \tilde{t}_2 = 7129 \) where 7129 is a prime number. We may choose \( b = (1) \) and \( a \) the unique degree one prime above 13 in \( K_8 \). We search for target ideals and find \( P_7 \) in the first case and \( P_{7129} \) in the second case. Both ideals \( qN_{P_7} / a \) and \( qN_{P_{7129}} / a \) are principal with admissible generators \( h_1 = 43z^3 - 820z^2 + 830z + 313 \) and \( h_2 = -230z^3 + 4406z^2 - 4825z - 1208 \). The corresponding levels will be \( l_1 = 3 \cdot 13 \cdot 7 \) and \( l_2 = 3 \cdot 13 \cdot 7129 \). Computations show that the correct orientations should be \( \mu = [1, 1] \) and \( \nu = [1, -1] \). Let us write the parameters

\[
\tau = \frac{2\varepsilon_1\varepsilon_2 + 454\varepsilon_1 - 3}{\varepsilon_1}, \quad \tau' = \frac{57488\varepsilon_1\varepsilon_2 + 3\varepsilon_2 + 16}{\varepsilon_1\varepsilon_2},
\]

\[
\sigma = \frac{-3443\varepsilon_1\varepsilon_2 + 367\varepsilon_1 - 47}{\varepsilon_1\varepsilon_2}, \quad \sigma' = \frac{47\varepsilon_1\varepsilon_2 - 328\varepsilon_2 + 485399}{1},
\]

\[
\rho = \frac{-\varepsilon_1\varepsilon_2 - 3\varepsilon_1 + 40}{1}, \quad \rho' = \frac{-57\varepsilon_1\varepsilon_2 + 188027\varepsilon_2 - 2}{\varepsilon_2}
\]

Then

\[
G_2 \left( \frac{1}{3}, \frac{\tau}{2773}; \frac{\sigma}{2773}, \frac{\rho}{2773} \right)^{13} \times \frac{G_2 \left( \frac{1}{3}, \frac{\tau'}{278031}; \frac{\sigma'}{278031}, \frac{\rho'}{278031} \right)^{-13}}{G_2 \left( \frac{13}{3}, \frac{\tau'}{21387}; \frac{\sigma'}{21387}, \frac{\rho'}{21387} \right)^{-1}}
\]

is close to the root \( \approx 10.6409709... - i \cdot 5.9332732... \) of the polynomial \( x^8 - 18x^7 + 83x^6 + 396x^5 + 597x^4 + 396x^3 + 83x^2 - 18x + 1 \) which defines a subextension of \( K_8(f) / K_8 \). The computation time for 1000 digits is 2 minutes and 15 seconds.
5.2.3 Example 9

Keep $\mathbb{K}_8, \mathfrak{f}, \mathfrak{b}, \mathfrak{a}$ as in example 8 and let us change our choice of fundamental units. We fix another set of fundamental units for $O_1^{1,*}$:

$$
\varepsilon_1 = \frac{z^3 - 21z^2 + 54z + 11}{9}, \quad \varepsilon_2 = -z^3 + 19z^2 - 18z - 8
$$

Then, we compute $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$, $\tilde{t}_1 = 1$, $\tilde{t}_2 = 25$. In the second computation, a target ideal is $\mathfrak{P}_5^2$ the square of the unique degree one prime above 5 and the ideals $qN/a$ and $qNP_5^2/a$ are generated by the admissible vectors $h_1 = (-44z^3 + 843z^2 - 927z - 232)/3$, $h_2 = (76z^3 - 1449z^2 + 1470z + 344)/3$. The corresponding levels will be $t_1 = 3 \cdot 13$ and $t_2 = 3 \cdot 5^2 \cdot 13$. Computations show that the correct orientations should be $\mu = [1, 1]$ and $\nu = [-1, 1]$. Let us write the parameters

$$
\tau = \frac{2\varepsilon_1\varepsilon_2 + 109\varepsilon_1 - 3}{\varepsilon_1}, \quad \tau' = \frac{842\varepsilon_1\varepsilon_2 + \varepsilon_2 + 3}{\varepsilon_1\varepsilon_2},
$$

$$
\sigma = \frac{-321\varepsilon_1\varepsilon_2 - 16\varepsilon_1 + 7}{\varepsilon_1\varepsilon_2}, \quad \sigma' = \frac{-7\varepsilon_1\varepsilon_2 + 4\varepsilon_2 - 63}{1},
$$

$$
\rho = \frac{-\varepsilon_1 - 47}{1}, \quad \rho' = -3\varepsilon_1\varepsilon_2 + 566\varepsilon_2 - 2
$$

Then

$$
G_2 \left( \frac{1}{3}, \frac{\tau}{39}, \frac{\sigma}{39}, \frac{\rho}{39} \right)^{-13} \times G_2 \left( \frac{1}{3}, \frac{\tau'}{777}, \frac{\sigma'}{777}, \frac{\rho'}{777} \right)^{13}
$$

is close to the same root $\approx 10.6409709... - i \cdot 5.9332732...$ of the polynomial $x^8 - 18x^7 + 83x^6 + 396x^5 + 597x^4 + 396x^3 + 83x^2 - 18x + 1$. The computation time for 1000 digits is 11 seconds. We see that a good choice for the fundamental units can have a strong impact on the computation time.

5.2.4 Example 10

This example falls outside of (*) and (**) because $\gcd(\tilde{t}_1, \tilde{t}_2, q = 2) = 2$. Still, we can find a pair of vectors $h_1, h_2$ for which the computations give interesting results. Let $z$ be the complex root of the polynomial $x^4 - 12$ lying in the upper half-plane. Then $\mathbb{K}_{10} = \mathbb{Q}(z)$ has class number 1 and contains the real quadratic field $\mathbb{Q}(\sqrt{3})$. Remember that this means that we must carefully choose the fundamental units in this case to avoid totally real elements. We choose $\mathfrak{f}_4 = (2)$. We choose the fundamental units

$$
\varepsilon_1 = \frac{z^2 + 2z + 2}{4}, \quad \varepsilon_2 = \frac{-z^3 + z^2 + 4z - 2}{4}
$$
for $\mathcal{O}_f^{+,x}$. We compute $\hat{\lambda}_1 = \hat{\lambda}_2 = 1$, $\hat{t}_1 = 2 \cdot 3$, $\hat{t}_2 = 2 \cdot 11$. We may choose $b = (1)$ and $a = \mathfrak{p}_{23}$ a degree one prime above 23 in $\mathbb{K}_{10}$. We search for target ideals and find $\mathfrak{p}_3$ the degree one prime ideal above 3 in the first case and $\mathfrak{p}_{11}$ one of the degree one primes above 11 in the second case. Then $qN\mathfrak{p}_3/a$ and $qN\mathfrak{p}_{11}/a$ are generated by the admissible vectors $h_1 = (-4z^3 + 11z^2 + 10z + 30)/2$ and $h_2 = (9z^3 + 4z^2 - 34z - 56)/2$. The corresponding levels will be $t_1 = 2 \cdot 3 \cdot 23$ and $t_2 = 2 \cdot 11 \cdot 23$. Fixing the orientations $\mu = [1, -1], \nu = [-1, 1]$ we compute $I_{2,1,b,a}(\varepsilon_1, \varepsilon_2, h, \mu, \nu, \sigma_C) \approx 13.9102308... - i \cdot 24.0932265...$ with $t_1 = t_2 = 2$, i.e. we compute a product of 8 ordinary elliptic smoothed $G_2$ functions. The result is close to a root of the polynomial $x^8 - 28x^7 + 778x^6 - 112x^5 - 749x^4 - 112x^3 + 778x^2 - 28x + 1$ which defines $\mathbb{K}_{10}^+(f)$. The computation time for 1000 digits is 17 seconds.

### 5.3 One Quintic example

We have already successfully produced a dozen of quintic examples and we give here a detailed presentation of our simplest one. Let $z$ be the complex root of the polynomial $x^5 - x^4 - x^3 - 2x^2 + x + 1$ lying in the upper half-plane. Then $\mathbb{K}_{11} = \mathbb{Q}(z)$ has class number 1. We choose $f = \mathfrak{p}_3$ the degree one prime above 3. We choose the fundamental units (once again take the fundamental units provided by Pari/GP and then compute the simplest totally positive ones):

$$\varepsilon_1 = z^4 - 2z^3 - z + 3,$$  
$$\varepsilon_2 = 2z^4 - 2z^3 - 3z + 3,$$  
$$\varepsilon_3 = 2z^4 - 3z^3 - 4z + 4$$

for $\mathcal{O}_f^{+,x}$. We fix the order $\{Id, (32), (21), (321), (312), (31)\}$ of $\mathfrak{S}_3$ given by Pari/GP. We compute $\hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_4 = \hat{\lambda}_5 = \hat{\lambda}_6 = 1$, $\hat{t}_1 = 7 \cdot 37 \cdot 137$, $\hat{t}_2 = 31 \cdot 53$, $\hat{t}_3 = 491$, $\hat{t}_4 = 107$, $\hat{t}_5 = 1$ and $\hat{t}_6 = 145637$ where we have written the prime number decomposition of all the overflows $t$. We may choose $b = (1)$ and $a$ the unique degree one prime above 11 in $\mathbb{K}_{11}$. We look for target ideals above the prime numbers listed above and find the ideals $\mathfrak{p}_7, \mathfrak{p}_37, \mathfrak{p}_137, \mathfrak{p}_31, \mathfrak{p}_53, \mathfrak{p}_491, \mathfrak{p}_{107}, \mathfrak{p}_{145637}$ in the order provided by Pari/GP version 2.15.4. The ideals $qN\mathfrak{p}_7, qN\mathfrak{p}_{37}, qN\mathfrak{p}_{137}/a$, $qN\mathfrak{p}_{31}, qN\mathfrak{p}_{53}/a$, $qN\mathfrak{p}_{491}/a$, $qN\mathfrak{p}_{107}/a, qN\mathfrak{p}_{145637}/a$ admit the following admissible generators:

- $h_1 = 147z^4 - 135z^3 - 90z^2 - 234z + 72$
- $h_2 = -15z^4 - 30z^3 + 90z^2 + 36z + 60$
- $h_3 = 42z^4 - 15z^3 - 87z^2 - 48z + 30$
- $h_4 = -21z^4 + 57z^3 - 6z^2 - 9z - 48$
- $h_5 = 3z^4 - 6z^3 + 18z^2 - 6z + 12$
- $h_6 = 108z^4 - 246z^3 + 111z^2 - 81z + 195$

46
The corresponding levels are $l_1 = 3\cdot11\cdot7\cdot37\cdot137$, $l_2 = 3\cdot11\cdot491$, $l_3 = 3\cdot11\cdot107$, $l_4 = 3 \cdot 11 \cdot 31 \cdot 53$, $l_5 = 3 \cdot 11$, $l_6 = 3 \cdot 11 \cdot 145637$. Computations give the orientations $\underline{\mu} = [1, -1, -1, 1, 1, -1]$, $\underline{\nu} = [1, -1, -1, 1, 1, -1]$. We compute the six quotients:

\[
G_3 \left( \frac{1}{3}, \frac{r_1}{l_1}, \frac{r_2}{l_2}, \frac{r_3}{l_3}, \frac{r_4}{l_4} \right) \cong -6.9846353 \ldots 10^{-18} + i \cdot 6.2764063 \ldots 10^{-17}
\]

\[
G_3 \left( \frac{-11}{3}, \frac{11\sigma_1}{l_1}, \frac{11\sigma_2}{l_2}, \frac{11\sigma_3}{l_3}, \frac{11\sigma_4}{l_4} \right) ^{-1} \cong 1.7434761 \ldots 10^{11} - i \cdot 1.7914686 \ldots 10^{11}
\]

\[
G_3 \left( \frac{-11}{3}, \frac{11\tau_1}{l_1}, \frac{11\tau_2}{l_2}, \frac{11\tau_3}{l_3}, \frac{11\tau_4}{l_4} \right) ^{-1} \cong 0.2815024 \ldots - i \cdot 0.0665163\ldots
\]

\[
G_3 \left( \frac{1}{3}, \frac{r_1}{l_1}, \frac{r_2}{l_2}, \frac{r_3}{l_3}, \frac{r_4}{l_4} \right) ^{-1} \cong 9.8457700 \ldots - i \cdot 2.3506772\ldots
\]

\[
G_3 \left( \frac{11}{3}, \frac{11\sigma_1}{l_1}, \frac{11\sigma_2}{l_2}, \frac{11\sigma_3}{l_3}, \frac{11\sigma_4}{l_4} \right) \cong -0.2518907 \ldots - i \cdot 0.2274847\ldots
\]

\[
G_3 \left( \frac{-11}{3}, \frac{11\tau_1}{l_1}, \frac{11\tau_2}{l_2}, \frac{11\tau_3}{l_3}, \frac{11\tau_4}{l_4} \right) ^{-1} \cong 100864.0193260 \ldots - i \cdot 767246.5816458\ldots
\]

where the parameters are given in the table below. The product of these six quotients is close to the root $\approx -11.6360077 \ldots + i \cdot 3.4634701 \ldots$ of the polynomial $x^{10} + 24x^9 + 164x^8 + 99x^7 - 62x^6 - 89x^5 - 62x^4 + 99x^3 + 164x^2 + 24x + 1$ which defines $K_{11}(f)$. The computation time for 1000 digits is 1 minute and 35 seconds, but the computation time for each of the individual computations is not uniform at all. The fifth computation requires 1 second whereas the sixth computation requires 58 seconds because of the level difference $l_5 = 33$ versus $l_6 = 4806021$. 

47
\[
\begin{array}{llllll}
\varepsilon_1 \tau_1 &= 229 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 1727 \varepsilon_1 \varepsilon_2 - 796386 \varepsilon_1 + 3385 \\
\varepsilon_1 \varepsilon_2 \sigma_1 &= -114 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 987551 \varepsilon_1 \varepsilon_2 + 718 \varepsilon_1 + 949 \\
\varepsilon_1 \varepsilon_2 \varepsilon_3 \rho_1 &= -595138 \varepsilon_1 \varepsilon_2 \varepsilon_3 + 12 \varepsilon_1 \varepsilon_2 + 1445 \varepsilon_1 - 907 \\
\varpi_1 &= -52 \varepsilon_1 \varepsilon_2 \varepsilon_3 + 857 \varepsilon_1 \varepsilon_2 - 295 \varepsilon_1 + 2067162 \\
\varepsilon_1 \tau_2 &= 99 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 356 \varepsilon_1 \varepsilon_2 + 19304 \varepsilon_1 + 57 \\
\varepsilon_1 \varepsilon_2 \sigma_2 &= -17 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 70870 \varepsilon_1 \varepsilon_2 - 145 \varepsilon_1 + 40 \\
\varepsilon_1 \varepsilon_2 \varepsilon_3 \rho_2 &= 324170 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 85 \varepsilon_1 \varepsilon_3 + 197 \varepsilon_1 - 111 \\
\varpi_2 &= 91 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 460 \varepsilon_1 \varepsilon_3 + 3 \varepsilon_1 - 28725 \\
\varepsilon_2 \varepsilon_3 \tau_4 &= -62 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 12636 \varepsilon_1 \varepsilon_2 + 20 \varepsilon_2 + 3 \\
\varepsilon_1 \varepsilon_2 \varepsilon_3 \sigma_4 &= 327915 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 37 \varepsilon_1 \varepsilon_2 - 1101 \varepsilon_2 + 154 \\
\rho_4 &= -44 \varepsilon_1 \varepsilon_2 \varepsilon_3 + 314 \varepsilon_1 \varepsilon_2 - 65 \varepsilon_2 - 52915 \\
\varepsilon_2 \varpi_4 &= -262 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 5 \varepsilon_1 \varepsilon_2 - 768869 \varepsilon_2 - 19 \\
\varepsilon_2 \varepsilon_3 \tau_3 &= -27 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 84 \varepsilon_2 \varepsilon_3 + 11 \varepsilon_2 + 1 \\
\varepsilon_1 \varepsilon_2 \varepsilon_3 \sigma_3 &= 2555 \varepsilon_1 \varepsilon_2 \varepsilon_3 + 19 \varepsilon_2 \varepsilon_3 - 391 \varepsilon_2 + 52 \\
\rho_3 &= -11 \varepsilon_1 \varepsilon_2 \varepsilon_3 + 49 \varepsilon_2 \varepsilon_3 - 51 \varepsilon_2 - 491 \\
\varepsilon_2 \varpi_3 &= -80 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 23 \varepsilon_2 \varepsilon_3 - 1402 \varepsilon_2 - 1 \\
\varepsilon_1 \varepsilon_2 \varepsilon_3 \tau_5 &= -456 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 7 \varepsilon_1 \varepsilon_3 - 2 \varepsilon_3 + 3 \\
\sigma_5 &= -4 \varepsilon_1 \varepsilon_2 \varepsilon_3 + 17 \varepsilon_1 \varepsilon_3 + 12 \varepsilon_3 + 566 \\
\varepsilon_3 \rho_5 &= 9 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 39 \varepsilon_1 \varepsilon_3 + 1265 \varepsilon_3 + 13 \\
\varepsilon_1 \varepsilon_2 \varepsilon_3 \sigma_5 &= \varepsilon_1 \varepsilon_2 \varepsilon_3 - 252 \varepsilon_1 \varepsilon_3 - 3 \varepsilon_3 + 2 \\
\varepsilon_1 \varepsilon_2 \varepsilon_3 \tau_6 &= 9785210 \varepsilon_1 \varepsilon_2 \varepsilon_3 + 2609 \varepsilon_2 \varepsilon_3 - 523 \varepsilon_3 - 729 \\
\sigma_6 &= 479 \varepsilon_1 \varepsilon_2 \varepsilon_3 + 987 \varepsilon_2 \varepsilon_3 - 13874 \varepsilon_3 - 1988676 \\
\varepsilon_3 \rho_6 &= 1713 \varepsilon_1 \varepsilon_2 \varepsilon_3 - 8328 \varepsilon_2 \varepsilon_3 - 1320473 \varepsilon_3 + 764 \\
\varepsilon_2 \varepsilon_3 \varpi_6 &= 4002 \varepsilon_1 \varepsilon_2 \varepsilon_3 + 773526 \varepsilon_2 \varepsilon_3 + 1749 \varepsilon_3 - 2296 \\
\end{array}
\]

References


