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Asymptotic properties of some ODE's flows on the 3D torus under a min-max condition

Marc Briane*

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Contents

Abstract

This paper deals with the asymptotic analysis of the family of the three-dimensional flows $X = X_{fgh}$ on the torus $\mathbb{T}^3$, solutions to the ODE's system

$$x' = f(x,y), \quad y' = g(x,y), \quad z' = h(x,y) \quad \text{in } [0,\infty),$$

where f, g, h are regular $\mathbb{Z}^3$-periodic fonctions satisfying the condition $f\,\partial_x h + g\,\partial_y h = 0$. Under the min-max condition satisfied by the first integral h for the flow X_{fgh}

$$\min_{x\in\mathbb{T}^1}\max_{y\in\mathbb{T}^1} h(x,y) \neq \max_{x\in\mathbb{T}^1}\min_{y\in\mathbb{T}^1} h(x,y),$$

we prove that one of the coordinates x or y of the flow is bounded in $[0,\infty)$. Restricting ourselves to the subclass of flows $X = X_{ab}$ with

$$f(x,y) := b'(y), \quad g(x,y) := b'(y), \quad h(x,y) := b(y) - a(x),$$

it turns out that both coordinates x and y are bounded when $\|a\|_\infty = \|b\|_\infty$, and that the Herman rotation set of the flow X_{ab} is then reduced to a closed line segment. When $\|a\|_\infty \neq \|b\|_\infty$, assuming some extra condition on the roots of the derivatives a', b', we prove that the Herman rotation set of X_{ab} is planar and contains non-degenerate triangles.

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Finally, we refine this result in the case of the Arnold-Beltrami-Childress flow ABC with $A = 0$, by showing that the planar Herman rotation set contains a "fusiform" shape.

Keywords: Integrable system, min-max, asymptotics, invariant probability measure, rotation set, ABC flow

Mathematics Subject Classification: 34E05, 34E10, 37C10, 37C40

1 Introduction

In this paper we study the three-dimensional ODE's flow $X : [0, \infty) \times \mathbb{T}^3 \to \mathbb{R}^3$ solution to the ODE's system

$$\begin{cases} X'(t,x) := \partial_t X(t,x) = F\big(X(t, X_0)\big), & t \in [0,\infty), \\ X(0, X_0) = X_0, & X_0 \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

for the class of three-dimensional vector fields $F := F_{fgh} : \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3 \to \mathbb{R}^3$ defined by

$$F_{fgh}(x,y,z) := \big(f(x,y), g(x,y), h(x,y)\big) \quad \text{for } (x,y,z) \in \mathbb{T}^3. \tag{1.2}$$

Here, f, g are $\mathbb{Z}^2$-periodic functions in $C^1(\mathbb{R}^2)$ and h is a $\mathbb{Z}^2$-periodic function in $C^2(\mathbb{R}^2)$ satisfying

$$(f,g) \cdot \nabla h = f\,\partial_x h + g\,\partial_y h = 0 \quad \text{in } \mathbb{T}^2, \tag{1.3}$$

so that h is a first integral for the flow X associated with F_{fgh}. A well-known example of such a flow is given by the so-called Arnold-Beltrami-Childress flow (denoted by ABC) ([1])

$$\text{ABC flow:} \quad \begin{cases} x' = A\sin z + C\cos y \\ y' = B\sin x + A\cos z \\ z' = C\sin y + B\cos x, \end{cases} \tag{1.4}$$

when one of the three parameters is equal to zero. Up to a circular permutation of the coordinates we can assume that $A = 0$. The case where $ABC \neq 0$ has been the object of several works as an example of chaotic dynamical system (see, *e.g.*, [8, 18] and the references therein) along with an example of non-integrable flow (see [19, 20]). When $A = 0$ the ABC flow is integrable and clearly satisfies equation (1.3) with

$$f(x,y) := C\cos y, \quad g(x,y) := B\sin x, \quad h(x,y) := B\cos x + C\sin y \quad \text{for } (x,y) \in \mathbb{T}^2.$$

In [18] the flow ABC with $A = 0$ has been explicitly computed thanks to Jacobi elliptic functions restricting the flow to the periodic box $[0, 2\pi)^3$. However, this cannot not allow us

[1] The ABC flow has been introduced by Arnold [1] for studying the steady-state solution of Euler equations.

to evaluate the asymptotics of the flow X (1.1) in terms of the limits (or more rigorously the limit points)

$$\text{asymptotics of the flow:} \quad \lim_{t\to\infty} \frac{X(t, X_0)}{t} \quad \text{for } X_0 \in \mathbb{R}^3, \tag{1.5}$$

(which can take actually non-zero values), and to characterize the possible orbits $X([0,\infty))$ whose projections $x([0,\infty))$ or/and $y([0,\infty))$ are bounded in $\mathbb{R}$ (they do exist). The derivation of the asymptotics (1.5) allows us to the Herman rotation set [11] (see Section 1.2) as shown by Misiurewicz-Ziemian [14] for general flows in the framework of ergodic theory. The ergodic approach has seen a spectacular rise for the knowledge of two-dimensional flows on the torus since the pioneer work of Franks-Misiurewicz [9] until the most recent developments (see, *e.g.*, [7, 13]). However, restricting ourselves to the field of ODE's flows there are very few results concerning the asymptotics (1.5). Up to our best knowledge, the seminal contribution in the topic is due to Peirone [15] (see, also [16, 17]) which has proved the remarkable result that for any two-dimensional regular vector-field F which does not vanish on the torus, limits (1.5) do hold for any $X_0 \in \mathbb{R}^3$ (without extraction of any subsequence). This has allowed Peirone to revisit the Franks-Misiurewicz [9] result for the ODE's flows. Moreover, Peirone's result does not hold in dimension three [15], nor in dimension two when the vector field F does vanish [16]. This pioneer work has been extended in [3, 4, 5], and more recently we have studied in [2] the asymptotics of a two-dimensional Euler flow. In the present paper, we investigate the class of flows (1.1) associated with the vector fields F_{fgh} in terms of the asymptotics (1.5) together with the Herman rotation set.

In Section 2.1, dealing with the general class of vector fields (1.2) we prove (see Theorem 2.1) that one of coordinates of the flow $X = (x, y, z)$, either x or y is bounded in $\mathbb{R}$, if the first integral h satisfies the min-max condition

$$\min_{x\in\mathbb{T}^1} \max_{y\in\mathbb{T}^1} h(x, y) \neq \max_{x\in\mathbb{T}^1} \min_{y\in\mathbb{T}^1} h(x, y). \tag{1.6}$$

This implies that the Herman rotation set lies either in the plane $\{x = 0\}$ or in $\{y = 0\}$. This general result seems to be original in the setting of the ODE's flows, and it is proved thanks to a connectedness argument.

In Section 2.2 we restrict ourselves to the subclass of

$$ab \text{ flows:} \quad \begin{cases} x' = b'(y) \\ y' = a'(x) \\ z' = b(y) - a(x), \end{cases} \tag{1.7}$$

where a, b are two 1-periodic functions in $C^2(\mathbb{R})$. In this case, the min-max condition (1.6) reads as

$$\|a\|_\infty := \|a\|_{L^\infty(\mathbb{T}^1)} \neq \|b\|_{L^\infty(\mathbb{T}^1)} =: \|b\|_\infty. \tag{1.8}$$

Moreover, we prove in addition to (1.8) (see Theorem 2.3) that both the coordinates x and y of the flow ab are bounded in $\mathbb{R}$, so that the Herman rotation set agrees with the closed line segment $[-2\|a\|_\infty, 2\|a\|_\infty]$ in the line $\{x = y = 0\}$.

In Section 3 we focus on the asymptotics of the ab flow under additional assumptions on the functions a, b. Assuming that the derivatives of the functions $c = a, b$ have a finite number of roots in $\mathbb{T}^1$, whose two roots, say $\gamma_1 \neq \gamma_2$, satisfy

$$\big\{c(\gamma) : c'(\gamma) = 0\big\} \cap \big(c(\gamma_1), c(\gamma_2)\big) = \emptyset, \tag{1.9}$$

we prove that the Herman rotation set is a planar compact convex set in $\mathbb{R}^3$, which contains $0_{\mathbb{R}^3}$ and a family of non-degenerate triangles whose vertices are parametrized by the open set

$$\Big\{(x_0, y_0) \in \mathbb{R}^2 : -\Big|\|a\|_\infty - \|b\|_\infty\Big| < a(x_0) - b(y_0) < \Big|\|a\|_\infty - \|b\|_\infty\Big|\Big\}. \tag{1.10}$$

In contrast, the subset of the Herman rotation set induced by the regular (with respect to Lebesgue's measure) invariant probability measures for the flow (see Section 1.2) is reduced to the closed line segment $\big[-\|a\|_\infty - \|b\|_\infty, \|a\|_\infty + \|b\|_\infty\big]$ in the line $\{x = y = 0\}$. Therefore, any point of the Herman rotation set without the line $\{x = y = 0\}$, is obtained through a singular (with respect to Lebesgue's measure) invariant probability measure for the ab flow.

Finally, in Section 4 we refine the previous results to the ABC flow with $A = 0$, by exploiting the particular symmetries of the vector field. When $|B| \neq |C|$, we prove (see Proposition 4.1) that the planar Herman rotation set contains a "fusiform" shape as shown in Figure 1 on page 18.

1.1 Notation

- $(e_1, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$, and $0_{\mathbb{R}^d}$ denotes the null vector of $\mathbb{R}^d$.
- " $\cdot$ " denotes the scalar product and $|\cdot|$ the euclidean norm in $\mathbb{R}^d$.
- conv(A) denotes the convex hull of the subset A of $\mathbb{R}^d$.
- $\mathbb{T}^d$ for $d \in \mathbb{N}$, denotes the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ (respectively $\mathbb{R}^d/(2\pi\mathbb{Z})^d$ in Sec. 4), which may be identified to the unit cube $[0,1)^d$ (respectively $[0, 2\pi)^d$) in $\mathbb{R}^d$, and $0_{\mathbb{T}^d}$ denotes the null vector of $\mathbb{T}^d$.
- $C_c^k(\mathbb{R}^d)$ for $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions in $C^k(\mathbb{R}^d)$ with compact support in $\mathbb{R}^d$.
- $C_\sharp^k(\mathbb{T}^d)$ for $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions $f \in C^k(\mathbb{R}^d)$ which are $\mathbb{Z}^d$-periodic, *i.e.*

$$\forall\, K \in \mathbb{Z}^d,\ \forall\, X \in \mathbb{R}^d, \quad f(X + K) = f(X). \tag{1.11}$$

- The Jacobian matrix of a C^1-mapping $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is denoted by the matrix-valued function $\nabla\Phi$ with entries $\partial_{x_j}\Phi_i$ for $i, j \in \{1, \ldots, d\}$.
- The abbreviation "a.e." for almost everywhere, will be used throughout the paper. The simple mention "a.e." refers to the Lebesgue measure on $\mathbb{R}^d$.

- dX denotes the Lebesgue measure on $\mathbb{R}^d$ and $\mathbb{T}^d$, and $\sigma(X)\,dX$ denotes the regular Lebesgue measure with density σ.

- $L^p_\sharp(\mathbb{T}^d)$ for $p \in [1,\infty]$, denotes the space of the Lebesgue measurable functions f in $L^p_{\rm loc}(\mathbb{R}^d)$, which are $\mathbb{Z}^d$-periodic dX-a.e. in $\mathbb{R}^d$.

$$\begin{cases} \|f\|_p := \Big(\displaystyle\int_{\mathbb{T}^d} |f(X)|^p\,dX\Big)^{1/p} < \infty & \text{if } p<\infty, \\ \|f\|_\infty := \operatorname{supess}_{\mathbb{T}^d}|f| < \infty & \text{if } p=\infty. \end{cases}$$

- $\mathscr{M}(\mathbb{T}^d)$ denotes the the space of the Radon measures on $\mathbb{T}^d$, and $\mathscr{M}_{\rm p}(\mathbb{T}^d)$ denotes the space of the probability measures on $\mathbb{T}^d$.

- $\mathscr{D}'(\mathbb{R}^d)$ denotes the space of the distributions on $\mathbb{R}^d$.

- For a Borel measure μ on $\mathbb{T}^d$ and for $f \in L^1_\sharp(\mathbb{T}^d,\mu)$, we denote

$$\mu(f) := \int_{\mathbb{T}^d} f(X)\,\mu(dX), \tag{1.12}$$

  which is simply denoted by $\overline{f}$ when μ is Lebesgue's measure. The same notation is used for a vector-valued function in $L^1_\sharp(\mathbb{T}^d,\mu)^d$.

- c denotes a positive constant which may vary from line to line.

1.2 A few tools of ergodic theory

Let $F \in C^1_\sharp(\mathbb{T}^d)^d$. Consider the ODE's flow X (1.1) associated with the vector field F. A probability measure $\mu \in \mathscr{M}_p(\mathbb{T}^d)$ on $\mathbb{T}^d$ is said to be *invariant* for the flow X defined by (1.1) if

$$\forall\, t \in \mathbb{R},\ \forall\, \psi \in C^0_\sharp(\mathbb{T}^d), \quad \int_{\mathbb{T}^d} \psi\big(X(t,y)\big)\,d\mu(y) = \int_{\mathbb{T}^d} \psi(y)\,d\mu(y). \tag{1.13}$$

Then, we may define the set of invariant probability measures

$$\mathscr{I}_F := \big\{\mu \in \mathscr{M}_p(\mathbb{T}^d) : \mu \text{ invariant for the flow } X\big\}. \tag{1.14}$$

Also define for any vector field $F \in C^1_\sharp(\mathbb{T}^d)^d$, the following non empty subsets of $\mathbb{R}^d$:

- According to [14, (1.1)] the set of all the limit points of the sequences $\big(X(n,\xi_n)/n\big)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ for any sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathbb{T}^d$, is defined by

$$\rho_F := \bigcap_{n\in\mathbb{N}} \left(\overline{\bigcup_{\xi\in\mathbb{T}^d} \left\{ \frac{X(k,\xi)}{k} : k \ge n \right\}} \right). \tag{1.15}$$

  By [14, Lem. 2.2, Thm. 2.3] it is a compact and connected set of $\mathbb{R}^d$.

Remark 1.1. From the semi-group property of the flow (1.1) combined with the following uniform bound

$$\forall\, t \in [0,\infty),\ \forall\, \xi \in \mathbb{T}^d \quad \begin{cases} X(t,\xi) = X\big([t], X(t-[t],\xi)\big) \\ \big|X(t-[t],\xi) - \xi\big| \leq \|F\|_\infty, \end{cases} \tag{1.16}$$

we deduce that the set ρ_F (1.15) agrees with the set all the limit points of the sequences $\big(X(t_n,\xi_n)/t_n\big)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ for any positive sequence $(t_n)_{n\in\mathbb{N}}$ tending to ∞ and for any sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathbb{T}^d$.

- The so-called Herman [11] rotation set is defined by

$$\mathsf{C}_F := \big\{\mu(F) : \mu \in \mathscr{I}_F\big\}. \tag{1.17}$$

  It is clearly a compact and convex set of $\mathbb{R}^d$.

An implicit consequence of [14, Thm. 2.4, Rem. 2.5, Cor. 2.6] shows that in any dimension

$$\mathsf{C}_F = \mathrm{conv}\,(\rho_F). \tag{1.18}$$

The connection (1.18) between the asymptotics of the flow (1.1) and the Herman rotation set for the flow is even stronger in dimension two, since by virtue of [14, Thm. 3.4 (b)] we get that

$$d = 2 \ \Rightarrow\ \mathsf{C}_F = \rho_F. \tag{1.19}$$

We have the following characterization of an invariant probability measure, known as Liouville's theorem (see, *e.g.*, [6, Thm 1, Sec. 2.2]) which can also be regarded as a divergence-curl result with measures (see [3, Rem. 2.2] for further details).

Proposition 1.2 (Liouville's theorem)**.** *Let $F \in C^1_\sharp(\mathbb{T}^d)^d$ and let $\mu \in \mathscr{M}(\mathbb{T}^d)$. Then, the two following assertions are equivalent:*

i) *μ is invariant for the flow X,* i.e. (1.13) *holds true.*

ii) *$\mu\, F$ is divergence free in $\mathbb{R}^d$,* i.e.

$$\mathrm{div}\,(\mu\, F) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^d), \tag{1.20}$$

or equivalently, $\mu\, F$ is divergence free in the torus $\mathbb{T}^d$, i.e.

$$\forall\, \psi \in C^1_\sharp(\mathbb{T}^d), \quad \int_{\mathbb{T}^d} F(Y)\cdot\nabla\psi(Y)\,\mu(dY) = 0. \tag{1.21}$$

2 Three-dimensional flows satisfying a min-max condition

2.1 The class of fgh flows

Let f, g be two functions in $C^1_\sharp(\mathbb{T}^2)$, and let h be a function in $C^2_\sharp(\mathbb{T}^2)$ the gradient of which is solution to the equation

$$f\,\partial_x h + g\,\partial_y h = 0 \ \text{ in } \mathbb{T}^2. \tag{2.1}$$

We consider the three-dimensional flow $X_{fgh} = (x, y, z)$ associated with the vector field $F_{fgh}(X)$ by

$$\begin{cases} X'_{fgh}(t, X_0) = \begin{Bmatrix} f\big(x(t,X_0), y(t,X_0)\big) \\ g\big(x(t,X_0), y(t,X_0)\big) \\ h\big(x(t,X_0), y(t,X_0)\big) \end{Bmatrix} =: F_{fgh}\big(X_{fgh}(t,X_0)\big), \quad t \geq 0, \\ \\ X_{fgh}(0, X_0) = X_0 = (x_0, y_0, z_0) \in \mathbb{R}^3. \end{cases} \tag{2.2}$$

Then, we have the following min-max result.

Theorem 2.1. *Let $f, g \in C^1_\sharp(\mathbb{T}^2)$ and let $h \in C^2_\sharp(\mathbb{T}^2)$ be a function satisfying* (2.1)*. Then, the following implications hold for any fixed $X_0 \in \mathbb{R}^3$,*

$$\begin{cases} \min\limits_{x\in\mathbb{T}^1} \max\limits_{y\in\mathbb{T}^1} h(x,y) < \max\limits_{x\in\mathbb{T}^1} \min\limits_{y\in\mathbb{T}^1} h(x,y) \ \Rightarrow \ \forall\, t \in [0,\infty), \ |x(t,X_0) - x_0| < 1, \\ \min\limits_{x\in\mathbb{T}^1} \max\limits_{y\in\mathbb{T}^1} h(x,y) > \max\limits_{x\in\mathbb{T}^1} \min\limits_{y\in\mathbb{T}^1} h(x,y) \ \Rightarrow \ \forall\, t \in [0,\infty), \ |y(t,X_0) - y_0| < 1. \end{cases} \tag{2.3}$$

Remark 2.2. In the two implications (2.3) the use of min and max, rather than inf and sup, is actually justified. Indeed, due to $h \in C^1_\sharp(\mathbb{T}^2)$ the functions defined by

$$h_M(x) := \max_{y\in\mathbb{T}^1} h(x,y) \ \text{ and } \ h_m(x) := \min_{y\in\mathbb{T}^1} h(x,y) \quad \text{for } x \in \mathbb{T}^1,$$

are Lispschitz with constant $\|\partial_x h\|_{L^\infty(\mathbb{T}^2)}$, thus continuous in $\mathbb{R}$. Therefore, the functions h_M and h_m do attain their bounds on $\mathbb{T}^1$ so that

$$\min_{x\in\mathbb{T}^1} \max_{y\in\mathbb{T}^1} h(x,y) = \min_{x\in\mathbb{T}^1} h_M(x) \quad \text{and} \quad \max_{x\in\mathbb{T}^1} \min_{y\in\mathbb{T}^1} h(x,y) = \max_{x\in\mathbb{T}^1} h_m(x).$$

Also note that since the extrema are taken independently with respect of the coordinates x, y, we have

$$\begin{cases} \min\limits_{x\in\mathbb{T}^1} \max\limits_{y\in\mathbb{T}^1} h(x,y) = \inf\limits_{x\in\mathbb{T}^1} \sup\limits_{y\in\mathbb{T}^1} h(x,y) = \sup\limits_{y\in\mathbb{T}^1} \inf\limits_{x\in\mathbb{T}^1} h(x,y) = \max\limits_{y\in\mathbb{T}^1} \min\limits_{x\in\mathbb{T}^1} h(x,y) \\ \max\limits_{x\in\mathbb{T}^1} \min\limits_{y\in\mathbb{T}^1} h(x,y) = \sup\limits_{x\in\mathbb{T}^1} \inf\limits_{y\in\mathbb{T}^1} h(x,y) = \inf\limits_{y\in\mathbb{T}^1} \sup\limits_{x\in\mathbb{T}^1} h(x,y) = \min\limits_{y\in\mathbb{T}^1} \max\limits_{x\in\mathbb{T}^1} h(x,y). \end{cases} \tag{2.4}$$

Proof of Theorem 2.1. Let $X_0 \in \mathbb{R}^3$. In the sequel we simply denote the flow X (2.2) by

$$X_{fgh}(t) = \big(x(t), y(t), z(t)\big) \quad \text{instead of} \quad X_{fgh}(t, X_0) = \big(x(t,X_0), y(t,X_0), z(t,X_0)\big). \tag{2.5}$$

The function $h \in C^2_\sharp(\mathbb{T}^2)$ is a Hamiltonian for the flow X (2.8), since by the chain rule and by (2.1) we have

$$\begin{aligned}\partial_t\big(h(X_{fgh}(t))\big) &= x'(t)\,\partial_x h(x(t),y(t)) + y'(t)\,\partial_y h(x(t),y(t)) \\ &= f(x(t),y(t))\,\partial_x h(x(t),y(t)) + g(x(t),y(t))\,\partial_y h(x(t),y(t)) = 0 \quad \text{for } t \in [0,\infty).\end{aligned}$$

Hence, it follows that

$$\forall\, t \in [0,\infty), \quad h(x(t),y(t)) = h(x_0,y_0). \tag{2.6}$$

Now, assume that $\min\limits_{x\in\mathbb{T}^1}\max\limits_{y\in\mathbb{T}^1} h(x,y) < \max\limits_{x\in\mathbb{T}^1}\min\limits_{y\in\mathbb{T}^1} h(x,y)$. Then, we have

$$h(x_0,y_0) > \min_{x\in\mathbb{T}^1}\max_{y\in\mathbb{T}^1} h(x,y) \quad \text{or} \quad h(x_0,y_0) < \max_{x\in\mathbb{T}^1}\min_{y\in\mathbb{T}^1} h(x,y). \tag{2.7}$$

First, we assume that the first inequality of (2.7) holds. There exists $x_m \in \mathbb{T}^1$ such that

$$\min_{x\in\mathbb{T}^1}\max_{y\in\mathbb{T}^1} h(x,y) = h_M(x_m) \quad \text{with} \quad h_M(x) := \max_{y\in\mathbb{T}^1} h(x,y).$$

This combined with equality (2.6) and the first inequality of (2.7) yields

$$\forall\, t \in [0,\infty), \quad h_M(x(t)) \geq h(x(t),y(t)) = h(x_0,y_0) > \min_{x\in\mathbb{T}^1}\max_{y\in\mathbb{T}^1} h(x,y) = h_M(x_m),$$

Hence, due to the 1-periodicity of h_M, we deduce that

$$x([0,\infty)) \subset \bigcup_{n\in\mathbb{Z}} (n + x_m, n + 1 + x_m).$$

Therefore, since by the intermediate value theorem $x([0,\infty))$ is an interval of $\mathbb{R}$, there exists an integer $n \in \mathbb{Z}$ such that

$$x([0,\infty)) \subset (n + x_m, n + 1 + x_m),$$

which implies that

$$\forall\, t \in [0,\infty), \;\; |x(t) - x_0| = |x(t) - x(0)| < 1.$$

The proof is quite similar if the second inequality of (2.7) holds using this time that there exists $x_M \in \mathbb{T}^1$ such that

$$\max_{x\in\mathbb{T}^1}\min_{y\in\mathbb{T}^1} h(x,y) = h_m(x_M) \quad \text{with} \quad h_m(x) := \min_{y\in\mathbb{T}^1} h(x,y),$$

which leads us to the existence of $n \in \mathbb{Z}$ such that

$$x([0,\infty)) \subset (n + x_M, n + 1 + x_M).$$

Therefore, we have just proved the first implication of (2.3).

Finally, in view of (2.4) the inequality of the second implication of (2.3) also reads as

$$\min_{y\in\mathbb{T}^1}\max_{x\in\mathbb{T}^1} h(x,y) < \max_{y\in\mathbb{T}^1}\min_{x\in\mathbb{T}^1} h(x,y),$$

so that it can be simply deduced from the previous proof by a permutation of x and y. The proof of Theorem 2.1 is thus complete. □

2.2 The subclass of ab flows

Let a, b be two functions in $C^2_\sharp(\mathbb{T}^1)$. We consider the flow $X_{ab} = (x, y, z)$ of type (1.1) associated with the vector field $F_{ab}(X)$ by

$$\left\{\begin{array}{l} X'_{ab}(t, X_0) = \left\{\begin{array}{l} b'\big(y(t, X_0)\big) \\ a'\big(x(t, X_0)\big) \\ b\big(y(t, X_0)\big) - a\big(x(t, X_0)\big) \end{array}\right\} =: F_{ab}\big(X(t, X_0)\big), \quad t \geq 0, \\ \\ X_{ab}(0, X_0) = X_0 = (x_0, y_0, z_0) \in \mathbb{R}^3. \end{array}\right. \tag{2.8}$$

Note that the projection of the flow X on the pair (x, y) turns out to be a two-dimensional Hamiltonian flow whose Hamiltonian is the fonction $(x, y) \mapsto b(y) - a(x)$.

We have the following result satisfied by the flow X_{ab}.

Theorem 2.3. *Let a and b be two functions in $C^2_\sharp(\mathbb{T}^1)$ satisfying the range condition*

$$a\big([0,1)\big) = \big[-\|a\|_\infty, \|a\|_\infty\big] \quad \textit{and} \quad b\big([0,1)\big) = \big[-\|b\|_\infty, \|b\|_\infty\big]. \tag{2.9}$$

Then, we have the following alternative.

i) The following implications hold for any fixed $X_0 \in \mathbb{R}^3$,

$$\left\{\begin{array}{lll} \|a\|_\infty > \|b\|_\infty & \Rightarrow & \forall\, t \in [0, \infty), \ \ |x(t, X_0) - x_0| < 1, \\ \|a\|_\infty < \|b\|_\infty & \Rightarrow & \forall\, t \in [0, \infty), \ \ |y(t, X_0) - y_0| < 1. \end{array}\right. \tag{2.10}$$

ii) If $\|a\|_\infty = \|b\|_\infty$, we have

$$\forall\, t \in [0, \infty), \quad |x(t, X_0) - x_0| \leq 1 \ \ \textit{and} \ \ |y(t, X_0) - y_0| \leq 1. \tag{2.11}$$

Moreover, the Herman rotation set (1.17) *is given by*

$$\mathsf{C}_{F_{ab}} = \big[-2\,\|a\|_\infty, 2\,\|a\|_\infty\big]\, e_3. \tag{2.12}$$

Proof of Theorem 2.3.

Proof of i). Define the functions f, g and h by

$$f(x, y) := b'(y), \ \ g(x, y) := a'(x) \ \text{ and } \ h(x, y) := b(y) - a(x) \quad \text{for } (x, y) \in \mathbb{T}^2, \tag{2.13}$$

so that equality (2.1) is satisfied. Moreover, by the range condition (2.9) we get that

$$\left\{\begin{array}{l} \min\limits_{x\in\mathbb{T}^1} \max\limits_{y\in\mathbb{T}^1} h(x, y) = \min\limits_{x\in\mathbb{T}^1} \max\limits_{y\in\mathbb{T}^1} \big(b(y) - a(x)\big) = \|b\|_\infty - \|a\|_\infty, \\ \max\limits_{x\in\mathbb{T}^1} \min\limits_{y\in\mathbb{T}^1} h(x, y) = \max\limits_{x\in\mathbb{T}^1} \min\limits_{y\in\mathbb{T}^1} \big(b(y) - a(x)\big) = \|a\|_\infty - \|b\|_\infty. \end{array}\right.$$

Therefore, the implications (2.3) of Theorem 2.1 imply the desired implications (2.10).

Proof of ii). Assume that $\|a\|_\infty = \|b\|_\infty > 0$, and define the function $a_\varepsilon := (1+\varepsilon)\,a$ for $\varepsilon > 0$. First of all, let X_ε be the flow defined by (2.8) with the functions a_ε and b, which is associated with the vector-field

$$F^\varepsilon_{ab}(X) := F_{ab}(X) + \varepsilon\,\big(0, a'(x), -a(x)\big) \quad \text{for } X = (x,y,z) \in \mathbb{T}^3.$$

Since $F_{ab} \in C^1_\sharp(\mathbb{T}^3)$ is a κ-Lipschitz function on $\mathbb{R}^3$ for some $\kappa > 0$, we have for any fixed $X_0 \in \mathbb{R}^3$, for any fixed $T > 0$ and for any $t \in [0,T]$,

$$\begin{aligned} 0 \le \big|X_\varepsilon(t,X_0) - X(t,X_0)\big| &= \int_0^t \big|F^\varepsilon_{ab}\big(X_\varepsilon(s,X_0)\big) - F_{ab}\big(X(s,X_0)\big)\big|\,ds \\ &\le \kappa \int_0^t \big|X_\varepsilon(s,X_0)\big) - X(s,X_0)\big)\big|\,ds + \varepsilon\,t\,\underbrace{\big(\|a'\|_\infty + \|a\|_\infty\big)}_{c:=} \\ &\le \kappa \int_0^t \big|X_\varepsilon(s,X_0)\big) - X(s,X_0)\big)\big|\,ds + c\,T\,\varepsilon. \end{aligned}$$

Hence, by Gronwall's inequality (see, *e.g.*, [12, Sec. 17.3]) we deduce that

$$\forall\, t \in [0,T], \quad \big|X_\varepsilon(t,X_0) - X(t,X_0)\big| \le c\,T\,\varepsilon\,e^{\kappa\,t}. \tag{2.14}$$

On the other hand, since

$$\|a_\varepsilon\|_\infty = (1+\varepsilon)\,\|a\|_\infty = (1+\varepsilon)\,\|b\|_\infty > \|b\|_\infty > 0,$$

by the first implication of (2.10) we get that

$$\forall\, t \in [0,\infty), \quad |x_\varepsilon(t,X_0) - x_0| < 1.$$

Therefore, passing to the limit $\varepsilon \to 0$ together with the uniform estimate (2.14) for an arbitrary $T > 0$, we obtain the enlarged inequality

$$\forall\, t \in [0,\infty), \quad |x(t,X_0) - x_0| \le 1,$$

i.e. the first inequality of (2.11). Finally, proceeding similarly with the functions a and the perturbed function $b_\varepsilon := (1+\varepsilon)\,b$, and using the second estimate of (2.10), we obtain the two desired inequalities (2.11).

Now, it remains to characterize the Herman rotation set $\mathsf{C}_{F_{ab}}$ when $\|a\|_\infty = \|b\|_\infty$. On the one hand, as an immediate consequence of (2.11) we have

$$\forall\, X_0 \in \mathbb{R}^3, \quad \lim_{t\to\infty} \frac{x(t,X_0)}{t} = \lim_{t\to\infty} \frac{y(t,X_0)}{t} = 0,$$

which by the set equality (1.18) (see also (1.15) and Remark 1.1) implies that

$$\mathsf{C}_{F_{ab}} \subset \mathbb{R}\,e_3. \tag{2.15}$$

On the other hand, since the function h defined by (2.13) is a first integral of the flow X_{ab}, we get that for any $X_0 \in \mathbb{R}^3$,

$$\forall\, t \in [0,\infty),\ \ z(t,X_0) = \big(b(y_0) - a(x_0)\big)\, t + z_0 \quad \text{so that} \quad \lim_{t\to\infty} \frac{z(t,X_0)}{t} = h(X_0), \tag{2.16}$$

which again by (1.18) implies that

$$\mathsf{C}_{F_{ab}} \cdot e_3 = \big\{h(X_0)),\, X_0 \in \mathbb{R}^3\big\}.$$

This combined with (2.15) thus gives

$$\mathsf{C}_{F_{ab}} = \big\{h(X_0),\, X_0 \in \mathbb{R}^3\big\}\, e_3 = \big\{b(y_0) - a(x_0)),\, (x_0,y_0) \in \mathbb{T}^2\big\}\, e_3.$$

Finally, using the range condition (2.9) combined with equality $\|a\|_\infty = \|b\|_\infty$, we obtain the equality

$$\big\{b(y_0) - a(x_0),\, (x_0,y_0) \in \mathbb{T}^2\big\} = \big[-2\,\|a\|_\infty, 2\,\|a\|_\infty\big],$$

which leads us to the expression (2.12) of the Herman rotation set.

The proof of Theorem 2.3 is now complete. □

3 Asymptotics of the ab flows

In this section we assume that the derivatives a', b' of the flow X (2.8) have a finite number of roots in the torus $\mathbb{T}^1$:

$$\begin{cases} \big\{\alpha \in [0,1) : a'(\gamma) = 0\big\} = \big\{0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_{p-1} < 1\big\}, & p \ge 2, \\ \big\{\beta \in [0,1) : b'(\delta) = 0\big\} = \big\{0 \le \beta_0 < \beta_1 < \cdots < \beta_{q-1} < 1\big\}, & q \ge 2. \end{cases} \tag{3.1}$$

When the 1-periodic functions a, b satisfy some extrema conditions, we can determine the regular (with respect to Lebesgue's measure) invariant probability measures and the Herman rotation set $\mathsf{C}_{F_{ab}}$ (1.17) in a rather precise way. This is the aim of the following result.

Theorem 3.1. *In addition to condition* (2.9) *assume that either*

$$\|a\|_\infty > \|b\|_\infty, \quad \exists\, i_0, j_0 \in \{0,\dots,m-1\},\ a(\alpha_{i_0}) < a(\alpha_{j_0}),\ \ \forall\, k,\ a(\alpha_k) \notin \big(a(\alpha_{i_0}), a(\alpha_{j_0})\big), \tag{3.2}$$

or

$$\|a\|_\infty < \|b\|_\infty, \quad \exists\, k_0, l_0 \in \{0,\dots,n-1\},\ \ b(\beta_{k_0}) < b(\beta_{l_0}),\ \ \forall\, j,\ b(\beta_j) \notin \big(b(\beta_{k_0}), b(\beta_{l_0})\big). \tag{3.3}$$

Then, the following results hold.

i) The closed subset D_{ab} *of the rotation set* $\mathsf{C}_{F_{ab}}$ (1.17) *induced by the regular (with respect to Lebesgue's measure) invariant probability measures for the flow* X_{ab} *is given by*

$$\mathsf{D}_{ab} := \overline{\big\{\overline{\sigma F_{ab}} : \sigma \in L^1_\sharp(\mathbb{T}^3) \text{ and } \sigma(Y)\,dY \in \mathscr{I}_{F_{ab}}\big\}} = \big[-\|a\|_\infty - \|b\|_\infty, \|a\|_\infty + \|b\|_\infty\big]\, e_3. \tag{3.4}$$

ii) For any pair $(x_0, y_0) \in \mathbb{T}^2$*, we have the two following implications:*

$$\left\{\begin{array}{c} (3.2) \quad \textit{and} \quad a(\alpha_{i_0}) + \|b\|_\infty < a(x_0) - b(y_0) < a(\alpha_{j_0}) - \|b\|_\infty \\ \Downarrow \\ \exists\, \gamma_0 \neq 0, \ \big(0, \gamma_0, b(y_0) - a(x_0)\big) \in \mathsf{C}_{F_{ab}}, \end{array}\right\} \tag{3.5}$$

and

$$\left\{\begin{array}{c} (3.3) \quad \textit{and} \quad b(\beta_{k_0}) + \|a\|_\infty < b(y_0) - a(x_0) < b(\beta_{l_0}) - \|a\|_\infty \\ \Downarrow \\ \exists\, \delta_0 \neq 0, \ \big(\delta_0, 0, b(y_0) - a(x_0)\big) \in \mathsf{C}_{F_{ab}}. \end{array}\right\} \tag{3.6}$$

Moreover, if there exists a pair $(x_0, y_0) \in \mathbb{T}^2$ *satisfying*

$$\left\{\begin{array}{l} (3.2) \quad \textit{and} \quad a(\alpha_{i_0}) + \|b\|_\infty < a(x_0) - b(y_0) < a(\alpha_{j_0}) - \|b\|_\infty \\ \qquad\qquad \textit{or} \\ (3.3) \quad \textit{and} \quad b(\beta_{k_0}) + \|a\|_\infty < b(y_0) - a(x_0) < b(\beta_{l_0}) - \|a\|_\infty, \end{array}\right. \tag{3.7}$$

then the Herman rotation set $\mathsf{C}_{F_{ab}}$ *is a planar compact convex set in* $\mathbb{R}^3$*, which contains* $0_{\mathbb{R}^3}$ *and a non-degenerate triangle.*
iii) Any invariant probability measure μ *for the flow* X_{ab} *satisfying*

$$\mu(F_{ab}) \cdot e_1 \neq 0 \quad \textit{or} \quad \mu(F_{ab}) \cdot e_2 \neq 0, \tag{3.8}$$

is singular with respect to Lebesgue's measure.

Remark 3.2. In view of Theorem 2.3 and Theorem 3.1 the geometrical nature of the Herman rotation set $\mathsf{C}_{F_{ab}}$ is very different depending on whether $\|a\|_\infty \neq \|b\|_\infty$ or $\|a\|_\infty = \|b\|_\infty$. If $\|a\|_\infty \neq \|b\|_\infty$, $\mathsf{C}_{F_{ab}}$ is a non-degenerate planar set of $\mathbb{R}^3$. On the contrary, if $\|a\|_\infty = \|b\|_\infty$, $\mathsf{C}_{F_{ab}}$ is a close line segment. This also shows the gap between dimension two and dimension three for the Herman rotation set. Indeed, Franks and Misiurewicz [9] have proved that the rotation set for any dimensional continuous flow is always a closed line segment, while according to [4, Thm. 4.1] any convex polyedra of $\mathbb{R}^3$ can be a Herman rotation set C_F for some suitable vector field F.

Proof of Theorem 3.1. In the sequel we again use the simplified notation (2.5).
Proof of i). Let σ be a non-negative function in $L^1_\sharp(\mathbb{T}^3)$ such that $\sigma(Y)\,dY$ is an invariant probability measure for the flow X_{ab}. Applying the div-curl equality (1.21) with the invariant probability measure $\mu(dY) := \sigma(Y)\,dY$ and the vector field F_{ab} of (2.8), using condition (3.1), and noting that the vector field F_{ab} is independent of coordinate z, we get that

$$\left\{\begin{array}{ll} \displaystyle\int_{\mathbb{T}^2} \widetilde{F}_{ab}(x,y) \cdot \nabla\varphi(x,y)\, \overline{\sigma}^z(x,y)\, dxdy = 0, & \forall\, \varphi \in C^1_\sharp(\mathbb{T}^2), \\ \widetilde{F}_{ab}(x,y) := \big(b'(y), a'(x)\big), \ \ \overline{\sigma}^z(x,y) := \displaystyle\int_{\mathbb{T}^2} \sigma(x,y,z)\, dz & \text{for } (x,y) \in \mathbb{T}^2. \end{array}\right. \tag{3.9}$$

By condition (3.1) the previous equality reads as

$$\forall\,\varphi\in C^1_\sharp(\mathbb{T}^2),\quad \sum_{i=0}^{p-2}\sum_{j=0}^{q-2}\int_{(\alpha_i,\alpha_{i+1})\times(\beta_j,\beta_{j+1})}\widetilde{F}_{ab}(x,y)\cdot\nabla\varphi(x,y)\,\overline{\sigma}^z(x,y)\,dxdy=0. \tag{3.10}$$

Now, fix a pair (i,j) in $\{0,\dots,p-2\}\times\{0,\dots,q-2\}$, and define the C^1-mapping Φ_{ij} by

$$\Phi_{ij}(x,y):=\frac{1}{2}\big(a(x)-b(y),a(x)+b(y)\big)\quad\text{for }(x,y)\in R_{ij}:=(\alpha_i,\alpha_{i+1})\times(\beta_j,\beta_{j+1}), \tag{3.11}$$

the Jacobian of which satisfies

$$J_{\Phi_{ij}}(x,y)=\frac{1}{4}\begin{vmatrix} a'(x) & -b'(y)\\ a'(x) & b'(y)\end{vmatrix}=\frac{1}{2}\,a'(x)\,b'(y)\neq 0\quad\text{for any }(x,y)\in R_{ij}.$$

Moreover, since the function a is injective in (α_i,α_{i+1}) (due to $a'\neq 0$ in the interval) and the function b is injective in $(\beta_j\beta_{j+1})$ (due to $b'\neq 0$ in the interval), the mapping Φ_{ij} is clearly injective in the open rectangle R_{ij}. Hence, by virtue of the (global) inverse function theorem, Φ_{ij} is a C^1-diffeomorphism from the open rectangle R_{ij} on its open range $\Phi_{ij}(R_{ij})$.
Then, making the change of variables $(u,v)=\Phi_{ij}(x,y)$ and $\psi_{ij}(u,v)=\varphi(x,y)$ in (3.10), we get that

$$\begin{aligned}0\;&=\sum_{i=0}^{p-2}\sum_{j=0}^{q-2}\int_{R_{ij}}\big(b'(y)\,\partial_x\varphi(x,y)+a'(x)\,\partial_y\varphi(x,y)\big)\,\overline{\sigma}^z(x,y)\,dxdy\\ &=\sum_{i=0}^{p-2}\sum_{j=0}^{q-2}\int_{\Phi_{ij}(R_{ij})}a'(x)\,b'(y)\,\partial_v\psi_{ij}(u,v)\,\overline{\sigma}^z(x,y)\,\Big|J_{\Phi_{ij}}(x,y)\Big|^{-1}\,dudv\\ &=\sum_{i=0}^{p-2}\sum_{j=0}^{q-2}\int_{\Phi_{ij}(R_{ij})}2\,\partial_v\psi_{ij}(u,v)\,\overline{\sigma}^z\big(\Phi_{ij}^{-1}(u,v)\big)\,dudv.\end{aligned}$$

Therefore, due to the arbitrariness of the function ψ_{ij} chosen in $C_c^\infty\big(\Phi_{ij}(R_{ij})\big)$ for any (i,j), we deduce from the previous equality that

$$\partial_v\left(\overline{\sigma}^z\big(\Phi_{ij}^{-1}(u,v)\big)\right)=0\quad\text{in }\mathscr{D}'\big(\Phi_{ij}(R_{ij})\big),$$

which is equivalent to the existence of some negative function $\theta_{ij}\in L^1\big(\Phi_{ij}^{-1}(u,v)\big)$ such that

$$\overline{\sigma}^z\big(\Phi_{ij}^{-1}(u,v)\big)=\theta_{ij}(u)\ \text{ in }\big(\Phi_{ij}(R_{ij})\big),\quad\text{for any }(i,j)\in\{0,\dots,p-2\}\times\{0,\dots,q-2\}.$$

This implies that the mean-value function $\overline{\sigma}^z$ reads as

$$\overline{\sigma}^z(x,y)=\theta(a(x)-b(y))\quad\text{for }(x,y)\in\mathbb{T}^2, \tag{3.12}$$

for some non-negative Lebesgue measurable function θ in $\mathbb{R}$ satisfying

$$\int_{\mathbb{T}^2}\theta(a(x)-b(y))\,dxdy=1. \tag{3.13}$$

Conversely, recalling equality (3.9) we can check that any non-negative function σ satisfying (3.12), (3.13) clearly defines the Lebesgue density of an invariant probability measure for the flow X_{ab}.

As a consequence of the representation (3.12), (3.13), we have

$$\begin{aligned}\overline{\sigma F_{ab}} &= \int_{\mathbb{T}^2} \overline{\sigma}^z(x,y)\, F_{ab}(x,y)\, dxdy \\ &= \left(\int_{\mathbb{T}^2} b'(y)\, \theta(a(x)-b(y))\, dxdy\right) e_1 + \left(\int_{\mathbb{T}^2} a'(x)\, \theta(a(x)-b(y))\, dxdy\right) e_2 \\ &- \left(\int_{\mathbb{T}^2} (a(x)-b(y))\, \theta(a(x)-b(y))\, dxdy\right) e_3\end{aligned} \tag{3.14}$$

Let Θ be the function in $C^0(\mathbb{R})$ defined by

$$\Theta(t) := \int_0^t \theta(s)\, ds \quad \text{for } t \in \mathbb{R}.$$

If θ is continuous in $\mathbb{R}$, we have $\theta = \Theta'$ and by the 1-periodicity of b,

$$\int_{\mathbb{T}^2} b'(y)\, \theta(a(x)-b(y))\, dxdy = -\int_0^1 \Big[\Theta(a(x)-b(y))\Big]_{y=0}^{y=1} dx = 0.$$

By a density argument, the previous equality still holds for any Lebesgue measurable function θ satisfying (3.12). Similarly, we have

$$\int_{\mathbb{T}^2} a'(x)\, \theta(a(x)-b(y))\, dxdy = 0.$$

Hence, the mean-value (3.14) is reduced to

$$\overline{\sigma F_{ab}} = -\left(\int_{\mathbb{T}^2} (a(x)-b(y))\, \theta(a(x)-b(y))\, dxdy\right) e_3, \tag{3.15}$$

for any non-negative Lebesgue measurable function θ in $\mathbb{R}$ satisfying equality (3.13).

Finally, let us prove equality (3.4). On the one hand, from the equality (3.15) combined with the representation (3.12), (3.13), we easily deduce that

$$\mathsf{D}_{ab} = \overline{\left\{\overline{\sigma F_{ab}} : \sigma \in L^1_\sharp(\mathbb{T}^3) \text{ and } \sigma(Y)\, dY \in \mathscr{I}_{F_{ab}}\right\}} \subset \big[-\|a\|_\infty - \|b\|_\infty, \|a\|_\infty + \|b\|_\infty\big]\, e_3. \tag{3.16}$$

On the other hand, by the range condition (2.9) there exist two pairs $(\gamma,\delta) \in \mathbb{T}^2$ such that

$$a(\gamma) = \mp \|a\|_\infty \quad \text{and} \quad b(\delta) = \pm \|b\|_\infty.$$

For such a pair (γ,δ) and for any $\varepsilon > 0$, define the non-negative function θ_ε by

$$\theta_\varepsilon(t) := \frac{\mathbb{1}_{[c-\varepsilon,c+\varepsilon]}(t)}{\int_{\mathbb{T}^2} \mathbb{1}_{[c-\varepsilon,c+\varepsilon]}(a(x)-b(y))\, dxdy} \quad \text{for } t \in \mathbb{R}, \quad \text{where } c := a(\gamma) - b(\delta),$$

so that equality (3.13) is fulfilled. Hence, the associated function σ_ε by (3.12) satisfies the equality (3.15) with θ_ε, *i.e.*, recalling the definition of D_{ab} in (3.16),

$$\overline{\sigma_\varepsilon F_{ab}} = -\,\frac{\int_{\mathbb{T}^2}(a(x)-b(y))\,\mathbb{1}_{[c-\varepsilon,c+\varepsilon]}(a(x)-b(y))\,dxdy}{\int_{\mathbb{T}^2}\mathbb{1}_{[c-\varepsilon,c+\varepsilon]}(a(x)-b(y))\,dxdy}\,e_3 \in \mathsf{D}_{ab}.$$

Hence, applying the C^1-diffeomorphism Φ_{ij} (3.11) in each open rectangle R_{ij}, and using the continuity of the functions a, b and the compactness of D_{ab}, we obtain that

$$\lim_{\varepsilon\to 0}\overline{\sigma_\varepsilon F_{ab}} = -\,c\,e_3 = \big(-a(\gamma)+b(\delta)\big)\,e_3 = \big(\pm\|a\|_\infty \pm \|b\|_\infty\big)\,e_3 \in \mathsf{D}_{ab}.$$

Therefore, since the set D_{ab} is clearly convex, the inclusion (3.16) and the previous limit yield equality (3.4), which ends the proof of part *i*).

Proof of ii). Let us prove implication (3.5). The proof of (3.6) is quite similar by permuting the roles of x and y.

First of all, note that the hypothesis of (3.5) implies that there exists $\varepsilon>0$ satisfying

$$\varepsilon + a(\alpha_{i_0}) + \|b\|_\infty \le a(x_0) - b(y_0) \le a(\alpha_{j_0}) - \varepsilon - \|b\|_\infty. \tag{3.17}$$

By property (2.6) and by the hypothesis of (3.5) we have for any $t\in[0,\infty)$,

$$\varepsilon + a(\alpha_{i_0}) \le \varepsilon + a(\alpha_{i_0}) + \|b\|_\infty + b(y(t)) \le a(x(t)) \le a(\alpha_{j_0}) - \varepsilon - \|b\|_\infty + b(y(t)) \le a(\alpha_{j_0}) - \varepsilon,$$

namely

$$\forall\, t\in[0,\infty),\quad \varepsilon + a(\alpha_{i_0}) \le a(x(t)) \le a(\alpha_{j_0}) - \varepsilon. \tag{3.18}$$

This combined with (3.1) and (3.2) implies that the function $a'(x) : t \mapsto a'(x(t))$ does not vanish in $[0,\infty)$. Then, by the continuity of $a'(x)$ we get that

$$\forall\, t\in[0,\infty),\ \ a'(x(t))>0 \quad\text{or}\quad \forall\, t\in[0,\infty),\ \ a'(x(t))<0. \tag{3.19}$$

First, assume that the first inequality of (3.19) holds true. Hence, we deduce the existence of a constant $\gamma>0$ such that

$$\forall\, t\in[0,\infty),\quad y'(t) = a'(x(t)) \ge \gamma. \tag{3.20}$$

Otherwise, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in $[0,\infty)$ such that $\lim_{n\to\infty} a'(x(s_n)) = 0$. However, due to $\|a\|_\infty > \|b\|_\infty$ the first implication of (2.10) shows that the sequence $(x(s_n))_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Hence, by (3.18) there exists a subsequence, still denoted by $(s_n)_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} x(s_n) = u$ with

$$a'(u) = 0 \quad\text{and}\quad a(\alpha_{i_0}) < \varepsilon + a(\alpha_{i_0}) \le a(u) \le a(\alpha_{j_0}) - \varepsilon < a(\alpha_{j_0}),$$

which contradicts (3.2). Hence, inequality (3.20) holds and implies that

$$\liminf_{t\to\infty}\ \frac{y(t)}{t} \ge \gamma > 0.$$

This combined with the first implication of (2.10) and the formula (2.16) satisfied by $z(t)$, shows the existence of a number $\gamma_0 > 0$ and of a positive sequence $(t_n)_{n\in\mathbb{N}}$ tending to ∞ such that

$$\lim_{n\to\infty} \frac{x(t_n)}{t_n} = 0, \quad \lim_{n\to\infty} \frac{y(t_n)}{t_n} = \gamma_0, \quad \lim_{n\to\infty} \frac{z(t_n)}{t_n} = b(y_0) - a(x_0). \tag{3.21}$$

Finally, using the set equality (1.18) (see also (1.15) and Remark 1.1) we obtain that

$$\big(0, \gamma_0, b(y_0) - a(x_0)\big) \in \mathsf{C}_{F_{ab}}.$$

Similarly, when the second inequality of (3.19) holds, there exists a number $\gamma_0 < 0$ satisfying (3.21) so that

$$\big(0, \gamma_0, b(y_0) - a(x_0)\big) \in \mathsf{C}_{F_{ab}}.$$

which ends the proof of implication (3.5).

On the other hand, taking into account the range condition (2.9) consider a pair (x_0, y_0) in $\mathbb{T}^2$ satisfying the extrema conditions

$$a(x_0) \in \big\{-\|a\|_\infty, \|a\|_\infty\big\} \text{ and } b(y_0) \in \big\{-\|b\|_\infty, \|b\|_\infty\big\} \quad \text{which imply} \quad a'(x_0) = b'(y_0) = 0.$$

Then, by the uniqueness of the solution to the ODE (2.8) we have

$$\forall\, t \in [0, \infty), \quad a'(x(t)) = b'(y(t)) = 0 \text{ and } z(t) = \big(\pm\|b\|_\infty \mp \|a\|_\infty\big)\, t + z_0.$$

Therefore, again using equality (1.18) we obtain that

$$\big(0, 0, \pm\|b\|_\infty \mp \|a\|_\infty\big) \in \mathsf{C}_{F_{ab}}.$$

This combined with the conclusion of (3.5) implies the existence of a number $\gamma_0 \neq 0$ and of some pair $(x_0, y_0) \in \mathbb{T}^2$ satisfying, under condition (3.2), the inclusion

$$\mathsf{T}_{ab} := \operatorname{conv}\big(\big\{\big(0, 0, \|a\|_\infty - \|b\|_\infty\big), \big(0, 0, \|b\|_\infty - \|a\|_\infty\big), \big(0, \gamma_0, b(y_0) - a(x_0)\big)\big\}\big) \subset \mathsf{C}_{F_{ab}}, \tag{3.22}$$

where T_{ab} is a non-degenerate triangle containing $0_{\mathbb{R}^2}$ in the plane $\{x = 0\}$. This concludes the proof of part *ii*).

Proof of iii). Assume that μ is a regular invariant probability measure for the flow X_{ab} with density $\sigma \in L^1_\sharp(\mathbb{T}^3)$. Then, the vector $\mu(F_{ab})$ in $\mathsf{C}_{F_{ab}}$ agrees with the mean-value $\overline{\sigma F_{ab}}$ which belongs to the line $\mathbb{R}\, e_3$ by (3.4). Therefore, condition (3.8) cannot hold, which establishes the part *iii*) by contraposition.

This concludes the proof of Theorem 3.1. □

4 The ABC flow with $A = 0$

The ABC flow $X_{BC} = (x, y, z)$ with parameters $A = 0$ and $B, C \in \mathbb{R}$ is solution to the ODE's system

$$\left\{ \begin{array}{l} X'_{BC}(t, X_0) = \left\{ \begin{array}{l} C\cos\big(y(t, X_0)\big) \\ B\sin\big(x(t, X_0)\big) \\ C\sin\big(y(t, X_0)\big) + B\cos\big(x(t, X_0)\big) \end{array} \right\} =: F_{BC}\big(X_{BC}(t, X_0)\big), \quad t \geq 0, \\ \\ X_{BC}(0, X_0) = X_0 = (x_0, y_0, z_0) \in \mathbb{R}^3. \end{array} \right. \tag{4.1}$$

In [18, Sec. 2] the parametric expressions of $x(t, X_0)$ and $y(t, X_0)$ are explicitly given in terms of some Jacobi elliptic functions modulo 2π in the two-dimensional torus $\mathbb{T}^2 := \mathbb{R}^2 \setminus (2\pi\mathbb{Z})^2$, but not in the whole space $\mathbb{R}^2$. However, when $|B| \neq |C|$ some orbits of the flow X_{BC} are not bounded in $\mathbb{R}^2$. More generally, we will characterize the Herman rotation set $\mathsf{C}_{F_{BC}}$ which is strongly connected to the asymptotics of the flow X_{BC} by the set equality (1.18).

On the one hand, apply the the results of Section 3 depending on the cases $|B| \neq |C|$ and $|B| = |C|$. In the sequel we still use the simplified notation (2.5).
The ABC flow X_{BC} (4.1) agrees with the flow X_{ab} (2.8) taking

$$a(x) := -B\cos(x) \;\text{ and }\; b(y) := C\sin(y) \quad \text{for } x, y \in \mathbb{T}^1 := \mathbb{R}/(2\pi\mathbb{Z}), \tag{4.2}$$

which satisfy the range condition

$$a\big([0, 2\pi)\big) = \big[-|B|, |B|\big] \quad \text{and} \quad b\big([0, 2\pi)\big) = \big[-|C|, |C|\big], \tag{4.3}$$

The roots condition (3.1) also holds since

$$\left\{ \begin{array}{ll} \big\{\alpha \in [0, 2\pi) : a'(\gamma) = 0\big\} = \big\{0 = \alpha_0 < \alpha_1 = \pi\big\} & \text{with } p = 2, \\ \big\{\beta \in [0, 2\pi) : b'(\delta) = 0\big\} = \big\{\pi/2 = \beta_0 < \beta_1 = 3\pi/2\big\} & \text{with } q = 2. \end{array} \right. \tag{4.4}$$

Moreover, condition (3.2) is satisfied since the derivatives a', b' have only two roots in $[0, 2\pi)$.

Therefore, equality (3.14) reads as

$$\overline{\big\{\overline{\sigma F_{BC}} : \sigma \in L^1_\sharp(\mathbb{T}^3) \text{ and } \sigma(Y)\, dY \in \mathscr{I}_{F_{BC}}\big\}} = \big[-|B|-|C|, |B|+|C|\big]\, e_3. \tag{4.5}$$

Next, taking into account the 2π-periodicity, the boundedness properties (2.10) are written as

$$\left\{ \begin{array}{lll} |B| > |C| & \Rightarrow & \forall\, t \in [0, \infty), \;\; |x(t) - x_0| < 2\pi, \\ |B| < |C| & \Rightarrow & \forall\, t \in [0, \infty), \;\; |y(t) - y_0| < 2\pi. \end{array} \right. \tag{4.6}$$

In addition, the inclusion (3.22) yields

$$\big(0, 0, \pm\,|B| \pm |C|\big) \in \mathsf{C}_{F_{BC}}. \tag{4.7}$$

On the other hand, a more careful analysis leads us to the following result.

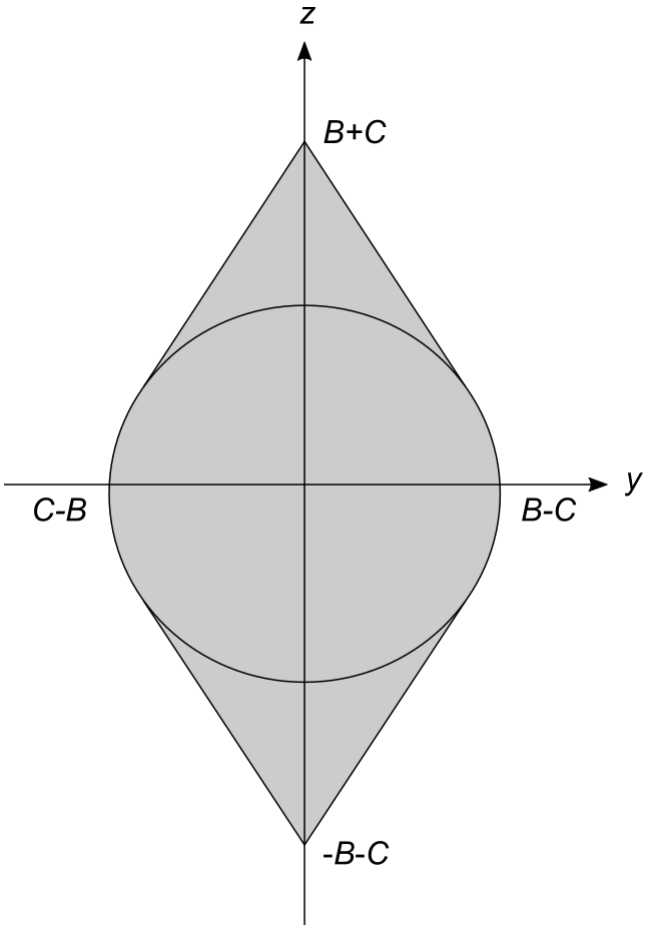

Figure 1: *The convex subset* D_{BC} *of* $\mathsf{C}_{F_{BC}}$ *for* $B > C > 0$

Proposition 4.1.
i) If $|B| \neq |C|$*, the Herman rotation set for the flow* X_{BC} *satisfies the inclusion*

$$\mathsf{D}_{BC} := \operatorname{conv}\big(D\big(0_{\mathbb{R}^2}, ||B|-|C||\big) \cup \big[-|B|-|C|, |B|+|C|\big]\, e_3\big) \subset \mathsf{C}_{F_{BC}}, \tag{4.8}$$

where D_{BC} *is the fusiform shape (see Figure 1) defined as the convex hull of the closed line segment* $\big[-|B|-|C|, |B|+|C|\big]$ *along the line* $\mathbb{R}\, e_3$*, together with the open disk* $D\big(0_{\mathbb{R}^2}, ||B|-|C||\big)$ *centered on* $0_{\mathbb{R}^2}$ *and of radius* $||B|-|C||$ *which lies either in the plane* $\{x = 0\}$ *(if* $|B| > |C|$*) or in the plane* $\{y = 0\}$ *(if* $|B| < |C|$*).*

In terms of the flow asymptotics, for any point $\mu(F_{BC})$ (1.12) *in the set* D_{BC}*, there exist a point* $U_0 \in \mathbb{T}^3$ *and a sequence* $(s_n)_{n\in\mathbb{N}}$ *tending to* ∞ *such that*

$$\Pi\big(\mu(F_{BC})\big) = \lim_{n\to\infty} \frac{\Pi\big(X_{BC}(s_n, U_0)\big)}{s_n}, \tag{4.9}$$

where $\Pi : (x, y, z) \mapsto (x, y)$ *is the orthogonal projection of* $\mathbb{R}^3$ *on the plane* $\{z = 0\}$*.*

Moreover, for any point $\mu(F_{BC})$ *in* $\mathsf{C}_{F_{BC}} \setminus (\mathbb{R}\, e_3)$*, the invariant probability measure* μ *for the flow* X_{BC} *is necessarily singular with respect to Lebesgue's measure.*

ii) If $|B| = |C|$, the Herman rotation set for the flow X_{BC} is given by

$$\mathsf{C}_{F_{BC}} = \big[-2|B|, 2|B|\,\big]\, e_3. \tag{4.10}$$

Proof of Proposition 4.1.

Proof of ii). First, let us prove that for any $(x_0, y_0) \in \mathbb{T}^2$, the implications (3.5) and (3.6) yield

$$\begin{cases} |C|-|B| < \underbrace{B\cos(x_0) + C\sin(y_0)}_{=:c_0} < \underbrace{|B|-|C|}_{>0} \;\Rightarrow\; \exists\, \gamma_0 \neq 0, \;\Big|\; \begin{array}{l}(0, \pm\gamma_0, c_0) \in \mathsf{C}_{F_{BC}} \\ (0, \gamma_0, -c_0) \in \mathsf{C}_{F_{BC}},\end{array} \\ |B|-|C| < B\cos(x_0) + C\sin(y_0) < \underbrace{|C|-|B|}_{>0} \;\Rightarrow\; \exists\, \delta_0 \neq 0, \;\Big|\; \begin{array}{l}(\pm\delta_0, 0, c_0) \in \mathsf{C}_{F_{BC}} \\ (-\delta_0, 0, -c_0) \in \mathsf{C}_{F_{BC}}.\end{array} \end{cases} \tag{4.11}$$

Compared to the general case of (3.5) and (3.6), we get the additional points of $\mathsf{C}_{F_{BC}}$ as follows: Assume for example that $|B| > |C|$. Since the vector field

$$Y_{BC} : t \mapsto \big(-x(t), \pi - y(t), z(t)\big) = X_{BC}\big(t, (-x_0, \pi - y_0, z_0)\big), \tag{4.12}$$

is solution to (4.1), and since by (3.21) there exists a sequence $(t_n)_{n\in\mathbb{N}}$ tending to ∞ such that

$$(0, \gamma_0, c_0) = \lim_{n\to\infty} \frac{X_{BC}(t_n)}{t_n},$$

we deduce from the set equality (1.18) (see also (1.15) and Remark 1.1) that

$$(0, -\gamma_0, c_0) = \lim_{n\to\infty} \frac{Y_{BC}(t_n)}{t_n} \in \mathsf{C}_{F_{BC}}. \tag{4.13}$$

Moreover, the vector field

$$Z_{BC} : t \mapsto \big(\pi - x(t), \pi + y(t), -z(t)\big) = X_{BC}\big(t, (\pi - x_0, \pi + y_0, -z_0)\big) \tag{4.14}$$

is also solution to (4.1), which implies that

$$(0, \gamma_0, -c_0) = \lim_{n\to\infty} \frac{Z_{BC}(t_n)}{t_n} \in \mathsf{C}_{F_{BC}}.$$

Similarly, assuming that $|B| < |C|$ and using the two fields (4.12), (4.14), we get the second implication of (4.11).

Then, collecting (4.11), (4.7) and using the convexity of the Herman rotation set $\mathsf{C}_{F_{BC}}$, we obtain that for any $(x_0, y_0) \in \mathbb{T}^2$ with $|c_0| < ||B|-|C||$, there exist $\gamma_0, \delta_0 \neq 0$ such that

- if $|B| > |C|$, $\mathsf{C}_{F_{BC}}$ contains the convex pentagon in the plane $\{x = 0\}$

$$\mathrm{conv}\left(\big\{(0, 0, -|B|-|C|), (0, 0, |B| + |C|), (0, \gamma_0, c_0), (0, -\gamma_0, c_0), (0, \gamma_0, -c_0)\big\}\right), \tag{4.15}$$

- if $|B| < |C|$, $\mathsf{C}_{F_{BC}}$ contains the convex pentagon in the plane $\{y = 0\}$

$$\mathrm{conv}\left(\big\{(0, 0, -|B|-|C|), (0, 0, |B| + |C|), (\delta_0, 0, c_0), (-\delta_0, 0, c_0), (-\delta_0, 0, -c_0)\big\}\right). \tag{4.16}$$

Next, let us prove the two constants γ_0, δ_0 satisfy the following estimate from below

$$|c_0| = \Big|B\cos(x_0) + C\sin(y_0)\Big| < ||B|-|C|| \quad \Rightarrow \quad |\gamma_0|, |\delta_0| \geq \sqrt{(|B|-|C|)^2 - c_0^2}. \tag{4.17}$$

Assume for example that $|B| > |C|$. Consider a pair $(x_0, y_0) \in \mathbb{T}^2$ satisfying $|c_0| < |B| - |C|$. Then, by property (2.6) we have in $[0, \infty)$

$$\begin{aligned} \big(B\sin(x) + C\cos(y)\big)^2 + c_0^2 &= \big(B\sin(x) + C\cos(y)\big)^2 + \big(B\cos(x) + C\cos(y)\big)^2 \\ &= B^2 + C^2 + 2BC\,\sin(x+y) \geq (|B|-|C|)^2, \end{aligned}$$

which implies that

$$|x' + y'| = \Big|B\sin(x) + C\cos(y)\Big| \geq \sqrt{(|B|-|C|)^2 - c_0^2} > 0 \quad \text{in } [0, \infty).$$

Then, by a continuity argument it follows that

$$x' + y' \geq \sqrt{(|B|-|C|)^2 - c_0^2} \text{ in } [0, \infty) \quad \text{or} \quad x' + y' \leq -\sqrt{(|B|-|C|)^2 - c_0^2} \text{ in } [0, \infty).$$

Hence, due to the boundedness of x (recall that $|B| > |C|$) and to the definition (3.21) of γ_0 we deduce that γ_0 satisfies (4.17).

Therefore, collecting (4.15), (4.17) and using the convexity of $\mathsf{C}_{F_{BC}}$ we get that

$$\{0\} \times \Big(\bigcup_{|c_0|<|B|-|C|} \Big[-\sqrt{(|B|-|C|)^2 - c_0^2}\,, \sqrt{(|B|-|C|)^2 - c_0^2}\Big] \times \{c_0\}\Big) \subset \mathsf{C}_{F_{BC}}. \tag{4.18}$$

Now, observe that the subset of $\mathbb{R}^2$ between the brackets in (4.18) agrees with the open disk $D(0_{\mathbb{R}^2}, |B|-|C|)$ of the plane $\{x = 0\}$. The two inclusions (4.7), (4.18) thus give (4.8).

On the other hand, the projection $\Pi(X_{BC}) = (x, y)$ of the flow X_{BC} (4.1) agrees with the two-dimensional flow solution to the ODE's system

$$\left\{ \begin{array}{l} x'(t) = C\cos\big(y(t)\big) \\ y'(t) = B\sin\big(x(t)\big) \end{array} \right\} = \Pi(F_{BC})(x(t), y(t)), \quad t \geq 0, \quad (x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^2. \tag{4.19}$$

Assume for example that $|B| > |C|$. Let $\mu \in \mathscr{I}_{F_{BC}}$ be an invariant probability measure for the flow X_{BC} such that the mass $\mu(F_{BC})$ belongs to the subset D_{BC} of C_{BC} defined by (4.8), which implies that

$$\Pi\big(\mu(F_{BC})\big) = (0, b_0) \quad \text{with} \quad |b_0| \leq |B| - |C|. \tag{4.20}$$

Then, using (3.21), (4.13) and applying the inequality (4.17) with $c_0 = 0$, there exist a number γ_0 with $|\gamma_0| \geq |B|-|C|$, a point $U_0 = (u_0, v_0, w_0) \in \mathbb{T}^3$ and a sequence $(t_n)_{n\in\mathbb{N}}$ tending to ∞ such that

$$\left\{ \begin{array}{l} (0, \gamma_0, 0) = \lim\limits_{n\to\infty} \dfrac{X_{BC}(t_n, U_0)}{t_n} \\ (0, -\gamma_0, 0) = \lim\limits_{n\to\infty} \dfrac{X_{BC}\big(t_n, (-u_0, \pi - v_0, w_0)\big)}{t_n}. \end{array} \right.$$

Applying the orthogonal projection Π in the previous equalities and using the set equality (1.18) for the flow $\Pi(X_{BC})$ associated with the vector field $\Pi(F_{BC})$ in (4.19), we get that

$$\begin{cases} (0,\gamma_0) = \lim\limits_{n\to\infty} \dfrac{\Pi(X_{BC})\big(t_n,(u_0,v_0)\big)}{t_n} \in \mathsf{C}_{\Pi(F_{BC})} \\ (0,-\gamma_0) = \lim\limits_{n\to\infty} \dfrac{\Pi(X_{BC})\big(t_n,(-u_0,\pi-v_0)\big)}{t_n} \in \mathsf{C}_{\Pi(F_{BC})}, \end{cases}$$

which by the convexity of $\mathsf{C}_{\Pi(F_{BC})}$ combined with (4.20), and thus $b_0 \in [-\gamma_0,\gamma_0]$, yields

$$\Pi\big(\mu(F_{BC})\big) = (0,b_0) \in \mathsf{C}_{\Pi(F_{BC})}.$$

Now, by virtue of the two-dimensional equality (1.19) (see also (1.15) and Remark 1.1) there exists a sequence $(s_n)_{n\in\mathbb{N}}$ tending to ∞ such that

$$\Pi\big(\mu(F_{BC})\big) = \lim_{n\to\infty} \frac{\Pi(X_{BC})\big(s_n,(u_0,v_0)\big)}{s_n},$$

which is equivalent to the desired projection equality (4.9).

Finally, in view of (4.5), for any invariant probability measure μ for the flow X_{BC} such that the mass $\mu(F_{BC})$ does not belongs to the line $\mathbb{R}\,e_3$, μ is necessarily singular with respect to Lebesgue's measure.

Proof of ii). Taking into account the 2π-periodicity, by (2.11) the flow X_{BC} satisfies the boundedness property

$$\forall\, t\in[0,\infty), \quad |x(t)-x_0| \le 2\pi \ \text{ and } \ |y(t)-y_0| \le 2\pi. \tag{4.21}$$

Moreover, by equality (2.12) the Herman rotation set for the flow X_{BC} is given by the closed line segment (4.10). □

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