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# Asymptotic properties of some ODEs flows on the 3D torus under a min-max condition

Marc Briane\*

July 8, 2024

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### Abstract

This paper deals with the asymptotic analysis of the family of the three-dimensional flows  $X = X_{fgh}$  on the torus  $\mathbb{T}^3$ , solutions to the ODEs system

$$x' = f(x, y), \quad y' = g(x, y), \quad z' = h(x, y) \quad \text{in } [0, \infty),$$

where  $f, g, h$  are regular  $\mathbb{Z}^3$ -periodic functions solutions to  $f \partial_x h + g \partial_y h = 0$ .

Under the min-max condition satisfied by the first integral  $h$  for the flow  $X_{fgh}$

$$\min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) \neq \max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} h(x, y),$$

we prove that one of the coordinates  $x$  or  $y$  of the flow is bounded in  $[0, \infty)$ .

Restricting ourselves to the subclass of flows  $X = X_{ab}$  with

$$f(x, y) := b'(y), \quad g(x, y) := b'(y), \quad h(x, y) := b(y) - a(x),$$

it turns out that both coordinates  $x$  and  $y$  are bounded when  $\|a\|_\infty = \|b\|_\infty$ , and that the Herman rotation set of the flow  $X_{ab}$  is then reduced to a closed line segment. When  $\|a\|_\infty \neq \|b\|_\infty$ , assuming some extra condition on the roots of the derivatives  $a', b'$ , we prove that the Herman rotation set of  $X_{ab}$  is planar and contains non-degenerate triangles. Finally, we refine this result in

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the case of the Arnold-Beltrami-Childress flow ABC with  $A = 0$ , by showing that the planar Herman rotation set contains a “fusiform” shape.

**Keywords:** Integrable system, min-max, asymptotics, invariant probability measure, rotation set, ABC flow

**Mathematics Subject Classification:** 34E05, 34E10, 37C10, 37C40

## 1 Introduction

In this paper we study the three-dimensional ODEs flow

$$X : [0, \infty) \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$$

solution to the ODEs system

$$\begin{cases} X'(t, x) := \partial_t X(t, x) = F(X(t, X_0)), & t \geq 0, \\ X(0, X_0) = X_0, & X_0 \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

for the three-dimensional vector fields  $F := F_{fgh}$  defined in  $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$  by

$$F_{fgh}(x, y, z) := (f(x, y), g(x, y), h(x, y)) \quad \text{for } (x, y, z) \in \mathbb{T}^3. \quad (1.2)$$

Here,  $f, g$  are  $\mathbb{Z}^2$ -periodic functions in  $C^1(\mathbb{R}^2)$  and  $h$  is a  $\mathbb{Z}^2$ -periodic function in  $C^2(\mathbb{R}^2)$  satisfying

$$(f, g) \cdot \nabla h = f \partial_x h + g \partial_y h = 0 \quad \text{in } \mathbb{T}^2, \quad (1.3)$$

so that  $h$  is a first integral for the flow  $X$  associated with  $F_{fgh}$ . A well-known example of such a flow is given by the so-called Arnold-Beltrami-Childress flow (denoted by ABC) <sup>(1)</sup>

$$\text{ABC flow: } \begin{cases} x' = A \sin z + C \cos y \\ y' = B \sin x + A \cos z \\ z' = C \sin y + B \cos x, \end{cases} \quad (1.4)$$

when one of the three parameters is equal to zero. Up to a circular permutation of the coordinates we can assume that  $A = 0$ . The case where  $ABC \neq 0$  has been the object of several works as an example of chaotic dynamical system (see, *e.g.*, [8, 18] and the references therein) along with an example of non-integrable flow (see [19, 20]). When  $A = 0$  the ABC flow is integrable and clearly satisfies equation (1.3) with

$$f(x, y) := C \cos y, \quad g(x, y) := B \sin x, \quad h(x, y) := B \cos x + C \sin y.$$

In [18] the flow ABC with  $A = 0$  has been explicitly computed thanks to Jacobi elliptic functions restricting the flow to the periodic box  $[0, 2\pi)^3$ . However, this

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<sup>1</sup>The ABC flow has been introduced by Arnold [1] for studying the steady-state solution of Euler equations.

cannot not allow us to evaluate the asymptotics of the flow  $X$  (1.1) in terms of the limits (or more rigorously the limit points)

$$\text{asymptotics of the flow: } \lim_{t \rightarrow \infty} \frac{X(t, X_0)}{t} \quad \text{for } X_0 \in \mathbb{R}^3, \quad (1.5)$$

(which can take actually non-zero values), and to characterize the possible orbits  $X([0, \infty))$  whose projections  $x([0, \infty))$  or/and  $y([0, \infty))$  are bounded in  $\mathbb{R}$  (they do exist). The derivation of the asymptotics (1.5) allows us to the Herman rotation set [11] (see Section 1.2) as shown by Misiurewicz-Ziemian [14] for general flows in the framework of ergodic theory. The ergodic approach has seen a spectacular rise for the knowledge of two-dimensional flows on the torus since the pioneer work of Franks-Misiurewicz [9] until the most recent developments (see, *e.g.*, [7, 13]). However, restricting ourselves to the field of ODEs flows there are very few results concerning the asymptotics (1.5). Up to our best knowledge, the seminal contribution in the topic is due to Peirone [15] (see, also [16, 17]) which has proved the remarkable result that for any two-dimensional regular vector-field  $F$  which does not vanish on the torus, limits (1.5) do hold for any  $X_0 \in \mathbb{R}^3$  (without extraction of any subsequence). This has allowed Peirone to revisit the Franks-Misiurewicz [9] result for the ODEs flows. Moreover, Peirone's result does not hold in dimension three [15], nor in dimension two when the vector field  $F$  does vanish [16]. This pioneer work has been extended in [3, 4, 5], and more recently we have studied in [2] the asymptotics of a two-dimensional Euler flow. In the present paper, we investigate the class of flows (1.1) associated with the vector fields  $F_{fgh}$  in terms of the asymptotics (1.5) together with the Herman rotation set.

In Section 2.1, dealing with the general class of vector fields (1.2) we prove (see Theorem 2.1) that one of coordinates of the flow  $X = (x, y, z)$ , either  $x$  or  $y$  is bounded in  $\mathbb{R}$ , if the first integral  $h$  satisfies the min-max condition

$$\min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) \neq \max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} h(x, y). \quad (1.6)$$

This implies that the Herman rotation set lies either in the plane  $\{x = 0\}$  or in  $\{y = 0\}$ . This general result seems to be original in the setting of the ODEs flows, and it is proved thanks to a connectedness argument.

In Section 2.2 we restrict ourselves to the subclass of

$$ab \text{ flows: } \begin{cases} x' = b'(y) \\ y' = a'(x) \\ z' = b(y) - a(x), \end{cases} \quad (1.7)$$

where  $a, b$  are two 1-periodic functions in  $C^2(\mathbb{R})$ . In this case, the min-max condition (1.6) reads as

$$\|a\|_\infty := \|a\|_{L^\infty(\mathbb{T}^1)} \neq \|b\|_{L^\infty(\mathbb{T}^1)} =: \|b\|_\infty. \quad (1.8)$$

Moreover, we prove in addition to (1.8) (see Theorem 2.3) that the coordinates  $x$  and  $y$  of the flow  $ab$  are both bounded in  $\mathbb{R}$ , so that the Herman rotation set agrees with the closed line segment  $[-2\|a\|_\infty, 2\|a\|_\infty]$  in the line  $\{x = y = 0\}$ .

In Section 3 we focus on the asymptotics of the  $ab$  flow under additional assumptions on the functions  $a, b$ . Assuming that the derivatives of the functions  $c = a, b$  have a finite number of roots in  $\mathbb{T}^1$ , whose two roots, say  $\gamma_1 \neq \gamma_2$ , have to satisfy

$$\{c(\gamma) : c'(\gamma) = 0\} \cap (c(\gamma_1), c(\gamma_2)) = \emptyset, \quad (1.9)$$

we prove that the Herman rotation set is a planar compact convex set in  $\mathbb{R}^3$ , which contains  $0_{\mathbb{R}^3}$  and a family of non-degenerate triangles whose vertices are parametrized by the open set

$$\{(x_0, y_0) \in \mathbb{R}^2 : -\|a\|_\infty - \|b\|_\infty < a(x_0) - b(y_0) < \|a\|_\infty - \|b\|_\infty\}.$$

In contrast, the subset of the Herman rotation set induced by the regular (with respect to Lebesgue's measure) invariant probability measures for the flow (see Section 1.2) is reduced to the closed line segment

$$[-\|a\|_\infty - \|b\|_\infty, \|a\|_\infty + \|b\|_\infty]$$

in the line  $\{x = y = 0\}$ . Therefore, any point of the Herman rotation set without the line  $\{x = y = 0\}$ , is obtained through a singular (with respect to Lebesgue's measure) invariant probability measure for the  $ab$  flow.

Finally, in Section 4 we refine the previous results to the ABC flow with  $A = 0$ , by exploiting the particular symmetries of the vector field. When  $|B| \neq |C|$ , we prove (see Proposition 4.1) that the planar Herman rotation set contains a “fusiform” shape as shown in Figure 1 on page 21.

## 1.1 Notation

- $(e_1, \dots, e_d)$  denotes the canonical basis of  $\mathbb{R}^d$ , and  $0_{\mathbb{R}^d}$  is the null vector of  $\mathbb{R}^d$ .
- “ $\cdot$ ” denotes the scalar product and  $|\cdot|$  the euclidean norm in  $\mathbb{R}^d$ .
- $\text{conv}(A)$  denotes the convex hull of the subset  $A$  of  $\mathbb{R}^d$ .
- $\mathbb{T}^d$  for  $d \in \mathbb{N}$ , denotes the  $d$ -dimensional torus  $\mathbb{R}^d/\mathbb{Z}^d$  (respectively  $\mathbb{R}^d/(2\pi\mathbb{Z})^d$  in Sec. 4), which may be identified to the unit cube  $[0, 1]^d$  (respectively  $[0, 2\pi]^d$ ) in  $\mathbb{R}^d$ , and  $0_{\mathbb{T}^d}$  denotes the null vector of  $\mathbb{T}^d$ .
- $C_c^k(\mathbb{R}^d)$  for  $k \in \mathbb{N} \cup \{\infty\}$ , denotes the space of the real-valued functions in  $C^k(\mathbb{R}^d)$  with compact support in  $\mathbb{R}^d$ .
- $C_\#^k(\mathbb{T}^d)$  for  $k \in \mathbb{N} \cup \{\infty\}$ , denotes the space of the real-valued functions  $f \in C^k(\mathbb{R}^d)$  which are  $\mathbb{Z}^d$ -periodic, *i.e.*

$$\forall K \in \mathbb{Z}^d, \forall X \in \mathbb{R}^d, \quad f(X + K) = f(X). \quad (1.10)$$

- The Jacobian matrix of a  $C^1$ -mapping  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is denoted by the matrix-valued function  $\nabla\Phi$  the entries of which are denoted by  $\partial_{x_j}\Phi_i$  for  $i, j \in \{1, \dots, d\}$ .

- The abbreviation “a.e.” for almost everywhere, will be used throughout the paper. The simple mention “a.e.” refers to the Lebesgue measure on  $\mathbb{R}^d$ .
- $dX$  denotes the Lebesgue measure on  $\mathbb{R}^d$  and  $\mathbb{T}^d$ , and  $\sigma(X) dX$  denotes the regular Lebesgue measure with density  $\sigma$ .
- $L^p_{\sharp}(\mathbb{T}^d)$  for  $p \in [1, \infty]$ , denotes the space of the Lebesgue measurable functions  $f$  in  $L^p_{\text{loc}}(\mathbb{R}^d)$ , which are  $\mathbb{Z}^d$ -periodic  $dX$ -a.e. in  $\mathbb{R}^d$ .

$$\begin{cases} \|f\|_p := \left( \int_{\mathbb{T}^d} |f(X)|^p dX \right)^{1/p} < \infty & \text{if } p < \infty, \\ \|f\|_{\infty} := \sup_{\mathbb{T}^d} |f| < \infty & \text{if } p = \infty. \end{cases}$$

- $\mathcal{M}(\mathbb{T}^d)$  denotes the the space of the Radon measures on  $\mathbb{T}^d$ , and  $\mathcal{M}_p(\mathbb{T}^d)$  denotes the space of the probability measures on  $\mathbb{T}^d$ .
- $\mathcal{D}'(\mathbb{R}^d)$  denotes the space of the distributions on  $\mathbb{R}^d$ .
- For a Borel measure  $\mu$  on  $\mathbb{T}^d$  and for  $f \in L^1_{\sharp}(\mathbb{T}^d, \mu)$ , we denote

$$\mu(f) := \int_{\mathbb{T}^d} f(X) \mu(dX), \quad (1.11)$$

which is simply denoted by  $\bar{f}$  when  $\mu$  is Lebesgue’s measure. The same notation is used for a vector-valued function in  $L^1_{\sharp}(\mathbb{T}^d, \mu)^d$ .

- $c$  denotes a positive constant which may vary from line to line.

## 1.2 A few tools of ergodic theory

Let  $F \in C^1_{\sharp}(\mathbb{T}^d)^d$ . Consider the ODEs flow  $X$  (1.1) associated with the vector field  $F$ . A probability measure  $\mu \in \mathcal{M}_p(\mathbb{T}^d)$  on  $\mathbb{T}^d$  is said to be *invariant* for the flow  $X$  defined by (1.1) if

$$\forall t \in \mathbb{R}, \forall \psi \in C^0_{\sharp}(\mathbb{T}^d), \quad \int_{\mathbb{T}^d} \psi(X(t, y)) d\mu(y) = \int_{\mathbb{T}^d} \psi(y) d\mu(y). \quad (1.12)$$

Then, we may define the set of invariant probability measures

$$\mathcal{I}_F := \{ \mu \in \mathcal{M}_p(\mathbb{T}^d) : \mu \text{ invariant for the flow } X \}. \quad (1.13)$$

Also define for any vector field  $F \in C^1_{\sharp}(\mathbb{T}^d)^d$ , the following non empty subsets of  $\mathbb{R}^d$ :

- According to [14, (1.1)] the set of all the limit points of the sequences  $(X(n, \xi_n)/n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^d$  for any sequence  $(\xi_n)_{n \in \mathbb{N}}$  in  $\mathbb{T}^d$ , is defined by

$$\rho_F := \bigcap_{n \in \mathbb{N}} \left( \overline{\bigcup_{\xi \in \mathbb{T}^d} \left\{ \frac{X(k, \xi)}{k} : k \geq n \right\}} \right). \quad (1.14)$$

By [14, Lem. 2.2, Thm. 2.3] it is a compact and connected set of  $\mathbb{R}^d$ .

**Remark 1.1.** From the semi-group property of the flow (1.1) combined with the following uniform bound

$$\forall t \geq 0, \forall \xi \in \mathbb{T}^d \quad \begin{cases} X(t, \xi) = X([t], X(t - [t], \xi)) \\ |X(t - [t], \xi) - \xi| \leq \|F\|_\infty, \end{cases} \quad (1.15)$$

we deduce that the set  $\rho_F$  (1.14) agrees with the set all the limit points of the sequences  $(X(t_n, \xi_n)/t_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^d$  for any positive sequence  $(t_n)_{n \in \mathbb{N}}$  tending to  $\infty$  and for any sequence  $(\xi_n)_{n \in \mathbb{N}}$  in  $\mathbb{T}^d$ .

- The so-called Herman [11] rotation set is defined by

$$\mathbf{C}_F := \{\mu(F) : \mu \in \mathcal{S}_F\}. \quad (1.16)$$

It is clearly a compact and convex set of  $\mathbb{R}^d$ .

An implicit consequence of [14, Thm. 2.4, Rem. 2.5, Cor. 2.6] shows that in any dimension

$$\mathbf{C}_F = \text{conv}(\rho_F). \quad (1.17)$$

The connection (1.17) between the asymptotics of the flow (1.1) and the Herman rotation set for the flow is even stronger in dimension two, since by virtue of [14, Thm. 3.4 (b)] we get that

$$d = 2 \Rightarrow \mathbf{C}_F = \rho_F. \quad (1.18)$$

We have the following characterization of an invariant probability measure, known as Liouville's theorem (see, *e.g.*, [6, Thm 1, Sec. 2.2]) which can also be regarded as a divergence-curl result with measures (see [3, Rem. 2.2] for further details).

**Proposition 1.2** (Liouville's theorem). *Let  $F \in C_{\sharp}^1(\mathbb{T}^d)^d$  and let  $\mu \in \mathcal{M}(\mathbb{T}^d)$ . Then, the two following assertions are equivalent:*

*i)  $\mu$  is invariant for the flow  $X$ , i.e. (1.12) holds true.*

*ii)  $\mu F$  is divergence free in  $\mathbb{R}^d$ , i.e.*

$$\text{div}(\mu F) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \quad (1.19)$$

*or equivalently,  $\mu F$  is divergence free in the torus  $\mathbb{T}^d$ , i.e.*

$$\forall \psi \in C_{\sharp}^1(\mathbb{T}^d), \quad \int_{\mathbb{T}^d} F(Y) \cdot \nabla \psi(Y) \mu(dY) = 0. \quad (1.20)$$

## 2 Three-dimensional flows satisfying a min-max condition

### 2.1 The class of $fgh$ flows

Let  $f, g$  be two functions in  $C_{\sharp}^1(\mathbb{T}^2)$ , and let  $h$  be a function in  $C_{\sharp}^2(\mathbb{T}^2)$  the gradient of which is solution to the equation

$$f \partial_x h + g \partial_y h = 0 \quad \text{in } \mathbb{T}^2. \quad (2.1)$$

We consider the three-dimensional flow  $X_{fgh} = (x, y, z)$  associated with the vector field  $F_{fgh}(X)$  by

$$X'_{fgh}(t, X_0) = F_{fgh}(X_{fgh}(t, X_0)) := \begin{cases} f(x(t, X_0), y(t, X_0)) \\ g(x(t, X_0), y(t, X_0)) \\ h(x(t, X_0), y(t, X_0)), \end{cases} \quad (2.2)$$

$$X_{fgh}(0, X_0) = X_0 = (x_0, y_0, z_0) \in \mathbb{R}^3.$$

Then, we have the following min-max result.

**Theorem 2.1.** *Let  $f, g \in C_{\sharp}^1(\mathbb{T}^2)$  and let  $h \in C_{\sharp}^2(\mathbb{T}^2)$  be a function satisfying (2.1). Then, the following implications hold for any  $X_0 \in \mathbb{R}^3$ ,*

$$\begin{cases} \min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) < \max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} h(x, y) \Rightarrow \forall t \geq 0, |x(t, X_0) - x_0| < 1, \\ \min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) > \max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} h(x, y) \Rightarrow \forall t \geq 0, |y(t, X_0) - y_0| < 1. \end{cases} \quad (2.3)$$

**Remark 2.2.** In the two implications (2.3) the use of min and max, rather than inf and sup, is actually justified. Indeed, due to  $h \in C_{\sharp}^1(\mathbb{T}^2)$  the functions defined by

$$h_M(x) := \max_{y \in \mathbb{T}^1} h(x, y) \quad \text{and} \quad h_m(x) := \min_{y \in \mathbb{T}^1} h(x, y) \quad \text{for } x \in \mathbb{T}^1,$$

are Lipschitz with constant  $\|\partial_x h\|_{L^\infty(\mathbb{T}^2)}$ , thus continuous in  $\mathbb{R}$ . Therefore, the functions  $h_M$  and  $h_m$  do attain their bounds on  $\mathbb{T}^1$  so that

$$\min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) = \min_{x \in \mathbb{T}^1} h_M(x) \quad \text{and} \quad \max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} h(x, y) = \max_{x \in \mathbb{T}^1} h_m(x).$$

Also note that since the extrema are taken independently with respect of the coordinates  $x, y$ , we have

$$\begin{aligned} \min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) &= \inf_{x \in \mathbb{T}^1} \sup_{y \in \mathbb{T}^1} h(x, y) = \sup_{y \in \mathbb{T}^1} \inf_{x \in \mathbb{T}^1} h(x, y) = \max_{y \in \mathbb{T}^1} \min_{x \in \mathbb{T}^1} h(x, y) \\ \max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} h(x, y) &= \sup_{x \in \mathbb{T}^1} \inf_{y \in \mathbb{T}^1} h(x, y) = \inf_{y \in \mathbb{T}^1} \sup_{x \in \mathbb{T}^1} h(x, y) = \min_{y \in \mathbb{T}^1} \max_{x \in \mathbb{T}^1} h(x, y). \end{aligned} \quad (2.4)$$



*Proof of Theorem 2.1.* Let  $X_0 \in \mathbb{R}^3$ . In the sequel we simply denote the flow  $X$  (2.2) by

$$\underbrace{X_{fgh}(t) = (x(t), y(t), z(t))}_{\text{instead of}} \quad (2.5)$$

$$\overbrace{X_{fgh}(t, X_0) = (x(t, X_0), y(t, X_0), z(t, X_0))}.$$

The function  $h \in C_{\sharp}^2(\mathbb{T}^2)$  is a Hamiltonian for the flow  $X$  (2.8), since by the chain rule and by (2.1) we have

$$\begin{aligned} \partial_t(h(X_{fgh}(t))) &= x'(t) \partial_x h(x(t), y(t)) + y'(t) \partial_y h(x(t), y(t)) \\ &= f(x(t), y(t)) \partial_x h(x(t), y(t)) + g(x(t), y(t)) \partial_y h(x(t), y(t)) = 0, \quad \forall t \geq 0. \end{aligned}$$

Hence, it follows that

$$\forall t \geq 0, \quad h(x(t), y(t)) = h(x_0, y_0). \quad (2.6)$$

Now, assume that  $\min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) < \max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} h(x, y)$ . Then, we have

$$h(x_0, y_0) > \min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) \quad \text{or} \quad h(x_0, y_0) < \max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} h(x, y). \quad (2.7)$$

First, we assume that the first inequality of (2.7) holds. There exists  $x_m \in \mathbb{T}^1$  such that

$$\min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) = h_M(x_m) \quad \text{with} \quad h_M(x) := \max_{y \in \mathbb{T}^1} h(x, y).$$

This combined with equality (2.6) and the first inequality of (2.7) yields

$$\forall t \geq 0, \quad h_M(x(t)) \geq h(x(t), y(t)) = h(x_0, y_0) > \min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) = h_M(x_m),$$

Hence, due to the 1-periodicity of  $h_M$ , we deduce that

$$x([0, \infty)) \subset \bigcup_{n \in \mathbb{Z}} (n + x_m, n + 1 + x_m).$$

Therefore, since by the intermediate value theorem  $x([0, \infty))$  is an interval of  $\mathbb{R}$ , there exists an integer  $n \in \mathbb{Z}$  such that

$$x([0, \infty)) \subset (n + x_m, n + 1 + x_m),$$

which implies that

$$\forall t \geq 0, \quad |x(t) - x_0| = |x(t) - x(0)| < 1.$$

The proof is quite similar if the second inequality of (2.7) holds by using this time the existence of  $x_M \in \mathbb{T}^1$  such that

$$\max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} h(x, y) = h_m(x_M) \quad \text{with} \quad h_m(x) := \min_{y \in \mathbb{T}^1} h(x, y),$$

which leads us to the existence of  $n \in \mathbb{Z}$  such that

$$x([0, \infty)) \subset (n + x_M, n + 1 + x_M).$$

Therefore, we have just proved the first implication of (2.3).

Finally, in view of (2.4) the inequality of the second implication of (2.3) also reads as

$$\min_{y \in \mathbb{T}^1} \max_{x \in \mathbb{T}^1} h(x, y) < \max_{y \in \mathbb{T}^1} \min_{x \in \mathbb{T}^1} h(x, y),$$

so that it can be deduced from the previous proof by a permutation of  $x$  and  $y$ .

The proof of Theorem 2.1 is thus complete.  $\square$

## 2.2 The subclass of $ab$ flows

Let  $a, b$  be two functions in  $C_{\sharp}^2(\mathbb{T}^1)$ . We consider the flow  $X_{ab} = (x, y, z)$  of type (1.1) associated with the vector field  $F_{ab}(X)$  by

$$X'_{ab}(t, X_0) = F_{ab}(X(t, X_0)) := \begin{cases} b'(y(t, X_0)) \\ a'(x(t, X_0)) \\ b(y(t, X_0)) - a(x(t, X_0)), \end{cases} \quad (2.8)$$

$$X_{ab}(0, X_0) = X_0 = (x_0, y_0, z_0) \in \mathbb{R}^3.$$

Note that the projection of the flow  $X$  on the pair  $(x, y)$  turns out to be a two-dimensional Hamiltonian flow whose Hamiltonian is the function  $(x, y) \mapsto b(y) - a(x)$ .

We have the following result satisfied by the flow  $X_{ab}$ .

**Theorem 2.3.** *Let  $a$  and  $b$  be two functions in  $C_{\sharp}^2(\mathbb{T}^1)$  satisfying the range condition*

$$a([0, 1)) = [-\|a\|_{\infty}, \|a\|_{\infty}] \quad \text{and} \quad b([0, 1)) = [-\|b\|_{\infty}, \|b\|_{\infty}]. \quad (2.9)$$

*Then, we have the following alternative:*

*i) The following implications hold for any fixed  $X_0 \in \mathbb{R}^3$ ,*

$$\begin{cases} \|a\|_{\infty} > \|b\|_{\infty} & \Rightarrow \quad \forall t \geq 0, \quad |x(t, X_0) - x_0| < 1, \\ \|a\|_{\infty} < \|b\|_{\infty} & \Rightarrow \quad \forall t \geq 0, \quad |y(t, X_0) - y_0| < 1. \end{cases} \quad (2.10)$$

*ii) If  $\|a\|_{\infty} = \|b\|_{\infty}$ , we have*

$$\forall t \geq 0, \quad |x(t, X_0) - x_0| \leq 1 \quad \text{and} \quad |y(t, X_0) - y_0| \leq 1. \quad (2.11)$$

*Moreover, the Herman rotation set (1.16) is given by*

$$\mathbf{C}_{F_{ab}} = [-2\|a\|_{\infty}, 2\|a\|_{\infty}] e_3. \quad (2.12)$$

*Proof of Theorem 2.3.*

*Proof of i).* Define the functions  $f, g, h$  in  $\mathbb{T}^2$  by

$$f(x, y) := b'(y), \quad g(x, y) := a'(x), \quad h(x, y) := b(y) - a(x), \quad (2.13)$$

so that equality (2.1) is satisfied. Moreover, by the range condition (2.9) we get that

$$\begin{cases} \min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} h(x, y) = \min_{x \in \mathbb{T}^1} \max_{y \in \mathbb{T}^1} (b(y) - a(x)) = \|b\|_\infty - \|a\|_\infty, \\ \max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} h(x, y) = \max_{x \in \mathbb{T}^1} \min_{y \in \mathbb{T}^1} (b(y) - a(x)) = \|a\|_\infty - \|b\|_\infty. \end{cases}$$

Therefore, the implications (2.3) of Theorem 2.1 imply the desired implications (2.10).

*Proof of ii).* Assume that  $\|a\|_\infty = \|b\|_\infty > 0$ , and define the function  $a_\varepsilon := (1 + \varepsilon)a$  for  $\varepsilon > 0$ .

First of all, let  $X_\varepsilon$  be the flow defined by (2.8) with the functions  $a_\varepsilon$  and  $b$ , which is associated with the vector-field

$$F_{ab}^\varepsilon(X) := F_{ab}(X) + \varepsilon (0, a'(x), -a(x)) \quad \text{for } X = (x, y, z) \in \mathbb{T}^3.$$

Since  $F_{ab} \in C_{\sharp}^1(\mathbb{T}^3)$  is a  $\kappa$ -Lipschitz function on  $\mathbb{R}^3$  for some  $\kappa > 0$ , we have for any fixed  $X_0 \in \mathbb{R}^3$ , for any fixed  $T > 0$  and for any  $t \in [0, T]$ ,

$$\begin{aligned} 0 &\leq |X_\varepsilon(t, X_0) - X(t, X_0)| = \int_0^t |F_{ab}^\varepsilon(X_\varepsilon(s, X_0)) - F_{ab}(X(s, X_0))| ds \\ &\leq \kappa \int_0^t |X_\varepsilon(s, X_0) - X(s, X_0)| ds + \varepsilon t \underbrace{(\|a'\|_\infty + \|a\|_\infty)}_{=: c} \\ &\leq \kappa \int_0^t |X_\varepsilon(s, X_0) - X(s, X_0)| ds + cT\varepsilon. \end{aligned}$$

Hence, by Gronwall's inequality (see, *e.g.*, [12, Sec. 17.3]) we deduce that

$$\forall t \in [0, T], \quad |X_\varepsilon(t, X_0) - X(t, X_0)| \leq cT\varepsilon e^{\kappa t}. \quad (2.14)$$

On the other hand, since

$$\|a_\varepsilon\|_\infty = (1 + \varepsilon)\|a\|_\infty = (1 + \varepsilon)\|b\|_\infty > \|b\|_\infty > 0,$$

by the first implication of (2.10) we get that

$$\forall t \geq 0, \quad |x_\varepsilon(t, X_0) - x_0| < 1.$$

Therefore, passing to the limit  $\varepsilon \rightarrow 0$  together with the uniform estimate (2.14) for an arbitrary  $T > 0$ , we obtain the enlarged inequality

$$\forall t \geq 0, \quad |x(t, X_0) - x_0| \leq 1,$$

*i.e.* the first inequality of (2.11). Finally, proceeding similarly with the functions  $a$  and the perturbed function  $b_\varepsilon := (1 + \varepsilon)b$ , and using the second estimate of (2.10), we obtain the two desired inequalities (2.11).

Now, it remains to characterize the Herman rotation set  $\mathbf{C}_{F_{ab}}$  when  $\|a\|_\infty = \|b\|_\infty$ . On the one hand, as an immediate consequence of (2.11) we have

$$\forall X_0 \in \mathbb{R}^3, \quad \lim_{t \rightarrow \infty} \frac{x(t, X_0)}{t} = \lim_{t \rightarrow \infty} \frac{y(t, X_0)}{t} = 0,$$

which by the set equality (1.17) (see (1.14) and Remark 1.1) implies that

$$\mathbf{C}_{F_{ab}} \subset \mathbb{R}e_3. \quad (2.15)$$

On the other hand, since the function  $h$  defined by (2.13) is a first integral of the flow  $X_{ab}$ , we get that for any  $X_0 \in \mathbb{R}^3$ ,

$$\forall t \geq 0, \quad z(t, X_0) = (b(y_0) - a(x_0))t + z_0, \quad \lim_{t \rightarrow \infty} \frac{z(t, X_0)}{t} = h(X_0), \quad (2.16)$$

which again by (1.17) implies that

$$\mathbf{C}_{F_{ab}} \cdot e_3 = \{h(X_0), X_0 \in \mathbb{R}^3\}.$$

This combined with (2.15) thus gives

$$\mathbf{C}_{F_{ab}} = \{h(X_0), X_0 \in \mathbb{R}^3\} e_3 = \{b(y_0) - a(x_0), (x_0, y_0) \in \mathbb{T}^2\} e_3.$$

Finally, using the range condition (2.9) combined with  $\|a\|_\infty = \|b\|_\infty$ , we obtain the equality

$$\{b(y_0) - a(x_0), (x_0, y_0) \in \mathbb{T}^2\} = [-2\|a\|_\infty, 2\|a\|_\infty],$$

which leads us to the expression (2.12) of the Herman rotation set.

The proof of Theorem 2.3 is now complete.  $\square$

### 3 Asymptotics of the $ab$ flows

In this section we assume that the derivatives  $a', b'$  of the flow  $X$  (2.8) have a finite number of roots in the torus  $\mathbb{T}^1$

$$\begin{aligned} \{\alpha \in [0, 1) : a'(\alpha) = 0\} &= \{0 \leq \alpha_0 < \alpha_1 < \dots < \alpha_{p-1} < 1\}, \quad p \geq 2, \\ \{\beta \in [0, 1) : b'(\beta) = 0\} &= \{0 \leq \beta_0 < \beta_1 < \dots < \beta_{q-1} < 1\}, \quad q \geq 2, \end{aligned} \quad (3.1)$$

When the 1-periodic functions  $a, b$  satisfy some extrema conditions, we can determine the regular (with respect to Lebesgue's measure) invariant probability measures and the Herman rotation set  $\mathbf{C}_{F_{ab}}$  (1.16) in a rather precise way. This is the aim of the following result.

**Theorem 3.1.** *In addition to condition (2.9) assume that either*

$$\begin{cases} \|a\|_\infty > \|b\|_\infty, \\ \exists i_0, j_0 \in \{0, \dots, m-1\}, a(\alpha_{i_0}) < a(\alpha_{j_0}), \\ \forall k \in \{0, \dots, m-1\}, a(\alpha_k) \notin (a(\alpha_{i_0}), a(\alpha_{j_0})), \end{cases} \quad (3.2)$$

or

$$\begin{cases} \|a\|_\infty < \|b\|_\infty, \\ \exists k_0, l_0 \in \{0, \dots, n-1\}, b(\beta_{k_0}) < b(\beta_{l_0}), \\ \forall j \in \{0, \dots, n-1\}, b(\beta_j) \notin (b(\beta_{k_0}), b(\beta_{l_0})). \end{cases} \quad (3.3)$$

Then, the following results hold.

i) The closed subset  $D_{ab}$  of the rotation set  $C_{F_{ab}}$  (1.16) induced by the regular (with respect to Lebesgue's measure) invariant probability measures for the flow  $X_{ab}$ , is given by

$$\begin{aligned} D_{ab} &:= \overline{\{\sigma F_{ab} : \sigma \in L^1_{\mathbb{R}^3}(\mathbb{T}^3) \text{ and } \sigma(Y) dY \in \mathcal{I}_{F_{ab}}\}} \\ &= [-\|a\|_\infty - \|b\|_\infty, \|a\|_\infty + \|b\|_\infty] e_3. \end{aligned} \quad (3.4)$$

ii) For any pair  $(x_0, y_0) \in \mathbb{T}^2$ , we have the two following implications:

$$\begin{aligned} \underbrace{(3.2) \text{ and } a(\alpha_{i_0}) + \|b\|_\infty < a(x_0) - b(y_0) < a(\alpha_{j_0}) - \|b\|_\infty}_{\downarrow} & \quad (3.5) \\ \underbrace{\exists \gamma_0 \neq 0, (0, \gamma_0, b(y_0) - a(x_0)) \in C_{F_{ab}}}_{\downarrow} & \end{aligned}$$

and

$$\begin{aligned} \underbrace{(3.3) \text{ and } b(\beta_{k_0}) + \|a\|_\infty < b(y_0) - a(x_0) < b(\beta_{l_0}) - \|a\|_\infty}_{\downarrow} & \quad (3.6) \\ \underbrace{\exists \delta_0 \neq 0, (\delta_0, 0, b(y_0) - a(x_0)) \in C_{F_{ab}}}_{\downarrow} & \end{aligned}$$

Moreover, if there exists a pair  $(x_0, y_0) \in \mathbb{T}^2$  satisfying

$$\begin{cases} (3.2) \text{ and } a(\alpha_{i_0}) + \|b\|_\infty < a(x_0) - b(y_0) < a(\alpha_{j_0}) - \|b\|_\infty \\ \text{or} \\ (3.3) \text{ and } b(\beta_{k_0}) + \|a\|_\infty < b(y_0) - a(x_0) < b(\beta_{l_0}) - \|a\|_\infty, \end{cases} \quad (3.7)$$

then the Herman rotation set  $C_{F_{ab}}$  is a planar compact convex set in  $\mathbb{R}^3$ , which contains  $0_{\mathbb{R}^3}$  and a non-degenerate triangle.

iii) Any invariant probability measure  $\mu$  for the flow  $X_{ab}$  satisfying (recall (1.11))

$$\mu(F_{ab}) \cdot e_1 \neq 0 \quad \text{or} \quad \mu(F_{ab}) \cdot e_2 \neq 0, \quad (3.8)$$

is singular with respect to Lebesgue's measure.

**Remark 3.2.** In view of Theorem 2.3 and Theorem 3.1 the geometrical nature of the Herman rotation set  $\mathbf{C}_{F_{ab}}$  is very different depending on whether  $\|a\|_\infty \neq \|b\|_\infty$  or  $\|a\|_\infty = \|b\|_\infty$ . If  $\|a\|_\infty \neq \|b\|_\infty$ ,  $\mathbf{C}_{F_{ab}}$  is a non-degenerate planar set of  $\mathbb{R}^3$ . On the contrary, if  $\|a\|_\infty = \|b\|_\infty$ ,  $\mathbf{C}_{F_{ab}}$  is a close line segment. This also shows the gap between dimension two and dimension three for the Herman rotation set. Indeed, Franks and Misiurewicz [9] have proved that the rotation set for any dimensional continuous flow is always a closed line segment, while according to [4, Thm. 4.1] any convex polyedra of  $\mathbb{R}^3$  can be a Herman rotation set  $\mathbf{C}_F$  for some suitable vector field  $F$ .

*Proof of Theorem 3.1.* In the sequel we again use notation (2.5).

*Proof of i).* Let  $\sigma$  be a non-negative function in  $L^1_{\#}(\mathbb{T}^3)$  such that  $\sigma(Y) dY$  is an invariant probability measure for the flow  $X_{ab}$ . Applying the div-curl equality (1.20) with the invariant probability measure  $\mu(dY) := \sigma(Y) dY$  and the vector field  $F_{ab}$  of (2.8), using condition (3.1), and noting that the vector field  $F_{ab}$  is independent of coordinate  $z$ , we get that

$$\left\{ \begin{array}{l} \int_{\mathbb{T}^2} \widetilde{F}_{ab}(x, y) \cdot \nabla \varphi(x, y) \bar{\sigma}^z(x, y) dx dy = 0, \quad \forall \varphi \in C^1_{\#}(\mathbb{T}^2), \\ \widetilde{F}_{ab}(x, y) := (b'(y), a'(x)), \quad \bar{\sigma}^z(x, y) := \int_{\mathbb{T}^2} \sigma(x, y, z) dz \quad \text{for } (x, y) \in \mathbb{T}^2. \end{array} \right. \quad (3.9)$$

By condition (3.1) the previous equality reads as

$$\forall \varphi \in C^1_{\#}(\mathbb{T}^2), \quad \sum_{i=0}^{p-2} \sum_{j=0}^{q-2} \int_{(\alpha_i, \alpha_{i+1}) \times (\beta_j, \beta_{j+1})} \widetilde{F}_{ab}(x, y) \cdot \nabla \varphi(x, y) \bar{\sigma}^z(x, y) dx dy = 0. \quad (3.10)$$

Now, fix a pair  $(i, j)$  in  $\{0, \dots, p-2\} \times \{0, \dots, q-2\}$ , and define the  $C^1$ -mapping  $\Phi_{ij}$  by

$$\begin{aligned} \Phi_{ij}(x, y) &:= \frac{1}{2} (a(x) - b(y), a(x) + b(y)) \\ \text{for } (x, y) &\in R_{ij} := (\alpha_i, \alpha_{i+1}) \times (\beta_j, \beta_{j+1}), \end{aligned} \quad (3.11)$$

the Jacobian of which satisfies

$$J_{\Phi_{ij}}(x, y) = \frac{1}{4} \begin{vmatrix} a'(x) & -b'(y) \\ a'(x) & b'(y) \end{vmatrix} = \frac{1}{2} a'(x) b'(y) \neq 0, \quad \forall (x, y) \in R_{ij}.$$

Moreover, since the function  $a$  is injective in  $(\alpha_i, \alpha_{i+1})$  (due to  $a' \neq 0$  in the interval) and the function  $b$  is injective in  $(\beta_j, \beta_{j+1})$  (due to  $b' \neq 0$  in the interval), the mapping  $\Phi_{ij}$  is clearly injective in the open rectangle  $R_{ij}$ . Hence, by virtue of the (global) inverse function theorem,  $\Phi_{ij}$  is a  $C^1$ -diffeomorphism from the open rectangle  $R_{ij}$  on its open range  $\Phi_{ij}(R_{ij})$ .

Then, for  $\varphi \in C^1_{\#}(\mathbb{T}^2)$  making the change of variables in (3.10):

$$(u, v) := \Phi_{ij}(x, y) \quad \text{and} \quad \psi_{ij}(u, v) := \varphi(x, y) \quad \text{for } (x, y) \in R_{ij},$$

we get that

$$\begin{aligned}
0 &= \sum_{i=0}^{p-2} \sum_{j=0}^{q-2} \int_{R_{ij}} (b'(y) \partial_x \varphi(x, y) + a'(x) \partial_y \varphi(x, y)) \bar{\sigma}^z(x, y) dx dy \\
&= \sum_{i=0}^{p-2} \sum_{j=0}^{q-2} \int_{\Phi_{ij}(R_{ij})} a'(x) b'(y) \partial_v \psi_{ij}(u, v) \bar{\sigma}^z(x, y) |J_{\Phi_{ij}}(x, y)|^{-1} dudv \quad (3.12) \\
&= \sum_{i=0}^{p-2} \sum_{j=0}^{q-2} \int_{\Phi_{ij}(R_{ij})} 2 \partial_v \psi_{ij}(u, v) \bar{\sigma}^z(\Phi_{ij}^{-1}(u, v)) dudv.
\end{aligned}$$

Therefore, by the arbitrariness of the function  $\psi_{ij}$  chosen in  $C_c^\infty(\Phi_{ij}(R_{ij}))$  for any  $(i, j)$ , we deduce from equality (3.12) that

$$\partial_v (\bar{\sigma}^z(\Phi_{ij}^{-1}(u, v))) = 0 \quad \text{in } \mathcal{D}'(\Phi_{ij}(R_{ij})),$$

which implies the existence of a non-negative function  $\theta_{ij}$  satisfying

$$\begin{cases} \bar{\sigma}^z(\Phi_{ij}^{-1}(u, v)) = \theta_{ij}(u) \quad \text{in } \Phi_{ij}(R_{ij}) \\ \frac{\theta_{ij}}{J_{\Phi_{ij}} \circ \Phi_{ij}^{-1}} \in L^1(\Phi_{ij}(R_{ij})), \end{cases} \quad \forall (i, j) \in \{0, \dots, p-2\} \times \{0, \dots, q-2\}. \quad (3.13)$$

Moreover, consider two pairs  $(i, j)$  and  $(k, l)$  such that  $\Phi_{ij}(R_{ij})$  and  $\Phi_{kl}(R_{kl})$  have a common boundary  $\Gamma_{ijkl}$  of positive  $\mathcal{H}^1$ -Hausdorff measure. Then, taking the test function  $\varphi \in C_0^\infty(\mathbb{T}^2)$  with compact support in  $\Phi_{ij}(R_{ij}) \cup \Phi_{kl}(R_{kl})$  and integrating by parts the last integrals of (3.12), we get that

$$\begin{aligned}
0 &= \int_{\partial\Phi_{ij}(R_{ij})} \psi_{ij}(u, v) \bar{\sigma}^z(\Phi_{ij}^{-1}(u, v)) \nu_{ij} \mathcal{H}^1(du, dv) \\
&\quad + \int_{\partial\Phi_{kl}(R_{kl})} \psi_{ij}(u, v) \bar{\sigma}^z(\Phi_{kl}^{-1}(u, v)) \nu_{kl} \mathcal{H}^1(du, dv) \\
&= \int_{\Gamma_{ijkl}} \varphi(\Phi_{ij}^{-1}(u, v)) (\theta_{ij}(u) - \theta_{kl}(u)) \nu_{ij} \mathcal{H}^1(du, dv),
\end{aligned}$$

where  $\nu_{ij} = -\nu_{kl}$  denotes the  $v$ -coordinate of the outside normal of  $\Gamma_{ijkl}$ . Due to the arbitrariness of  $\varphi$  we get that

$$(\theta_{ij}(u) - \theta_{kl}(u)) \nu_{ij} = 0 \quad \text{on } \Gamma_{ijkl}.$$

This combined with (3.13) implies that the mean-value function  $\bar{\sigma}^z$  reads as

$$\bar{\sigma}^z(x, y) = \theta(a(x) - b(y)) \quad \text{for } (x, y) \in \mathbb{T}^2, \quad (3.14)$$

where  $\theta$  is a non-negative Lebesgue measurable function in  $\mathbb{R}$  satisfying

$$\int_{\mathbb{T}^2} \theta(a(x) - b(y)) dx dy = 1. \quad (3.15)$$

Conversely, recalling equality (3.9) we can check that any non-negative function  $\sigma$  satisfying (3.14), (3.15) clearly defines the Lebesgue density of an invariant probability measure for the flow  $X_{ab}$ .

As a consequence of the representation (3.14), (3.15), we have

$$\begin{aligned}
\overline{\sigma F_{ab}} &= \int_{\mathbb{T}^2} \bar{\sigma}^z(x, y) F_{ab}(x, y) dx dy \\
&= \left( \int_{\mathbb{T}^2} b'(y) \theta(a(x) - b(y)) dx dy \right) e_1 \\
&\quad + \left( \int_{\mathbb{T}^2} a'(x) \theta(a(x) - b(y)) dx dy \right) e_2 \\
&\quad - \left( \int_{\mathbb{T}^2} (a(x) - b(y)) \theta(a(x) - b(y)) dx dy \right) e_3.
\end{aligned} \tag{3.16}$$

Moreover, using Fubini's theorem, Chasles' rule and making the change of variable  $z = b(y)$  in each interval  $(\beta_j, \beta_{j+1})$  by denoting  $\beta_{-1} := 0$ ,  $\beta_q := 1$ , we get that

$$\begin{aligned}
\int_{\mathbb{T}^2} b'(y) \theta(a(x) - b(y)) dx dy &= \int_0^1 dx \left( \sum_{j=-1}^{q-1} \int_{\beta_j}^{\beta_{j+1}} b'(y) \theta(a(x) - b(y)) dy \right) \\
&= \int_0^1 dx \left( \sum_{j=-1}^{q-1} \int_{b(\beta_j)}^{b(\beta_{j+1})} \theta(a(x) - z) dz \right) \\
&= \int_0^1 dx \left( \int_{b(\beta_{-1})}^{b(\beta_q)} \theta(a(x) - z) dz \right) = 0,
\end{aligned}$$

since by the 1-periodicity of  $b$  we have  $b(\beta_{-1}) = b(0) = b(1) = b(\beta_q)$ . Similarly, we obtain that

$$\int_{\mathbb{T}^2} a'(x) \theta(a(x) - b(y)) dx dy = 0.$$

Hence, the mean-value (3.16) is reduced to

$$\overline{\sigma F_{ab}} = - \left( \int_{\mathbb{T}^2} (a(x) - b(y)) \theta(a(x) - b(y)) dx dy \right) e_3, \tag{3.17}$$

for any non-negative Lebesgue measurable function  $\theta$  in  $\mathbb{R}$  satisfying equality (3.15).

Finally, let us prove equality (3.4). On the one hand, from the equalities (3.15) and (3.17) we easily deduce that

$$\begin{aligned}
D_{ab} &= \overline{\left\{ \overline{\sigma F_{ab}} : \sigma \in L^1_{\#}(\mathbb{T}^3) \text{ and } \sigma(Y) dY \in \mathcal{S}_{F_{ab}} \right\}} \\
&\subset [-\|a\|_{\infty} - \|b\|_{\infty}, \|a\|_{\infty} + \|b\|_{\infty}] e_3.
\end{aligned} \tag{3.18}$$

On the other hand, by the range condition (2.9) there exists a pair  $(\gamma, \delta)$  in  $[0, 1]^2$  such that

$$a(\gamma) = \pm \|a\|_{\infty} \quad \text{and} \quad b(\delta) = \mp \|b\|_{\infty}, \tag{3.19}$$



which thus satisfies  $a'(\gamma) = b'(\delta) = 0$ . Note that by (3.1) we have

$$(\gamma, \delta) \in \{\alpha_0, \alpha_1, \dots, \alpha_{p-1}\} \times \{\beta_0, \beta_1, \dots, \beta_{q-1}\},$$

and for any  $(x, y) \in [0, 1]^2$ ,

$$\begin{aligned} a(x) - b(y) &= \pm \|a\|_\infty \pm \|b\|_\infty \\ \Rightarrow a(x) &= \pm \|a\|_\infty \text{ and } b(y) = \mp \|b\|_\infty \\ \Rightarrow (x, y) &\in \{\alpha_0, \alpha_1, \dots, \alpha_{p-1}\} \times \{\beta_0, \beta_1, \dots, \beta_{q-1}\}. \end{aligned} \quad (3.20)$$

For such a pair  $(\gamma, \delta)$  and for any  $\varepsilon > 0$ , the non-negative function  $\theta_\varepsilon$  defined by

$$\theta_\varepsilon := \frac{\mathbb{1}_{(c-\varepsilon, c+\varepsilon)}}{\int_{\mathbb{T}^2} \mathbb{1}_{(c-\varepsilon, c+\varepsilon)}(a(x) - b(y)) \, dx dy}, \quad \text{where } c := a(\gamma) - b(\delta), \quad (3.21)$$

satisfies equality (3.15). Then, any invariant probability measure  $\sigma_\varepsilon(Y) \, dY$  for the flow  $X_{ab}$ , with  $\sigma_\varepsilon \in L^1_\#(\mathbb{T}^3)$ , satisfies the equalities (3.14), (3.17) with the function  $\theta_\varepsilon$ , which (recalling (3.18)) yields

$$\overline{\sigma_\varepsilon F_{ab}} = - \left( \int_{\mathbb{T}^2} (a(x) - b(y)) \theta_\varepsilon(a(x) - b(y)) \, dx dy \right) e_3 \in \mathbf{D}_{ab}. \quad (3.22)$$

Moreover, by (3.19) and the implications (3.20), the set  $\{(x, y) : a(x) - b(y) = c\}$  is a finite subset of  $[0, 1]^2$  containing the pair  $(\gamma, \delta)$ , which thus reads as

$$\begin{aligned} &\{(x, y) \in [0, 1]^2 : a(x) - b(y) = c\} \\ &= \{(x_k, y_k) \in [0, 1]^2 : a(x_k) - b(y_k) = c, k = 1, \dots, N\}. \end{aligned} \quad (3.23)$$

The following result whose proof is given below, shows the concentration effect of the sequence  $\theta_\varepsilon(a(x) - b(y))$  on the points  $(x_k, y_k)$  as  $\varepsilon \rightarrow 0$ .

**Lemma 3.3.** *There exist a positive sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  which converges to 0 and  $N$  non-negative numbers  $\omega_1, \dots, \omega_N$  whose sum is equal to 1, such that the following convergence holds in the weak\* sense of measures*

$$\theta_{\varepsilon_n}(a(x) - b(y)) \xrightarrow[n \rightarrow \infty]{} \sum_{k=1}^N \omega_k \delta_{(x_k, y_k)} \quad \text{weakly* in } \mathcal{M}(\mathbb{T}^2), \quad (3.24)$$

where  $\delta_\xi$  denotes the Dirac mass at the point  $\xi \in \mathbb{R}^2$ .

Hence, by the continuity of the functions  $a, b$  combined with the compactness of  $\mathbf{D}_{ab}$ , we deduce from (3.22), (3.23), (3.24) and (3.19) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \overline{\sigma_{\varepsilon_n} F_{ab}} &= - \sum_{k=1}^N \omega_k \int_{\mathbb{T}^2} (a(x) - b(y)) \delta_{(x_k, y_k)}(dx, dy) = -c e_3 \\ &= (-a(\gamma) + b(\delta)) e_3 = (\mp \|a\|_\infty \mp \|b\|_\infty) e_3 \in \mathbf{D}_{ab}. \end{aligned}$$

Therefore, since the set  $D_{ab}$  is clearly convex, the inclusion (3.18) and the previous limit imply the desired equality (3.4), which ends the proof of part *i*).

*Proof of ii*). Let us prove implication (3.5). The proof of (3.6) is quite similar by permuting the roles of  $x$  and  $y$ .

First of all, note that the hypothesis of (3.5) implies that there exists  $\varepsilon > 0$  satisfying

$$\varepsilon + a(\alpha_{i_0}) + \|b\|_\infty \leq a(x_0) - b(y_0) \leq a(\alpha_{j_0}) - \varepsilon - \|b\|_\infty. \quad (3.25)$$

By the equality (2.6) together with (3.25) we get that for any  $t \geq 0$ ,

$$\begin{cases} a(x(t)) \geq \varepsilon + a(\alpha_{i_0}) + \|b\|_\infty + b(y(t)) \geq \varepsilon + a(\alpha_{i_0}) \\ a(x(t)) \leq a(\alpha_{j_0}) - \varepsilon - \|b\|_\infty + b(y(t)) \leq a(\alpha_{j_0}) - \varepsilon, \end{cases}$$

namely

$$\forall t \geq 0, \quad \varepsilon + a(\alpha_{i_0}) \leq a(x(t)) \leq a(\alpha_{j_0}) - \varepsilon. \quad (3.26)$$

This combined with (3.1) and (3.2) implies that  $a'(x) : t \mapsto a'(x(t))$  does not vanish in  $[0, \infty)$ . Then, by the continuity of  $a'(x)$  we get that

$$\forall t \geq 0, \quad a'(x(t)) > 0 \quad \text{or} \quad \forall t \geq 0, \quad a'(x(t)) < 0. \quad (3.27)$$

First, assume that the first inequality of (3.27) holds true. Hence, we deduce the existence of a constant  $\gamma > 0$  such that

$$\forall t \geq 0, \quad y'(t) = a'(x(t)) \geq \gamma. \quad (3.28)$$

Otherwise, there exists a sequence  $(s_n)_{n \in \mathbb{N}}$  in  $[0, \infty)$  such that

$$\lim_{n \rightarrow \infty} a'(x(s_n)) = 0.$$

However, due to  $\|a\|_\infty > \|b\|_\infty$  the first implication of (2.10) shows that the sequence  $(x(s_n))_{n \in \mathbb{N}}$  is bounded in  $\mathbb{R}$ . Hence, by (3.26) there exists a subsequence, still denoted by  $(s_n)_{n \in \mathbb{N}}$ , such that  $\lim_{n \rightarrow \infty} x(s_n) = u$  with

$$a'(u) = 0 \quad \text{and} \quad a(\alpha_{i_0}) < \varepsilon + a(\alpha_{i_0}) \leq a(u) \leq a(\alpha_{j_0}) - \varepsilon < a(\alpha_{j_0}),$$

which contradicts (3.2). Hence, inequality (3.28) holds and implies that

$$\liminf_{t \rightarrow \infty} \frac{y(t)}{t} \geq \gamma > 0.$$

This combined with the first implication of (2.10) and the formula (2.16) satisfied by  $z(t)$ , shows the existence of a number  $\gamma_0 > 0$  and of a positive sequence  $(t_n)_{n \in \mathbb{N}}$  tending to  $\infty$  such that

$$\lim_{n \rightarrow \infty} \frac{x(t_n)}{t_n} = 0, \quad \lim_{n \rightarrow \infty} \frac{y(t_n)}{t_n} = \gamma_0, \quad \lim_{n \rightarrow \infty} \frac{z(t_n)}{t_n} = b(y_0) - a(x_0). \quad (3.29)$$

Finally, using the set equality (1.17) (see (1.14) and Remark 1.1) we obtain that

$$(0, \gamma_0, b(y_0) - a(x_0)) \in \mathbf{C}_{F_{ab}}.$$

Similarly, when the second inequality of (3.27) holds, there exists a number  $\gamma_0 < 0$  satisfying (3.29) so that

$$(0, \gamma_0, b(y_0) - a(x_0)) \in \mathbf{C}_{F_{ab}}.$$

which ends the proof of implication (3.5).

On the other hand, taking into account the range condition (2.9) consider a pair  $(x_1, y_1)$  in  $\mathbb{T}^2$  satisfying the extrema conditions

$$a(x_1) \in \{-\|a\|_\infty, \|a\|_\infty\} \quad \text{and} \quad b(y_1) \in \{-\|b\|_\infty, \|b\|_\infty\}$$

which imply that  $a'(x_1) = b'(y_1) = 0$ . Then, by the uniqueness of the solution to the ODE (2.8) we get the following four solutions

$$\forall t \geq 0, \quad a'(x(t)) = b'(y(t)) = 0 \quad \text{and} \quad z(t) = (\pm\|a\|_\infty \pm \|b\|_\infty)t + z_0.$$

Therefore, again using equality (1.17) we obtain that

$$(0, 0, \pm\|a\|_\infty \pm \|b\|_\infty) \in \mathbf{C}_{F_{ab}}. \tag{3.30}$$

This combined with the conclusion of (3.5) implies the existence of a number  $\gamma_0 \neq 0$  and of some pair  $(x_0, y_0) \in \mathbb{T}^2$  satisfying under condition (3.2), the inclusion

$$\mathbf{T}_{ab} \subset \mathbf{C}_{F_{ab}}, \tag{3.31}$$

where  $\mathbf{T}_{ab}$  is the non-degenerate triangle with vertices

$$(0, 0, -\|a\|_\infty - \|b\|_\infty), \quad (0, 0, \|a\|_\infty + \|b\|_\infty), \quad (0, \gamma_0, b(y_0) - a(x_0)),$$

containing  $0_{\mathbb{R}^2}$  in the plane  $\{x = 0\}$ . This concludes the proof of part *ii*).

*Proof of iii*). Assume that  $\mu$  is a regular invariant probability measure for the flow  $X_{ab}$  with density  $\sigma \in L^1_{\mathbb{R}^+}(\mathbb{T}^3)$ . Then, the vector  $\mu(F_{ab})$  in  $\mathbf{C}_{F_{ab}}$  agrees with the mean-value  $\overline{\sigma F_{ab}}$  which belongs to the line  $\mathbb{R}e_3$  by (3.4). Therefore, condition (3.8) cannot hold, which establishes the part *iii*) by contraposition.

This concludes the proof of Theorem 3.1. □

*Proof of Lemma 3.3*. First, let us prove by contradiction that (recall (3.21), (3.23))

$$\begin{aligned} & |a(x) - b(y) - c| < \varepsilon \\ \forall \delta > 0, \exists \varepsilon \in (0, \delta), \forall (x, y) \in \mathbb{T}^2, & \quad \downarrow \\ & \exists k = 1, \dots, N, \|(x, y) - (x_k, y_k)\|_{\mathbb{T}^2} < \delta, \end{aligned} \tag{3.32}$$

where  $\|\xi\|_{\mathbb{T}^2} := \min_{(k_1, k_2) \in \mathbb{Z}^2} (|\xi_1 - k_1| + |\xi_2 - k_2|)$  for  $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ .

If (3.32) does not hold, there exist  $\delta > 0$  and a sequence  $((u_n, v_n))_{n \in \mathbb{N}}$  in  $\mathbb{T}^2$  such that

$$\forall n \in \mathbb{N}, \quad |a(u_n) - b(v_n) - c| < \frac{1}{n} \quad \text{and} \quad \forall k = 1, \dots, N, \quad \|(u_n, v_n) - (x_k, y_k)\|_{\mathbb{T}^2} \geq \delta.$$

Up to extract a subsequence, the sequence  $(u_n, v_n)$  converges to some  $(u, v)$  in  $\mathbb{T}^2$  which satisfies

$$a(u) - b(v) = c \quad \text{and} \quad \forall k = 1, \dots, N, \quad \|(u, v) - (x_k, y_k)\|_{\mathbb{T}^2} \geq \delta,$$

and leads us to a contradiction with (3.23).

Now, take in (3.32)  $\delta := 1/n$  for  $n \in \mathbb{N}$ , and choose a suitable  $\varepsilon_n \in (0, 1/n)$ . Hence, it follows from the definitions (3.21), (3.23) that for any  $n \in \mathbb{N}$  and for any pair  $(x, y) \in [0, 1]^2$ ,

$$\theta_{\varepsilon_n}(a(x) - b(y)) \neq 0 \Rightarrow (x, y) \in \bigcup_{k=1}^N \underbrace{\left(x_k - \frac{1}{n}, x_k + \frac{1}{n}\right) \times \left(y_k - \frac{1}{n}, y_k + \frac{1}{n}\right)}_{=: R_n^k},$$

where the rectangles  $R_n^k$  are mutually disjoint for any large enough  $n$ .

Let  $\varphi \in C_{\sharp}^0(\mathbb{T}^2)$ . Then, using the uniform continuity of  $\varphi$  in  $\mathbb{T}^2$  combined with the boundedness (3.15) of  $\theta_{\varepsilon_n}(a(x) - b(y))$  in  $L_{\sharp}^1(\mathbb{T}^2)$ , we have

$$\begin{aligned} \int_{\mathbb{T}^2} \theta_{\varepsilon_n}(a(x) - b(y)) \varphi(x, y) \, dx dy &= \int_{[0,1]^2} \theta_{\varepsilon_n}(a(x) - b(y)) \varphi(x, y) \, dx dy \\ &= \sum_{k=1}^N \int_{R_n^k} \theta_{\varepsilon_n}(a(x) - b(y)) \varphi(x, y) \, dx dy \\ &= \sum_{k=1}^N \left( \int_{R_n^k} \theta_{\varepsilon_n}(a(x) - b(y)) \, dx dy \right) \varphi(x_k, y_k) + o(1). \end{aligned}$$

Therefore, up to extract a subsequence of  $n$ , we deduce that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}^2} \theta_{\varepsilon_n}(a(x) - b(y)) \varphi(x, y) \, dx dy = \sum_{k=1}^N \omega_k \varphi(x_k, y_k)$$

$$\text{where } \omega_k := \lim_{n \rightarrow \infty} \int_{R_n^k} \theta_{\varepsilon_n}(a(x) - b(y)) \in [0, 1] \quad \text{for } k = 1, \dots, N.$$

By (3.15) we also have

$$\sum_{k=1}^N \omega_k = \lim_{n \rightarrow \infty} \sum_{k=1}^N \int_{R_n^k} \theta_{\varepsilon_n}(a(x) - b(y)) \, dx dy = \lim_{n \rightarrow \infty} \int_{\mathbb{T}^2} \theta_{\varepsilon_n}(a(x) - b(y)) \, dx dy = 1.$$

This establishes the weak \*-convergence (3.24), and concludes the proof.  $\square$

## 4 The ABC flow with $A = 0$

The ABC flow  $X_{BC} = (x, y, z)$  with parameters  $A = 0$  and  $B, C \in \mathbb{R}$  is solution to the ODEs system

$$X'_{BC}(t, X_0) = F_{BC}(X_{BC}(t, X_0)) := \begin{cases} C \cos(y(t, X_0)) \\ B \sin(x(t, X_0)) \\ B \cos(x(t, X_0)) + C \sin(y(t, X_0)), \end{cases} \quad (4.1)$$

$$X_{BC}(0, X_0) = X_0 = (x_0, y_0, z_0) \in \mathbb{R}^3.$$

In [18, Sec. 2] the parametric expressions of  $x(t, X_0)$  and  $y(t, X_0)$  are explicitly given in terms of some Jacobi elliptic functions modulo  $2\pi$  in the two-dimensional torus  $\mathbb{T}^2 := \mathbb{R}^2 \setminus (2\pi\mathbb{Z})^2$ , but not in the whole space  $\mathbb{R}^2$ . However, when  $|B| \neq |C|$  some orbits of the flow  $X_{BC}$  are not bounded in  $\mathbb{R}^2$ . More generally, we will characterize the Herman rotation set  $\mathcal{C}_{F_{BC}}$  which is strongly connected to the asymptotics of the flow  $X_{BC}$  by the set equality (1.17).

On the one hand, apply the the results of Section 3 depending on the cases  $|B| \neq |C|$  and  $|B| = |C|$ . In the sequel we still use the simplified notation (2.5). The ABC flow  $X_{BC}$  (4.1) agrees with the flow  $X_{ab}$  (2.8) taking

$$a(x) := -B \cos(x), \quad b(y) := C \sin(y), \quad \text{for } x, y \in \mathbb{T}^1 := \mathbb{R}/(2\pi\mathbb{Z}), \quad (4.2)$$

which satisfy the range condition

$$a([0, 2\pi)) = [-|B|, |B|] \quad \text{and} \quad b([0, 2\pi)) = [-|C|, |C|]. \quad (4.3)$$

Here, equality (2.6) reads as

$$B \cos(x) + C \sin(y) = B \cos(x_0) + C \sin(y_0) \quad \text{in } [0, \infty). \quad (4.4)$$

The roots condition (3.1) also holds since

$$\begin{cases} \{\alpha \in [0, 2\pi) : a'(\alpha) = 0\} = \{0 = \alpha_0 < \alpha_1 = \pi\} & p = 2, \\ \{\beta \in [0, 2\pi) : b'(\beta) = 0\} = \{\pi/2 = \beta_0 < \beta_1 = 3\pi/2\} & q = 2. \end{cases} \quad (4.5)$$

Moreover, conditions (3.2) and (3.3) are satisfied since the derivatives  $a'$ ,  $b'$  have only two roots in the interval  $[0, 2\pi)$ .

Therefore, equality (3.4) reads as

$$\begin{aligned} & \overline{\{\sigma F_{BC} : \sigma \in L^1_{\sharp}(\mathbb{T}^3) \text{ and } \sigma(Y) dY \in \mathcal{I}_{F_{BC}}\}} \\ & = [-|B| - |C|, |B| + |C|] e_3. \end{aligned} \quad (4.6)$$

Next, taking into account the  $2\pi$ -periodicity the boundedness properties (2.10) are written as

$$\begin{cases} |B| > |C| \Rightarrow \forall t \geq 0, \quad |x(t) - x_0| < 2\pi, \\ |B| < |C| \Rightarrow \forall t \geq 0, \quad |y(t) - y_0| < 2\pi. \end{cases} \quad (4.7)$$

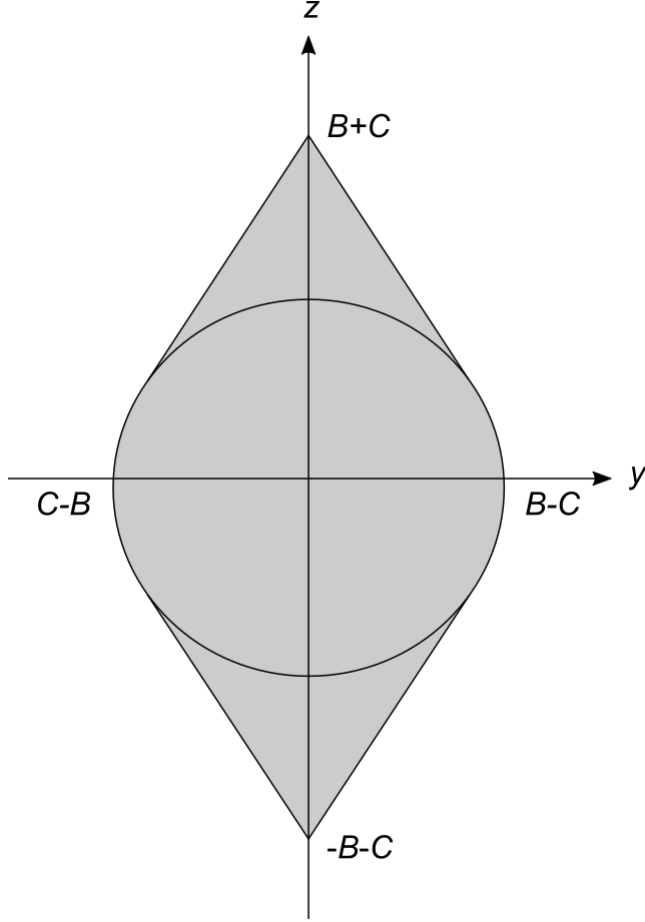


Figure 1: The convex subset  $D_{BC}$  of  $C_{F_{BC}}$  for  $B > C > 0$

In addition, the inclusion (3.30) yields

$$(0, 0, \pm|B| \pm |C|) \in C_{F_{BC}}. \quad (4.8)$$

On the other hand, a more careful analysis leads us to the following result.

**Proposition 4.1.**

i) If  $|B| \neq |C|$ , the Herman rotation set for the flow  $X_{BC}$  satisfies the inclusion

$$\begin{aligned} D_{BC} &:= \text{conv} \left( D(0_{\mathbb{R}^2}, ||B|-|C||) \cup [-|B|-|C|, |B|+|C|] e_3 \right) \\ &\subset C_{F_{BC}}, \end{aligned} \quad (4.9)$$

where  $D_{BC}$  is the fusiform shape (see Figure 1) defined as the convex hull of the closed line segment  $[-|B|-|C|, |B|+|C|]$  along the line  $\mathbb{R}e_3$ , together with the open disk  $D(0_{\mathbb{R}^2}, ||B|-|C||)$  centered on  $0_{\mathbb{R}^2}$  and of radius  $||B|-|C||$  which lies either in the plane  $\{x = 0\}$  (if  $|B| > |C|$ ) or in the plane  $\{y = 0\}$  (if  $|B| < |C|$ ).

In terms of the flow asymptotics, for any point  $\mu(F_{BC})$  (1.11) in the set  $D_{BC}$ , there exist a point  $U_0 \in \mathbb{T}^3$  and a sequence  $(s_n)_{n \in \mathbb{N}}$  tending to  $\infty$  such that

$$\Pi(\mu(F_{BC})) = \lim_{n \rightarrow \infty} \frac{\Pi(X_{BC}(s_n, U_0))}{s_n}, \quad (4.10)$$

where  $\Pi$  is the orthogonal projection of  $\mathbb{R}^3$  on the plane  $\{z = 0\}$ .

Moreover, for any point  $\mu(F_{BC})$  in  $\mathbf{C}_{F_{BC}} \setminus (\mathbb{R}e_3)$ , the invariant probability measure  $\mu$  for the flow  $X_{BC}$  is necessarily singular with respect to Lebesgue's measure.

ii) If  $|B| = |C|$ , the Herman rotation set for the flow  $X_{BC}$  is given by

$$\mathbf{C}_{F_{BC}} = [-2|B|, 2|B|] e_3. \quad (4.11)$$

*Proof of Proposition 4.1.*

*Proof of ii).* First, let us prove that for any  $(x_0, y_0) \in \mathbb{T}^2$ , the implications (3.5) and (3.6) yield

$$\left\{ \begin{array}{c} |C| - |B| < c_0 := B \cos(x_0) + C \sin(y_0) < |B| - |C| \\ \Downarrow \\ \exists \gamma_0 \neq 0, \left| \begin{array}{l} (0, \pm\gamma_0, c_0) \in \mathbf{C}_{F_{BC}} \\ (0, \gamma_0, -c_0) \in \mathbf{C}_{F_{BC}} \end{array} \right. \end{array} \right\} \quad (4.12)$$

$$\left\{ \begin{array}{c} |B| - |C| < B \cos(x_0) + C \sin(y_0) < |C| - |B| \\ \Downarrow \\ \exists \delta_0 \neq 0, \left| \begin{array}{l} (\pm\delta_0, 0, c_0) \in \mathbf{C}_{F_{BC}} \\ (-\delta_0, 0, -c_0) \in \mathbf{C}_{F_{BC}} \end{array} \right. \end{array} \right\}$$

Compared to the general case of (3.5) and (3.6), we get the additional points of  $\mathbf{C}_{F_{BC}}$  as follows:

Assume for example that  $|B| > |C|$ . Since the vector field

$$Y_{BC}(t) := (-x(t), \pi - y(t), z(t)) = X_{BC}(t, (-x_0, \pi - y_0, z_0)), \quad (4.13)$$

is solution to (4.1), and since by (3.29) there exists a sequence  $(t_n)_{n \in \mathbb{N}}$  tending to  $\infty$  such that

$$(0, \gamma_0, c_0) = \lim_{n \rightarrow \infty} \frac{X_{BC}(t_n)}{t_n},$$

we deduce from the set equality (1.17) (see (1.14) and Remark 1.1) that

$$(0, -\gamma_0, c_0) = \lim_{n \rightarrow \infty} \frac{Y_{BC}(t_n)}{t_n} \in \mathbf{C}_{F_{BC}}. \quad (4.14)$$

Moreover, the vector field

$$Z_{BC}(t) := (\pi - x(t), \pi + y(t), -z(t)) = X_{BC}(t, (\pi - x_0, \pi + y_0, -z_0)) \quad (4.15)$$

is also solution to (4.1), which implies that

$$(0, \gamma_0, -c_0) = \lim_{n \rightarrow \infty} \frac{Z_{BC}(t_n)}{t_n} \in \mathbf{C}_{F_{BC}}.$$

Similarly, assuming that  $|B| < |C|$  and using the two fields (4.13), (4.15), we get the second implication of (4.12).

Then, collecting (4.12), (4.8) and using the convexity of the Herman rotation set  $\mathbf{C}_{F_{BC}}$ , we get that for any  $(x_0, y_0) \in \mathbb{T}^2$  with  $|c_0| < ||B| - |C||$ , there exist  $\gamma_0, \delta_0 \neq 0$  such that:

- if  $|B| > |C|$ ,  $\mathbf{C}_{F_{BC}}$  contains the convex pentagon with vertices

$$(0, 0, -|B| - |C|), (0, 0, |B| + |C|), (0, \gamma_0, c_0), (0, -\gamma_0, c_0), (0, \gamma_0, -c_0) \quad (4.16)$$

in the plane  $\{x = 0\}$ ,

- if  $|B| < |C|$ ,  $\mathbf{C}_{F_{BC}}$  contains the convex pentagon with vertices

$$(0, 0, -|B| - |C|), (0, 0, |B| + |C|), (\delta_0, 0, c_0), (-\delta_0, 0, c_0), (-\delta_0, 0, -c_0), \quad (4.17)$$

in the plane  $\{y = 0\}$ .

Next, let us prove the two constants  $\gamma_0, \delta_0$  satisfy the following estimate from below

$$\begin{aligned} |c_0| &= |B \cos(x_0) + C \sin(y_0)| < ||B| - |C|| \\ &\Downarrow \\ |\gamma_0|, |\delta_0| &\geq \sqrt{(|B| - |C|)^2 - c_0^2}. \end{aligned} \quad (4.18)$$

Assume for example that  $|B| > |C|$ . Consider  $(x_0, y_0) \in \mathbb{T}^2$  satisfying  $|c_0| < |B| - |C|$ . Then, by equality (4.4) we have in  $[0, \infty)$

$$\begin{aligned} &(B \sin(x) + C \cos(y))^2 + c_0^2 \\ &= (B \sin(x) + C \cos(y))^2 + (B \cos(x) + C \cos(y))^2 \\ &= B^2 + C^2 + 2BC \sin(x + y) \geq (|B| - |C|)^2, \end{aligned}$$

which implies that

$$|x' + y'| = |B \sin(x) + C \cos(y)| \geq \sqrt{(|B| - |C|)^2 - c_0^2} > 0 \quad \text{in } [0, \infty).$$

Then, by a continuity argument it follows that

$$\begin{aligned} x' + y' &\geq \sqrt{(|B| - |C|)^2 - c_0^2} \quad \text{in } [0, \infty) \\ \text{or } x' + y' &\leq -\sqrt{(|B| - |C|)^2 - c_0^2} \quad \text{in } [0, \infty). \end{aligned}$$

Hence, due to the boundedness of  $x$  (recall that  $|B| > |C|$ ) and to the definition (3.29) of  $\gamma_0$  we deduce that  $\gamma_0$  satisfies (4.18).

Therefore, collecting (4.16), (4.18) and using the convexity of  $\mathbf{C}_{F_{BC}}$  we get that

$$\{0\} \times \left[ \bigcup_{|c_0| < |B| - |C|} \left[ -\sqrt{(|B| - |C|)^2 - c_0^2}, \sqrt{(|B| - |C|)^2 - c_0^2} \right] \times \{c_0\} \right] \subset \mathbf{C}_{F_{BC}}. \quad (4.19)$$

Now, observe that the set of the left-hand side of (4.19) agrees with the open disk  $D(0_{\mathbb{R}^2}, |B| - |C|)$  in the plane  $\{x = 0\}$ . The two inclusions (4.8) and (4.19) thus give the desired inclusion (4.9).



On the other hand, the projection  $\Pi(X_{BC}) = (x, y)$  of the flow  $X_{BC}$  (4.1) agrees with the two-dimensional flow solution to the ODEs system

$$\left. \begin{aligned} x'(t) &= C \cos(y(t)) \\ y'(t) &= B \sin(x(t)) \end{aligned} \right\} = \Pi(F_{BC}(x(t), y(t))), \quad t \geq 0, \quad (4.20)$$

$$(x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^2.$$

Assume for example that  $|B| > |C|$ . Let  $\mu \in \mathcal{I}_{F_{BC}}$  be an invariant probability measure for the flow  $X_{BC}$  such that the mass  $\mu(F_{BC})$  belongs to the subset  $\mathbf{D}_{BC}$  of  $\mathbf{C}_{BC}$  defined by (4.9). We have

$$\Pi(\mu(F_{BC})) = (0, b_0) \quad \text{with} \quad |b_0| \leq |B| - |C|. \quad (4.21)$$

Then, using (3.29), (4.14) and applying the inequality (4.18) with  $c_0 = 0$ , there exist  $\gamma_0 \in \mathbb{R}$  with  $|\gamma_0| \geq |B| - |C|$ , a point  $U_0 = (u_0, v_0, w_0) \in \mathbb{T}^3$  and a sequence  $(t_n)_{n \in \mathbb{N}}$  tending to  $\infty$  such that

$$\left\{ \begin{aligned} (0, \gamma_0, 0) &= \lim_{n \rightarrow \infty} \frac{X_{BC}(t_n, U_0)}{t_n} \\ (0, -\gamma_0, 0) &= \lim_{n \rightarrow \infty} \frac{X_{BC}(t_n, (-u_0, \pi - v_0, w_0))}{t_n}. \end{aligned} \right.$$

Applying the orthogonal projection  $\Pi$  in the previous equalities and using the set equality (1.17) for the flow  $\Pi(X_{BC})$  associated with the vector field  $\Pi(F_{BC})$  in (4.20), we get that

$$\left\{ \begin{aligned} (0, \gamma_0) &= \lim_{n \rightarrow \infty} \frac{\Pi(X_{BC})(t_n, (u_0, v_0))}{t_n} \in \mathbf{C}_{\Pi(F_{BC})} \\ (0, -\gamma_0) &= \lim_{n \rightarrow \infty} \frac{\Pi(X_{BC})(t_n, (-u_0, \pi - v_0))}{t_n} \in \mathbf{C}_{\Pi(F_{BC})}, \end{aligned} \right.$$

which by the convexity of the Herman rotation set  $\mathbf{C}_{\Pi(F_{BC})}$  combined with (4.21), and thus  $b_0 \in [-\gamma_0, \gamma_0]$ , yields

$$\Pi(\mu(F_{BC})) = (0, b_0) \in \mathbf{C}_{\Pi(F_{BC})}.$$

Now, by virtue of the two-dimensional equality (1.18) (see (1.14) and Remark 1.1) there exists a sequence  $(s_n)_{n \in \mathbb{N}}$  tending to  $\infty$  such that

$$\Pi(\mu(F_{BC})) = \lim_{n \rightarrow \infty} \frac{\Pi(X_{BC})(s_n, (u_0, v_0))}{s_n},$$

which is equivalent to the desired projection equality (4.10).

Finally, in view of (4.6), for any invariant probability measure  $\mu$  for the flow  $X_{BC}$  such that the mass  $\mu(F_{BC})$  does not belongs to the line  $\mathbb{R}e_3$ ,  $\mu$  is necessarily singular with respect to Lebesgue's measure.

*Proof of ii).* Taking into account the  $2\pi$ -periodicity, by (2.11) the flow  $X_{BC}$  satisfies the boundedness property

$$\forall t \geq 0, \quad |x(t) - x_0| \leq 2\pi \quad \text{and} \quad |y(t) - y_0| \leq 2\pi. \quad (4.22)$$

Moreover, by equality (2.12) the Herman rotation set for the flow  $X_{BC}$  is given by the closed line segment (4.11).

This concludes the proof of Proposition 4.1. □

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