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NONLINEAR NORMAL MODES OF VIBRATING MECHANICAL SYSTEMS: 10 YEARS OF PROGRESS

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Abstract

This paper contains a review of the theory and applications of nonlinear normal modes, which are developed during last decade. This review has more than 200 references. It is a continuation of two previous review papers by the same authors. The following theoretical issues of nonlinear normal modes are treated: basic concepts and definitions; application of the normal form theory for nonlinear modes construction; nonlinear modes in finite degrees of freedom systems; resonances and bifurcations; reduced-order modelling; nonlinear modes in stochastic dynamical systems; numerical methods; identification of mechanical systems using nonlinear modes. The following applied issues of this theory are treated in this review: experimental measurement of nonlinear modes; nonlinear modes in continuous systems; engineering applications (aerospace engineering, power engineering, piecewise-linear systems and structures with dry friction); nonlinear modes in nanostructures and physical systems; targeted energy transfer and absorption problem.

1. Introduction

1.1. Basic Concepts and Main Definitions

This review is a continuation of the previous ones [1] and [2]. Over the last decade, a large number of publications have appeared relating to the theoretical and applied aspects of the nonlinear normal vibrations modes (NNMs). This indicates the importance of this area in nonlinear dynamics. There has been a significant increase of studies devoted to different dynamical systems including applied systems, whose dynamics are studied using NNMs. The analytical and numerical methods for NNMs analysis are significantly developed during the last decade. Moreover, the studies of NNMs in a stochastic dynamical system, the identification of mechanical systems using NNMs, are initiated. The NNMs analysis is a useful tool for the first step of nonlinear system complex behavior study.

In this review, we touch the historical aspects of the NNMs, which are a generalization of the well-known linear normal vibrations. Turning to the origin of NNMs, we note the contribution by A.M. Lyapunov [3], who at the end of the 19th century developed the method for construction of the periodic solutions in finite-dimensional dynamical systems with a first analytic integral. Such solutions tend to normal modes of linear systems, if the vibration amplitudes tend to zero. The generalization of the Lyapunov's results is made for Hamiltonian systems by A. Weinstein [4]. Later it is shown [5] that the Lyapunov solutions have properties of the NNMs.

H. Seifert [6] proved the existence at least of one periodic solution of the conservative system, which trajectory twice intersects the maximal equipotential surface during one period. Later such solutions were called BB-solutions ("boundary-boundary"), or BOB-solutions, when the trajectory passes through the origin. The Seifert's theorem is generalized in [7, 8] where the existence at least n BOB-solutions in the finite DOF systems with even potential energy is proven.

H. Kauderer [9] was the first who constructed NNMs of two-DOF conservative system with cubic nonlinearity, using the equations of motions in the system configuration space. R. Rosenberg in the papers [10-13] proposed the first definition of NNM as "vibrations in unison," i.e., synchronous periodic motions, when all generalized coordinates of the finite-DOF system reach their maximum and minimum values at the same instants of time. Rosenberg obtains several classes of nonlinear systems, including nonlinearized ones, which have NNMs with straight modal lines (so-called "similar" NNMs). These NNMs are a direct generalization of the linear normal modes to a nonlinear case. The Kauderer-Rosenberg (K-R) concept of the NNMs is based on the construction of the trajectories (modal lines) in the configuration space.

The K-R concept can be used in systems with significant and even predominant non-linearity. It seems that the K-R NNMs are an effective tool to study free and forced large amplitude vibrations of low dimensional essential nonlinear systems without dissipation. Note that the construction of modal lines is most efficient when they are close to the rectilinear ones [14]. This concept is applied to analyze the mechanical systems having nonlinear absorbers, shallow shells and arches with snap-through motions, other finite-DOF mechanical systems [1,2].

Different concept of the NNMs was proposed by S.Shaw and C.Pierre [15-17]. They considered NNMs as invariant manifolds in the system phase space, which are tangent to the linear normal modes. Note that a similar approach is suggested to construct periodic solutions

in phase space by A.M. Lyapunov [3]. The Shaw-Pierre (S-P) approach can be used for dissipative dynamical systems and does not imply binding to conservative condition as in the K-R concept. At present, both K-R NNMs and S-P NNMs can be used for mechanical systems with significant nonlinearity.

S-P NNMs can be used to analyze the continuous mechanical systems, which are reduced to nonlinear ordinary differential equations (ODEs) by means of finite element methods, Galerkin approach, or assumed-mode method. This S-P concept is successfully used to analyze both conservative and dissipative mechanical systems [1,2]. Moreover, this technique can be applied to analyze free and forced nonlinear vibrations of mechanical systems with dozens and hundreds DOF. The detailed description of the S-P NNMs and their use in different theoretical and applied problems are made in the book chapter [18].

A continuation and generalization of the S-P concept of NNMs is made in recent publications by Haller with co-authors. The authors introduced spectral submanifold (SSM) in phase space. They treated it as the smoothest invariant manifold tangent to a nonresonant spectral subspace at a fixed point under the condition that there are no low-order resonances with the rest of the linearized spectrum. A corresponding strict mathematical definition of SSMs is given in [19, Definition 3]. Using this definition, the authors explore the existence and uniqueness of SSMs from prior abstract results on mappings on Banach spaces, as well the smoothness and robustness of these manifolds. The existence and uniqueness of SSM depends on the spectral quotient, which is calculated from the real part of the linearized system spectrum. Such definition of SSMs has a general form, which can be **also (5)** used for the systems under the action of a periodic or quasi-periodic external excitation. The fast, intermediate and slow SSMs are obtained. Connection of the proposed SSMs with the K-R NNMs is explained.

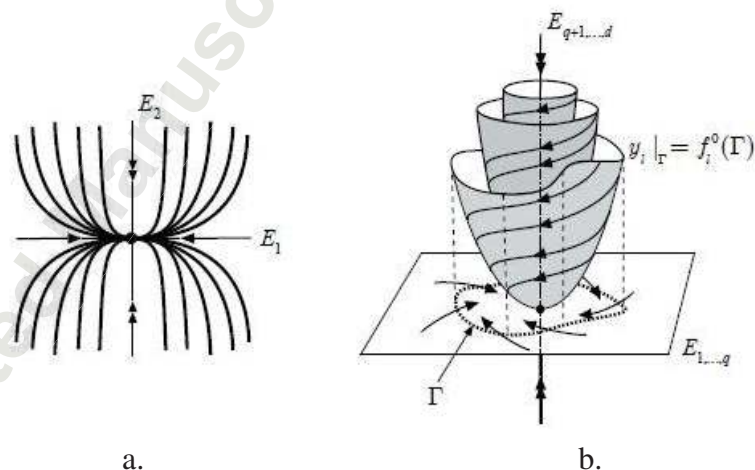


Fig.1 Non-unique invariant manifolds, which are tangent to the slower-decaying spectral subspace in the planer and finite-dimensional cases. Figure courtesy of G. Haller

The non-unique invariant manifolds tangent to the slower-decaying spectral subspace of a planar linear dynamical system (Fig. 1a). But the invariant manifold, which is tangent to the faster-decaying spectral subspace, is unique. The non-unique invariant manifolds of the finite dimensional dynamical system, which are tangent to the individual spectral subspace, are shown in Fig. 1b. Both the choice of the co-dimension one boundary surface Γ and the boundary values $f_i^o(\Gamma)$ of the invariant manifold are arbitrary, as long as Γ is transverse to the linear vector field (Fig.1b).

Other publications by G. Haller with co-authors, where the concept of the SSMS is developed, are presented in next sections¹.

At the end of this Section, we note that the different aspects of the theory of NNMs and their applications are reviewed over the past years in several papers [28-31].

1.2. NNMs and theory of normal forms

There are many studies that compare the NNMs theory in the framework of the S-P concept with the normal form (NF) theory originated in classical works by Poincaré and Birkhoff. A part of the NF theory is treated in the section devoted to reduced-order modeling.

Consider the following dynamical system in the form of a system of ordinary differential equations (ODEs):

$$\dot{\mathbf{q}} = \Lambda \mathbf{q} + N_q(\mathbf{q}), \mathbf{q} \in C^p, \quad (1)$$

where \mathbf{q} is a vector of the phase coordinates. The principal transform of the NF method is the following near-identity one to the new variables $\mathbf{u} \in C^p$:

$$\mathbf{q} = \mathbf{u} + \mathbf{h}(\mathbf{u}). \quad (2)$$

This transformation permits to remove the non-resonant terms of the successively higher powers in the Taylor expansion of the nonlinear terms $N_q(\mathbf{q})$. As a result, the new more simple dynamical system is derived in the following form:

$$\dot{\mathbf{u}} = \Lambda \mathbf{u} + N_u(\mathbf{u}), \quad (3)$$

where $N_u(\mathbf{u})$ contains only resonant terms. Schematic representation of the NF mapping transform (2) from the phase coordinates \mathbf{q} to a curvilinear coordinate system \mathbf{u} is shown for a four-dimensional phase space in Fig.2. Here M_1, M_2 are two-dimensional invariant manifolds. Components of the vector \mathbf{q} are presented by the modal coordinates (Y_1, Y_2, Z_1, Z_2) . The normal coordinates corresponding to the vector \mathbf{u} are denoted by $(G_1, G_2, \Lambda_1, \Lambda_2)$. These coordinates generate the curved grid, which is associated with the invariant-based span of the phase space.

¹ The NNMs concept was generalized previously to non-conservative systems. Namely, the K-R NNMs together with Rauscher method are used to analyze the forced and self-sustained vibrations in [17, 20-23]. The S-P NNMs and the Rauscher method are used together to analyze the forced vibrations in [24] and parametric vibrations in [25]. Moreover, NNMs of forced vibrations with smooth trajectories in the configuration space and chaotic behavior in time are presented in [26] for two-DOF systems with more than one equilibrium positions. Localized mode with chaotic behavior in time is described in the paper [27].

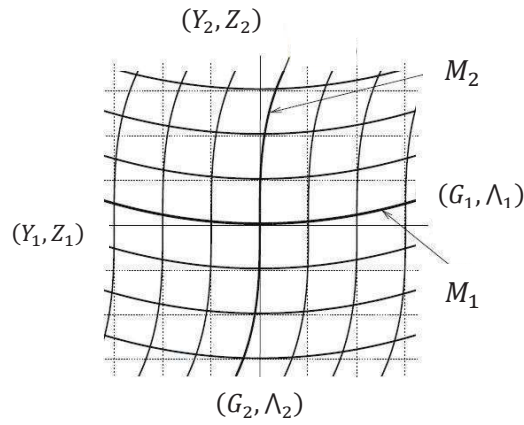


Fig. 2 Schematic representation of the NF mapping transform in a four-dimensional phase space. Figure courtesy of C. Touzé

The combination of the NF approach, singularity theory and variational technique are used to prove the NNMs existence in [32]. It seems to us, that the relationship between the NNMs and NF construction is first discussed in [33].

A detailed analysis of the connection between the NNMs, the NF theory and a use of NNMs for reduced-order modeling is made in [34, 35, 36]. Large amplitude vibrations of thin-walled structures are considered in these papers.

The application of the NF theory for the NNMs analysis is treated in [37]. The NF theory is applied to obtain the NNMs of two-DOF cable model with account of the internal resonance between in-phase and out-of-phase modes. The transition from planar to whirling motions is described by the second-order NF.

Note that the effectiveness of the NF approach, as one of the most developed in nonlinear dynamics, is beyond doubt. NF theory is applied to wide classes of nonlinear mechanical systems. In particular, nonlinear vibrations of the thin-walled structures, rotors, etc., are analyzed by this approach.

2. NNMs of finite-DOF systems

In this section the publications devoted to NNMs in finite DOF mechanical systems are analyzed by different approaches.

The paper [38] is dedicated to a nonlinear modal analysis of dissipative mechanical systems using the *complex nonlinear modes* (CNMs), when the complex non-linear eigenvalue problem in the frequency-domain is formulated. The corresponding eigenfunctions are determined in the form of the generalized Fourier series. The NNMs in the turbomachinery bladed disk with dry-friction contacts are analyzed.

The resonant NNMs in the self-excited and parametrically coupled oscillators under the action of the periodic excitation are analyzed by the multiple scales method (MSM) in [39].

The K-R NNMs of the spring pendulum and the mechanical system with pendulum absorber are analyzed in [21]. The combination of the NNM approach and the Rauscher method, when the generalized coordinate associated with the NNM is chosen as new

independent variable instead of time, is used to describe the forced vibrations of such two DOF systems.

The existence of the in-phase and the out-of-phase NNMs in two coupled nonlinear oscillators is proved using the comparison principle for ODEs in [40].

The extended definition of NNMs as a family of periodic motions is proposed for autonomous systems in [41]. These periodic motions are excited by the mass-proportional auto-oscillations force. The harmonic balance method (HBM) are used to construct these periodic motions.

New representation of the NNMs using eigenfunctions of the Koopman operator, which is associated with the system observables. In this case, the relative displacements, potential or kinetic energies are discussed in [42, 43]. The expansions by the eigenfunctions of the Koopman operator are used. The analogy with the S-P NNMs is discussed. The numerical technique for the construction of invariant manifolds in larger domains of the phase space is proposed to overcome local properties of the expansions.

The NF technique and the energy balance method (EBM) are used to determine the forced responses of two-DOF nonlinear dynamical system in [44, 45]. The resonance crossing points between the forced responses and the backbone curves of the system free vibrations are obtained.

We can note that the EBM approach is based on the assumption, that the total vibration energy for the periodic steady-state response is equal to zero over one period. Moreover, this assumption is used in different perturbation techniques and can be expressed by the following equation:

$$\int_0^T f(x_0, \dot{x}_0, t) \dot{x}_0 dt \equiv \oint f(x_0, \dot{x}_0, t) dx_0 = 0, \quad (4)$$

where $f(\dots)$ is the perturbation vector function for ODEs; x_0 is a solution of the unperturbed ODEs. The integral (4) is calculated along the unperturbed solution x_0 . In particular, the condition (4) is used to study the bifurcations of NNMs in [14]. The well-known subharmonic Melnikov function also has the form of the integral (4). The authors of [46] prove a generalized version of the equation (4) that is valid for finite DOFs system.

Nonlinear dynamics of two DOF system with a cubic nonlinearity, which is obtained from the Galerkin projections of the simply supported geometrically nonlinear model of beam, is analyzed in [47]. NNMs of the systems are studied using the continuation technique. The main attention is paid to the branches of NNMs, which correspond to different internal resonances. The phase-locked and phase-unlocked branches of the NNMs are obtained. Then the modes are studied in the presence of the external excitation. It is assumed that this excitation and damping mutually balance each other. Results of the numerical analysis show that the phase-locked NNMs are strongly attractive, while the non-phase-locked NNMs are weakly attractive. As follows from the numerical analysis, the energy transfer mechanism is much less intensive for the phase-unlocked modes.

S-P NNMs are used to describe the spacecraft relative motion on Kepler orbits in [48]. The Lyapunov-Floquet transformation, invariant manifold-based order reduction and time-dependent NF are used to obtain NNMs.

The vibration localization of the cyclic symmetric chain of six externally excited weakly-coupled Duffing oscillators is analyzed in [49]. The system periodic motions have fifty two

different modes (including the trivial one) due to the system symmetry. Twenty two of these solutions can be obtained from the linear modes. These modes undergo the bifurcations, which are observed if the amplitudes of vibrations are varied from low to high values. The others isolated thirty branches of the periodic oscillations exist after a certain energy threshold that is these branches disappear at low energy. If all oscillators are excited by the same force, spatial localized vibrations take place. The degree of this localization is strongly dependent on the excitation level.

The localized NNMs in a system of the linearly coupled Duffing oscillators under the action of a harmonic excitation are treated in [27]. Such localization is obtained in the form of the intrinsic localized modes, which can be both symmetric modes and asymmetric ones. The free and forced responses are approximated by a single harmonic. As follows from the analysis, if linear coupling disappear, then these localized modes are stable, but they are chaotic in time.

The SSM concept is used to analyze the forced nonlinear vibrations of finite-DOF mechanical systems under the non-resonance conditions in [50]. The two-dimensional SSMs for non-autonomous system are obtained in vicinity of the origin using the small parameter method. These SSMs can be used for the forced system analysis, if the forcing frequency does not satisfy the resonance condition with an imaginary part of the eigenvalues of the linearized system. Amplitudes of the forced response are obtained as roots of polynomial. For the cases of internal resonances, the system dynamics is not observed on two-dimensional SSM. In this case, the higher-dimensional SSM must be derived. Such SSMs are also used to derive analytically the response of periodically forced finite DOF mechanical systems in [51]. The proposed algorithm permits to obtain the analytical solution. The analytical predictions include an explicit criterion for the existence of the isolated forced response (“isola”) (Fig.3), which belongs to the time-periodic SSM. As follows from this Figure, three limit cycles for the forcing frequency $\Omega = \Omega_0$ are observed. The proposed approach is used to analyze two DOF system with cubic nonlinearity and the Euler-Bernoulli beam transversal vibrations with a cubic spring and damper attached to the beam end.

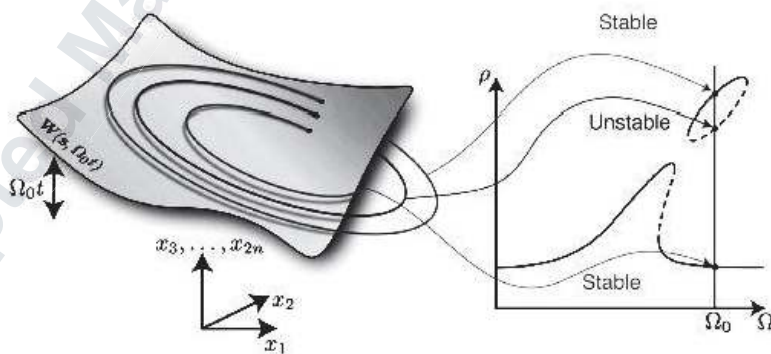


Fig. 3 Periodic SSM and forced frequency periodic response. Figure courtesy by G. Haller

The paper [46] is dedicated to study of stability and bifurcations of the periodic forced vibrations in the dissipative mechanical system beginning from the backbone curves analysis in the conservative limit. This problem is reduced to analysis of the generalized subharmonic Melnikov function, which is a leading term in the equation of the energy balance over one

period of motion. The proposed approach allows obtaining maximal amplitudes of the forced vibrations and the saddle-node bifurcation points.

The modal superposition method based on, so-called, the hybrid mode shape concept is developed to obtain the steady state forced response of nonlinear systems in [52]. This mode shape is presented as linear combination of the modal vectors of the limiting linear systems. This approach is applied to the nonlinear system with dry friction.

The S-P NNMs are obtained using the Gröbner polynomials basis in [53]. This permits to increase accuracy of the NNMs approximation.

A modification of the EBM is applied to the dissipative forced nonlinear systems using a concept of the *damped nonlinear normal modes* (dNNMs) [54]. This modification of the EBM allows to carry out the balance between the energy losses from the damping and the external excitation energy. The EMB approach predicts resonances of forced NNMs. The self-sustained vibrations of two-DOF system interacting with a moving belt are considered.

The nonlinear mechanical subsystems with linear attachments are considered in [55]. The numerical simulations of the cantilevered beam with cubic spring are treated.

A stability of NNMs, standing and traveling waves is analyzed using the method of the Ince algebraization in [56]. The positional coordinate describing the unperturbed NNM is used as the independent variable instead of time. In this case, equations in variations are transformed into the equations with singular points. Then solutions corresponding to boundaries of the stability/instability regions are obtained. Another analytical-numerical approach of the NNM stability analysis, which is associated with the classical Lyapunov definition of stability, is also used.

The dynamics of two coupled pendulums under a magnetic field are considered in [57]. The small parameter method is used to construct two nonlinear normal modes one of them is the coupled vibration mode, and the second is the localized one. The influence of the masses ratio and other system parameters to these NNMs for small and not small initial angles of the pendulums is studied.

In publications presented in this Section, various methods for NNMs analysis are used, in particular, the K-R approach, S-P expansions, SSMs, the harmonic balance method (HBM), the multiple scales method (MSM), the energy balance method (EBM), the Rauscher method, the Fourier expansions etc. At the same time, it can be noticed that a number of new kinds of NNMs have been introduced, in particular, complex, damped, transient, phase-locked NNMs, are considered. We also believe that it is not always possible to establish a direct correspondence of the obtained vibration modes to the basic concepts of NNMs, which are presented at the beginning of this review. It seems to us that the significance of certain definitions and their connection with the main definitions of NNMs will be estimated only by further research.

3. Resonances and bifurcations of NNMs

This section is devoted to free and forced NNMs in the vicinity of resonances of various origins.

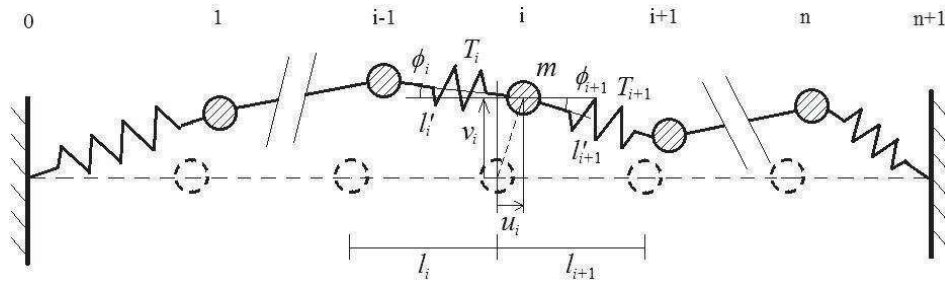


Fig.4 The chain of particles. Reprinted with permission from Society for Industrial and Applied Mathematics

The papers [58, 59] are dedicated to study of the in-plane oscillations of the strongly nonlinear finite chain of particles with clamped-clamped boundary conditions (Fig.5). In the low energy case, transversal geometrically nonlinear oscillations are predominant. This case is called "nonlinear acoustic vacuum" and the nonlinear terms are predominant in the equations of motion. The continuum limit of the system is described by the following partial differential equations (PDEs):

$$\begin{aligned} \varepsilon^2 \frac{\partial^2 u(i, \tau)}{\partial \tau^2} &= \frac{\partial}{\partial i} \left[\frac{\partial u(i, \tau)}{\partial i} + \frac{1}{2} \left(\frac{\partial v(i, \tau)}{\partial i} \right)^2 \right] + \dots, \\ \frac{\partial^2 v(i, \tau)}{\partial \tau^2} &= \frac{\partial}{\partial i} \left[\left(\frac{\partial u(i, \tau)}{\partial i} + \frac{1}{2} \left(\frac{\partial v(i, \tau)}{\partial i} \right)^2 \right) \frac{\partial v(i, \tau)}{\partial i} \right] + \dots, \end{aligned} \quad (5)$$

where ε is a small parameter, which describes the order of the axial and transverse displacements. The main variables $u(i, \tau)$ and $v(i, \tau)$ are shown on Fig.5. In this continuum limit the nonlinear sonic vacuum leads to absence of any linear acoustics and zero speed of sound. The NNMs of the system are identical to ones of the linear chain. The case of the 1:1 resonance is studied asymptotically. The strong energy exchange between the modes is observed. As follows from [59], all NNMs excepting one with high wavelength are unstable. Two different cases are considered for low system energies. In the first case, the predominantly transverse NNMs are excited by the spatially extended transverse loads, which act on the lattice. In the second case, the localized transverse impulse excitations occur. An intense and repetitive energy exchange between the directly excited NNM and other NNMs with higher wave numbers is observed.

The *phase-locking* between the modes in the case of the internal resonance is analyzed using the NF technique in [60]. Both phase-locked and phase-unlocked backbone curves of the weakly nonlinear cable are obtained. The phase-locking occurs between in-plane and out-of-plane motions. For the phase-locked and the phase-unlocked cases the mixed-mode backbone curves appear due to the bifurcation of the single-mode motions. The EBM is used to determine the system forced responses from the backbone curves analysis. A stability analysis is used to show that the phase locking is associated with the internal resonance. It is also shown that the phase-unlocked modes may lead to stiffening effects in the system.

The NNMs backbone curves of free nonlinear vibrations are used to describe forced responses of the dissipative systems in the next two papers. The EBM and the continuation

technique are used to describe the forced resonance vibrations using the backbone curves in [61]. As follows from the numerical analysis, the “isola” frequency response curve appears due to interactions between the NNMs.

The NNMs interaction in a three-DOF system with weak cubic nonlinearity is considered in [62]. The system with the internal resonance, when all eigenfrequencies are close, is treated. Then the NF approach is used to obtain the backbone curves of the dissipative system forced response.

Free and forced vibrations of the finite-DOFs dissipative mechanical systems in the case of internal and/or forced resonance conditions are considered in [63-65]. The NNMs interactions are analyzed using the MSM. The modulation equations are transformed to the so-called reduced system with respect to three variables (the system energy, the arc tangent of the amplitudes ratio and phases difference). The analysis of the reduced system permits to describe transition from unstable nonlinear modes to stable ones. The conditions of the vibrations energy localization are obtained. Besides, the *transient NNMs* (TNNMs) are observed. Such modes exist, if the energy exceeds the threshold values for some time instant. These transient NNMs attract other motions of the system near this time value. If the transient mode disappears, the motions of the system move to some stable NNM.

The nonlinear mode localization in the boundary-interior coupled structures with cubic nonlinearity (Fig.6) is considered in [66]. Resonant and non-resonant boundary motions are described by the *boundary modulation operator*. An asymptotic approach permits to describe three types of NNMs. In the case of the localized NNM (Fig.6a), the displacements of the cable A are essentially larger than the displacements of the bridge desk C. Therefore, the nonlinearity of the component C is not accounted. The global NNM (Fig.6b) has comparable amplitudes of the substructures A and C. The NNM frequency is close to one of the component C. The frequencies of both components are close for the hybrid mode. In this case, the weak boundary motions of the A-C coupling interface excite the motions of the component A. Nonlinear frequencies of these three modes and their invariant manifolds are obtained. The localized and global modes are induced as NNMs pairs, if frequencies of the linearized system are not close. Four hybrid NNMs are obtained with two different frequencies. The localized/global/hybrid NNMs of the beam-beam coupled structure are obtained by the same asymptotic method. The phenomenon of *abrupt asymptote exchange* caused by the structural nonlinearity, has been found for the frequencies of hybrid NNMs.

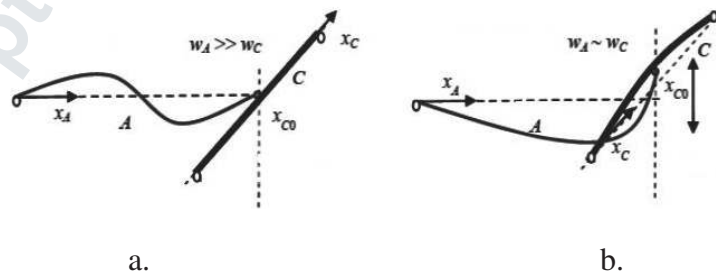


Fig.5 NNM of boundary-interior coupled structures. Reprinted with permission from Elsevier

The bifurcation analysis of a system of two coupled cubic oscillators in a case of the 1:1 internal resonance is considered in [67]. MSM is used for such analysis. The bifurcations curves are presented on the system parametric plane. An experimental analysis is also performed. The geometrically nonlinear circular plate vibrations obtained experimentally are compared with the nonlinear dynamics of two coupled oscillators. Imperfect bifurcations have been discussed to explain the slight differences between the theory and the experiments.

4. Reduced-order modeling

Reduced-order modeling (ROM) based on the NNM theory is a rapidly growing area of research. The number of publications on this subject has increased significantly over the last decade, as the NNM approach is a very useful tool to solve this problem. In most of the presented papers the ROM of the thin-walled structures with geometrical nonlinearity are treated. Various definitions of NNMs are used here.

The NNMs used in the ROM is the subject of a detailed, recently published reviews [35, 36, 68, 69]. We also refer to several first publications [15, 70, 71, 72] devoted to ROM.

In some subsequent papers by C. Touzé with co-authors, the computational procedures based on the NF method are used to perform ROMs in several geometrically nonlinear thin-walled structures.

The moderate vibrations amplitudes of the thin shells and the doubly-curved panels are studied with account of the in-plane inertia by using the ROM in [34]. Two approaches (S-P NNMs and proper orthogonal decomposition (POD)) are compared for ROM. The author notes that the NNM method is directly applicable to the original dynamical system. The obtained results have better accuracy, than the modes obtained by POD. If the vibrations amplitudes are increased, the S-P NNM-based ROM may lose accuracy, as the used S-P NNMs are local.

The ROMs of the geometrically nonlinear thin shell free and forced vibrations are obtained in [72]. The expansions of the shell displacements by the eigenmodes are used to discretize the continuous system. The obtained finite-DOF dynamical system with quadratic and cubic nonlinearities is analyzed by the NF method. Two methods for ROM of the geometrically nonlinear structure vibrations are compared in [73]. The first method is based on the NF theory to obtain the reduced-order dynamics on the invariant manifolds. The second method is based on the modal derivatives approach, which is suggested to derive the quadratic manifold (QM). The NF theory is used to obtain ROMs of the geometrically nonlinear structure in [74]. This structure is discretized by the finite element method (FEM). The suggested procedure is used the direct NFs, which has not limitation on the number of the master coordinates in the resonance case. The third-order approximation of the S-P NNM is derived. The proposed approach is applied to analyze both the clamped-clamped beam nonlinear dynamics and the FE model of the fan blade.

NF theory is used for ROM of the mechanical systems with quadratic and cubic nonlinearities in [75]. The direct NF approach leads to the homological equations, which permit to obtain the reduced system parameters. A special emphasis has been put on the treatment of the second-order internal resonance, which induces the strong energy exchange between the resonance modes. A comparison of the implicit condensation, expansion method and the NNM approach is suggested to construct ROM in geometrically nonlinear structures in [76]. It is

shown that NNMs in the form of invariant manifolds generate accurate ROM. The simplified procedure, which is based on the NF approach to identify only resonant terms in the nonlinear stiffness, is developed. The obtained results are illustrated on the beams and plates nonlinear vibrations.

A comparison of the implicit condensation and expansion method and the NNM approach to construct ROM in geometrically nonlinear structures is suggested in [72]. It is shown that NNMs in the form of invariant manifolds generate accurate ROM. The simplified procedure, which is based on the NF approach and identification of only resonant terms in the nonlinear stiffness, is developed. The obtained results are illustrated on the beams and plates nonlinear vibrations.

The direct parametrization of the invariant manifolds for multiple DOF finite element approximation of continuous system with geometric nonlinearity is suggested in [77-79]. The main idea of this method is to consider the coordinate nonlinear change applied to the basic mechanical system generalized coordinates and velocities. New coordinates describe the evolution of dynamical systems along a low-dimensional invariant-based manifold in the system phase space. For the non-autonomous case the dimension of the dynamical system is enlarged to make it autonomous and the added coordinates related to the forcing is treated. Different types of parametrization are introduced, namely, the graph style, the complex normal form and the real normal form. The proposed approach is applied to models of shallow arch, cantilever beam, twisted plate (fan blade) and MEMS. It is shown that the parametrization method generates efficient ROMs for forced vibrations, including superharmonic resonances.

The different approaches for ROM of the geometrically nonlinear structures, which are approximated by FE models, are compared in the following three publications. The authors of the papers [80, 81] study the convergence of the ROM of nonlinear systems, which are obtained by the FEM and nonlinear modal analysis. The implicit condensation approaches are applied to estimate ROM in such systems. NNM of the full dimension model is used as a metric to estimate a convergence of the ROM equations. Two FE models of the clamped-clamped beam and the exhaust panel are analyzed using the above-mentioned approaches. A convergence of different ROM is analyzed using the frequency-energy plots constructed from NNMs. The thin-walled structure, which consists of the coupled sub-structures, is considered in [82]. The implicit condensation and expansion method using fixed-interface modes and system-level characteristic constraint modes chosen as a basis (the Craig-Bampton basis [83]) is used. As a result, the dimension of the obtained FE model is significantly reduced. The structure NNMs are obtained with high precision.

The extension of the component mode synthesis methods to NNMs is suggested in [84, 85]. The concept of the CNMs [38] is used for the structures, which are divided on substructures. The complex modes of each substructure are computed by the HBM. The responses of the substructures are approximated by the multi-harmonic sum of nonlinear eigenvectors and by a set of linear static modes. The suggested approach is applied to mistuned cyclic structure with dry friction under the action of the external harmonic excitation. The FE model of the bladed disk vibrations with dry frictions nonlinear contacts is studied. Forced responses of the randomly mistuning bladed disk are analyzed too.

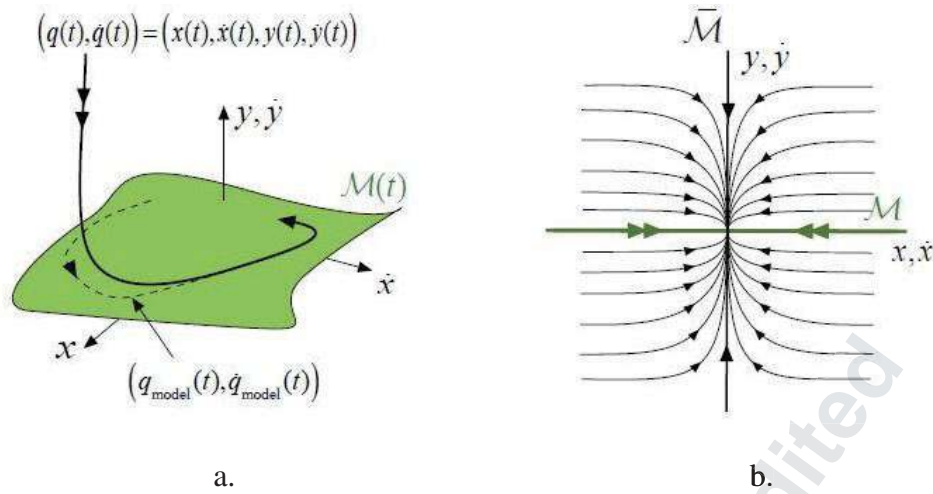


Fig. 6 Geometrical sense of requirements (R1) and (R2) for ROM. Figure courtesy by G. Haller

In the subsequent papers the slow-fast decomposition and the SSMs are used for model reduction. The methodology for ROM in finite DOF mechanical systems is developed in [86]. Two requirements for ROM in nonlinear non-autonomous dynamical systems are suggested: (R1). An attracting lower-dimensional invariant manifold $\mathbf{M}(t)$ exists. The system generalized coordinates and velocities are smooth functions of the master generalized coordinates and the velocities along the manifold $\mathbf{M}(t)$ (Fig. 6a); (R2). The general trajectories approaching $\mathbf{M}(t)$ is faster than the rates within $\mathbf{M}(t)$. Fig.6b shows the case, when the requirement R2 is not satisfied.

Then the attracting invariant manifold \mathbf{M} does not provide accurate ROM for the full nonlinear dynamical system. The explicit expressions for the slow manifold are obtained. A boundary of the slow manifold instability is derived. The obtained analytical results are illustrated by several examples, including a three-DOF system with a pendulum damper.

The SSM reduction is applied to the model of the von Kármán beam with viscoelastic damping in [87]. This model is reduced up to two-DOF system with cubic nonlinearities. The forced dynamics of the finite DOF dissipative mechanical system is reduced using time-periodic SSMs in [88]. The forced response of the discretized cantilever Euler- Bernoulli beam with nonlinear spring attachment is treated. The SSM reduction technique advantages are discussed. As follows from [89], the backbone curves and the forced responses of finite DOF mechanical systems can be described by the Lyapunov subcenter manifolds (LSMs) and the SSMs, respectively. The LSM reduction is effective for conservative systems. The more general SSM reduction is suitable to analyze the forced response. It is shown that the third-order LSM reduced model of the beam leads to the Duffing oscillator in the case of the non-resonance condition. As follows from this paper, the near-identity transformation used in the NF based reduction method is not needed.

SSM theory is applied for ROM of forced vibrations of nonlinear mechanical systems with internal resonances in [90, 91]. The dimension of ROM is obtained from the mode number taking place in internal resonances. The finite element models of nonlinear beams and plates are used to show the effectiveness of suggested reduction approaches.

The quadratic manifold approach for the reduction of the geometrically nonlinear thin-walled structures is considered in [92, 93]. The quadratic transform of the vector of initial physical displacements $\mathbf{u} \in R^n$ is applied in the following form:

$$\mathbf{u} = \mathcal{L}(\mathbf{z}) = \mathbf{V}\mathbf{z} + 0.5\mathbf{\Theta}\mathbf{z}\mathbf{z}, \quad (6)$$

where $\mathbf{z} \in R^{\mathcal{N}}$ is a vector of the reduced generalized coordinates. Note, that $\mathcal{N} \ll n$. Here $\mathbf{\Theta}$ is a symmetric third order tensor. The approach for the derivation of the reduced nonlinear dynamical system is suggested. The clamped-clamped bending oscillations of the plate are treated under the action of the multiple harmonic periodic force. The shell finite elements and the von Karman theory are used. The thin-walled wing structure with NACA 0012 airfoil, which is stiffened by ribs in longitudinal and lateral directions, is treated in [92]. This structure is discretized by the triangular finite elements with 18 DOFs. The transient responses of this structure under the action of pulse pressure are analyzed using the reduced order finite DOF mechanical systems.

The paper [94] is dedicated to the ROM implementation using dNNMs. The forced vibrations of the nonlinear system with friction joints are analyzed. The truncated Fourier series are used to approximate such damped modes. This ROM approach permits to obtain the resonance response with high accuracy in wide range of the vibration amplitudes. Besides, the dNNMs are used to predict the forced resonance responses using the nonlinear modal synthesis and the EBM.

5. NNMs of stochastic dynamical systems

Several recent papers initiate a new direction in the NNM theory associated with the dynamics of systems under a stochastic excitation.

The influence of the random excitation on NNMs of mechanical system free vibrations is analyzed in [95, 96]. The power spectrum of random vibrations can be predicted using the frequency-energy plot of the NNMs. In particular, such frequency-energy plots for a two-DOF system with cubic nonlinearity are shown in Fig.8. Here vertical lines indicate the energy percentile levels in the random response. The connection between the undamped NNMs and the spectrum of its response to high amplitude random forcing is investigated. A spring-mass system and the random model of a clamped-clamped beam are considered. The authors conclude that the NNMs are a global property of mechanical systems that do not depend on the form of random excitation².

The stability of the similar NNMs in two DOF mechanical system under the stochastic excitation is analyzed in [97] using the approach proposed in [26] which is an implementation of the Lyapunov stability criterion. Taking into account specific properties of the stability problem in the system under stochastic excitation, a modification to the approach is suggested. Namely, it is allowed that small part of perturbations of NNMs can be out of their given neighborhood during each fixed interval of time. The stability analysis is performed for four different kinds of the stochastic excitation. Boundaries of the stability/ instability regions are

² The authors of this review suppose that this result significantly depends on the intensity of the random excitation.

obtained in the plane of the system parameters. Similar results on the NNMs stability are obtained for all considered types of the stochastic excitation.

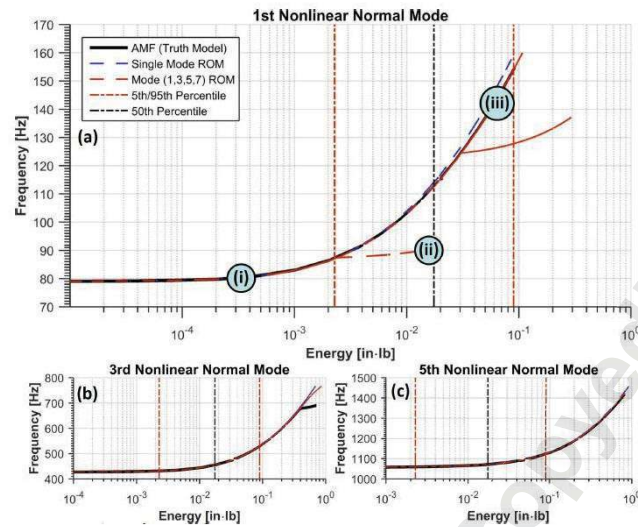


Fig.7 NNM frequency-energy plots of the clamped-clamped beam. Reprinted with permission from Elsevier

The system of two coupled Duffing oscillators under the action of resonance harmonic and random excitation is treated in [98]. The possible transition from one stable vibrations mode to another one is analyzed, when the short duration Gaussian white noise acts on the system. The transient probability density functions at discrete time points are obtained using the modification of the path integral method. It is shown that, the localized mode with a high probability collapses faster for a strong connection between oscillators than for a weak connection. Probability values in each basin of attraction of the vibration modes and the shifts of the probability density at different energy levels are obtained.

As follows from the paper [99], the invariant manifolds of deterministic dynamical system can be used, when small white noise excitation acts on dynamical system. The authors show the importance of normally hyperbolic invariant manifolds and the SSMS for analysis of the random dynamical systems.

As follows from the results presented in this Section, the study of NNMs in systems under the action of random excitations is just beginning. We emphasize the importance to study the influence of the random fluctuations for various engineering applications, for example, self-sustained vibrations of thin-walled structures under the action of turbulent boundary layer fluctuations et al. The significance of the study of NNMs in such problems, as a kind of "framework" of dynamic behavior, can be clarified in subsequent studies.

6. Numerical methods for NNMs

The use of numerical methods in problems related to NNMs has received significant development in the last decade.

Different numerical methods for NNMs analysis are described in the review paper [100]. First of all, the authors present definitions and the frequency-energy dependence of

NNMs. The shooting technique and methods of discretization for NNMs analysis are presented. The pseudo-arclength or asymptotic continuation can be used to calculate NNMs. NNMs in the form of two-dimensional invariant manifolds, which are described by governing PDEs, are suggested to calculate numerically by the Galerkin based approach, the transport method associated with a fluid dynamics and FEM. Besides, the boundary-value formulation is possible for such NNMs, when the trajectory-based method, the graph transform and complex nonlinear modes are used. An assessment and comparison of different approaches for NNMs numerical analysis is presented.

The papers [101, 102] are dedicated to use of the shooting technique and continuation method for analysis of NNMs. The shooting technique permits to solve the boundary-value problem defined for NNMs by the following periodicity condition:

$$\mathbf{H}(T, \mathbf{z}_{p0}) = \mathbf{z}_p(T, \mathbf{z}_{p0}) - \mathbf{z}_{p0} = \mathbf{0}, \quad (7)$$

where $\mathbf{z}_p = \mathbf{z}_p(t, \mathbf{z}_{p0})$ is a vector of the phase coordinates; $\mathbf{z}_p(\mathbf{0})$ is a vector of the initial conditions; $\mathbf{H}(T, \mathbf{z}_{p0})$ is the so-called shooting function; T is a period of the solution. The relation (7) is supplemented by the so-called phase condition. The Newton method is used to solve the nonlinear equations (7). The combination of the shooting technique and pseudo-arclength continuation method are used too. The proposed approaches permit to calculate the backbone curves of the NNMs and to analyze their stability and bifurcations.

The numerical method for S-P NNMs analysis is suggested in the paper [103]. The PDEs of the S-P NNMs are rewritten in the form of the transport problem:

$$(\vec{\mathbf{V}} \cdot \vec{\nabla})\mathbf{P} = \mathbf{Q}; (\vec{\mathbf{V}} \cdot \vec{\nabla})\mathbf{Q} + \omega^2\mathbf{P} = \mathbf{f}, \quad (8)$$

where $\mathbf{P}(a, \phi) = (P_2, \dots, P_N)$; $\mathbf{Q}(a, \phi) = (Q_2, \dots, Q_N)$ are vector-functions with respect to the slave coordinates; a, ϕ are master coordinates; $\vec{\mathbf{V}} = (\dot{a}, \dot{\phi})$; ω^2 is diagonal matrix of the squared eigenfrequencies; \mathbf{f} is a vector of nonlinear terms in equations of motions. The method of flux analysis is used to solve the equations (8). The transport problem is supplemented by the periodicity constraint as

$$\begin{aligned} \mathbf{P}(a, 0) &= \mathbf{P}_0(a); \mathbf{Q}(a, 0) = \mathbf{Q}_0(a); \\ \mathbf{P}(a, 2\pi) &= \mathbf{P}_0(a); \mathbf{Q}(a, 2\pi) = \mathbf{Q}_0(a), \end{aligned} \quad (9)$$

where $\mathbf{P}_0(a), \mathbf{Q}_0(a)$ are the vector-functions, which are obtained from the minimization of the cost functional:

$$J(\mathbf{P}_0, \mathbf{Q}_0) = 0.5 \int_0^A \{[\mathbf{P}(a, 2\pi) - \mathbf{P}_0(a)]^2 + [\mathbf{Q}(a, 2\pi) - \mathbf{Q}_0(a)]^2\} da. \quad (10)$$

Then iterative procedure is proposed to minimize the functional (10). The finite difference numerical scheme is used to solve the invariant manifold equations (8).

The FEM is used to compute the S-P NNMs of non-conservative system in [104]. The NNMs are described by the PDEs (8). Then the streamline upwind Petrov-Galerkin method is applied to solve such PDEs

$$\int_{\Omega} (\vec{\mathbf{V}} \cdot \vec{\nabla} P_i - Q_i) \delta Q_i d\Omega = 0;$$

$$\int_{\Omega} [\vec{\mathbf{V}} \cdot \vec{\nabla} Q_i + \omega_i^2 P_i - f_{2,i}] \delta P_i d\Omega = 0, i = 2, \dots, N, \quad (11)$$

where $\delta \tilde{Q}_i, \delta \tilde{P}_i$ are variations of the corresponding slave variables. The finite element discretization of the equations (11) permits to transform PDEs into the coupled nonlinear algebraic equations, which are solved numerically.

The HBM with the multi harmonic approximations of NNMs is applied to calculate forced vibrations in [105]. The system of the nonlinear algebraic equations with respect to the amplitudes of the Fourier series harmonics is obtained and solved by the Newton method.

The dynamical models, which are described by the system of nonlinear ODEs, are considered in [106]. As follows from this paper, such models describe the nonlinear vibrations of the thin-walled structures (beams, plates and shells). The authors obtain that many of the included modes lead to a static effect on the response. Therefore, the generalized coordinates describing these modes can be considered statically³. NNMs in [106] are analyzed by using the combination of the shooting technique and the pseudo-arclength continuation.

The dynamical models, which are described by the system of nonlinear ODEs, are considered in [97]. As follows from this paper, such models describe the nonlinear vibrations of the thin-walled structures (beams, plates and shells). The authors discovered that many of the included modes lead to a static effect on the response. Therefore, the generalized coordinates describing these modes, can be considered statically. NNMs are analyzed by using the combination of the shooting technique and the pseudo-arclength continuation.

A numerical approach is proposed to study a slow dynamics of nonlinear modes in [107]. The complexification-averaging approach suggested by L. Manevitch [108] is used. Two computational approaches for the modal analysis are suggested. One of these approaches is based on the HBM and the second one is based on the shooting technique. The proposed approaches are restricted to the nonlinear modes analysis. Both approaches are applicable to generic nonlinearities, including non-smooth forces.

A modification of the S-P NNMs calculations is suggested in [109]. A use of the proposed approach makes it possible to halve the dimension of the systems of nonlinear algebraic equations obtained for NNMs calculation in comparison with the systems from [16]. This modification is also applied for modal analysis of the forced vibrations in [110], where the Rauscher method is used together with S-P NNMs.

A modification of the shooting technique is suggested to analyze NNMs of forced vibrations in [111, 112]. The Jacobian matrix is used to implement the Newton method for shooting technique. The special approach, which improves the Jacobian matrix calculation, is suggested to guarantee fast convergence of the Newton method. The systems of ODEs, which solutions are elements of the Jacobian matrix, are derived.

³ The authors of this review draw attention that the static approximation of vibration modes results in loss of some properties of precisely dynamic modes.

The method of the K-R NNMs numerical calculations is suggested in the paper [113]. The basis of this method is an analysis of ODEs, which describe the modal lines in configurational space of the n-DOF conservative system. Then all generalized coordinates are presented as the single-valued functions of the single selected generalized coordinate x_1 : $x_j = x_j(x_1); j = 2, \dots, n$. Two subspaces are considered in the form: $\mathbf{z}_d = (x_2, \dots, x_n)$; $\mathbf{z}_v = \left(\frac{dx_2}{dx_1}; \dots; \frac{dx_n}{dx_1}\right)$. The differential of the arc length s of the modal line is

$$ds = \sqrt{2T_2} dt = \sqrt{S(x_1, \mathbf{z}_d, \mathbf{z}_v)} dx_1, \quad (12)$$

where T_2 is kinetic energy of the mechanical system; $\sqrt{S(x_1, \mathbf{z}_d, \mathbf{z}_v)}$ is a metric of the configurational space. Applying the least – action variational principle, the dynamical system, which describes the motions in the configurational space, is presented as

$$\begin{aligned} \mathbf{z}'_d &= \mathbf{z}_v; \\ \mathbf{M}(x_1, \mathbf{z}_v, \mathbf{z}_d) \mathbf{z}'_v &= -\mathbf{K}(x_1, \mathbf{z}_v, \mathbf{z}_d) \mathbf{z}_v - \mathbf{N}_c(x_1, \mathbf{z}_v, \mathbf{z}_d). \end{aligned} \quad (13)$$

The i^{th} row of the matrices $\mathbf{M}^{(i)}(x_1, \mathbf{z}_v, \mathbf{z}_d)$; $\mathbf{K}^{(i)}(x_1, \mathbf{z}_v, \mathbf{z}_d)$ and i^{th} element of $\mathbf{N}_c^{(i)}(x_1, \mathbf{z}_v, \mathbf{z}_d)$ take the following forms:

$$\begin{aligned} \mathbf{M}^{(i)} &= (h - V) \left(2S \frac{\partial^2 S}{\partial z_{vi} \partial \mathbf{z}_v} - \frac{\partial S}{\partial \mathbf{z}_v} \frac{\partial S}{\partial z_{vi}} \right); \\ \mathbf{K}^{(i)} &= (h - V) \left(2S \frac{\partial^2 S}{\partial \mathbf{z}_d \partial z_{vi}} - \frac{\partial S}{\partial \mathbf{z}_d} \frac{\partial S}{\partial z_{vi}} \right); \\ \mathbf{N}_c^{(i)} &= (h - V) \left[2S \left(\frac{\partial^2 S}{\partial x_1 \partial z_{vi}} - \frac{\partial S}{\partial z_{di}} \right) - \frac{\partial S}{\partial x_1} \frac{\partial S}{\partial z_{vi}} \right] + S \left[2S \frac{\partial V}{\partial z_{di}} - \left(\frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial \mathbf{z}_d} \cdot \mathbf{z}_d \right) \frac{\partial S}{\partial z_{vi}} \right]; i \\ &= 1, \dots, n - 1, \end{aligned}$$

where h is the system total energy; V is the system potential energy. Then the numerical shooting technique is used to calculate the NNM modal lines. Two end boundary points of the NNMs are located at the equipotential surface $V = h$. The numerical calculations start inside the energy enclosure surface, when $x_1 = 0$. From this point the integration of differential equations (13) performs forward to reach the boundary point $x_1 = x_{1A}$. Then the integrations of the equations (13) perform backward to reach the other boundary point $x_1 = x_{1B}$. The modal line in the configurational space consists of two intervals $x_1 \in (x_{1B}; 0]$ и $x_1 \in [0; x_{1A})$. This shooting technique is based on the Runge-Kutta and the Newton- Raphson methods.

The parametrization method for invariant manifolds calculations is developed in [114]. This approach is used in [115] to calculate the two-dimensional SSMs of finite DOFs autonomous nonlinear systems. The parameterized SSM is approximated around the fixed point by polynomials. The proposed approach permits to obtain reduced dynamics and the backbone curves with required precision. The implementation of the SSM tool helps to analyze near-resonances cases. In the presence of the forced resonance, the calculation of SSM is not possible, whereas the presence of the internal resonances leads to the nonlinear terms in the dynamics on the SSMs. The developed numerical method is used to obtain the ROM for the

nonlinear vibrations of the Timoshenko beam. The fully automated Matlab code which is available to compute SSMs of any dimension for autonomous or periodically forced systems of arbitrary dimension and nonlinearities is used in this paper. The various capabilities of this code (including built-in bifurcation analysis and handling of mechanical systems with constraints) have been developed in the series of publications [90, 91, 116, 117].

Vibrations of the geometrically nonlinear thin-walled structures are treated in [118]. These vibrations are described by the autonomous systems of high dimension nonlinear ODEs, which is obtained by the Galerkin method. The NNMs of the obtained dynamical system are calculated by the shooting technique, which is incorporated with the pseudo-arclength continuation. Numerical analysis is implemented for frame, rod and arch shows that interacting of vibrations modes describes accurately the nonlinear effects associated with nonlinear coupling. The comparison between the NNMs computed from the full and reduced order models is made.

The generalized continuation method incorporating with proper generalized decomposition is used to calculate NNMs in [119]. This procedure permits to decrease significantly a number of generalized coordinates for the NNMs description. The proposed approach is applied to two models of the Euler-Bernoulli cantilever beam with cubic spring or unilateral contact.

The finite DOF nonlinear mechanical systems under the action of a single-harmonic excitation are studied in [120]. The harmonic balance-based computational method is suggested to predict the mode shapes and the oscillation frequencies at phase resonance. The obtained motions are called phase resonance nonlinear modes. The repeated resonance dynamic behaviors of superharmonic, subharmonic and ultra-subharmonic motions are reported in two DOF system with cubic nonlinearity.

As follows from this section, various numerical methods are applied to analyze NNMs. In our opinion, the most effective method for numerical calculations of NNMs is the shooting technique in combination with the continuation method. This approach is used in most of the works presented in this section. The shooting technique permits to calculate any periodic motions, which maybe both NNMs and others periodic motions. In particular, the modified shooting technique is used to study NNMs modal lines in the system configurational space. The numerical methods for PDEs, which describe S-P NNMs, is developed using the FEM and finite difference methods. The HBM is also used to calculate NNMs. Finally, we can note that recent publications describe mathematical procedures for calculating the SSMs in various classes of dynamic systems.

7. Identification of mechanical systems

The essential advance in identification of nonlinear dynamical system by means of NNMs is achieved during the last decade. Note that, mainly, the dynamical system parameters are identified from the experimental data. Therefore, this section is closely connected with the subsequent one, which is devoted to experimental analysis.

Different problems of identification of mechanical system, including the application of NNMs, are discussed in two reviews [121, 122].

The time-frequency signal processing approach based on the Hilbert-Huang transformation and the conjugate-pair decomposition is applied for an analysis of NNMs and

the parametric identification of finite DOF nonlinear mechanical systems in [123]. Finite DOF nonlinear dynamical system is studied to illustrate the proposed methodology and to analyze NNMs, vibration localization, and nonlinear modal coupling. Numerical analysis shows that the suggested method gives accurate time-frequency characterization of the NNMs and parametric identification of the considered system.

The nonlinear modal identification of the four-storey steel frame under the action of pendulum-like exciter is considered in [124]. This structure performs nonlinear longitudinal vibrations, which are analyzed both experimentally and numerically. Four generalized coordinates describe the storey motions. Experiments show that mode shapes can be considered as linear one in the acceleration range. Thus, the modal curves are straight lines in configuration space, that is, these motions belong to K-R NNMs. The backbone curves of the NNMs are also obtained experimentally. The behavior of the structure is nonlinear for both low and large displacements. The harmonic linearization technique is used for the structure parameters identification by fitting the numerical results to the experimental data.

The time-frequency signal processing approach, which is used the Hilbert-Huang transformation and the conjugate-pair decomposition, is applied for an analysis of NNMs and the parametric identification of finite DOF nonlinear mechanical systems in [112]. Finite DOF nonlinear dynamical system is studied to illustrate the proposed methodology and to analyze NNMs, vibration localization, and nonlinear modal coupling. Numerical analysis shows that the suggested method gives accurate time-frequency characterization of the NNMs and parametric identification of the considered system.

The method for identification of the backbone curves of the finite DOF mechanical system is suggested in [125]. The used technique is based on estimation of the instantaneous frequency and the envelope amplitude corresponding to the steady-state oscillations of the system. The experimental data can be used to obtain damping ratio and backbone curves with a good accuracy.

The approach for identification of the backbone curve of finite DOF mechanical system from the experimental data is suggested in [126]. The control-based continuation is used in this approach. The backbone curves are directly traced from the experiment. The NNMs of the underlying conservative system are used to construct such backbone curves using the continuation technique. Then the proposed approach is used to NNMs analysis in multi DOF nonlinear mechanical systems.

The forced damped nonlinear mechanical system is described by the following system of the ODEs [127, 128]:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{f}(\mathbf{q}) = \mathbf{p}(\mathbf{t}), \quad (14)$$

where $\mathbf{p}(\mathbf{t})$ is vector of the external forces. The nonlinear elastic forces $\mathbf{f}(\mathbf{q})$ are expanded by using linear combination of the trial functions $\mathbf{h}_j(\mathbf{q})$ as:

$$\mathbf{f}(\mathbf{q}) = \sum_{j=1}^s \mathbf{c}_j \mathbf{h}_j(\mathbf{q}). \quad (15)$$

Knowing the measurements of $\mathbf{q}(\mathbf{t})$, the frequency domain nonlinear subspace identification method developed in is used to identify the vectors \mathbf{c}_j . In order to perform the identification, the nonlinear terms are transferred into the right-hand side of the ODEs (14):

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(\mathbf{t}) - \sum_{j=1}^s \mathbf{c}_j \mathbf{h}_j(\mathbf{q}).$$

Then the nonlinear terms are considered as the addition excitation and the obtained pseudo-linear dynamical system is analyzed. The transfer function matrices are used to determine the coefficients \mathbf{c}_j . The shooting technique is used to extract numerically NNMs in modal space of the system from measurements collected under broadband forcing. Knowing the measurements of $\mathbf{q}(\mathbf{t})$, the frequency domain nonlinear subspace identification method developed in [119] is used to identify the vectors \mathbf{c}_j . In order to perform the identification, the nonlinear terms are transferred into the right-hand side of the ODEs (14):

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(\mathbf{t}) - \sum_{j=1}^s \mathbf{c}_j \mathbf{h}_j(\mathbf{q}).$$

Then the nonlinear terms are considered as the addition excitation. Thus, such pseudo-linear dynamical system is analyzed. The transfer function matrices are used to obtain the coefficients \mathbf{c}_j . The shooting technique is used to calculate the NNMs numerically.

The methodology of the parametric identification based on comparison of experimental and analytical NNMs backbone curves is proposed in [129]. The K-R NNMs concept with the extension to the internal resonances is used. An experimental setup has the clamped beam with a small beam at the tip. The main beam is locally excited with an electrodynamic shaker. The HBM is used for NNM backbone curves calculation. Then the difference between experimental data and numerical calculations of backbone curves is minimized, which permits to obtain all unknown system parameters included in the identification.

The method of two-dimensional SSMs identification is proposed in [130]. The dynamical system (14) is presented with respect to the phase coordinates \mathbf{x} :

$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}), \quad (16)$$

where $\mathbf{x} \in R^{2N}$. The solution vector $\mathbf{x}(\mathbf{t})$ gives rise to the flow map:

$$\Theta_{\mathbf{t}}: \mathbf{x}_0 \rightarrow \mathbf{x}(t). \quad (17)$$

The scalar observable parameter along the trajectories of the mechanical system (16) is considered. This is accounted by a scalar function of the phase coordinates $\varphi(\mathbf{x}): R^{2N} \rightarrow R$. The new state vector $\xi \in R^{2\nu}$, which contains the 2ν subsequent observations of the parameter, take the following form:

$$\xi = \Phi(\mathbf{x}) = \left(\varphi(\mathbf{x}), \varphi(\Theta_{\mathbf{t}}(\mathbf{x})), \dots, \varphi(\Theta_{\mathbf{t}}^{2\nu-1}(\mathbf{x})) \right). \quad (18)$$

The second dynamical system is the sampling map \mathbf{F} , which is used to describe the experimental data for identification. This sampling map takes the following form:

$$\xi_{k+1} = \mathbf{F}(\xi_k); \xi_0 = \Phi(\mathbf{x}_0). \quad (19)$$

The experimental sampling of the system vibrations is described by the map (19). The sampling map $\mathbf{F}(\xi_k)$ is computed by the approach, suggested in [130]. The significant stages of methodology of SSMs computation from the data fitted measured vibrations are treated.

The general approach for the reduced order model identification of the nonlinear mechanical system from the observable data is treated in [131]. This method constructs the NF of the attracting SSM. The most general NF of $2m$ dimensional SSM is derived. The suggested

algorithm uses observable data to identify model of nonlinear dynamics. The Takens delay embedding theorem is applied to identify the SSM.

The open-source Matlab package (<https://github.com/haller-group/SSMLearn>) is used for identification of the SSMs from experimental and numerical data in [132,133].

The approach for NNMs analysis, which is used a generalization of the principal orthogonal decomposition, is suggested in [134, 135]. The data-based approach measures the statistically independence (or orthogonality) of the derived modes from the random data. It permits to optimize a parametric form of the NNMs mapping. The equations of motions are not used, as the method is applied for the experimentally measured nonlinear system data. The proposed approach is illustrated by few examples based on both simulated and experimental data. The analysis of the statistically-independent NNMs using the nonlinear system identification is presented in [136]. These NNMs are investigated in nonlinear cubic-stiffness systems using the neural-network approach. The nonlinear system identification models are fitted to physical displacements. Statistically independent NNMs are analyzed to study nonlinear structures under the action of broadband random excitation. The obtained NNMs are successful for a practical nonlinear extension to modal analysis and single-input-single-output decomposition.

The experimental identification of NNMs of the structure, which consists of the main beam and the cross beam, which is welded in the middle to the main beam, is studied in [137, 138]. The main beam is clamped at both ends. The NNMs are measured experimentally, when a shaker is attached to the main beam. Then this beam performs pure bending vibrations. The structure dynamics, when natural frequencies of the bending and torsion modes are close, is considered. The finite element model of the beam structure in Abacus is generated taking into account the geometrical nonlinearity [137]. The implicit condensation method (ICM) is applied to obtain the finite DOF reduced order dynamical system with quadratic and cubic nonlinearities. NNMs of this system are obtained by the combination of the shooting technique and the pseudo-arclength continuation. The comparison between the NNMs and the forced response is made. The considered system is approximated by two DOF dynamical model in [138]. Then NNMs are obtained by HBM. The energy transfer between such modes is considered.

Identification of finite-DOF nonlinear models of the thin-walled structures is performed in [139]. The considered structures have several sources of nonlinearities: piezoelectric material properties and geometrical nonlinearity. The concept of NNMs and NF theory are used for the theoretical analysis. Experimental approach to measure NNMs backbone curve is suggested. The measurement method based on the phase-locked loop is used for identification of the ROM. The whole procedure is used for a circular plate, a Chinese gong and a piezo-electric cantilever beam.

The NNMs identification method, which is applied to a nonlinear beam model, is proposed in [140]. The beam with one clamped end and the other one connected to a thin beam is studied experimentally. The frequency-domain nonlinear subspace identification method is applied for NNMs identification from the simulated data. The shooting method combined with pseudo-arclength continuation technics is used to compute NNMs. The Bayesian approach is used to obtain the posterior probability distribution of updating parameters by means the stochastic simulation methods. The numerical simulations to the nonlinear beam dynamics show that both deterministic and Bayesian approaches are capable to estimate the updating

parameters with a small error. The Bayesian approach is used for the wing-engine system, which consists of wing plate and two pylons in [141]. The finite element model is generated and simplified by the Craig-Bampton reduction method (CBM) [83]. HBM is used to calculate NNMs. Good agreement exists between the calculated and experimentally identified NNMs.

The polynomial nonlinearity state-space model is based on multiple-input-multiple-output formulation in [142]. This model takes the following form:

$$\begin{aligned}\mathbf{x}(t_{k+1}) &= \mathbf{A}\mathbf{x}(t_k) + \mathbf{B}\mathbf{u}(t_k) + \mathbf{E}\boldsymbol{\varepsilon}(\mathbf{x}(t_k), \mathbf{u}(t_k)); \\ \mathbf{y}(t_k) &= \mathbf{C}\mathbf{x}(t_k) + \mathbf{D}\mathbf{u}(t_k) + \mathbf{F}\boldsymbol{\chi}(\mathbf{x}(t_k), \mathbf{u}(t_k)),\end{aligned}\quad (20)$$

where t_k is a discrete time; $\mathbf{x} \in R^q$ is a state vector. The linear part of the model is determined by the \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} matrices. Nonlinear part of the model is characterized by the \mathbf{E} , \mathbf{F} matrixes. The nonlinear vectors functions $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ are polynomials. The input $\mathbf{u} \in R^q$, which describe the excitation force, is generated by shaker. The output vibrations response is obtained experimentally. The matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} , \mathbf{E} , \mathbf{F} are fitted from experimental data. The nonlinear optimization problem is formulated with the cost function in the following form:

$$\min_{(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F})} \sum_k \|\xi(t_k)\|^2, \quad (21)$$

where $\xi(t_k) = y_{ref}(t_k) - y_{sim}(t_k)$; $y_{ref}(t_k)$ is experimentally measured output; $y_{sim}(t_k)$ is numerically simulated data. The aim of the identification is to minimize the difference between the numerically simulated data and the experimentally measured output by correct choosing the model (20) parameters. The nonlinear vibrations of the cantilever beam with repelling magnets are analyzed. The nonlinearities are approximated by polynomial functions. The nonlinear mode model shows a vicinity of the numerical results and the experimental measurements.

The approach for identification of nonlinear multi DOF system is suggested in [143]. The basis of this algorithm is the single nonlinear resonant mode method. Then the one DOF approximation of the system solution in the vicinity of the principle resonances is used. Several measurements with different forcing amplitudes are used for identification of nonlinear motions and the dependence of damping on amplitude. The suggested approach can be used to analyze the structure vibrations without internal resonances. The models of clamped-clamped curved and flat beams are treated. The approach uses measurements of the NNMs backbone curves.

8. Experimental measurement of NNMs

Till recently majority researches of NNMs are performed numerically and analytically on the basis of nonlinear ODEs analysis. The first studies on NNMs experimental analysis originate during the last decade. The approaches and results of NNMs experimental analysis are treated in this Section.

The aim of the papers [144, 145] is developing the methodology of the practical experimental modal analysis of nonlinear mechanical structures. The new approach for extracting NNM modal curves from the experimental time series is suggested. The finite DOF nonlinear dynamical system has the form (14). It is assumed that the extracted NNM $\mathbf{q}_{nnm}(t)$ satisfy this system and, simultaneously, this NNM satisfies the equations of the corresponding conservative system. As a result, the following equation can be derived from (14):

$$\mathbf{C}\dot{\mathbf{q}}_{nnm} = \mathbf{p}_{nnm}(\mathbf{t}). \quad (22)$$

Thus, the excitation is compensated by the structure damping. The NNMs, which satisfy the Rosenberg definition, are obtained as the following truncated Fourier series:

$$\mathbf{q}_{\text{nnm}}(t) = \sum_{k=1} \mathbf{x}_k^{(\text{nnm})} \cos(k\omega t). \quad (23)$$

Then the special type of the excitation satisfied the relation (22) takes the form:

$$\mathbf{p}_{\text{nnm}}(\mathbf{t}) = - \sum_{k=1} \mathbf{c} \mathbf{x}_k^{(\text{nnm})} k \omega \sin(k\omega t).$$

Thus, the NNM \mathbf{q}_{nnm} is excited by special type of the external force. The proposed methodology is used in experimental studies of NNMs. The forced NNMs approximation is expressed according the above-considered method. Nonlinear system oscillates according the NNM of the conservative system, if the response is monophase periodic motions with the phase lag of 90^0 with respect to the excitation.

The periodic response of the nonlinear system (14) is expressed as the following complex Fourier series:

$$\mathbf{q}(t) = \sum_k \text{Re}(\mathbf{z}_k \exp(ik\omega t)), \quad (24)$$

The mode indicator function Δ is used to define the NNM approximation as

$$\Delta = \frac{1}{N} \sum_{k=1}^N \frac{\text{Re}(\mathbf{z}_k)^* \text{Re}(\mathbf{z}_k)}{\mathbf{z}_k^* \mathbf{z}_k}, \quad (25)$$

where N is number of essential harmonics in (24). The function Δ indicates the accuracy of the NNM appropriation. The parameter Δ is used for identification of NNM, which is experimentally observed.

The paper [146] is devoted to experimental and numerical analysis of the perforated circular plate dynamics. Two finite element models are obtained. One of them is constructed with zero initial curvature over the center of the plate, and the second one is constructed without this assumption. The authors use the assumption that the perforated plate behaves identically to non-perforated one with reduced mechanical properties. The electrodynamic shaker is used to excite the perforated plate vibrations. The reduced elastic modulus and density are updated from the experimental analysis. The frequency responses of the plate nonlinear vibrations are measured and the NNMs backbone curves are extracted. Then the experimentally measured NNMs are compared with numerically calculated NNMs.

The NNMs can be measured experimentally by the action of the multi-harmonic excitation even in the case of the internal resonance [147]. The system (14) is analyzed using the EBM when the following equation is satisfied for NNMs:

$$E_{\text{dis/cyc}} = E_{\text{force/cyc}}, \quad (26)$$

where $E_{\text{dis/cyc}} = \int_0^T \dot{\mathbf{q}}(\mathbf{t})^* \mathbf{C} \dot{\mathbf{q}}(\mathbf{t}) dt$ is the total dissipated energy of per one cycle of vibrations; T is period of vibrations; the asterisk denotes transposition; $E_{\text{force/cyc}} = \int_0^T \dot{\mathbf{q}}(\mathbf{t})^* \mathbf{p}(\mathbf{t}) dt$ is the energy of the external force $\mathbf{p}(\mathbf{t})$. The nonlinear vibrations of the clamped-clamped beam with concentrated magnetic force are studied numerically and experimentally. The indicator (25) is used to isolate accurately the NNMs in the experimental analysis and to obtain their backbone curves. The nonlinear vibrations of the clamped-clamped beam with concentrated magnetic force are studied experimentally. As follows from experimental analysis, NNM even in the case of the internal resonance can be isolated by the action of the multi-harmonic excitation. The

indicator (25) is used in experimental analysis to isolate accurately the NNMs. Two NNMs, which are obtained experimentally, are shown on backbone curves. As follows from the experimental analysis, these backbone curves are very sensitive to the beam initial shape and the boundary conditions.

The structure consisting of two beams is treated in [148]. The main clamped-clamped beam has non-linear behavior due to the axial stresses, which are generated by the moderate transversal displacements. The cross beam with a movable mass at each end is attached to the main beam. Changing the position of one mass with respect to the others masses positions leads to the modal interaction. A nonlinear ROM is constructed and the NNMs are used to describe the resonance nonlinear behavior of the system. The NNMs backbone curves, which are obtained numerically and experimentally, are compared. The EBM helps to predict the points, where the resonance forced responses cross the backbone curves. Then the forced vibrations of the structure are studied using several analytical and numerical approaches. These results demonstrate a good agreement with the experimental measurements. The NNMs considered in [148] are analyzed in the general 2-DOF system with the cubic nonlinearity and 1:3 internal resonance condition in [149]. Such system can be obtained as ROM from the continuous geometrically nonlinear cross-beams structure that exhibits a coupling between its bending and torsional modes. Origination of the NNM from a set of the bifurcations is analyzed numerically. The comparison of the experimental measurements of NNMs with the numerical results is performed.

The new experimental method to measure the backbone curve using the phase-locked-loop (PLL) is suggested in [150,151]. Such method permits to track the energy dependent backbone curves associated with NNMs. The EBM is used to study the forced vibrations by balancing forcing with damping. The NNMs of the beam with essentially nonlinear end spring are analyzed experimentally. The structure is excited by the electrodynamic shaker. The mode indicator function (25) is used to identify the forced NNMs, which are obtained experimentally. The effect of internal resonance on the experimentally measured PLL is discussed. The beam structure excited by the electrodynamic shaker with the small beam to adjust the pretension of the beam is considered in [151]. The PLL controller permits to tune excitation phase to measure the backbone curves of the NNM using a series of steady-state measurements.

The approach suggested in [152,153] is an experimental procedure for nonlinear modal testing of damped structures. It is assumed that the local phase resonance corresponding to a single general coordinate is taken place. The excitation frequency is recorded when the PLL has the locked state. The verification of the experimental tests is carried out on the friction-damped system. The extractions of the frequencies, damping ratio, deflection shapes are performed due to this method. The backbone curves of mechanical system are extracted. The NNMs of lightly damped slender beam with geometric nonlinearity excited by the electrodynamic shaker is analyzed experimentally in [153]. Seven accelerometers are mounted on the beam point to measure the structure nonlinear response. The synthesis method relies on a superposition of one nonlinear mode with several linear modal contributions and is limited to NNMs without internal resonances. For lower vibration energy the linear modal analysis can be used, but a phase-controlled tracking of the backbone curves associated with NNMs permits to obtain a good estimation of the nonlinear frequency response.

The mechanical system with a rotating nonlinear energy sink (NES) is analyzed in [154]. The NES is attached to a linear two floors structure modeled by the two-DOF system

with strongly nonlinear coupling. The transient dynamics of the structure under shock excitation is analyzed both numerically by the NNM energy-frequency plot calculation and experimentally. It is shown that the oscillatory mode is not appropriate for initial energy dissipation and the rotational mode guarantees the fast shock mitigation. The NES absorbs a significant portion of the input energy.

The approach for detection of nonlinear modal interactions from the transient time series response is suggested in [155]. The POD is used to extract the modes shapes, which is applied in Rayleigh quotient for the system eigenfrequencies calculations. The method is used to analyze the response of the cantilever beam with local strongly nonlinear spring. The suggested method can be used to detect the strongly nonlinear interactions from measured transient response.

The machine learning scheme is used to perform nonlinear modal analysis in [156]. The one-to-one mapping from modal space to the natural coordinate space is applied. The generative network and the neural networks, which support the orthogonality properties, are used. Three-storey frame with the bumper nonlinearity in the form of bilinear stiffness is studied experimentally. Four sensors are used to measure the accelerations of all storeys and the base.

As follows from this section, NNMs are analyzed experimentally a little. The authors of this review believe that this direction will be developed due to importance of relevant research for the NNM theory.

9. Continuous systems

Note that geometrically nonlinear deformation is inherent in thin-walled structures, such as rods, plates and shells. The use of NNMs to analyze the dynamics of such distributed systems is very useful.

The flexural-longitudinal vibrations of isotropic rotating beam are described by the system of the nonlinear PDEs, which is obtained from the Hamiltonian principle in [157]. Both three to one and two to one internal resonances are accounted by the MSM. The expressions for the backbone curves are derived. The backbone curves, which are obtained for the first and the second internal resonances, are soft and hard, respectively.

In the next three publications, the dynamics of isotropic and anisotropic rods is studied using the Cosserat theory.

The paper [158] is devoted to study NNMs of the beam vibrations, when internal forces/moments and translational/angular velocities are introduced as primary degrees of freedom. The system of PDEs obtained by Hodges [159] is used. This system contains the equations of motion and the compatibility equations:

$$\mathbf{m}\dot{\mathbf{x}}_1 - \mathbf{x}'_2 - \mathbf{e}\mathbf{x}_2 + L_1(\mathbf{x}_1)\mathbf{m}\mathbf{x}_1 + L_2(\mathbf{x}_2)\mathbf{c}\mathbf{x}_2 = \mathbf{f}_1; \quad (27)$$

$$\mathbf{c}\dot{\mathbf{x}}_2 - \mathbf{x}'_1 + \mathbf{e}^T \mathbf{x}_1 - L_1^T(\mathbf{x}_1)\mathbf{c}\mathbf{x}_2 = 0.$$

Here \mathbf{x}_1 is a vector of the velocities, \mathbf{x}_2 is a vector of the internal-forces/moments, \mathbf{m} is the cross-section mass matrix, \mathbf{c} is the cross-section flexibility matrix, \mathbf{e} presents the shear/bending coupling coefficients in equilibrium equations; L_1, L_2 are matrix operators of the nonlinear

equilibrium equations. The variables are expanded in terms of the linear space mode shapes. Then the S-P NNMs are used to analyse the isotropic and composite beams in-plane vibrations without internal resonances.

The nonlinear PDEs of the pre-twisted rotating composite beams vibrations are derived by using the Cosserat theory in [160, 161]. The constitutive equations of the composite beam material are obtained from three dimension theory of elasticity. The MSM is applied directly to the nonlinear PDEs in [160]. As follows from the analysis, the lowest flapping mode of the composite beam has soft backbone curve at low angular velocity and the backbone curve of the second flapping mode is hard. The nonlinear modal interaction due to the 2:1 internal resonance is studied. The Galerkin technique is applied to the mentioned PDEs in [161]. Then MSM is applied to the system of nonlinear ODEs to analyze the nonlinear flapping modes.

The nonlinear planar oscillations of the functionally graded box beam in the case of internal resonance are considered in [162]. The combination of thermal and mechanical harmonic transverse loads act on the structure. The beam is rotated with constant angular velocity Ω . It is assumed that the beam cross section remains plane during the structure flexure vibrations. The Euler-Bernoulli beam theory is used. The flexural vibrations of beam are described by the following nonlinear integro-differential equations:

$$\ddot{v} - \chi v'' + v^{iv} + 2\mu\dot{v} - \gamma v'' \int_0^L v'^2 dx + \lambda v' = f \cos(\omega t), \quad (28)$$

where $\gamma = \frac{EA}{2EI}$; $\lambda = \bar{\Omega}^2 \left(0.5 + \frac{R_0}{L}\right)$; $f = \frac{F(x)L^4}{EI}$; $\bar{\Omega}^2 = \Omega^2 \frac{J_{11}^\rho L^4}{EI}$; $\chi = \frac{J_{11}^\rho L^2}{EI} \left(\frac{L^2}{6} + \frac{R_0 L}{2}\right) \Omega^2 - L^2 \frac{EA \bar{Q}_T}{EI}$; EA, EI are axial and flexural stiffness; L is length of the beam; R_0 is radius of the hub; $\bar{Q}_T = \alpha \Delta T$; α is thermal expansion coefficient; $F(x)$ is spatial distribution of the harmonic load; $J_{11}^\rho = \int_A \rho(n) ds dn$; ρ is graded mass density; n is normal axis to the beam cross section contour; A is area of the beam cross section. The PDE (28) is solved directly by MSM. As a result, the system of four modulation equations is derived. Numerical analysis is used to obtain a steady-state response. Frequency responses and bifurcation behavior in the case of internal resonances are obtained. In particular, the saddle-node and the Hopf bifurcations are studied. It is shown that chaotic behavior of the system is possible due to the period-doubling bifurcations.

The offshore compliant articulated tower consists of stiffened inverted column and mass at the end of the column [163]. This column is pinned at the base. Dynamics of the system is described by two-DOF nonlinear system with respect to two direction angles. Four stable similar NNMs of free vibrations are obtained. The Poincaré sections are used to describe global dynamics of the system. The resonance forced responses are obtained by the combination of the HBM and the arc-length continuation method. Jump phenomena, pitchfork and saddle-node bifurcations are analyzed.

The paper [164] is dedicated to the gradient-based optimization using NNMs. Such optimization is proposed for tailoring the hardening/softening backbone curves behavior of the geometrically nonlinear thin-walled structures, which are discretized by FEM. The HBM with multi harmonic approximation is used to study periodic free vibrations. The system of the nonlinear algebraic equations with respect to the amplitudes of the harmonics is solved numerically by the iterative Newton- Raphson method. In order to obtain the frequency

response, the arclength continuation method with adaptive steps is used. The one-floor frames of two different configurations consisting of three beams are studied.

The nonlinear dynamics of simply supported beams is studied by computing the NNMs and to predict damage in such structure [165]. The structure damage is modeled by a reduction of the flexural stiffness within a small segment of the beam span. MMS is used to compute the NNMs and the corresponding backbone curves. The comparison between the damaged and undamaged beams shows the high sensitivity of the effective nonlinearity coefficients on the damaged structure. An effective strategy for the identification of the damage position is proposed.

NNMs of simply supported beams are considered in [166]. The integro-differential equation of the lateral displacement motions is approximated by the system of nonlinear ODEs. The S-P NNMs of the obtained ODEs are analyzed using the perturbation technique. The obtained backbone curves are compared with the results of the finite element calculations.

The vibrations of the geometrically nonlinear circular perforated plate with rolled ends are discussed in [167]. The finite element model of the perforated plate is obtained using the implicit condensation method with extremely small mesh. The perforated structure vibrates identically to non-perforated one with the same dimension and effective modulus of elasticity, which is calculated from the perforated structure using linear natural frequencies. The obtained finite DOF nonlinear dynamical system contains quadratic and cubic nonlinearities. The shooting method and pseudo arc-length continuation are used together to calculate the NNM backbone curves. Moreover, the NNMs of the perforated plate are analyzed experimentally when the structure vibrations are excited by the periodic force. Then the obtained numerical results are compared with the experimental data.

The method for computing of invariant manifolds in high dimensional nonlinear mechanical system, which is derived from finite element discretization of partial differential equations, is suggested in [116]. SSMs are used to analyze frequency responses and backbone curves of nonlinear damped mechanical systems. Lyapunov subcenter manifolds are applied to study conservative nonlinear systems. The computation method is applied to the nonlinear dynamics of aircraft wing with hundred-thousand DOF.

The geometrical nonlinear vibrations of the simply supported cylindrical shell are described by the system of three PDEs of the Donnell theory [168]. Three displacement projections of the shell middle surfaces are expanded using the eigenmodes and the generalized coordinates. The interactions of two conjugate modes are considered. The Galerkin technique is used to derive the system of nonlinear ODEs with respect to the generalized coordinates. K-R NNMs are analyzed. The results are shown on the backbone curves. The obtained NNMs describe the standing waves in the cylindrical shell.

The nonlinear vibrations of clamped shallow shell and plates with cuts are considered in [169, 170]. The thin shallow shell with two cuts is treated in [169]. The lateral displacements are assumed to be commensurable with the structure thickness. The clamped shallow shell flexural vibrations are described by the Donnell-Mushtari equations with the following boundary conditions:

$$u|_{\partial D} = 0; v|_{\partial D} = 0; w|_{\partial D} = \frac{\partial w}{\partial n}|_{\partial D} = 0, \quad (29)$$

where u, v, w are displacements projections on (x, y, z) ; ∂D is the shallow shell boundary; n is a normal to the boundary. The shell base has the two cuts. Therefore, the boundary of this shallow shell has complex shape. The R-function $\Omega(x, y)$ [171] is used to describe such complex shape boundary. The function Ω satisfies the following relations:

$$\begin{aligned}\Omega(x, y) &= 0; \forall (x, y) \in \partial D; \\ \Omega(x, y) &> 0; \forall (x, y) \in D.\end{aligned}\tag{30}$$

In order to satisfy the boundary conditions, the shell linear vibrations are presented using function Ω in the form:

$$\begin{aligned}u(x, y, t) &= \Omega(x, y)P_1(x, y) \cos(\omega t); \\ v(x, y, t) &= \Omega(x, y)P_2(x, y) \cos(\omega t); \\ w(x, y, t) &= \Omega^2(x, y)P_3(x, y) \cos(\omega t),\end{aligned}$$

where ω is a frequency of linear vibrations. Then the Rayleigh-Ritz method is used to obtain trial functions. The functions $P_1(x, y), P_2(x, y), P_3(x, y)$ are presented as truncation series with trial functions and unknown coefficients. These coefficients and eigenfrequencies are obtained from the eigenvalue problem. Using the Galerkin technique, the nonlinear dynamical system with respect to the generalized coordinates is derived. The backbone curves of free vibrations are obtained using S-P NNMs. The clamped circular isotropic plate with two cutouts is treated in [170]. The R-function is used to describe the plate boundary conditions. Nonlinear vibrations of the plate are described by von Kármán geometrically nonlinear plate theory. The system of three nonlinear PDEs with respect to displacements projections is used. The vibrational eigenmodes are obtained by the Rayleigh-Ritz method. Applying the Galerkin technique, the nonlinear system of ODEs with small parameter is derived and analyzed by MSM. The NNMs backbone curves are described by the system of algebraic nonlinear modulation equations.

The isotropic shallow shell with complex boundaries is considered in [172]. The part of the boundary is clamped and the rest part is free. The geometrically nonlinear vibrations of the shell interacting with the motionless fluid are treated. The wet vibrational eigenmodes of the shell are expanded into truncation series of the dry eigenmodes. The coefficients of this series are calculated from the eigenvalue problem, which are obtained using the Galerkin technique. The interaction of the shell with the fluid is described by the following singular integral equation:

$$\frac{1}{4\pi} \iint_S \Gamma_k(\zeta) \frac{\partial^2}{\partial \mathbf{n}_x \partial \mathbf{n}_\zeta} \left(\frac{1}{|\mathbf{x} - \zeta|} \right) dS_\zeta = W_k(\mathbf{x}),\tag{31}$$

where W_k is wet eigenmodes of the shell vibrations; Γ_k is circulations, which is induced by wet eigenmode W_k ; \mathbf{x}, ζ are spatial coordinates vectors; S_ζ is shell surface; $\mathbf{n}_x, \mathbf{n}_\zeta$ are normals to the shell surface. The hypersingular equation (31) is solved by the boundary element method [173]. Namely, the surface of the shell S is divided into J triangles $S_j; j = 1, \dots, J$. The coordinates of the triangles gravity centers $\mathbf{x}_{0j}; j = 1, \dots, J$ are used. The circulation is assumed constant in

every triangular element $\Gamma_{k,j} = \Gamma_k(\mathbf{x}_{0j})$. Then the hypersingular equation (31) is transformed into the system of the linear algebraic equations, which has the following form:

$$\sum_{\mu=1}^J H_{j\mu} \Gamma_{k,\mu} = W_k(\mathbf{x}_{0j}); j = 1, \dots, J, \quad (32)$$

where $H_{j\mu}$ is determined from the hypersingular integral. In order to study the nonlinear vibrations of the shell/fluid system with account of the structure geometrical nonlinearity, the assumed-mode method is used. The potential and kinematic energies of the structure are derived. The system of the nonlinear ODEs of the structure motions is obtained. Then S-P NNMs are used to study the backbone curves of nonlinear vibrations.

The simply supported isotropic cylindrical shell under the action of the parametric excitation $N_0 + N_1 \cos(2vt)$ is considered in [174]. The geometrical nonlinearity is accounted to study the parametric vibrations, which are described by the Donnell-Mushtari equations. Three conjugate modes are taken into account in the expansion of the shell radial displacements. After discretization using the Galerkin technique, the parametric vibrations are described by the following eight DOF nonlinear dynamical system with respect to the generalized coordinates (f_1, \dots, f_8) :

$$\begin{aligned} \ddot{f}_i + \omega_i^2 f_i + f_i R_i + G_i + \chi_i(N_0 + N_1 \cos(2vt)) f_i &= 0; i = 1, \dots, 6; \\ \ddot{f}_7 + \frac{4}{3} \ddot{f}_8 + \omega_7^2 f_7 + \tilde{\omega}_8^2 f_8 + \sum_{j=1}^6 \gamma_{7j} f_j^2 &= 0; \\ \ddot{f}_8 + \frac{1}{2} \ddot{f}_7 + \omega_8^2 f_8 + \tilde{\omega}_7^2 f_7 + \sum_{j=1}^6 \gamma_{8j} f_j^2 &= 0, \end{aligned} \quad (33)$$

where $R_i(f_1, \dots, f_7), G_i(f_1, \dots, f_6)$ are nonlinear functions of the generalized coordinates; $\gamma_{7j}, \gamma_{8j}, \omega_7^2, \tilde{\omega}_8^2, \omega_8^2, \tilde{\omega}_7^2$ are constants, which depend on the structure parameters. Two K- R NNMs in the form

$$f_{2i-1} = \pm f_{2i}; i = 1, 2, 3;$$

exist in the system (33). The parametric vibrations on these NNMs are described by three DOF nonlinear dynamical systems, which are studied by the HBM.

The nonlinear vibrations of the simply supported rectangular plate with the moderate amplitudes are treated in [175]. The plate potential energy is derived using the von Karman nonlinear plate theory. The assumed-mode method is used to derived the system of nonlinear ODEs from kinetic and potential energies. The S-P NNMs are used to obtain the backbone curves of the free nonlinear vibrations.

NNMs are studied mainly in roods, certain plates and shallow shells. The NNMs in cylindrical and conical shells are not analyzed. Preferably the internal resonance 1:1 is observed in these structures and circular plates and circular shallow shells. We think that the multi-mode invariant manifold [16] can be used to study NNMs. The most of thin-walled structures have internal resonances; therefore, a use of the multi-mode invariant manifolds is

useful. We emphasize that such studies with rare exceptions have not been carried out. Note also that NNMs of shell structures from composite, functionally gradient materials and nanocomposites are not carried out, although the authors of this review believe that this direction is promising.

10. Engineering applications

10.1. Aerospace engineering

Linear normal vibration modes are very popular tools to analyze a dynamics of the engineering structures. Unfortunately, the NNMs approach has been used in a few studies related to engineering problems.

Note, that the CBM [83] is used to reduce a dimension of the finite element structure model in the several subsequently considered papers.

The NNMs of the Morane–Saulnier Paris aircraft airframe without its jet engine are analyzed in [176, 177]. It stands on the ground through landing gears. The soft backbone curve is observed experimentally by the ground vibration test. The connection between fuel tanks and wings are the source of the nonlinearity, which is piecewise linear. The finite element model of the full aircraft is generated. The wings, fuselage, tails are modeled using the beams and the shells finite elements. The shooting technique and the pseudo-arclength continuation, which are mentioned in Section 6, are used to compute the NNMs from the reduced-order nonlinear finite element model.

The papers [178,179] are dedicated to study of nonlinear dynamics of the spacecraft structure with nonlinear mechanical stops. This spacecraft structure supports telescope, which is mounted on the base. The telescope plate is connected to the floor by shock attenuators, which exhibit nonlinear properties. The bracket connects to wheel mounting system to absorb the high-frequency excitation. The experimental identification of this structure including detecting of important nonlinear effects is made in [178]. The composite telescope structure is discretized using orthotropic shell elements. The floor wheel supports are modeled using shell finite elements. The nonlinearity of the wheel mounting system is modeled by the trilinear elastic characteristics. The ROM is obtained using the CBM. Then NNMs of the obtained trilinear nonlinear reduced system are analyzed numerically by using the shooting technique. The frequency-energy plots of the NNMs are calculated.

The nonlinear oscillations of the feed line of the launch vehicle (Fig.9) are treated in [180]. The fuel is moved into the pump 3 from the tank 1. The gas-liquid damper 2, 3 absorbs the fuel oscillations. The fuel dynamics are described by the pressure oscillations, which are modeled by one linear partial differential equation and one nonlinear boundary condition. This mathematical problem is reduced to the system of nonlinear ODEs using the weighted residual method. Then the S-P NNMs are used to obtain the backbone curves.

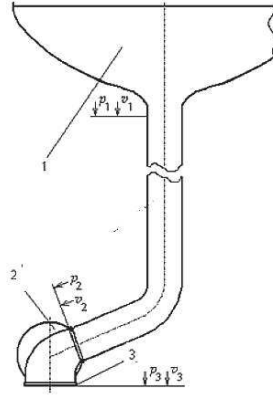


Fig.8 Feed line of launch vehicle

The turbofan engine, which is a part of the aircraft thrust, consists of blades and discs on the rotating shaft [181]. The nonlinearity of this structure is concentrated in the dovetail joints. The friction joints are the source of nonlinearity. The FEM is used to obtain the blade model. 3D node to node contact elements are used to model the contact interface. The CBM is used to obtain the ROM. Then the HBM and the alternating frequency-time technique, which permits to calculate the nonlinear contact friction force, are applied to study the damped NNMs of the ROM. The obtained backbone curves of NNMs are hard.

10.2. Power engineering

The modal analysis of the turbomachinery compressor blades with dry-friction nonlinearity is considered in [38]. The concept of the complex nonlinear modes (CNMs), which is considered in the Section 2, is used to study mechanical systems with dry friction. The reduced order model for the blade is obtained by FEM and then essentially simplified by the CBM. The influence of the dry friction on the compressor blades vibrations is analyzed.

The self-sustained vibrations of one disc elastic rotor in two identical journals bearing are analyzed in [182]. The rotor is rotated with constant angular velocity Ω around z axis. The disc motions are described by four coordinates, namely, two plane translations x, y and two rotation angles $\theta_1; \theta_2$. The journal motions are described by four variables x_1, y_1, x_2, y_2 , which are obtained as a function of four generalized coordinates x, θ_1, y, θ_2 . The rotor motions are described by four DOF dynamical system, which has the following matrix form:

$$\ddot{\mathbf{q}} + \mathbf{F}\dot{\mathbf{q}} + \mathbf{Q}\mathbf{q} = \widetilde{\mathbf{W}}(\mathbf{q}, \dot{\mathbf{q}}), \quad (34)$$

where $\mathbf{q} = [x, \theta_1, y, \theta_2]^T$ is a vector of the generalized coordinates; $\mathbf{F}\dot{\mathbf{q}}$ describes the linear damping forces of the journal bearings fluid films and the gyroscopic forces; $\mathbf{Q}\mathbf{q}$ presents the shaft elastic forces and linear elastic forces of the journal bearing fluid film; $\widetilde{\mathbf{W}}(\mathbf{q}, \dot{\mathbf{q}})$ is the nonlinear part of the journal bearing forces. The self-sustained oscillations of the rotor are arisen due to the Hopf bifurcation of trivial equilibrium. The S-P NNMs are developed to analyze these oscillations.

The motions of the rotor with nonlinear stator contact are described by finite DOF piecewise smooth system in [183]. Such motions are analyzed analytically using the NNMs. As follows from the analysis, NNMs can be used to obtain the self-excited whirl motions. These motions are observed due to the Hopf bifurcation.

The paper [184] is devoted to study of the spar floating platform dynamics. The nonlinear dynamics of this platform is described by 2-DOF model. The heave and pitch dynamical responses of free and forced vibrations are analyzed. The wave motions are described by the single harmonic approximation. The S-P NNMs are obtained by Galerkin approach using the polynomial series approximations. The stability of the NNMs is analyzed by the Floque theory. As follows from the numerical simulations, the similar and non-similar K-R NNMs exist in the system configurational space.

The torsional vibrations of the power train with piecewise-linear elastic moments are studied in the paper [185]. The clutches are described by these moments. The system dynamics is approximated by finite-DOF nonlinear dynamical system. NNMs of the torsional vibrations are studied by HBM. The power train parameters are chosen so, that the resonance forced vibrations turn away from the operation frequency range of the crankshaft. The linearized model and the theorem about sensitivity of the eigenfrequencies are used to detune the forced vibrations. After detuning the linearized system, the backbone curves of the NNMs are calculated to verify the choice of the system parameters.

The rubbing rotor, which consists of the stator and the Jeffcott rotor, is analyzed in [186, 187]. The rotor massless model has elastic shaft and solid disc. The mass center of the disc and its geometrical center are located at the distance e . The rotor is moved with constant angular velocity $\bar{\omega}$. Governing equations for modified Jeffcott rotor system free vibrations with a rub-impact take the following non-dimensional form:

$$X'' + 2\zeta X' + \gamma X + H(R - R_0) \left(1 - \frac{R_0}{R}\right) (X - \text{sign}(V_{rel})\mu Y) = \Omega^2 \cos\Omega\tau; \quad (35)$$

$$Y'' + 2\zeta Y' + \gamma Y + H(R - R_0) \left(1 - \frac{R_0}{R}\right) (Y + \text{sign}(V_{rel})\mu X) = \Omega^2 \sin\Omega\tau.$$

Here $X = x/e$; $Y = y/e$; x, y are horizontal and vertical displacements of the disc; $R = \sqrt{x^2 + y^2}/e$; $R_0 = r_0/e$ and r_0 is the distance between stator and disc; $H(\cdot)$ is the Heaviside

function; μ is the friction coefficient; $\gamma = k/k_c$; k is the shaft stiffness; k_c is the stator stiffness;

$V_{rel} = \Omega R_{disc} + R\omega$; $\Omega = \sqrt{m\bar{\omega}}/\sqrt{k_c}$; $R_{disc} = r_{disc}/e$; r_{disc} is radius of the disc; $\Omega =$

$\sqrt{m}\omega_w/\sqrt{k_c}$; ω_w is the whirl angular velocity of the rotor. The concept of CNMs is used to

analyze corresponding autonomous system, that is the unbalance forces on the right side of the governing equations (35) are ignored. Then nonlinear free oscillations and the nonlinear rubbing forces are presented in the form of the truncated Fourier series. The system of algebraic equations for determination of these series coefficients is solved numerically by the Newton method. As follows from the numerical analysis, the rotor has backward and forward whirl motions. The whirl amplitude and whirl frequency of the dry whip are approximately equal to

the modal amplitude and modal frequency of the backward whirl mode motions. The existence boundary of the dry whip can be predicted by this mode.

The paper [188] is devoted to study the nonlinear dry friction dampers in aircraft engine. The influence of the contact parameters on the system dynamics is analyzed numerically. NNMs calculations are based on the implementation of the HBM together with the continuation technique. The backbone curves of the NNMs have soft behavior. The robustness of the frictional damper is discussed.

The hybrid damper, coupling the dry-frictional mechanism and the piezoelectric shunting circuit is used to absorb the turbine blade vibrations in [189]. The concept of CNMs and the multi-HBM permit to calculate the steady-state forced vibrations of the cyclic symmetric system. Two indicators (the modal damping ratio and the nonlinear modal electromechanical coupling factor) are used to estimate both the frictional and the piezo-electric damping effects. The FEM analysis is shown that the proposed hybrid damper is effective for absorption of the dominant blade and disc-blade vibrations modes.

The wing-engine system, which consists of wing plate and two pylons, is analyzed experimentally and numerically in [190]. The pylons have thin plates, which are clamped at the thin wing plate (Fig.9a). The plate restoring force has nonlinear dependence of the pylon displacements. The software Abaqus is used to generate finite element model (Fig.9b), which has huge dimension. The CBM is applied to obtain ROM. Then the HBM is used to calculate NNMs. The NNMs backbone curves show hard behavior.

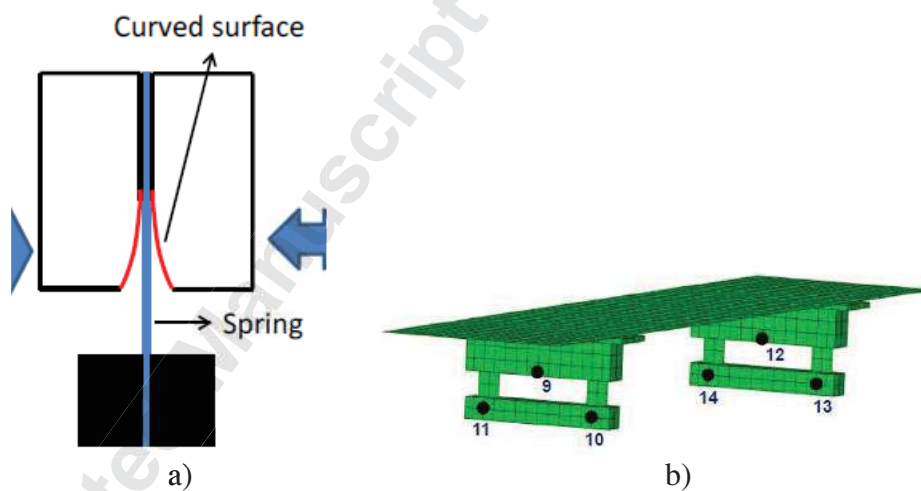


Fig.9 The wing-engine structure. Reprinted with permission from Elsevier

10.3. Piecewise-linear systems and structures with dry friction

Piecewise linear systems are often encountered in mechanical engineering. Thus, piecewise linear forces and moments are used to describe the dynamics of splined and elastic couplings, gears, rotors and aircraft structures. Piecewise linear systems are models of beams with cracks, etc. In such systems, both regular and complex dynamic behavior can be observed. We believe, and this is confirmed by the papers presented in this sub-Section, that the study of NNMs is important to study the dynamics of the piecewise linear systems.

Papers [191-193] are devoted to analysis of the NNMs in the two-DOF piecewise-linear dynamical systems, which represent beams with a breathing crack. Numerical procedures and the Poincaré map are used for the NNMs analysis. The non-linear frequencies are independent of the energy level but depend on the crack parameters. It is shown that the influence of crack leads to bifurcations of the basic NNMs due to existence of internal resonances. Then additional branches of the NNM backbone curve are arisen. The influence of the crack on the NNM frequencies is studied. The two-DOF structure under the action of the harmonic excitation is analyzed experimentally. It is possible to match the NNM with the forced response of the system, which gives the ability to determine the crack position. All observed NNMs are divided into two types: persistent and ghost. If the NNM of free vibration exists in the forced response, then this NNM is called persistent. If the NNM of free vibration is lost under the action of the periodic force, then this NNM is called ghost.

As follows from [194, 195], the NNMs are useful tools to analyze the dynamics of the finite granular homogenous chains. The Hertzian contact between neighboring beads leads to essentially nonlinear dynamical system with discontinuous. As follows from the numerical analysis, the in-phase NNM is non-synchronous and discontinuity in slope of the phase plane. The in-phase NNM is similar to the traveling wave propagating backward and forward through the granular chain. The out-of phase NNM is smooth. The interaction of each bead with the wall or with the other bead is observed on this NNM. The nonlinear forced response of the structure under the harmonic excitation is analyzed numerically. As follows from the analysis, this granular media can be used as the shock protector.

Two DOF system with two particles and bilinear elastic force is treated in [196]. The NNMs are considered as periodic motions in unison according to the Rosenberg definition. The two impacts motions are analyzed as the modal lines in the configurational plane. The HBM and the continuation asymptotic numerical method are used to solve nonlinear algebraic equations. The energy-frequency plots are obtained numerically. The NNMs stability is studied.

Free vibrations of finite DOF system with the piecewise-linear elastic force are considered in [109]. This system is described by the ODEs in the matrix form,

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{F}(\mathbf{z}) = 0; \quad (36)$$

$$\mathbf{F}(\mathbf{z}) = \begin{cases} \mathbf{K}_1\mathbf{z}; & \Delta_2 \leq \mathbf{h}^T\mathbf{z} \leq \Delta_3; \\ \mathbf{K}_2\mathbf{z} - \mathbf{b}_2; & \mathbf{h}^T\mathbf{z} < \Delta_2; \\ \mathbf{K}_3\mathbf{z} - \mathbf{b}_3; & \Delta_3 < \mathbf{h}^T\mathbf{z}, \end{cases}$$

where $\mathbf{M} = \text{diag}(\mathbf{M}_1, \dots, \mathbf{M}_N)$; \mathbf{z} is vector of the generalized coordinates; $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ are matrices of $N \times N$ dimension; $\mathbf{b}_2, \mathbf{b}_3$ are vectors, which allow to obtain continuity of the elastic force. The following change of the variables is used $\mathbf{z} = \mathbf{Q}\boldsymbol{\eta}$, where the matrix \mathbf{Q} is the set of the eigenvectors of the matrix $\mathbf{M}^{-1}\mathbf{K}_1$. The S-P NNMs are used to study the dynamical system (36). The suggested modification of NNMs calculations allows reducing twice the dimension of the system of nonlinear algebraic equations in comparison with the approach suggested in [15]. The proposed approach is used to analyze the torsional vibrations of the internal combustion engine transmission. This transmission consists of gear train and two crankshafts. The crankshafts are joined by gear train, which has five spur gears (Fig.10a). One spur gear has a clutch, which is described by the piecewise-linear elastic characteristic (Fig.10b). The model

consists of the rigid discs and massless shafts and it has fifteen DOFs (Fig.10c). The S-P NNMs are used to obtain backbone curves of the torsional vibrations.

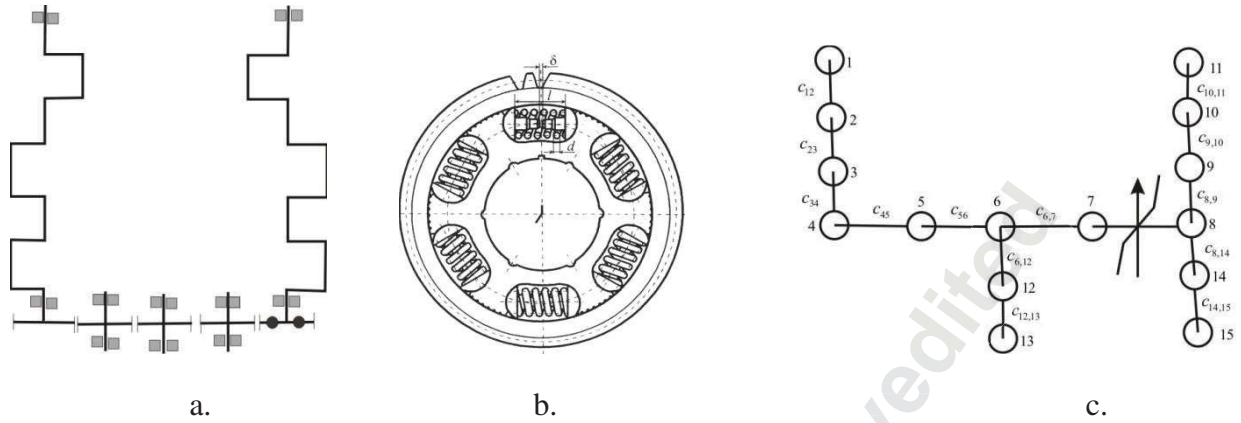


Fig.10 Sketch of diesel engine transmission: a). kinematic diagram; b). outline of elastic clutch; c) dynamical model

The forced vibrations of the piecewise linear mechanical systems are described by the ODEs similar to (36) with external periodic excitation [110]:

$$\ddot{\boldsymbol{\eta}} + \Lambda_1 \boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) + \mathbf{Q}_{*,k}^{-1} A \cos(\omega t), \quad (37)$$

where k is number the particle, which is excited by the periodic force $A \cos(\omega t)$; $\mathbf{Q}_{*,k}^{-1} = (Q_{1,k}^{-1}, \dots, Q_{N,k}^{-1})$ is k^{th} column of the matrix \mathbf{Q}^{-1} . The coordinates $\eta_1, \dot{\eta}_1$ are chosen as master ones. All the rest coordinates are slave. The motions are taken in the form of the single harmonic approximation ($\eta_1 = a \cos \phi$; $\dot{\eta}_1 = -a \omega_1 \sin \phi$). The periodic motions in the vicinity of the principle resonance ($\omega \approx \omega_1$) has the following form:

$$\eta_1 = a \cos(\omega t). \quad (38)$$

The periodic force is rewritten in the following form:

$$A \cos(\omega t) = A \cos \phi = \frac{A \eta_1}{\sqrt{\eta_1^2 + \omega_1^{-2} \dot{\eta}_1^2}}$$

The nonautonomous dynamical system (37) is transformed into the pseudo-autonomous one as

$$\ddot{\boldsymbol{\eta}} + \Lambda_1 \boldsymbol{\eta} = \mathbf{g}(\boldsymbol{\eta}), \quad (39)$$

where the elements of the vector $\mathbf{g}(\boldsymbol{\eta})$ are the following:

$$g_i(\boldsymbol{\eta}) = f_i(\boldsymbol{\eta}) + \frac{A Q_{i,k}^{-1} \eta_1}{\sqrt{\eta_1^2 + \omega_1^{-2} \dot{\eta}_1^2}}$$

The pseudo-autonomous dynamical system (39) is analyzed by the S-P NNMs. The proposed procedure permits to study nonlinear torsional vibrations of the fifteen DOF model of the diesel engine transmission (Fig.10c). The forced vibrations of the piecewise linear dynamical system (37) is considered in the vicinity of the superharmonic resonance in [197]. This resonance satisfies the following condition: $\omega = \frac{\omega_\mu}{M} + \varepsilon\gamma$, where μ is eigenfrequency number; M is integer number; $0 < \varepsilon \ll 1$ is small parameter; γ is detuning parameter. As follows from the resonance condition, the generalized coordinate η_μ is active. Therefore, it takes as the master coordinates $(\eta_\mu, \dot{\eta}_\mu) = (a \cos \phi; -a \omega_\mu \sin \phi)$, $\phi = \omega_\mu t$.

The Raucher's method is also used to study the NNMs of the forced superharmonic vibrations. The external force (37) is presented in the following form:

$$\cos(\omega t) = \cos\left(\frac{\phi}{M}\right) + O(\varepsilon) = \cos\left[\frac{1}{M} \arccos\left(\frac{\eta_\mu \omega_\mu}{\sqrt{\omega_\mu^2 \eta_\mu^2 + \dot{\eta}_\mu^2}}\right)\right] + O(\varepsilon) = G(\eta_\mu, \dot{\eta}_\mu) + O(\varepsilon). \quad (40)$$

Then pseudo-autonomous dynamical system (39), which describes superharmonic resonance vibrations, is obtained using the relation (40).

The torsional vibrations of the transmission are treated in the vicinity of the superharmonic resonance. These torsional vibrations are described by a two DOF mechanical system. The rotation angles of the discs $\theta_1; \theta_2$ are chosen for the generalized coordinates. The shaft with clutch connects two discs. This clutch is described by the tri-linear elastic moment [198]:

$$f(\theta_1 - \theta_2) = \begin{cases} c_1(\theta_1 - \theta_2); & -\Delta < \theta_1 - \theta_2 < \Delta; \\ c_2(\theta_1 - \theta_2) + \Delta(c_2 - c_1); & \theta_1 - \theta_2 \geq \Delta; \\ c_2(\theta_1 - \theta_2) - \Delta(c_2 - c_1); & \theta_1 - \theta_2 \leq -\Delta. \end{cases}$$

The dynamical system has the following form where moments of inertia of the discs are denoted by I_1 and I_2 :

$$\begin{aligned} I_1 \ddot{\theta}_1 + f(\theta_1 - \theta_2) &= A \cos(\omega t); \\ I_2 \ddot{\theta}_2 - f(\theta_1 - \theta_2) + c_1 \theta_2 &= 0. \end{aligned} \quad (41)$$

The superharmonic resonance vibrations are analyzed in the system configuration space. The resonance vibrations are presented at the frequency response.

Nonlinear oscillations of thin-walled structures with unilateral contact forces are considered in [199]. The beam interacts with unilateral elastic stops due to vibrations. The discrete lattice method is used to transform the continuous structure to finite DOF piecewise-linear dynamical system. Then the shooting technique is used to calculate NNMs. The frequency-stiffness plots and evolution of the NNM trajectories in the system configuration space are presented. The Floquet multipliers are calculated to analyze stability of these modes. Bifurcations of NNMs are analyzed numerically. NNMs in the tensionless granular chain with non-smooth Hertzian contact between the identical particles are studied numerically.

The research [200] is devoted to detection of thin beam health through the analysis of the nonlinear vibrations. The simply supported thin-walled beam with one crack is described by

nonlinear PDE of the beam flexural vibrations. This equation solution is expanded by eigenmodes of the simply supported beam without crack to obtain a multi-DOF nonlinear dynamical system using the Galerkin technique. The MSM is applied to study NNMs of the forced vibrations in the vicinity of the principle resonance. The frequency responses of the forced vibrations are obtained. The criteria of the crack existence are deduced from the analysis of the system nonlinear vibrations.

Two DOF nonlinear mechanical system with vibro-impact energy sink is treated in [201, 202]. This system is studied by using MSM to obtain the steady-state response, when the ratio of masses is used as a small parameter. The system forced vibrations in the vicinity of the resonance are analyzed. The closed form approximation of the NNM is obtained. The energy pumping mode takes place only at non-conservative impact conditions. PLL is used to obtain an appropriate excitation, where the frequency and phase are synchronized with respect to a reference signal. The parameters of the impact nonlinear energy sink, which results in low amplitude vibrations of the main subsystem, are chosen from the numerical analysis.

The self – sustained friction induced oscillations of the mass-spring chain with a generalized Rayleigh’s dissipative term are analyzed in [203]. Equations of motion for the two-DOF chain are transformed to new model with respect to the following variables: total excitation level, its distribution between the oscillators and coherency of the oscillations. Then the excitation threshold is observed, when the self-sustained oscillations are taken place. The threshold value of the nonlinearity corresponds to the out-of-phase NNM, when instability through saddle –node bifurcation is observed. The numerical simulations show the existence of the self-sustained waves propagating in the chain of many particles.

11. Nanostructures and physical systems

The application of NNMs for the nanostructures and physical systems analysis is rapidly developed.

Two DOF mechanical model of vocal fold is derived in [204]. This model describes the behavior of vocal fold during voice production. The external forces act on the system masses. In order to estimate the pressure distribution along the vocal folds area, the Bernoulli equations are applied. This model helps to study NNMs taking into account tissue stress-strain curve nonlinearity. The NNMs are calculated numerically using the combination of the shooting technique and arclength continuation algorithm. The simulation of NNMs helps to analyze some phenomena obtained during phonation. Vocal folds undergo bifurcation behavior in vicinity of internal resonances.

The geometrical nonlinear vibrations of the micro-beams with moderate transverse deflection are described by one nonlinear PDE in [205]. The dominant damping is described using well-known thermo-elastic damping mechanism. The S-P continuous NNMs are used to study the micro-beam dynamic behavior.

Micro-electro mechanical structures with geometric nonlinearity are used as sensors [206]. The deep learning based ROMs and invariant manifolds are applied to analyze nonlinear vibrations of microstructures. The internal resonances in microstructures result in interaction between different modes. Nonlinear autoparametric effects are studied.

A concept of the NNMs bushes is used to study of the dynamics of the molecular and crystal structures in a series of papers. Every bush possesses its own symmetry group. It is an

invariant manifold containing the set of NNMs. If the bush dimension is greater than unity, it describes a quasi-periodic dynamical mode. The technique for the symmetry group is used for the bushes analysis. Applying this concept, it is possible to find exact solutions using the harmonic series. The detailed description of the bushes of NNMs theory is presented in [207]. An analysis of such bushes for N particles mechanical systems with the symmetry of 230 space groups is made using the analytical potentials. Such bushes in SF_6 molecule are analyzed numerically using the group-theoretical technique in [208]. The density functional theory is applied to analyze the SF_6 molecule with symmetry. The bushes of NNMs are used to study the in-plane atomic vibrations of the graphene monolayer in [209]. The low-dimensional bushes of the graphene are calculated using the density functional theory. The transfer between NNMs of different symmetry belonging to the same bush is studied.

The paper [210] is dedicated to study of vibrations of the graphene, which is considered as a crystal with a hexagonal lattice. Three-component delocalized nonlinear vibration modes (DNVMs), which describe atomic vibrations in the lattice plane, are considered. Such modes are exact solutions to the atomic equations of motions with arbitrary values of amplitudes. They are determined using the lattice symmetry. The dependences of the mode frequencies, energy per atom, and average stresses on the root mode amplitudes are obtained. One- and two-component DNVMs in graphene are analyzed in [211]. The mode excitation results in the appearance of negative in-plane pressure. The DNVMs of the nonlinear lattices are analyzed using two different interatomic potentials in [212]. Using space symmetry group of the fcc lattice, all one-component DNVMs are treated. Stability of DNVMs in graphene lattice is studied in [213]. DNVM instability results in formation of large amplitude localized vibrational modes. If the DNVM loses stability, the atom trajectories deviate from the mode exponentially in time.

The influence of the large amplitudes, short-wavelength modes on the elastic properties of the nanostructure is analyzed using the molecular dynamics simulations in [214]. The structure particles interact due to the β -Fermi-Pasta-Ulam pair potential. Spatially localized modes (discrete breather) are treated too. The instability of the modes is studied by the direct numerical simulations of the finite DOF nonlinear ODEs. The modulation instability of delocalized short-wave vibrations modes is analyzed in [215] for the two-dimensional hexagonal lattice using the molecular dynamics simulations.

The large amplitude oscillations of harmonically coupled pendulums are considered in [216]. An example of such structure is the Frenkel–Kontorova model, which describes the dislocations in the crystal lattice. Multiple resonances between NNMs exist even with different wave numbers. It is shown that the modes interaction in the vicinity of the long wavelength leads to the oscillations localization.

A finite DOF model of the nanobeam is obtained using the Euler-Bernoulli beam theory, nonlocal elasticity theory and finite element discretization in [217]. The von Karman type of nonlinearity is accounted. The first three NNMs are analyzed by HBM and continuation technique. The system of three PDEs is derived in [218] using the Timoshenko's beam theory and the von Karman's nonlinear theory to study oscillations of electrostatically actuated nanostructure. The Hamilton's principle is used to derive the system of PDEs. As a result of the finite element discretization with the Galerkin technique, the finite DOF nonlinear dynamical system is derived. The electrostatic force is derived by the nonlinear function of the nanobeam transversal displacements. In order to calculate the nonlinear oscillations, the HBM and arc-

length continuation are used jointly to obtain NNMs. Nonlinear dynamic response in the case of the internal resonance contains the bifurcations points. The influence of the electrostatic force on the system dynamics is analyzed.

Nonlinear modal dynamics of two gas bubbles located in a distance from stationary sound stream is analyzed in [219]. The bubbles oscillations are excited by time periodic pressure field. The oscillations are described by the variations of bubbles radii. The obtained two-DOF nonlinear dynamical system is analyzed using the MSM. The NNMs and the steady-state motions are studied. The results of the bifurcations analysis are treated.

The system of three nonlinear PDEs of the isotropic single-walled carbon nanotubes free oscillations is derived accounting nonlocal elasticity and the Sanders-Koiter nonlinear shell theory [220]. The system of nonlinear ODEs with quadratic and cubic nonlinearities describes the shell like carbon nanotubes nonlinear vibrations. The NNMs of the system are obtained by HBM with a single harmonic approximation. The NNM backbone curves are soft. The system free nonlinear vibrations losses stability due to the Neimark-Sacker bifurcation. As a result, the almost periodic oscillations are observed.

The Sanders-Koiter nonlinear shell theory is used to obtain ODEs of the nanotube oscillations in [221]. The nonlinear resonance interactions between bending and circumferential modes in single-walled nanotube are considered. The model of the NNMs interactions is analyzed. As a result of this study, the energy localization over nanotube surface is observed.

The β -Fermi-Pasta-Ulam-Tsingou chain with finite number of the Duffing type coupled oscillators is considered in [222]. The two DOF limit of the system has similar NNMs in the following form:

$$x_i(t) = A_i sn(\omega t, k), \quad (42)$$

where $sn(\omega t, k)$ is the Jacobi elliptic function; ω is the oscillation frequency; A_i are oscillation amplitude. The relation $A_2 = \zeta A_1$ defines similar K-R NNMs. The general N DOFs chain has such similar NNMs for the case of purely nonlinear couplings. The localized NNM of the considered system is analyzed numerically.

12. Targeted energy transfer and absorption problem

Variety of passive and active absorbers of mechanical vibrations is used in engineering. The description of such systems and their analysis are not subject of the present study. Thus, only few publications, where the targeted energy transfer (TET) and absorption problems are analyzed by means of NNMs, are included in this Section.

The target energy transfer (TET) has been analyzed intensively in the last decade. Most of these studies are devoted to analyze the target energy transfer in the mechanical systems with the nonlinear energy sink (NES). The current state of the art and perspectives of TET analysis are discussed in the review [223].

The method of complexification-averaging developed by L.I. Manevitch, the MSM and numerical simulations are used to analyze two-DOF systems, which consists of the primary linear subsystem and the NES in the case of the 1:1 internal resonance [224]. Two DOF mechanical system with NES in the form of the eccentric rotator attached to the primary subsystem is considered. The sequence of the resonances captures is observed in the system, if

the system energy is decreased due to the dissipation⁴. The study of the rotator nonlinear dynamics shows that it is predominantly chaotic for sufficiently large values of the system total energy in the case 1:1 resonance capture. The motions in each resonance capture are described by the system NNM. As follows from the simulations, the full energy transfer to the NES occurs after a threshold value of the primary mass oscillations. These motions depend on the initial system energy and dissipation parameters.

The essentially nonlinear weakly damped attachment is used for NES in [225]. Steady-state responses are analyzed for three cases: Hamiltonian, periodically forced and forced-damped systems. An approximate slow-flow analysis permits to obtain the in-phase and out-of-phase NNMs of the system. The frequency-energy plots are obtained for a comparative evaluation of the NES and the linear oscillator amplitudes. The numerical simulations show that the considered NES permits to significantly reduce the vibrations amplitudes.

The absorption of torsional vibrations of the single-cylinder diesel engine with crankshaft is performed by NES in [226]. The analytical study of the system is made using the complexification-averaging method. An analysis of modulation equations shows that the in-phase NNM can be eliminated by NES at very low energies. The out-of-phase strongly localized NNM is effective to reduce the crankshaft system vibrations at sufficiently high energies.

The nine-story building primary substructure containing a more rigid core (secondary substructure) is considered in [227]. The blast excitation is modeled by initial velocities applied to each floor. Explosive energy transfer is achieved by energy transition of explosion-excited low-frequency vibration modes in the linear primary structure to high-frequency ones. Such transition is provided by strong nonlinear Hertzian vibro-impacts between the primary and secondary structures. The distribution of clearance between the primary and secondary structures is chosen to ensure the fast absorption of the main structure vibrations. The simulation results show that TET provides fast and irreversible dissipation of the explosion energy.

The book [228] is devoted to the study of resonance energy transfer problem in finite-DOF nonlinear systems based on the concept of limiting phase trajectories (LPT) proposed by L.I. Manevitch. An important elements of the NNM theory, as well a comparison and principal differences between NNMs and LPT concepts are discussed in details.

Conclusion

The review of the NNMs, which are considered in the previous two papers by the same authors [1,2], is developed in this paper. The NNMs theory is developed significantly during the last decade. The following problems are discussed both in this review and in the previous two reviews: concepts by Kauderer–Rosenberg and Shaw- Pierre, new definitions of NNMs, resonances and bifurcations of NNMs, NNMs in finite degrees-of-freedom systems, NNMs in continuous dynamical systems, reduce order modelling, numerical methods for nonlinear modes.

⁴ It seems to the authors, that the NNMs described in the paper [224] correspond to the “transient NNMs” described in publications [63-65] which are considered in the Section 3.

The problems, which are initiated to study during the last decade, are reviewed here. Namely, they are the following: NNMs in stochastic dynamical systems, identification of mechanical systems using NNMs, experimental measurement of NNMs, applications of NNMs in aerospace engineering and power engineering, applications of NNMs for nanostructures.

New trend of the NNMs theory, which is developed during the last decade by Haller with co-authors, is the spectral sub-manifolds.

The essential development of the numerical methods for all issues of NNMs is the important basis of the theory during the last decade. In the previous decades, the researches gave attention mainly for development of analytical methods for NNMs.

As follows from this review, NNMs is started to use for real engineering problems, which are originated in aerospace engineering and power engineering. We can note that the FEM, which transforms the nonlinear continuous system to the system of nonlinear ODEs, is developed significantly for discretization of real nonlinear engineering structures. In this case, the reduced order modelling is used.

The NNMs experimental analysis and identification of mechanical systems from NNMs analysis are very important for progress of the NNMs theory. The authors of this review believe that these areas will be significantly developed in future research.

As follows from this review the application of NNMs for nanostructure nonlinear dynamics is developed slightly. We suppose that this subject will be also developed in future investigations.

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