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# A dominance approach for the analysis of emergent patterns in networks

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*Summary.* This note focuses on the analysis of emergent behaviors in networks of dynamical systems, where their interactions are of activation-type locally and inhibitory-type in the long-range. Such interplay between activation and inhibition is reminiscent of the classical formula for oscillations in lumped systems, *i.e.*, local positive feedback and long-range negative feedback. Dominance tools are used for the analysis of the resulting behavior, obtaining certificates in the form of linear matrix inequalities that allow to study the existence of emergent static patterns or oscillations in the network. A numerical example illustrates the proposed approach.

## Emergent oscillations in dynamical networks

Complex dynamical systems are well known for having convoluted interactions between large amounts of agents. These interactions promote the appearance of new behaviors that are not present in the individual components. Such *emergent behaviors* are challenging to study as they are intrinsically linked to nonlinear phenomena. Among all possible emergent behaviors, periodic rhythms hold an special place. From biological cells to the movement of planets, oscillations appear everywhere. From a control perspective, it is well known that self-sustained oscillations arise from the combination of positive and negative feedback loops acting at different scales. Such intuition extends to other settings, as for instance, neural fields, see *e.g.*, [1, 2, 3]. In there, interactions of the type local activation/long-range inhibition are the responsible for the appearance of static patterns and spatio-temporal oscillations in distributed parameter systems described by integro-differential equations. In this note, we study such mixed interaction paradigm for the case of dynamical networks.

### Network model under study

Consider the following family of linear systems

$$\nu_i : \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad y_i(t) = Cx_i(t), \quad (1)$$

where  $x_i(t) \in \mathbb{R}^n$ ,  $u_i(t), y_i(t) \in \mathbb{R}^m$  are the state, input and output of the  $i$ -th agent at time  $t$ ,  $i \in \{0, \dots, N-1\}$ , whereas the matrices  $A, B$ , and  $C$  are constant and of the appropriate dimensions. Let  $V = \{\nu_0, \dots, \nu_{N-1}\}$  be the set of all agents and let  $S_+ = \{\pm 1, \dots, \pm r\}$  and  $S_- = \{\pm(r+1), \dots, \pm \lfloor \frac{N}{2} \rfloor\}$ , where  $\lfloor \cdot \rfloor$  denotes the conventional floor function. The exchange of information between agents, characterizing the configuration of active and inhibitory links, is established by the circulant graphs,  $\mathcal{G}_+(V, S_+)$  and  $\mathcal{G}_-(V, S_-)$ , respectively. Recall that in a circulant graph  $\mathcal{G}(V, S)$  the vertices  $\nu_i$  and  $\nu_j$  are connected if and only if  $(j-i) \bmod N \in S$  (see *e.g.*, [5, Section 1.5] for more details and properties on circulant graphs). The graphs  $\mathcal{G}_+(V, S_+)$  and  $\mathcal{G}_-(V, S_-)$  are edge disjoint on the same vertex set  $V$ , and their union is the complete graph  $\mathcal{K}_N$ , see Figure 1. It is worth to emphasize that the analysis below is not limited to such configurations, which we keep here for the sake of simplicity. Thus, the local activation/long-range inhibition iteration is given as

$$u_i = \sigma^+ \sum_{(j-i) \bmod N \in S_+} (\varphi^+(y_i) - \varphi^+(y_j)) - \sigma^- \sum_{(j-i) \bmod N \in S_-} (\varphi^-(y_i) - \varphi^-(y_j)), \quad (2)$$

where  $\sigma^+, \sigma^-$  are positive constants, the map  $\varphi^+ : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is continuous and  $\rho$ -strongly monotone, that is, for any  $v_1, v_2 \in \text{Dom } \varphi^+$ , the inequality  $\langle v_1 - v_2, \varphi^+(v_1) - \varphi^+(v_2) \rangle \geq \rho \|v_1 - v_2\|^2$  holds, and the map  $\varphi^- : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is assumed monotone and Lipschitz continuous. The interaction rule (2) provides the claimed interplay, as for the  $i$ -th node, its closest  $2r$ -neighbors (*i.e.*, nodes  $\nu_{i \pm j \bmod N}$ ,  $j \in \{1, \dots, r\}$ ) provide a positive stimulation (local activation) promoting the difference between nodes. In contrast, the remaining  $N-1-2r$  neighbors provide a restraint effect (long-range inhibition) on the mismatch between nodes. In this way, the conventional recipe for oscillations for lumped systems is extrapolated to the case of networks of systems.

## Uniform dominance of networks

Let  $\Theta_c \in \mathbb{R}^{N \times \frac{N(N-1)}{2}}$  be an oriented incidence matrix associated to the complete graph  $\mathcal{K}_N$  and let  $x \in \mathbb{R}^{Nn}$  be the aggregated state of the family of systems (1)-(2), the mismatch between agents is the vector  $\Delta x := (\Theta_c^\top \otimes I_n)x \in \mathbb{R}^{\frac{N(N-1)}{2}}$ . For individual systems, the property of dominance constraints the asymptotic behavior of the closed-loop, as for a  $p$ -dominant system, the state converges towards a  $p$ -dimensional manifold of the space state. The consequences of such restriction are formally stated in [4, Corollary 1]. Concisely, bounded solutions of a 0-dominant, 1-dominant, and 2 dominant systems converge, respectively, towards an unique equilibrium state, some equilibrium state, and a simple attractor (either: an equilibrium point, or a connecting arc between equilibria, or a limit cycle).

**Definition 1.** *The family of systems (1)-(2) is uniformly  $q$ -dominant with rate  $\mu > 0$ , if there exists a nonsingular matrix  $P = P^\top$ , with  $q$  negative eigenvalues, and  $\varepsilon \geq 0$  such that the map  $V : \mathbb{R}^{Nn} \rightarrow \mathbb{R}$ , mapping  $\Delta x \mapsto \Delta x^\top (I_N \otimes P) \Delta x$  satisfies*

$$\frac{d}{dt} V(\Delta x) + 2\mu V(\Delta x) \leq -\varepsilon \|\Delta x\|^2. \quad (3)$$

Uniform dominance is robust with respect to the network structure. Uniformity is central for drawing conclusions on the resulting behavior of the entire network, as all of the mismatches in the vector  $\Delta x$  can be taken as coming from the same  $n$ -dimensional system. Thus, allowing to project the  $Nn$  state mismatches vector  $\Delta x$  into the  $n$ -dimensional space. This last observation makes possible to draw conclusions on the behavior of the complete network in an analogous way as is done for lumped systems, see *e.g.*, [4].

**Theorem 1.** *If for some  $\mu \geq 0$ , there exists a nonsingular matrix  $P = P^\top$ , with  $q$  negative eigenvalues, and  $\varepsilon \geq 0$  such that*

$$A^\top P + PA - (\sigma^+ \rho \lambda_2 - \sigma^- L_\varphi - \lambda_N^-) C^\top C + 2\mu P + \varepsilon I_n \prec 0, \text{ and } PB = -C^\top, \quad (4)$$

where  $\lambda_2^+$  and  $\lambda_N^-$  are the second smallest eigenvalue and the largest eigenvalue of the graph Laplacian matrices associated with  $\mathcal{G}_+(V, S_+)$  and  $\mathcal{G}_-(V, S_-)$ , respectively. Then, the network (1)-(2) is  $q$ -dominant with rate  $\mu \geq 0$ . In particular, if  $q = 2$  and in addition: *i)* all trajectories of (1)-(2) remain bounded and *ii)* the synchronization manifold is unstable. Then, the network shows either: a static pattern or a spatio-temporal oscillation.

*Proof.* (Sketch) The first part of the proof follows by direct computation of (3) along solutions of (1)-(2). For the second part, the proof follows by noticing that for the case of  $q = 2$ , (4) is a sufficient condition for signed 2-passivity (see [6]) of the Lur'e system:  $\dot{x}(t) = Ax(t) + B(\tilde{\varphi}^+(Cx(t)) - \tilde{\varphi}^-(Cx(t)))$ , where  $\tilde{\varphi}^+$  is  $\sigma^+ \rho \lambda_2$ -strongly monotone and  $\tilde{\varphi}^-$  is  $\sigma^- L_\varphi - \lambda_N^-$ -Lipschitz continuous. Such that the asymptotic behavior of the mismatches  $\Delta x$  agrees with that of a 2-dominant system. The conclusion thus follows from Corollary 1 in [4].  $\square$

### Numerical example

As an example let us consider a family of  $N = 20$  linear systems with the following parameters  $A = \begin{bmatrix} 0 & -1 \\ 5 & -\alpha \end{bmatrix}$ ,  $C = B^\top = [1 \ 0]$ , where  $\alpha > 0$ . Notice that each agent in (1), with  $u_i = 0$ , is globally asymptotically stable, since the matrix  $A$  is Hurwitz for all values of  $\alpha > 0$ . For the interconnections,  $S_+ = \{1, \dots, 5\}$ ,  $S_- = \{6, \dots, 10\}$ ,  $\sigma^+ = 0.5$ ,  $\sigma^- = 0.1$ ,  $\varphi^+ = \text{Id}$ , the identity map, and  $\varphi^-(\cdot) = (\cdot)^3$ , which is Lipschitz continuous in any compact set. Thus, with these parameters and by setting  $\alpha = 1$ , the network (1)-(2) is uniformly 2-dominant with rate  $\mu > 4.2$ . For instance, with  $\mu = 4.5$ ,  $P = \text{Diag}\{-1, -0.101\}$  solves (4) with  $\varepsilon = 1 \times 10^{-9}$  and periodic oscillations emerge. Figure 1 (center) illustrates the resulting behavior depicting an spatio-temporal oscillation. On the other hand, if  $\alpha = 10$ , whereas the remaining parameters are left unchanged, then the network is no longer 2-dominant, but 1-dominant instead with rate  $\mu \in (5.3, 8.9)$ . In this case no oscillation appears and a static pattern emerges, see Figure 1 (right).

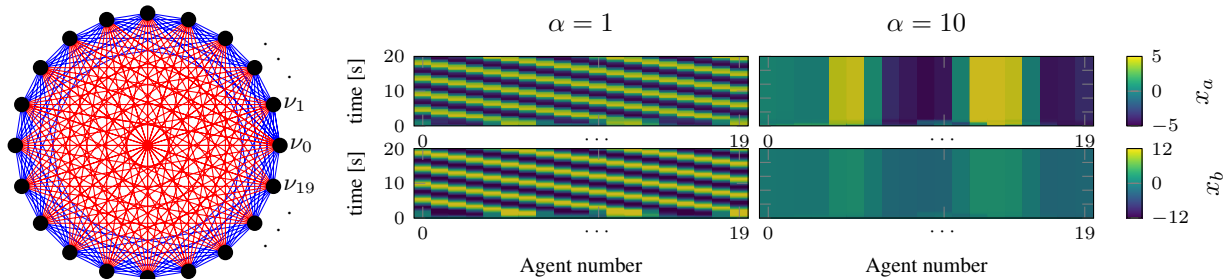


Figure 1: Left. Network of  $N = 20$  agents with local activation (blue links) and long-range inhibition (red links). Center. Space-time evolution of state components  $x_{a,i}$  (upper plot) and  $x_{b,i}$  (lower plot),  $i \in \{0, \dots, N\}$  for  $\alpha = 1$ , showing an emergent oscillatory behavior. Right. Space-time evolution of state components  $x_{a,i}$  (upper plot) and  $x_{b,i}$  (lower plot),  $i \in \{0, \dots, N\}$  for  $\alpha = 10$ , showing the emergence of a static pattern.

### Conclusions

Uniformly dominant networks were introduced in this note. It was shown that dominance analysis becomes useful for the analysis of the asymptotic behavior in the same way as is done for individual systems. Future work consists into exploiting the structure of the network for further constraining the behavior of the system.

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