



HAL
open science

In-duct flow computation and acoustic propagation using the admittance multimodal formulation

Bruno Mangin, Gwenael Gabard, Majd Daroukh

► **To cite this version:**

Bruno Mangin, Gwenael Gabard, Majd Daroukh. In-duct flow computation and acoustic propagation using the admittance multimodal formulation. *Journal of the Acoustical Society of America*, 2024, 155 (5), pp.3461-3474. 10.1121/10.0026091 . hal-04606595

HAL Id: hal-04606595

<https://hal.science/hal-04606595v1>

Submitted on 10 Jun 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

In-duct flow computation and acoustic propagation using the admittance multimodal formulation

B. Mangin^{a,1,2} G. Gabard,¹ and M. Daroukh²

¹*Laboratoire d'Acoustique de l'Université du Mans (LAUM), UMR 6613,
Institut d'Acoustique - Graduate School (IA-GS), CNRS, Le Mans Université,
France*

²*Department of Aerodynamics, Aeroelasticity, and Aeroacoustics (DAAA),
ONERA, Institut Polytechnique de Paris, 92320 Châtillon,
France*

(Dated: 28 May 2024)

^a bruno.mangin@safrangroup.com

1 A multimodal method for computing the potential base flow and propagating acoustic
2 perturbations inside axisymmetric ducts is presented. Instead of using the standard
3 modal basis, a polynomial basis is used in the radial direction to reduce the com-
4 putational cost of the method, but this introduces non-physical high-order modes.
5 The impact of these modes on the stability of the calculation is examined, and for
6 the acoustic computation, a modification of the axial integration is proposed to im-
7 prove the conditioning of the matrices involved. The flow computation is achieved
8 by applying the method (initially devoted to acoustics) at a zero frequency without
9 convective effects, by modifying the definition of the admittance at the exit of the
10 duct and by performing an induction process on the density. The method is validated
11 against a finite element method for ducts with hard walls or lined walls. The results
12 show that the proposed multimodal method is very efficient in computing the mean
13 flow and propagating the sound disturbances inside axisymmetric ducts.

14 I. INTRODUCTION

15 The accurate prediction of sound wave propagation in ducts is a crucial challenge in tur-
16 bofan engineering, and various methods are employed when a mean flow is present. These
17 methods can be broadly classified as numerical or semi-analytical/semi-numerical. Numer-
18 ical methods, such as finite difference 1–3 and finite element 4–7, are versatile and can
19 accommodate general geometries and mean flow profiles. Still, they are computationally
20 expensive as they do not take advantage of the ducted aspect of the problem. In contrast,
21 semi-analytic methods, such as WKB 8–10, pseudo-spectral 11 or multimodal 12–16, can
22 be fast, but they face difficulties in handling irregular geometry or non-uniform mean flow.
23 These methods are based on the idea of dissociating the propagation in the axial direction
24 from the acoustic shape in the transverse direction. Until recently, multimodal methods
25 were limited to cases with no flow or with a uniform flow 12–14, but a recent extension
26 to account for a multiple-scale flow has been proposed 16. Even if this method appeared
27 promising for low frequencies and Mach numbers, the results became less accurate for high
28 Mach numbers because of the multiple-scale approximation of the flow.

29 This work therefore aims to extend the multimodal methodology to account for an accu-
30 rate flow description. As for the previous studies, the method is intended to compute the
31 acoustic field in the presence of an already-known base flow (the novelty being here that it
32 can be heterogeneous as long as it remains potential). But in order to obtain fast predictions
33 on the whole, the multimodal method is here also used to compute the base flow by analogy
34 with an acoustic field at a zero frequency and without convective effects. In the proposed

35 method, Chebyshev polynomials are used to represent the acoustics in the radial direction.
36 The polynomial basis offers excellent convergence properties for smooth geometries. How-
37 ever, as this basis is non-physical, high-order modes (not well captured by the polynomial
38 basis) are present in the calculation and must be handled with care 17, 18. Here the au-
39 thors propose to modify the Magnus–Möbius scheme 14, 19 traditionally used for the axial
40 computation. This modification is based on an eigendecomposition of the Magnus matrix.
41 The involved matrices are then rewritten to avoid the summation of high values associated
42 with right-running non-physical eigenmodes with low values associated with left-running
43 non-physical eigenmodes. To validate our method for axisymmetric cases, we compare it to
44 the classical finite element method (FEM), which is commonly used for predicting sound
45 wave propagation in ducts. We focus on the accuracy and stability of our predictions for
46 various modes and frequencies.

47 The paper is organized as follows. Section II provides a brief overview of the equations
48 governing the flow and acoustic fields, which serve as a basis for constructing the proposed
49 formulation. Section III describes the developed multimodal method for computing the
50 acoustic field in the presence of a heterogeneous potential flow (supposed to be known at this
51 step), and section IV details the adaptation of the method for computing the aforementioned
52 potential flow. The developed method is then validated against a FEM code in section V
53 for a realistic engine geometry with hard walls and lined walls. Conclusions and discussions
54 are finally provided in section VI.

55 **II. GOVERNING EQUATIONS**

56 In this work, we study the linear propagation of sound in a waveguide with an axially
 57 varying circular or annular cross-section, defined by the inner and outer radii R_1 and R_2 .
 58 The analysis is performed in the frequency domain, with ω the angular frequency, and all
 59 the variables are dimensionless (see Reference 8 for more details). We express the velocity
 60 vector, density and speed of sound variables as follows

$$\begin{aligned}\tilde{\mathbf{v}} &= \mathbf{V} + \text{Re}(\mathbf{v} e^{i\omega t}) = (U, V, W) + \text{Re}((u, v, w) e^{i\omega t}), \\ \tilde{\rho} &= D + \text{Re}(\rho e^{i\omega t}), \\ \tilde{c} &= C + \text{Re}(c e^{i\omega t}).\end{aligned}\tag{1}$$

61 Capital letters denote time-averaged quantities, and lower-case letters represent unsteady
 62 harmonic perturbations.

63 The flow is assumed to be a potential perfect gas flow. Therefore, the mean flow velocity
 64 and the acoustic velocity derive from scalar potentials, denoted Φ and ϕ , respectively. The
 65 equations for the mean flow and the perturbations may be written

$$\nabla \cdot (D\nabla\Phi) = 0,\tag{2a}$$

$$\frac{D^{\gamma-1}}{\gamma-1} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi = E,\tag{2b}$$

$$\nabla \cdot (D\nabla\phi) - D\frac{D}{Dt} \left(\frac{1}{C^2} \frac{D\phi}{Dt} \right) = 0,\tag{2c}$$

$$p = -D\frac{D\phi}{Dt},\tag{2d}$$

66 with $D/Dt = i\omega + \mathbf{V} \cdot \nabla$ the convective derivative, $\mathbf{V} = \nabla\Phi$, $\mathbf{v} = \nabla\phi$ and where γ is the
 67 ratio of specific heats and E a Bernoulli constant. The walls of the duct are impermeable

68 to the mean flow

$$\mathbf{V} \cdot \mathbf{n} = 0. \quad (3)$$

69 The duct walls can be acoustically lined, using a locally reacting treatment. The Ingard-
70 Myers impedance condition is used for this purpose 20

$$i\omega \nabla \phi \cdot \mathbf{n} = - \left[\frac{D}{Dt} - \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \mathbf{V}) \right] \left(\frac{D}{Z} \frac{D\phi}{Dt} \right), \quad (4)$$

71 with \mathbf{n} the unit outgoing vector normal to the surface and Z the local impedance of the
72 liner.

73 To apply the multimodal method, the governing equation must be rearranged into a
74 system with only first-order derivatives in the axial direction x . The chosen variables are
75 the acoustic potential ϕ and its axial derivative u . The system then becomes

$$\frac{\partial \phi}{\partial x} = u, \quad (5a)$$

$$D \frac{\partial}{\partial x} ((1 - M_x^2) u) = D \frac{D}{Dt} \left(\frac{1}{C^2} \frac{D_{\perp} \phi}{Dt} \right) + D \frac{D_{\perp}}{Dt} \left(\frac{Uu}{C^2} \right) + \frac{D}{2C^2} \frac{\partial}{\partial x} (\mathbf{V}_{\perp} \cdot \mathbf{V}_{\perp}) u - \nabla_{\perp} \cdot (D \nabla_{\perp} \phi), \quad (5b)$$

76 in which $M_x = U/C$ is the axial Mach number. The symbol \perp refers to the transverse
77 direction, so that ∇_{\perp} is the gradient operator restricted to the cross-section. In addition,
78 \mathbf{V}_{\perp} is the mean flow velocity along the duct cross-section, and $D_{\perp}/Dt = i\omega + \mathbf{V}_{\perp} \cdot \nabla$.

79 III. MULTIMODAL FORMULATION FOR THE ACOUSTIC FIELD

80 This section presents the formulation of a multimodal method to solve the model de-
81 scribed above, assuming that the mean flow is known.

82 **A. Variational formulation**

83 The weak formulation of equations (5a) and (5b) is obtained by multiplying them by
 84 the complex conjugate of a test function g and by integrating over the duct cross-section,
 85 denoted S , whose boundary is the contour Λ . After some manipulations, one can write

$$\int_S D(1 - M_x^2) \frac{\partial \phi}{\partial x} g^* dS = \int_S D(1 - M_x^2) u g^* dS, \quad (6a)$$

$$\begin{aligned} \frac{d}{dx} \left(\int_S D(1 - M_x^2) u g^* - \frac{DU}{C^2} \frac{D_{\perp} \phi}{Dt} g^* dS \right) &= \int_S D(1 - M_x^2) u \frac{\partial g^*}{\partial x} + D \nabla_{\perp} g^* \cdot \nabla_{\perp} \phi \\ - \frac{D}{C^2} \left(\frac{D_{\perp} \phi}{Dt} \left(\frac{D_{\perp} g}{Dt} \right)^* + U u \left(\frac{D_{\perp} g}{Dt} \right)^* + U \frac{D_{\perp} \phi}{Dt} \frac{\partial g^*}{\partial x} \right) dS &- \int_{\Lambda} D g^* \nabla \phi \cdot \mathbf{n} d\Lambda. \end{aligned} \quad (6b)$$

86 in which $*$ denotes the complex conjugate.

87 The impedance condition (4) is used for the last term in (6b). Following Eversmann 21,
 88 we use Stokes' theorem to rewrite this term as follows

$$\begin{aligned} \int_{\Lambda} D g^* \nabla \phi \cdot \mathbf{n} d\Lambda &= \frac{1}{i\omega} \int_{\Lambda} \frac{D^2}{Z} \left(U u \left(\frac{Dg}{Dt} \right)^* + \frac{D_{\perp} \phi}{Dt} \left(\frac{Dg}{Dt} \right)^* \right) d\Lambda - \\ &\frac{1}{i\omega} \frac{d}{dx} \left[\int_{\Lambda} \frac{D^2}{Z} \left(U u + \frac{D_{\perp} \phi}{Dt} \right) g^* (\mathbf{V} \cdot \boldsymbol{\tau}) d\Lambda \right], \end{aligned} \quad (7)$$

89 where $\boldsymbol{\tau}$ is the unit vector tangential to the duct wall. In the remainder of the paper, $\mathbf{V} \cdot \boldsymbol{\tau}$
 90 is simply denoted V_{τ} . In the special case of a circular duct ($R_1 = 0$), the last term in (6b)
 91 vanishes on the inner duct wall since $d\Lambda_1 = R_1 d\theta = 0$.

92 **B. Modal decomposition**

93 To apply the multimodal method, the governing equation (6) must be rearranged into
 94 a system with only first-order derivatives in the axial direction x . To that end, the cho-

sen variables are the acoustic potential ϕ and its axial derivative $u = \partial\phi/\partial x$. They are represented using a set of linearly independent transverse cross-section functions, denoted $(\varphi_j)_{j \in \mathbb{N}}$. Using this basis, the acoustic potential and its axial derivative are written

$$\phi = \sum_j \phi_j(x) \varphi_j(x, r, \theta), \quad \text{and } u = \sum_j u_j(x) \varphi_j(x, r, \theta). \quad (8)$$

The same basis is used for the test functions associated to ϕ and u . The equations (6) and (7) then become

$$\begin{aligned} \int_S D(1 - M_x^2) \varphi_i^* \varphi_j dS \frac{d\phi_j}{dx} &= - \int_S D(1 - M_x^2) \varphi_i^* \frac{\partial \varphi_j}{\partial x} dS \phi_j + \int_S D(1 - M_x^2) \varphi_i^* \varphi_j dS u_j, \\ \frac{d}{dx} \left(\int_S D(1 - M_x^2) \varphi_j \varphi_i^* dS u_j - \int_S \frac{DU}{C^2} \frac{D_{\perp} \varphi_j}{Dt} \varphi_i^* dS \phi_j \right) &= \\ + \left(\int_S D \nabla_{\perp} \varphi_i^* \cdot \nabla_{\perp} \varphi_j - \frac{D}{C^2} \frac{D_{\perp} \varphi_j}{Dt} \left(\frac{D \varphi_i}{Dt} \right)^* dS - \frac{1}{i\omega} \int_{\Lambda} \frac{D^2}{Z} \frac{D_{\perp} \varphi_j}{Dt} \left(\frac{D \varphi_i}{Dt} \right)^* d\Lambda \right) \phi_j & (9) \\ + \left(\int_S D(1 - M_x^2) \varphi_j \frac{\partial \varphi_i^*}{\partial x} - \frac{DU}{C^2} \varphi_j \left(\frac{D_{\perp} \varphi_i}{Dt} \right)^* dS - \frac{1}{i\omega} \int_{\Lambda} \frac{D^2 U}{Z} \varphi_j \left(\frac{D \varphi_i}{Dt} \right)^* d\Lambda \right) u_j & \\ + \frac{1}{i\omega} \frac{d}{dx} \left(\int_{\Lambda} \frac{D^2 V_{\tau}}{Z} \frac{D_{\perp} \varphi_j}{Dt} \varphi_i^* d\Lambda \phi_j + \int_{\Lambda} \frac{D^2 V_{\tau} U}{Z} U \varphi_j \varphi_i^* d\Lambda u_j \right). & \end{aligned}$$

These expressions can be written in vector form by introducing the vectors $\boldsymbol{\phi}$ and \mathbf{u} containing the unknowns $\phi_j(x)$ and $u_j(x)$. The equations governing the axial variation of these vectors are

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \frac{d}{dx} \begin{pmatrix} \boldsymbol{\phi} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi} \\ \mathbf{u} \end{pmatrix}. \quad (10)$$

The detailed expressions for the matrices A_{11} , A_{12} , A_{21} , A_{22} , M_{11} , M_{12} , M_{21} and M_{22} are given in Appendix A. The two vectors $\boldsymbol{\phi}(x)$ and $\mathbf{u}(x)$ are linked through the admittance matrix Y such that $\mathbf{u}(x) = Y(x)\boldsymbol{\phi}(x)$.

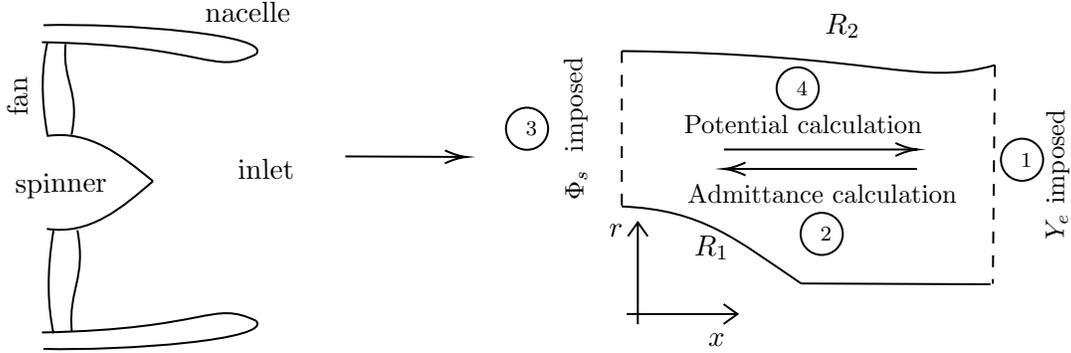


FIG. 1. Schematic of the calculation performed.

106 **C. Overview of the calculation**

107 Before going deeper into the numerical details of the method, an overview of the procedure
 108 to perform a multimodal computation is given in the following and is depicted in Figure 1.
 109 Let us suppose that we want to compute the acoustic propagation inside the ducted part of an
 110 aero-engine. This region is considered to be a duct whose cross-section varies axially. In the
 111 following, the source denotes the fan plane location while the exit refers to the inlet plane.
 112 First, the admittance Y_e at the exit is imposed. Using this initial value, the admittance
 113 Y is calculated from the exit to the source using a Magnus–Möbius scheme. Next, the
 114 potential at the source Φ_s is calculated using the specified injected wave and the previously
 115 obtained admittance. Finally, the Magnus–Möbius scheme is once more employed to obtain
 116 the potential throughout the entire engine.

117 **D. Transverse mode basis**

118 In multimodal methods, the basis functions φ_j are generally the local transverse mode of
 119 a hard-walled duct (i.e. Fourier-Bessel modes for an axisymmetric duct)^{12–14, 22}. However,

120 these bases tend to yield slow convergence rates, and additional modes are often required
 121 to improve the accuracy of the calculations 15, 23. In this paper, we focus on axisymmetric
 122 ducts, and therefore, a Fourier transform is used to represent the acoustic field in the cir-
 123 cumferential direction. However, Chebyshev functions are preferred in the radial direction
 124 since they provide good convergence properties 24–26. As in Reference 16, for each basis
 125 function index j , we associate a pair $(m, p) \in (\mathbb{Z}, \mathbb{N})$ such that

$$\varphi_p^m = \left(\frac{r}{R_2 - R_1} \right)^{\min(|m|, 1)} T_p \left(\frac{r - R_1}{R_2 - R_1} \right) e^{-im\theta}, \quad (11)$$

126 where T_p is the shifted Chebyshev polynomial of order p , related to the Chebyshev polyno-
 127 mials T_p^* by the relation $T_p(x) = T_p^*(2x - 1)$. The inclusion of the factor $r^{\min(|m|, 1)}$ ensures
 128 that all the matrices given in Appendix A are well-defined. This factor is included based on
 129 the property of propagating modes, which exhibit a behaviour of r^m as $r \rightarrow 0$ 27. Here we
 130 use the $\min(|m|, 1)$ exponent so that the multimodal matrices are the same for all non-zero
 131 azimuthal wavenumbers. Note that this reduces the number of polynomials needed to rep-
 132 resent the acoustic field if the transverse section is circular without increasing it when it is
 133 annular.

134 For the remainder of the paper, the duct is considered axisymmetric, so there is no
 135 possible coupling between different circumferential Fourier modes. Therefore we consider
 136 that the value of the azimuthal order m is fixed, and, in the rest of the paper, no distinction
 137 is made between φ_p^m and φ_p .

138 **E. Admittance calculation**

139 Introducing the definition of the admittance matrix in Equation (10), it is possible to
 140 show that this matrix satisfies a Riccati equation of the form

$$\frac{dY}{dx} = -Y A_{11}^{-1} M_{11} - Y A_{11}^{-1} M_{12} Y - A_{22}^{-1} A_{21} A_{11}^{-1} (M_{11} + M_{12} Y) + A_{22}^{-1} M_{21} + A_{22}^{-1} M_{22} Y . \quad (12)$$

141 This equation is solved numerically using a Magnus–Möbius scheme starting from an initial
 142 value Y_e . This numerical procedure is described in detail in References 14 and 19.

143 We use the same procedure as in References 16, 22 to define the matrix Y_e . We assume an
 144 infinite duct termination with constant cross-section with only forward waves. To determine
 145 the admittance at the exit, the acoustic variables are expressed as a summation of modes
 146 that either propagate or decay exponentially with respect to the axial distance. By doing
 147 so, the problem of equation (10) simplifies to the following eigenvalue problem

$$i\lambda_i \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \mathbf{w}_i^\phi \\ \mathbf{w}_i^u \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \mathbf{w}_i^\phi \\ \mathbf{w}_i^u \end{pmatrix}, \quad (13)$$

148 the solution of which gives the eigenvalues λ_i and the eigenvectors associated with the
 149 acoustic potential \mathbf{w}_i^ϕ and axial velocity \mathbf{w}_i^u waves. These waves are separated into the
 150 forward (+) and backward (−) directions based on the sign of $\text{Re}(\lambda_i) - \omega/\overline{C} \overline{M}_x / (1 - \overline{M}_x^2)$
 151 for cut-on modes and $\text{Im}(\lambda_i)$ for cut-off modes, where the overline denotes the mean value
 152 of the flow variable over the cross-section. The resulting forward admittance matrix is given
 153 by

$$Y_e = W_+^u (W_+^\phi)^{-1} = W_+^\phi \Lambda_+ (W_+^u)^{-1}, \quad (14)$$

154 where $\Lambda_+ = \text{diag}(\lambda_0, \lambda_1, \lambda_2, \dots)$ is the matrix of eigenvalues, and $W_+^\phi = (\mathbf{w}_0^\phi, \mathbf{w}_1^\phi, \dots)$ and
 155 $W_+^u = (\mathbf{w}_0^u, \mathbf{w}_1^u, \dots)$ are the matrices of potential and axial velocity eigenvectors associated
 156 with forward waves. Note that the matrix W_+^ϕ is the transfer matrix from the transverse
 157 eigenmode basis to the polynomial basis.

158 F. Potential calculation

159 The admittance is calculated from the exit to the source by integrating the equation (12)
 160 (the details of this axial integration will be addressed in section III G). Injecting the ex-
 161 pression of the admittance into equation (10) gives the following equation for the acoustic
 162 potential

$$A_{11} \frac{d}{dx} (\phi) = M_{11} \phi + M_{12} Y \phi. \quad (15)$$

163 It can then be calculated from the source to the exit (more details in section III G). Given
 164 an incoming acoustic wave ϕ_s^+ , the potential at the injection is $\phi_s = (I_d + R)\phi_s^+$ with
 165 $R = (Y - Y_-)^{-1}(Y_+ - Y)$ the local reflection matrix and Y_\pm the local forward/backward
 166 admittance matrix.

167 G. Modification of the Magnus–Möbius scheme

168 High-order modes, that are not well captured by the polynomial basis, are present in
 169 the calculations when using polynomials for the basis functions, as reported by Wilson et
 170 al. 17 and Guennoc et al. 18. Their axial wavenumbers are far from the ones associated with
 171 physical modes, and they tend to be strongly cut-off. It will be shown later that the matrices

172 involved in the Magnus–Möbius scheme are directly related to these wavenumbers. While
 173 the physical eigenvalues (obtained when using the hard-wall duct modes as an approximation
 174 basis) do not pose a problem, the non-physical eigenvalues obtained when using a polynomial
 175 basis lead to poorly conditioned matrices due to their strongly cut-off nature. To avoid this
 176 conditioning issue, we propose a modified formulation of the Magnus–Möbius scheme.

177 Firstly, the classical Magnus–Möbius scheme is as follows. We introduce the exit location
 178 x_e and the source position x_s . Considering an axial discretization $(x_n)_{n \in [0, N]}$ such that
 179 $x_N = x_s$ and $x_0 = x_e$, the potential and acoustic velocity at positions x_n and x_{n+1} are linked
 180 using the following equation

$$\begin{pmatrix} \phi_{n+1} \\ \mathbf{u}_{n+1} \end{pmatrix} = e^{\Omega_n} \begin{pmatrix} \phi_n \\ \mathbf{u}_n \end{pmatrix}, \quad (16)$$

181 where the matrix Ω_n is the Magnus matrix, the expression of which can be found using the
 182 matrices defined in section III B 14. After splitting the matrix exponential into four blocks

$$e^{\Omega_n} = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}, \quad (17)$$

183 the evolution of the admittance and the potential are written

$$Y_{n+1} = (E_3 + E_4 Y_n)(E_1 + E_2 Y_n)^{-1} \quad (18)$$

$$\phi_n = (E_1 + E_2 Y_n)^{-1} \phi_{n+1}.$$

184 The difficulty is that the matrix $(E_1 + E_2 Y_n)$ can be ill-conditioned 18, which makes the
 185 scheme not robust for high frequencies or strong flow gradients. We use an eigen decom-
 186 position of the matrix Ω_n to avoid using these matrices. Since the transfer matrix $\exp(\Omega_n)$
 187 represents the transfer of information from the axial location x_n to its neighbouring value

188 x_{n+1} , the eigenvalues (Λ_+, Λ_-) of Ω_n , are characteristic of the evolution of right-running
 189 (ϕ_+, \mathbf{u}_+) and left-running (ϕ_-, \mathbf{u}_-) eigenvectors. The exponential matrix can be rewritten
 190 as:

$$e^{\Omega_n} = \begin{pmatrix} \phi_+ & \phi_- \\ \mathbf{u}_+ & \mathbf{u}_- \end{pmatrix} \begin{pmatrix} e^{\Lambda_+} & 0 \\ 0 & e^{\Lambda_-} \end{pmatrix} \begin{pmatrix} \phi_+ & \phi_- \\ \mathbf{u}_+ & \mathbf{u}_- \end{pmatrix}^{-1}. \quad (19)$$

191 In addition, by assuming that all the sub-matrices are invertible, the last matrix can be
 192 rewritten

$$\begin{pmatrix} \phi_+ & \phi_- \\ \mathbf{u}_+ & \mathbf{u}_- \end{pmatrix}^{-1} = \begin{pmatrix} (\phi_+ - \phi_- \mathbf{u}_-^{-1} \mathbf{u}_+)^{-1} & 0 \\ 0 & (\mathbf{u}_- - \mathbf{u}_+ \phi_+^{-1} \phi_-)^{-1} \end{pmatrix} \begin{pmatrix} I_d & -\phi_- \mathbf{u}_-^{-1} \\ -\mathbf{u}_+ \phi_+^{-1} & I_d \end{pmatrix}. \quad (20)$$

193 By noting $Y_{\pm} = \mathbf{u}_{\pm} \phi_{\pm}^{-1}$ the admittance matrix associated to respectively right/left run-
 194 ning modes, the exponential matrices can be written

$$\begin{aligned} E_1 &= \phi_+ e^{\Lambda_+} \phi_+^{-1} (I_d - Y_-^{-1} Y_+)^{-1} + \phi_- e^{\Lambda_-} \phi_-^{-1} (I_d - Y_+^{-1} Y_-)^{-1}, \\ E_2 &= \phi_+ e^{\Lambda_+} \phi_+^{-1} (Y_+ - Y_-)^{-1} + \phi_- e^{\Lambda_-} \phi_-^{-1} (Y_- - Y_+)^{-1}, \\ E_3 &= \mathbf{u}_+ e^{\Lambda_+} \mathbf{u}_+^{-1} (Y_+^{-1} - Y_-^{-1})^{-1} + \mathbf{u}_- e^{\Lambda_-} \mathbf{u}_-^{-1} (Y_-^{-1} - Y_+^{-1})^{-1}, \\ E_4 &= \mathbf{u}_+ e^{\Lambda_+} \mathbf{u}_+^{-1} (I_d - Y_- Y_+^{-1})^{-1} + \mathbf{u}_- e^{\Lambda_-} \mathbf{u}_-^{-1} (I_d - Y_+ Y_-^{-1})^{-1}. \end{aligned} \quad (21)$$

195 Here the problem associated with the presence of the high-order modes is clearly visible.
 196 As these modes are strongly cut-off, the matrices E_1, E_2, E_3, E_4 involve adding terms in
 197 e^{Λ_+} to other terms in e^{Λ_-} while these two exponentials have entirely different orders of
 198 magnitudes. Depending on the direction of the axial integration, one of those two always
 199 corresponds to exponentially increasing terms, say here e^{Λ_+} , while the other is linked to
 200 exponentially decreasing terms, say here e^{Λ_-} . This means that the terms in e^{Λ_+} are very

201 large while the ones in e^{Λ_-} are very small, leading to round-off errors. The goal is to
 202 reformulate this expression so as to avoid this summation, which is difficult to compute
 203 accurately.

204 The previous expressions are therefore inserted into the equation (18) to give

$$\begin{aligned}
 Y_{n+1} &= Y_+ + (Y_- - Y_+)P_-RP_+^{-1}(I_d + P_-RP_+^{-1})^{-1} \\
 \phi_n &= (I_d + R)P_+^{-1}(I_d + P_-RP_+^{-1})^{-1}\phi_{n+1},
 \end{aligned}
 \tag{22}$$

205 where $P_- = \phi_-e^{\Lambda_-}\phi_-^{-1}$, $P_+^{-1} = \phi_+e^{-\Lambda_+}\phi_+^{-1}$ and $R = (Y_n - Y_-)^{-1}(Y_+ - Y_n)$.

206 Note that with these expressions, contrary to the ones in (21), there is no summation of
 207 the term e^{Λ_+} and e^{Λ_-} . The impact of the high-order mode interaction is included in the
 208 term $P_-RP_+^{-1}$, with the high-order modes which are small in both $e^{-\Lambda_+}$ and e^{Λ_-} .

209 A quick analysis of Y_{n+1} can give us a better understanding of the role of each of the
 210 previous matrices. The terms $\phi_{\pm}e^{\Lambda_{\pm}}\phi_{\pm}^{-1}$ are characteristic of the propagation of the left and
 211 right running modes between the axial position x_n and x_{n+1} . By analogy to the reflection
 212 matrix found when deriving the expression for the potential at the source, the matrix R
 213 can be associated with the local reflection between left- and right-running modes. When
 214 $Y_n = Y_+$ (constant cross-section duct), $R = 0$ and the formulation reduces to $Y_{n+1} = Y_+$.
 215 This is expected because the admittance is constant inside a duct when there is no wall
 216 or flow variation. In such a duct $\phi_n = \phi_+e^{-\Lambda_+}\phi_+^{-1}\phi_{n+1}$ which means that there is also no
 217 possible scattering.

218 Note that the expression obtained for the potential in equation (22) is similar to the ones
 219 derived by Wilson 17 or Félix & Pagneux 13, but here we have not neglected the contribution
 220 of high-order modes.

221 Although this methodology is useful, it has two drawbacks. First, computing the eigen-
 222 values of the matrix Ω_n can be numerically costly. Secondly, as explained previously, it is
 223 necessary to distinguish between left and right-running modes. Failing to sort these modes
 224 correctly can cause some numerical difficulties. In theory, it would be possible to use the
 225 group velocity to differentiate between them, but evaluating this term can be challenging in
 226 practice. As an alternative, the "transverse-mode" criteria, defined in Section III E, can be
 227 used to classify these modes. This method for ordering the different categories of eigenmodes
 228 is used in the remainder paper and reliably distinguishes left and right-running modes for
 229 the studied cases.

230 Finally, note that a non-reflective boundary condition at the exit ($R = 0$) becomes
 231 available with the newly derived expression. However, it is not used in the remainder of
 232 the paper for comparison reasons with validation methodologies and because it causes some
 233 phenomena that are not yet explained.

234 H. Impedance discontinuities

235 The equation (10) obtained from the multimodal approach is valid as long as the
 236 impedance is represented by a differentiable function. However, in practice, the liner is
 237 only applied to a finite portion of the duct, which leads to impedance discontinuities. Let
 238 us now suppose that one of these discontinuities is located at x_d . We denote x_d^- and x_d^+ the
 239 limit when $x \rightarrow x_d$ from the left and from the right, respectively. In a previous paper 16,
 240 we demonstrated that the jump in the admittance matrix between these two axial locations
 241 can be expressed as

$$Y(x_d^-) = (\mathcal{A} - \mathcal{N}^-)^{-1} (\mathcal{P}^- - \mathcal{P}^+ + (\mathcal{A} - \mathcal{N}^+) Y(x_d^+)), \quad (23)$$

242 with

$$\begin{aligned} \mathcal{A}_{ij} &= \int_S D(1 - M_x^2) \varphi_j \varphi_i^* dS, \\ (\mathcal{N}^\pm)_{ij} &= \frac{1}{i\omega} \int_\Lambda \frac{D^2 V_\tau}{Z(x_d^\pm)} U \varphi_j \varphi_i^* d\Lambda, \\ (\mathcal{P}^\pm)_{ij} &= \frac{1}{i\omega} \int_\Lambda \frac{D^2 V_\tau}{Z(x_d^\pm)} \left(i\omega \varphi_j + V \frac{\partial \varphi_j}{\partial r} \right) \varphi_i^* d\Lambda. \end{aligned} \quad (24)$$

243 IV. MULTIMODAL FORMULATION FOR FLOW CALCULATIONS

244 It is interesting to note that solving the acoustic propagation equation (2c) for $\omega \rightarrow 0$,
 245 $m = 0$ and $\mathbf{V} = \mathbf{0}$ amounts to solving equation (2a) for the potential mean flow, with ϕ
 246 replaced by Φ . In this section, we show how to apply the multimodal method developed for
 247 acoustic propagation to calculate the mean flow. The same procedure as for the acoustic
 248 calculation is followed, but with two important differences, described in the next sections.

249 A. Iterative procedure

250 An iterative procedure is required because equation (2a) is non-linear due to the depen-
 251 dence of the mean density D on the velocity potential Φ , as defined by (2b). Starting from
 252 an initial value for the density, we solve equation (2c) using the same multimodal method
 253 as above to calculate the velocity potential and the axial velocity. A new density field D
 254 is then calculated using (2b). This process is repeated until the change in D between two
 255 iterations is smaller than a chosen threshold.

256 **B. Inflow condition**

257 The other specificity for using the multimodal method for computing the mean flow lies
 258 in the definition of the inflow condition. Indeed, a problem arises when we need to define an
 259 admittance at the exit where we consider a constant cross-section duct end. The analogy
 260 with the acoustics shows that the first eigenvalue associated with the mean flow potential and
 261 mean flow axial velocity is $\lambda = 0$. Equation (14) then indicates that the mean flow potential
 262 is not proportional to the mean flow axial velocity. Consequently, the equation (14) cannot
 263 be used as such anymore.

264 To address this limitation, we propose representing the mean flow in constant ducts as a
 265 combination of a mean value U_0 and perturbations that vary exponentially. In a constant
 266 cross-section duct, the vector of flow axial velocity coefficients is therefore written

$$\mathbf{U}(x) = U_0 \mathbf{w}_0^u + \sum_{i \neq 0} \alpha_i \mathbf{w}_i^u e^{\lambda_i x}, \quad (25)$$

267 where $\lambda_i \in \mathbb{R}^*$. The term of \mathbf{w}_0^u associated with φ_0 is 1 and all the other ones are equal to
 268 zero. This implies that Φ writes

$$\Phi(x) = U_0 \mathbf{w}_0^\phi + \sum_{i \neq 0} \alpha_i \mathbf{w}_i^\phi e^{\lambda_i x}, \quad (26)$$

269 where the term of \mathbf{w}_0^ϕ associated with the mode φ_0 is $x + cst$. At the exit, this term equals
 270 $x_e + cst$.

271 By noting that the flow perturbations cannot increase in a constant cross-section duct,
 272 and therefore by only selecting the eigenvalues associated with exponentially decreasing
 273 perturbations, the exit boundary condition for the admittance is

$$Y_e = W_+^u (W_+^\phi)^{-1}. \quad (27)$$

274 For the flow cases, the admittance matrix plays a double role at the duct exit. On the one
 275 hand, it sets the potential constant, and on the other hand, it avoids spurious reflections.

276 With this value, it is possible to find the admittance everywhere inside the duct using the
 277 standard Magnus–Möbius scheme. Unfortunately, the technique to stabilize this scheme for
 278 high-order modes cannot be used for the flow since the matrix Ω_n is no more diagonalizable.

279 Finally, by imposing an “incoming” velocity U_s^+ at the source location and by using the
 280 local admittance matrix, it is possible to define an injection condition. The “incoming”
 281 velocity is then adjusted to get the desired mass flow.

282 V. VALIDATION FOR A VARYING DUCT WITH A CIRCULAR CROSS-SECTION

283 A. Validation methodology

284 1. Description of the test case

285 For all the following results, we use the axisymmetric geometry from Reference 28 which
 286 is representative of the intake duct of the CFM56 engine. It is entirely defined by the spinner
 287 radius R_1 and the radius R_2 of the inner wall of the intake

$$\begin{aligned} R_1(x) &= \max\left(0, 0.64212 - (0.04777 + 0.98234 y^2)^{0.5}\right), \\ R_2(x) &= 1 - 0.18453 y^2 + 0.10158 \frac{e^{-11(1-y)} - e^{-11}}{1 - e^{-11}}, \end{aligned} \quad (28)$$

288 for $0 \leq x \leq L$ and with $y = x/L$ and $L = 2$. The fan is located at the axial position $x = 0$,
 289 and the duct exit at $x = L$. The flow properties are specified at the fan location, with a
 290 prescribed axial velocity, density and speed of sound. The density and speed of sound are

291 set to 1 in all cases of the paper. The acoustic source is defined as an incoming acoustic
 292 mode at the fan plane, where the duct has hard walls. This mode is defined by its azimuthal
 293 and radial wavenumber orders m and n , respectively. At both ends of the duct, we assume
 294 infinite duct conditions with zero reflection coefficients. For test cases with a liner, we apply
 295 a constant impedance $Z_2 = 2 - i$ on the outer wall between $x = 0.2$ and $x = 1.8$.

296 *2. Numerical convergence*

297 For a given two-dimensional variable $f(x, r)$, we define the variables f_p referring to the
 298 approximation obtained using the multimodal method with p polynomials. The relative
 299 numerical error is defined as

$$\epsilon_p = \left(\frac{\int_{R_1}^{R_2} \int_0^L |f_p - f_{\text{ref}}|^2 r dx dr}{\int_{R_1}^{R_2} \int_0^L |f_{\text{ref}}|^2 r dx dr} \right)^{1/2}, \quad (29)$$

300 where f_{ref} is the reference solution obtained using an FEM code 29 with a fine grid. This
 301 FEM code is used to compute both the steady potential flow and the acoustic field. For flow
 302 calculations, the error is calculated on the axial velocity, while it is based on the amplitude
 303 of complex pressure for acoustic results.

304 The FEM solutions are computed on an unstructured, triangular mesh generated using
 305 Gmsh 30. Quadratic elements (6 node triangles) are used to represent the solutions. To
 306 avoid spurious reflections at the inlet and outlet boundary conditions, where a representation
 307 over transverse hard-wall modes is used, the duct ends should be located far enough from
 308 regions of non-uniformities. Therefore, the duct is extended by straight parts of length 0.5 at
 309 the inlet and the outlet to tend to a uniform flow at boundary conditions. The counterpart

310 is that the sharp junctions between the real duct and the straight duct extensions can also
 311 cause reflections, that should nevertheless be captured by the two solvers (note that a non-
 312 reflective boundary condition with the original duct geometry could have only been achieved
 313 with the multimodal method as discussed at the end of Section III G). A mesh convergence
 314 study is performed to evaluate if the acoustic field is accurately represented. When using
 315 the FEM solver, there is also a need to refine the mesh near the sharp edges and the liner
 316 discontinuities to have an accurate solution.

317 **3. Axial discretization**

318 The multimodal method involves a numerical integration along the x axis, to calculate
 319 both the admittance matrix and the sound field. It is important to identify a robust criterion
 320 to define the number of points in the axial direction to achieve a trade-off between accuracy
 321 and computational efficiency. For this purpose, we follow the approach of Guennoc 18 and
 322 define a density, D_{ex} , as the number of points per characteristic length of decay of the most
 323 evanescent mode. However, we have to ignore non-physical high-order modes since they are
 324 highly evanescent and would lead to a dramatically oversampled mesh. We therefore only
 325 consider the first two thirds of the eigenvalues, which is considered to be a good criterion
 326 to isolate physical modes 31. If N is the total number of eigenvalues, this means that the
 327 discretization is set to correctly capture the axial variation of all modes with a radial order
 328 inferior to $2N/3$. In practice, we use the eigenvalues found at the duct exit, λ_i , to define the
 329 number of axial grid points N_x

$$N_x = D_{ex} \lambda_{2N/3} L. \quad (30)$$

330 In the rest of this paper, we use $D_{ex} = 1$ and a second order Magnus–Möbius scheme 14, 19.

331 **B. Flow computation**

332 To validate the computation of the mean flow, three test cases are considered, with a low,
333 medium and high velocity defined at the fan face by $M_s = -0.2, -0.4$ and -0.6 , respectively.
334 For each of these cases, the number of Chebyshev polynomials is varied: $p = 3, 6$ or 9 . The
335 results are shown in Figure 2. While three polynomials are not sufficient to accurately
336 represent the mean flow, a good representation of the flow can already be achieved with
337 only six polynomials. With nine polynomials, the agreement is excellent. The areas where
338 the accuracy is reduced are near the sharp corners along the duct walls, which are located
339 at the transition from the spinner to the duct axis near $x = 1.25$, and at the exit of the
340 duct $x_e = 2$, where a straight duct is used for the exit condition. At these points, the flow
341 velocity indeed reaches zero to satisfy the zero normal velocity conditions (wall boundary)
342 along the two different normal directions of the sharp corner, which results in strong local
343 gradients in the solution. Yet, these localized losses of accuracy do not impact the rest of the
344 solution, and the method is able to compute the mean flow accurately even with a limited
345 number of polynomials.

346 The results in Figure 2 also indicate that the required number of polynomials does not
347 depend on the specified Mach number value, since there is no visible loss of accuracy when
348 increasing it. The difference, which is not visible in the graph, is the number of iterations
349 required to converge to the solution of the non-linear equation (2a). For the low-velocity
350 case, only three steps are required to reach a converged solution (with a tolerance of 10^{-5}),

351 while five steps are needed for the medium-velocity case and seven for the high-velocity case.
 352 This is expected, since compressible effects become more significant as the Mach number
 353 increases.

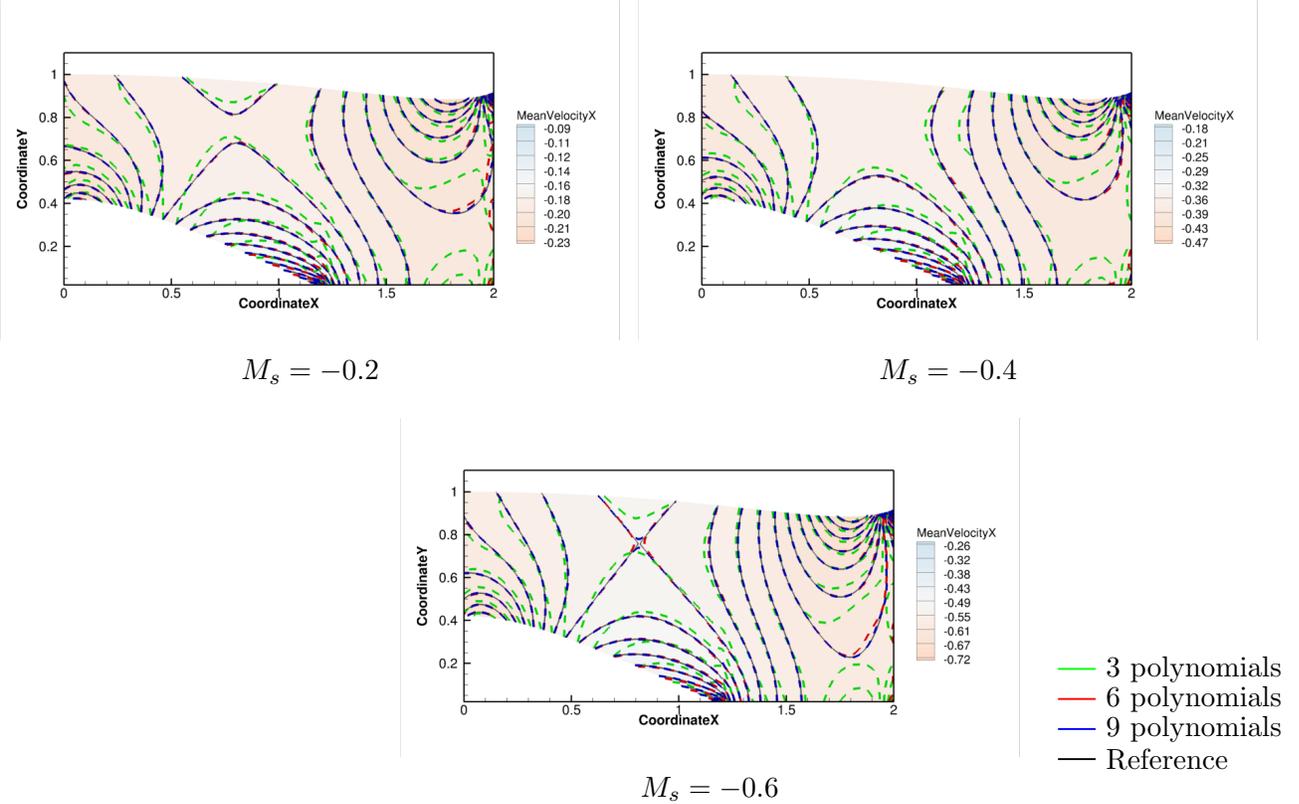


FIG. 2. Contours of normalized axial velocity, overlaid on the reference FEM solution, for varying numbers of polynomials.

354 Figure 3 presents a convergence analysis using the FEM computations as the reference
 355 solutions. The number of polynomials used in the multimodal method is varied between
 356 1 and 50. The results confirm the qualitative observations based on Figure 2. Firstly, an
 357 accurate representation of the flow can be obtained even with a small number of polynomials.

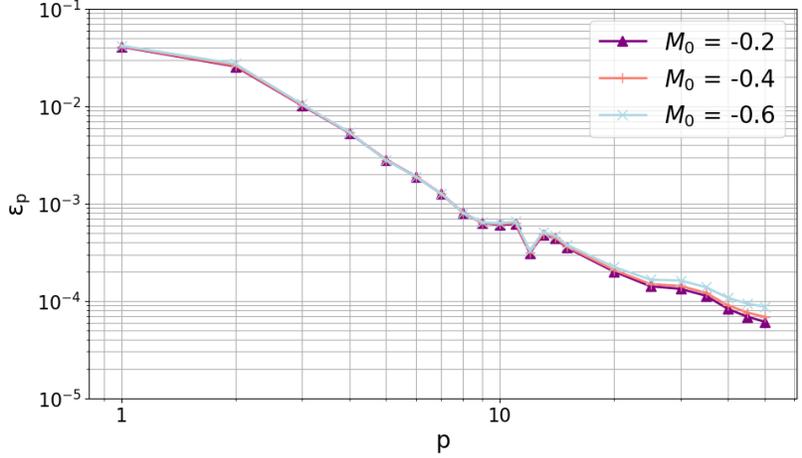


FIG. 3. Relative error on the axial velocity field as a function of the number of polynomials used in the multimodal method and for the three flow velocities.

358 For example, with just nine polynomials, the relative error is only $\epsilon_p \simeq 6 \times 10^{-4}$. Secondly,
 359 the specified Mach number only has a small influence on the accuracy.

360 To provide an indication of the computational cost of the flow calculations with the
 361 multimodal method (based on the Python implementation), we report the CPU time for
 362 a single iteration needed to solve Equation (2a). With five polynomials, the time is 0.09
 363 seconds, with ten polynomials it is 0.25 seconds, with twenty polynomials it is 0.97 seconds,
 364 and it is 7.2 seconds with fifty polynomials. While the method is advantageous with a small
 365 number of polynomials, the calculation time increases rapidly as the number of polynomials
 366 increases. Still, the multimodal method can provide useful results with reasonably low
 367 computational times.

368 **C. Hard-wall acoustic calculations**

369 The method is designed with high-velocity calculations in mind, since it is precisely in
 370 such cases that the multimodal method with a multiple-scale flow presented in a previous
 371 paper 16 encounters difficulties. Consequently, the high-velocity flow ($M_s = -0.6$) is used
 372 for acoustic computation. Forty polynomials are used for the flow computations to prevent
 373 the numerical error on the flow from affecting the acoustic calculations. Note that axial
 374 grid points used in the multimodal calculations of the mean flow might differ from those
 375 used in the multimodal calculation of the sound field. For this purpose, the amplitudes of
 376 the Chebyshev polynomials representing the flow properties are interpolated along the axial
 377 direction using a cubic interpolation. The interpolated amplitudes are then used to compute
 378 the flow properties at the desired radial locations.

379 To analyse the acoustic response, we first perform a qualitative test by analyzing contour
 380 maps of the sound pressure levels (SPL) defined as

$$p_{SPL} = 20 \log_{10} \left(\frac{|p|}{20 \times 10^{-6}} \right). \quad (31)$$

381 This is done for a mode $(m, n) = (13, 1)$ at $\omega = 20$ and $\omega = 30$, and for the modes
 382 $(m, n) = (1, 1)$ and $(m, n) = (13, 3)$ at $\omega = 20$. These modes include a case where the injected
 383 mode encounters a transition from cut-on to cut-off inside the duct ($(m, n) = (13, 3)$) and
 384 cases where the injected modes are far from a transition phenomena (all other modes). A
 385 unit value is specified for the outer wall pressure of the incoming wave at the source location
 386 for all cases. The SPL contours inside the duct are calculated for each case with the FEM,

387 in order to have a reference solution, and with the multimodal method with various numbers
 388 of polynomials $p = 10, 15$ and 25 . The results are shown in figure 4.

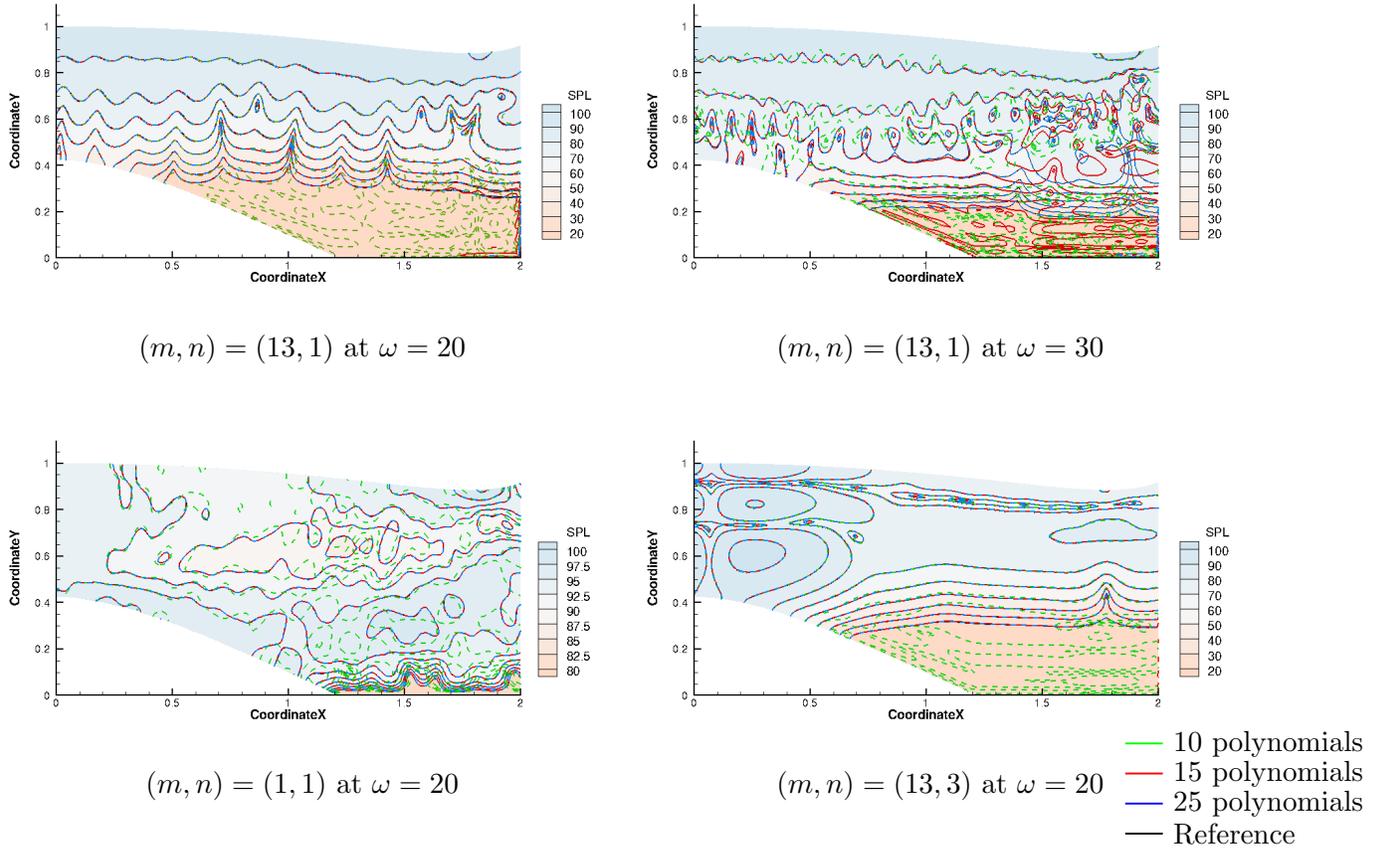


FIG. 4. Contours of sound pressure levels, overlaid on the reference FEM solution, for different numbers of polynomials.

389 When the mode $(13, 1)$ is injected at $\omega = 20$, even with 10 polynomials, the agreement is
 390 already good for SPL between 50 and 110 dB, but the agreement remains poor for low SPL.
 391 With 15 or 25 polynomials, the agreement is excellent for all SPL shown in the graph. It
 392 should be noted that it is in the regions where the flow speed is higher ($x \in [1.8, 2]$) that the

393 precision is the lowest. Similar conclusions can be made for the higher radial order mode
394 $(13, 3)$.

395 When the frequency is increased for the azimuthal mode $(m, n) = (13, 1)$, the agreement is
396 degraded. With 10 polynomials, the pressure field is inaccurate everywhere in the duct. With
397 15 polynomials, the agreement improves markedly, and for high SPL values, the agreement is
398 acceptable. With 25 polynomials, the agreement is once again excellent. As expected, more
399 polynomials are needed when increasing the frequency since the number of cut-on modes
400 also increases.

401 To further confirm these qualitative observations, we perform a convergence analysis by
402 plotting, in figure 5, the relative error on the amplitude of acoustic pressure as a function
403 of the number of polynomials. We consider the influence of the azimuthal mode order, the
404 frequency, as well as the first three radial orders. It is important to note that we exclude
405 cases in which the mode is strongly cut-off at the source plane since their strong decay is
406 hard to compute accurately using the FEM.

408 We first consider the azimuthal mode $m = 1$ for a wide range of polynomials. We observe
409 that the convergence graphs can be divided into three parts for this test case. First, a
410 pre-asymptotic region where the error is almost equal to one and where including further
411 polynomials does not modify the error. Then a sharp decay region, where the convergence is
412 almost exponential (typical of a polynomial basis), is obtained. Lastly, a slower convergence
413 is observed. The authors assume that this last region is linked to the axial discretization, and
414 that the observed convergence rate is related to the order of the Magnus–Möbius scheme.
415 Note that for the pulsation $\omega = 5$, the pre-asymptotic behaviour is not observed mainly

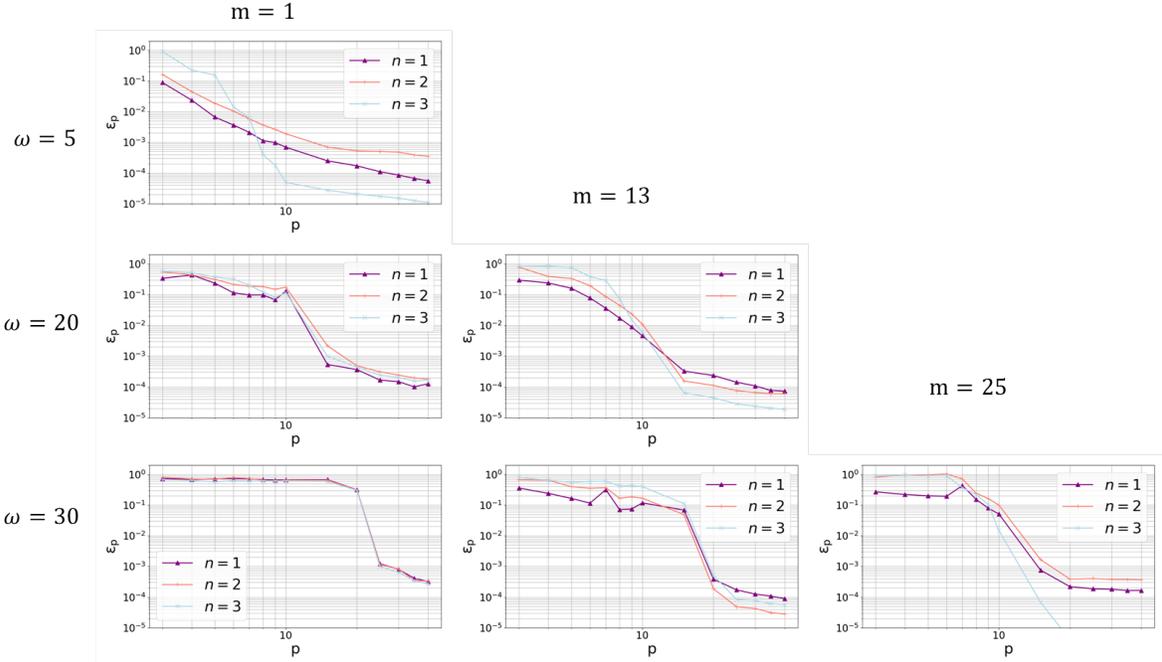


FIG. 5. Relative error on the acoustic pressure amplitude field as a function of the number of polynomials.

416 because very few polynomials are necessary to accurately represent the acoustic field. For
 417 the radial order $n = 3$ of the same frequency, the error decreases more sharply than for
 418 the other radial orders for a low number of polynomials $p < 10$, before converging again
 419 with the previously observed convergence rates. One explanation for this behaviour is that
 420 the Magnus–Möbius scheme is highly adapted to compute cut-off modes. Therefore the
 421 convergence rate at the beginning corresponds to the improvement in the representation of
 422 the mode $n = 3$. After that, the convergence is more linked to the representation of the
 423 modes $n = 1, 2$ (which are cut-on). In fact, the mode $n = 3$ will create modes $n = 1, 2$ by
 424 scattering mechanisms. Therefore, the convergence rate after the strong decay is the one of
 425 these newly created modes.

426 When the azimuthal mode increases ($m = 13$ and $m = 25$), all the observations made
 427 before remain true except for the size of the pre-asymptotic region. The decrease starts
 428 further in the graph since a higher azimuthal mode means that the acoustic energy is mainly
 429 concentrated near the tip wall, and higher-order polynomials are necessary to capture it.
 430 However, the error remains small even with a limited number of polynomials. In fact, for
 431 all frequencies and radial orders, the error is less than 10^{-3} with 30 polynomials.

432 Next, to assess the effectiveness of the stabilization proposed for the Magnus–Möbius
 433 scheme in Section III G, we compare the conditioning of the matrices that need to be inverted
 434 for both the standard and improved schemes: namely, $(E_1 + E_2Y)$ for the standard scheme
 435 and $(I_d + P_-RP_+^{-1})$ for the improved scheme. We consider two cases: the mode $(m, n) =$
 436 $(13, 1)$ at $\omega = 20$ and the mode $(m, n) = (1, 1)$ at $\omega = 30$ (both cases use 40 polynomials).
 437 The conditioning does not depend on the axial discretization with the modified scheme, while
 438 it is highly sensitive to it in the standard scheme. Therefore, to evaluate the improvement
 439 brought by the new formulation, four axial integration steps are used with the standard
 440 scheme: $D_{ex} = 0.5, 1, 2, 10$. The results are shown in figure 6. As expected, the conditioning
 441 of the improved scheme is better than the one of the standard scheme almost everywhere.
 442 Some peaks appear in both formulations near axial positions where transition phenomena
 443 occur (cut-on to cut-off, or vice versa). For the improved formulation, additional peaks also
 444 appear where an error in the sorting of the modes is made. Note that the conditioning of
 445 the matrix $(E_1 + E_2Y)$ is suddenly worsened when the duct cross section goes from annular
 446 to circular (around $x = 1.2$). In this region, the proposed stabilized scheme improves the
 447 conditioning by several orders of magnitudes.

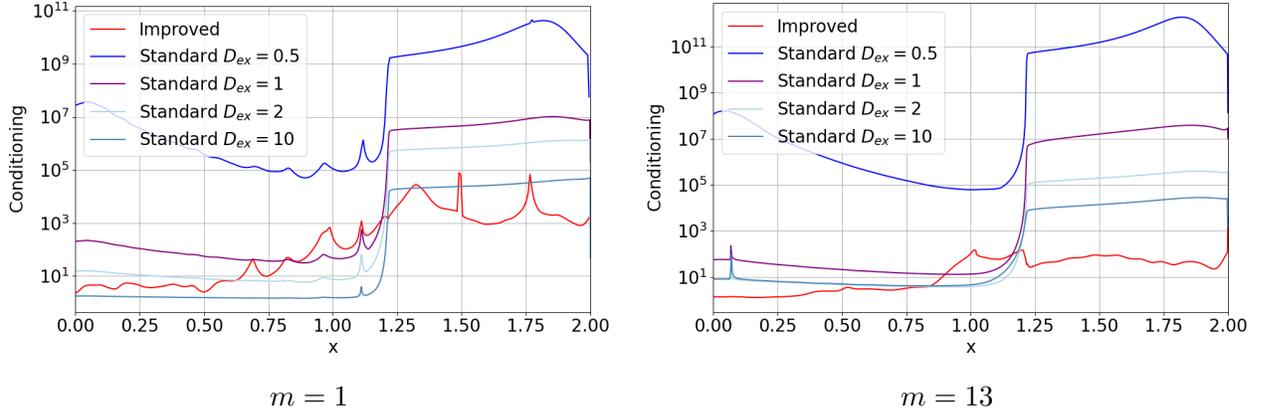


FIG. 6. Conditioning of the matrix to invert in the Magnus-Möbius scheme with the standard and improved formulation.

448 Finally, we report the computational time required for each calculation. These are mea-
 449 sured using a Python implementation of the multimodal method running on a desktop
 450 computer. The CPU time is not given for each calculation, but instead we provide a range
 451 of runtimes, from the shortest to the longest runtimes for a fixed number of polynomials.
 452 With five polynomials, the runtime ranges from 0.02 to 0.9 seconds, while with ten poly-
 453 nomials, it ranges from 0.09 to 3 seconds. Similarly, with twenty polynomials, the runtime
 454 ranges from 0.5 to 7 seconds, and with forty polynomials, it ranges from 1.5 to 33 seconds. It
 455 is important to note that unlike for the flow calculations in Section VB, the computational
 456 time for a given number of polynomials varies significantly depending on the studied mode
 457 and frequency since they influence the number of points in the axial direction. Note that
 458 the criterion defined for D_{ex} is rather conservative, and sufficiently accurate results can be
 459 obtained with smaller values of D_{ex} .

460 **D. Lined acoustic computation**

461 The same test cases are now considered with a lined wall (see section [V A 1](#)). It should be
 462 recalled that when there is an impedance discontinuity along the wall, the jump condition
 463 derived in Section [III H](#) is used. In practice, three multimodal computations are done: one
 464 before the liner, one in the liner region and the last one after the liner. The admittance
 465 jump condition of equation [\(23\)](#) is used between each part. Note that the density criterion
 466 D_{ex} is redefined in each region, leading to different axial discretizations. Contours of the
 467 sound pressure levels in the duct are given in [Figure 7](#). All the conclusions made for the
 468 hard-wall cases remain valid, and an additional observation can be noted. At the junctions
 469 between hard and lined walls, the jump in surface impedance induces sharp gradients in
 470 the velocity potential, and hence large jumps in the acoustic pressure. This type of weak
 471 singularity is difficult to capture with a spectral method [32](#) such as the multimodal method.
 472 It is therefore expected that the rate of convergence of the numerical model will be reduced
 473 when compared to the hard-wall test cases. To mitigate this, in the presence of a liner, high-
 474 order polynomials are required to accurately represent the large pressure gradients near the
 475 end points of the liner. This effect is relatively weak for the first liner discontinuity at
 476 $x = 0.2$, but is clearly visible at $x = 1.8$. Two reasons can explain why the effect seems
 477 to vary between both discontinuities. First, the fact that there are low SPL values close to
 478 the duct axis for the second discontinuity makes the errors more visible. Then, the higher
 479 errors at the second discontinuity could also be attributed to the slower convergence of the
 480 Chebyshev basis in the circular regions compared to annular ones.

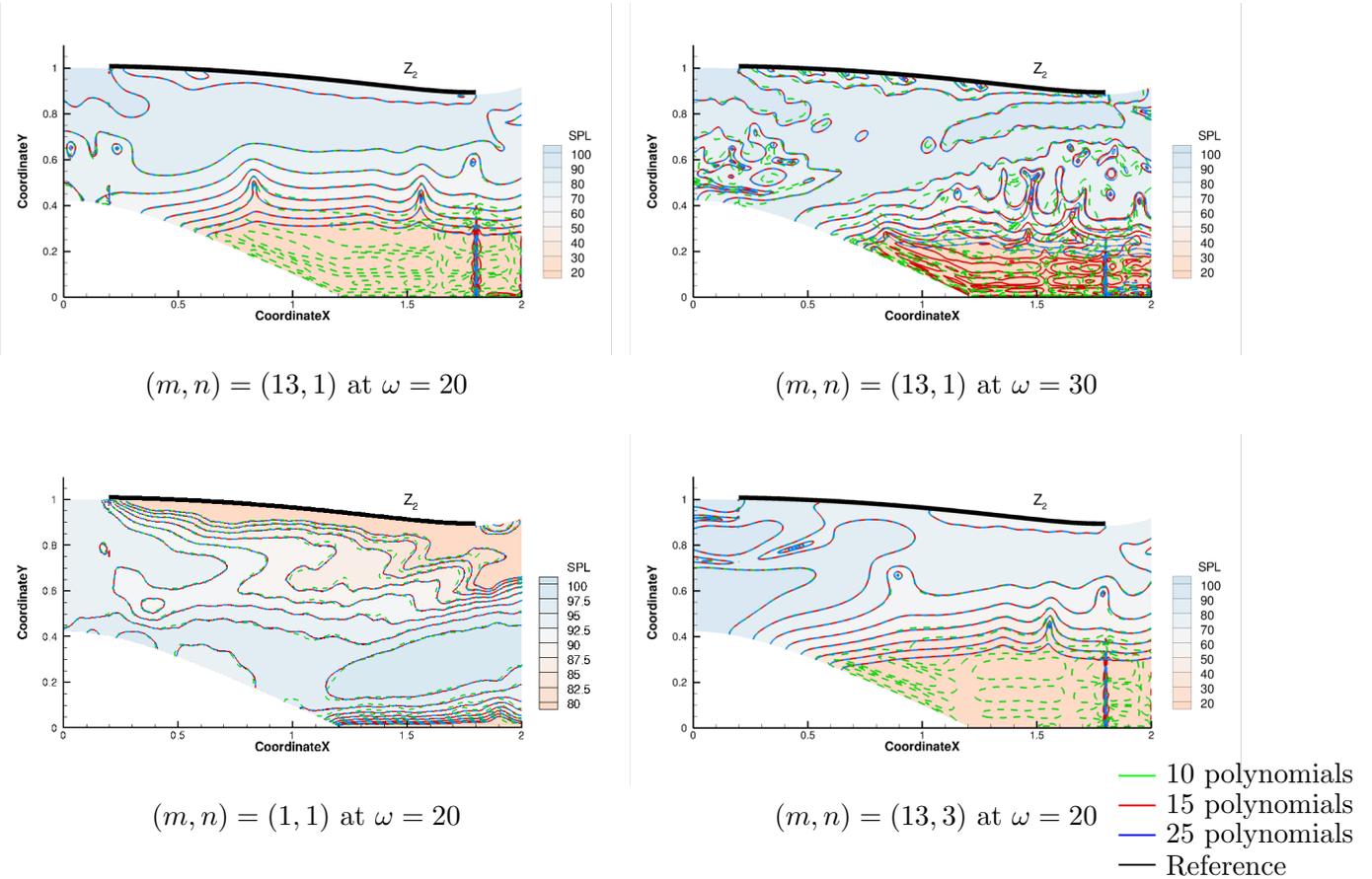


FIG. 7. Contours of the sound pressure level in the lined duct, overlaid on the reference FEM solution, for different numbers of polynomials.

481 Once again, a convergence study is performed, and the same flow condition and acoustic
 482 modes as the ones used for the hard-wall validation are taken. Figure 8 presents the results.
 483 As expected, achieving a precision of 10^{-3} requires more polynomials than the hard-wall
 484 case, and the convergence rate is lower. This slower convergence rate makes the method less
 485 appealing for lined wall computations. It would be necessary to add supplementary modes
 486 to address the singularities in the pressure field and avoid this slow convergence. Another
 487 key observation is that the liner tends to smooth the convergence plots by decreasing the
 488 size of the pre-asymptotic region, and for the most difficult hard-wall cases ($\omega = 30$), the

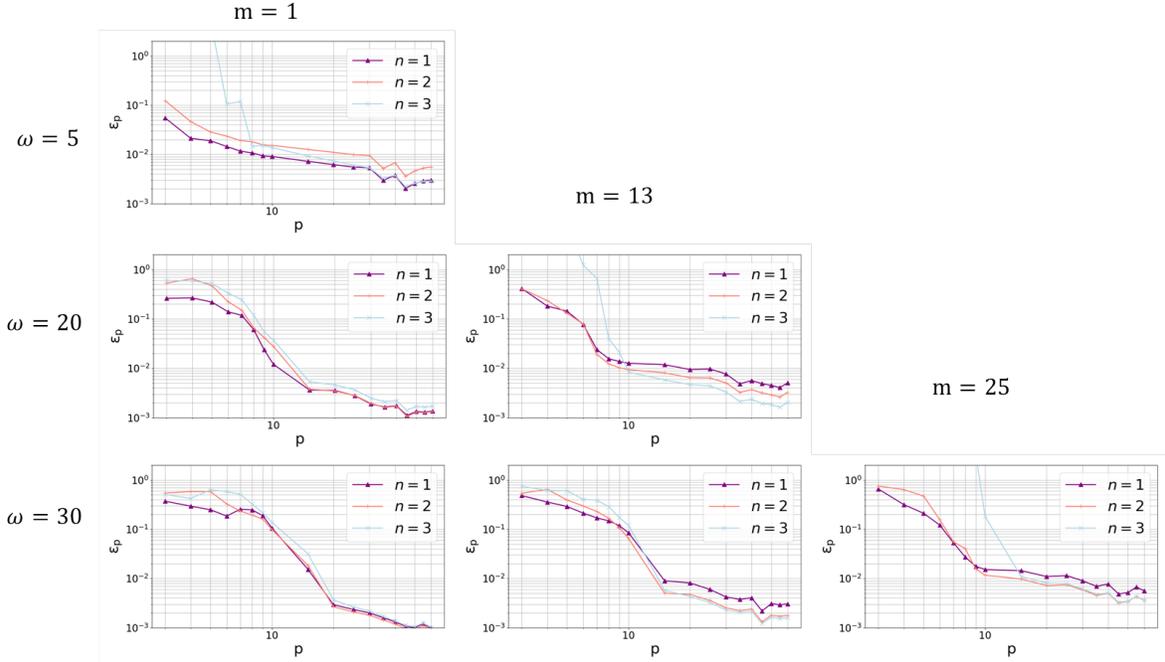


FIG. 8. Relative error on the acoustic pressure amplitude field as a function of the number of polynomials for the lined duct cases.

489 error goes faster to a value of 10^{-2} . This could be due to the fact that the method captures
 490 well the attenuation of the liner with few polynomials ($p < 20$), even if the prediction of the
 491 pressure singularity is erroneous. These singularities also pose problems with finite element
 492 methods, where it is necessary to refine the region around them. Note that the calculation
 493 times required by the multimodal method are similar to the ones obtained for the hardwalled
 494 test cases.

495 VI. CONCLUSION

496 This paper presents a novel multimodal method to efficiently compute the acoustic field
 497 in a duct with a potential mean flow. The acoustic potential equation is rearranged into a

498 system with only first-order derivatives in the axial direction governing the evolution of the
499 acoustic potential and acoustic axial velocity. Fourier functions and Chebyshev polynomials
500 provide the approximation basis in the duct cross-section, while a Magnus–Möbius scheme
501 is used to integrate the equations in the axial direction. The use of a high-order polynomials
502 basis results in the presence of spurious, non-physical modes. This is a problem common
503 to multimodal schemes using polynomial bases, and is not related to the presence of a
504 mean flow. A modification of the Magnus–Möbius scheme is proposed to avoid the poor
505 conditioning associated with the spurious modes. Results demonstrate that this modification
506 of the integration scheme significantly improves the conditioning of the model. Consequently,
507 larger axial steps can be used for Magnus–Möbius scheme, hence reducing the computational
508 cost of the method. This method, initially devoted to acoustics, has been, in addition,
509 adapted to compute the potential mean flow required for acoustic computations by modifying
510 the boundary conditions and applying an iterative method to solve the non-linear flow
511 equation.

512 The proposed method has been compared against a finite-element method on a model
513 of a turbofan inlet with hard and lined walls for several frequencies and modes. Validation
514 test cases demonstrate excellent agreement between the multimodal method and the FEM
515 solutions when a sufficient number of polynomials is used. For the mean flow, the proposed
516 method provides quick and accurate results, even for high flow velocities and a number of
517 polynomials as low as 9. For acoustic propagation with hard-wall conditions, a relative
518 error smaller than 10^{-3} can be achieved for all cases with only 25 polynomials, resulting
519 in very short computational times. However, for cases with lined ducts, the presence of

520 strong gradients in the solutions near impedance discontinuities requires the use of more
521 polynomials to achieve a similar level of accuracy. The computational cost therefore increases
522 for lined ducts, but the method remains efficient compared to a traditional finite-element
523 model.

524 While validations were conducted on axisymmetric ducts and flows, the method is also
525 applicable to non-axisymmetric flows, but it would require solving the acoustic equation
526 for several azimuthal orders simultaneously since these are coupled together. It would be
527 interesting to use this method to study phenomena such as flow distortion generated by an
528 angle of incidence into a turbofan intake.

529 **APPENDIX A: MULTIMODAL MATRICES**

530 The detailed expressions for the matrices introduced in the multimodal method are as
 531 follows

$$\begin{aligned}
 (A_{11})_{ij} &= \int_S D(1 - M_x^2) \varphi_j \varphi_i^* dS, \\
 (A_{12})_{ij} &= 0, \\
 (A_{21})_{ij} &= - \int_S \frac{DU}{C^2} \frac{D_{\perp} \varphi_j}{Dt} \varphi_i^* dS - \frac{1}{i\omega} \int_{\Lambda} \frac{D^2 V_{\tau}}{Z} \frac{D_{\perp} \varphi_j}{Dt} \varphi_i^* d\Lambda, \\
 (A_{22})_{ij} &= \int_S D(1 - M_x^2) \varphi_j \varphi_i^* dS - \frac{1}{i\omega} \int_{\Lambda} \frac{D^2 V_{\tau} U}{Z} \varphi_j \varphi_i^* d\Lambda, \\
 (M_{11})_{ij} &= - \int_S D(1 - M_x^2) \frac{\partial \varphi_j}{\partial x} \varphi_i^* dS, \\
 (M_{12})_{ij} &= \int_S D(1 - M_x^2) \varphi_j \varphi_i^* dS, \\
 (M_{21})_{ij} &= \frac{d}{dx} \left(\int_S \frac{DU}{C^2} \frac{D_{\perp} \varphi_j}{Dt} \varphi_i^* dS \right) + \int_S D \nabla_{\perp} \varphi_i^* \cdot \nabla_{\perp} \varphi_j dS \\
 &\quad - \int_S \frac{D}{C^2} \left(\frac{D_{\perp} \varphi_j}{Dt} \left(\frac{D_{\perp} \varphi_i}{Dt} \right)^* + U \frac{D_{\perp} \varphi_j}{Dt} \frac{\partial \varphi_i^*}{\partial x} \right) dS \\
 &\quad - \frac{1}{i\omega} \int_{\Lambda} \frac{D^2}{Z} \frac{D_{\perp} \varphi_j}{Dt} \left(\frac{D \varphi_i}{Dt} \right)^* d\Lambda + \frac{1}{i\omega} \frac{d}{dx} \left(\int_{\Lambda} \frac{D^2 V_{\tau}}{Z} \frac{D_{\perp} \varphi_j}{Dt} \varphi_i^* d\Lambda \right), \\
 (M_{22})_{ij} &= - \frac{d}{dx} \left(\int_S D(1 - M_x^2) \varphi_j \varphi_i^* dS \right) \\
 &\quad + \int_S D(1 - M_x^2) \varphi_j \frac{\partial \varphi_i^*}{\partial x} dS - \int_S \frac{DU}{C^2} \varphi_j \left(\frac{D_{\perp} \varphi_i}{Dt} \right)^* dS \\
 &\quad - \frac{1}{i\omega} \int_{\Lambda} \frac{D^2}{Z} U \varphi_j \left(\frac{D \varphi_i}{Dt} \right)^* d\Lambda + \frac{1}{i\omega} \frac{d}{dx} \left(\int_{\Lambda} \frac{D^2 V_{\tau} U}{Z} \varphi_j \varphi_i^* d\Lambda \right).
 \end{aligned}$$

532 **Acknowledgments**

533 ONERA carried out these activities in the framework of the ADEC project. This project
 534 has received funding from the Clean Sky 2 Joint Undertaking within the European Union's

535 Horizon 2020 research and innovation program, under grant agreement GA ID No. 945583
536 - LPA IADP 2020-2021.

537 **Author Declarations**

538 The authors declare no competing interests. The data that support the findings of this
539 study are available within the article.

540 **References**

541 ¹S. Redonnet, E. Manoha, and P. Sagaut, “Numerical simulation of propagation of small
542 perturbations interacting with flows and solid bodies,” in *7th AIAA/CEAS Aeroacoustics
543 Conference and Exhibit* (2001), doi: [mu10.2514/6.2001-2223](https://doi.org/10.2514/6.2001-2223).

544 ²S. Redonnet and Y. Druon, “Computational aeroacoustics of aft fan noises characterizing
545 a realistic coaxial engine,” *AIAA Journal* **50**(5), 1029–1046 (2012) doi: [mu10.2514/1.
546 J050730](https://doi.org/10.2514/1.J050730).

547 ³S. Redonnet and G. Cunha, “An advanced hybrid method for the acoustic prediction,”
548 *Advances in Engineering Software* **88**, 30–52 (2015) doi: [mu10.1016/j.advengsoft.2015.
549 05.006](https://doi.org/10.1016/j.advengsoft.2015.05.006).

550 ⁴M. C. Duta and M. B. Giles, “A three-dimensional hybrid finite element/spectral analysis
551 of noise radiation from turbofan inlets,” *Journal of Sound and Vibration* **296**(3), 623–642
552 (2006) doi: [mu10.1016/j.jsv.2006.03.006](https://doi.org/10.1016/j.jsv.2006.03.006).

553 ⁵I. Achunche, J. Astley, R. Sugimoto, and A. Kempton, “Prediction of forward fan noise
554 propagation and radiation from intakes,” *15th AIAA/CEAS Aeroacoustics Conference
555 (30th AIAA Aeroacoustics Conference)* (2009) doi: [mu10.2514/6.2009-3239](https://doi.org/10.2514/6.2009-3239).

556 ⁶R. J. Astley, R. Sugimoto, and P. Mustafi, “Computational aero-acoustics for fan duct
557 propagation and radiation. Current status and application to turbofan liner optimisation,”
558 *Journal of Sound and Vibration* **330**(16), 3832–3845 (2011) doi: [mu10.1016/j.jsv.2011.
559 03.022](https://doi.org/10.1016/j.jsv.2011.03.022).

560 ⁷A. G. Prinn, R. Sugimoto, and R. J. Astley, “The effect of steady flow distortion on
561 noise propagation in turbofan intakes,” 22nd AIAA/CEAS Aeroacoustics Conference, 2016
562 (2016) doi: [mu10.2514/6.2016-3028](https://doi.org/10.2514/6.2016-3028).

563 ⁸S. W. Rienstra, “Sound transmission in slowly varying circular and annular lined ducts with
564 flow,” *Journal of Fluid Mechanics* **380**, 279–296 (1999) doi: [mu10.2514/6.1998-2311](https://doi.org/10.2514/6.1998-2311).

565 ⁹N. Peake and A. J. Cooper, “Acoustic propagation in ducts with slowly varying elliptic
566 cross-section,” *Journal of Sound and Vibration* **243**(3), 381–401 (2001) doi: [mu10.1006/
567 jsvi.2000.3411](https://doi.org/10.1006/jsvi.2000.3411).

568 ¹⁰N. C. Ovenden, “A uniformly valid multiple scales solution for cut-on cut-off transition
569 of sound in flow ducts,” *Journal of Sound and Vibration* **286**(1-2), 403–416 (2005) doi:
570 [mu10.1016/j.jsv.2004.12.009](https://doi.org/10.1016/j.jsv.2004.12.009).

571 ¹¹R. P. Dougherty, “A parabolic approximation for flow effects on sound propagation in
572 nonuniform, softwall, ducts,” in *5th AIAA/CEAS Aeroacoustics Conference* (1999), doi:
573 [mu10.2514/6.1999-1822](https://doi.org/10.2514/6.1999-1822).

574 ¹²V. Pagneux, N. Amir, and J. Kergomard, “A study of wave propagation in varying cross-
575 section waveguides by modal decomposition. Part I. Theory and validation,” *The Journal*
576 *of the Acoustical Society of America* **100**(4), 2034 (1996) doi: [mu10.1121/1.417913](https://doi.org/10.1121/1.417913).

577 ¹³S. Félix and V. Pagneux, “Multimodal analysis of acoustic propagation in
578 three-dimensional bends,” *Wave Motion* **36**(2), 157–168 (2002) doi: [mu10.1016/
579 S0165-2125\(02\)00009-4](https://doi.org/10.1016/S0165-2125(02)00009-4).

- 580 ¹⁴V. Pagneux, “Multimodal admittance method in waveguides and singularity behavior at
581 high frequencies,” *Journal of Computational and Applied Mathematics* **234**(6), 1834–1841
582 (2010) doi: [mu10.1016/j.cam.2009.08.034](https://doi.org/10.1016/j.cam.2009.08.034).
- 583 ¹⁵T. Guennoc, J. B. Doc, and S. Félix, “Improved multimodal formulation of the wave
584 propagation in a 3D waveguide with varying cross-section and curvature,” *The Journal of*
585 *the Acoustical Society of America* **149**(1), 476–486 (2021) doi: [mu10.1121/10.0003336](https://doi.org/10.1121/10.0003336).
- 586 ¹⁶B. Mangin, M. Daroukh, and G. Gabard, “Propagation of Acoustic Waves in Ducts with
587 Flow Using the Multimodal Formulation,” *AIAA Journal* **61**(6) (2023) doi: [mu10.2514/](https://doi.org/10.2514/1.J062659)
588 [1.J062659](https://doi.org/10.2514/1.J062659).
- 589 ¹⁷A. Wilson, “Propagation of acoustic perturbations in non-uniform ducts with non-uniform
590 mean flow using eigen analysis in general curvilinear coordinate systems,” *Journal of Sound*
591 *and Vibration* **443**, 605–636 (2019) doi: [mu10.1016/j.jsv.2018.11.023](https://doi.org/10.1016/j.jsv.2018.11.023).
- 592 ¹⁸T. Guennoc, “Modélisation du rayonnement des guides de forme complexe par une méthode
593 multimodale,” Ph.D. thesis, Le Mans Université, 2021.
- 594 ¹⁹Y. Y. Lu, “A fourth-order Magnus scheme for Helmholtz equation,” *Journal of Compu-*
595 *tational and Applied Mathematics* **173**(2), 247–258 (2005) doi: [mu10.1016/j.cam.2004.](https://doi.org/10.1016/j.cam.2004.03.010)
596 [03.010](https://doi.org/10.1016/j.cam.2004.03.010).
- 597 ²⁰M. K. Myers, “On the acoustic boundary condition in the presence of flow,” *Journal of*
598 *Sound and Vibration* **71**(3), 429–434 (1980) doi: [mu10.1016/0022-460X\(80\)90424-1](https://doi.org/10.1016/0022-460X(80)90424-1).
- 599 ²¹W. Eversman, “The boundary condition at an impedance wall in a non-uniform duct
600 with potential mean flow,” *Journal of Sound and Vibration* **246**(1), 63–69 (2001) doi:

601 [mu10.1006/jsvi.2000.3607](https://doi.org/10.1006/jsvi.2000.3607).

602 ²²J. P. McTavish and E. J. Brambley, “Nonlinear sound propagation in two-dimensional
603 curved ducts: A multimodal approach,” *Journal of Fluid Mechanics* **875**, 411–447 (2019)
604 doi: [mu10.1017/jfm.2019.497](https://doi.org/10.1017/jfm.2019.497).

605 ²³J. F. Mercier and A. Maurel, “Improved multimodal method for the acoustic propagation
606 in waveguides with a wall impedance and a uniform flow,” *Proceedings of the Royal Society
607 A: Mathematical, Physical and Engineering Sciences* **472** (2016) doi: [mu10.1098/rspa.
608 2016.0094](https://doi.org/10.1098/rspa.2016.0094).

609 ²⁴C. Canuto, M. Y. Houssanini, A. Quarteroni, and T. A. Zang, *Springer series in compu-
610 tational physics* (Springer, New York, 1988).

611 ²⁵E. J. Brambley and N. Peake, “Sound transmission in strongly curved slowly varying
612 cylindrical ducts with flow,” *Journal of Fluid Mechanics* **596**, 387–412 (2008) doi: [mu10.
613 1017/S0022112007009603](https://doi.org/10.1017/S0022112007009603).

614 ²⁶G. Boyer, E. Piot, and J. P. Brazier, “Theoretical investigation of hydrodynamic surface
615 mode in a lined duct with sheared flow and comparison with experiment,” *Journal of
616 Sound and Vibration* **330**(8), 1793–1809 (2010) doi: [mu10.1016/j.jsv.2010.10.035](https://doi.org/10.1016/j.jsv.2010.10.035).

617 ²⁷H. Ralph Lewis and P. M. Bellan, “Physical constraints on the coefficients of Fourier
618 expansions in cylindrical coordinates,” *Journal of Mathematical Physics* **31**(11), 2592–
619 2596 (1990) doi: [mu10.1063/1.529009](https://doi.org/10.1063/1.529009).

620 ²⁸N. C. Ovenden, W. Eversman, and S. W. Rienstra, “Cut-on cut-off transition in flow ducts:
621 comparing multiple-scales and finite-element solutions,” in *10th AIAA/CEAS Aeroacous-*

622 *tics Conference* (2004), doi: [mu10.2514/6.2004-2945](https://doi.org/10.2514/6.2004-2945).

623 ²⁹G. Gabard, “PFE Software” (2021), [muhttps://github.com/GwenaelGabard/pfe](https://github.com/GwenaelGabard/pfe).

624 ³⁰C. Geuzaine and J. F. Remacle, “Gmsh: A 3-D finite element mesh generator with built-in
625 pre-and post-processing facilities,” *International journal for numerical methods in engi-*
626 *neering* **79**(11), 1309–1331 (2009) doi: [mu10.1002/nme.2579](https://doi.org/10.1002/nme.2579).

627 ³¹A. J. Cooper and N. Peake, “Propagation of unsteady disturbances in a slowly varying
628 duct with mean swirling flow,” *Journal of Fluid Mechanics* **445**, 207–234 (2001) doi: [mu10.](https://doi.org/10.1017/s0022112001005559)
629 [1017/s0022112001005559](https://doi.org/10.1017/s0022112001005559).

630 ³²J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, 2nd ed., rev ed. (Springer Berlin
631 Heidelberg, Mineola, N.Y, 1989).