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# Extension of the bang-bang principle for piecewise affine dynamics under state constraints <sup>★</sup>

Ruben Chenevat <sup>a</sup>, Bruno Cheviron <sup>b</sup>, Sebastien Roux <sup>a</sup>, Alain Rapaport <sup>a</sup>

<sup>a</sup>MISTEA, Univ. Montpellier, INRAE, Institut Agro, 34060 Montpellier, France

<sup>b</sup>G-EAU, Univ Montpellier, AgroParisTech, BRGM, CIRAD, INRAE, Institut Agro, IRD, Montpellier, France

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## Abstract

As an extension of the classical bang-bang principle for linear systems, we show that for non-autonomous regional optimal control problems with state constraints, the bang-bang principle does not hold globally even when the dynamics is continuous with respect to the state variable. However, we show that there exist minimal time trajectories to reach a target with extreme controls at the loci where the dynamics is differentiable. We give two examples which exhibit singular arcs exactly at the loci of non-differentiability, i.e. at the boundaries of the regions.

*Key words:* Optimal control; Bang-Bang principle; Piecewise affine dynamics.

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## 1 Introduction

The Bang-Bang Principle is a pillar of the theory of linear controlled systems (see [17,22,12,18]). Consider a linear dynamical system

$$\dot{X} = A(t)X + B(t)u, \quad X \in \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^r \quad (1)$$

where  $A(\cdot)$ ,  $B(\cdot)$  are time-measurable matrices of adequate dimensions, and  $U$  is a convex compact set. Denote  $E(U)$  the set of extreme points of the set  $U$ , i.e. points that cannot be written as a convex combination of two distinct points of  $U$  (see [18]). For any initial condition  $(t_0, X_0)$  in  $\mathbb{R} \times \mathbb{R}^n$ , the attainability set  $\mathcal{A}(t)$ , resp.  $\mathcal{A}_{BB}(t)$ , is defined as the set of  $X(t)$  where  $X(\cdot)$  is the solution of (1) for the initial condition  $(t_0, X_0)$  and any time-measurable controls  $u(\cdot)$  that takes values in  $U$ , resp. in  $E(U)$ . The Bang-Bang Principle states that  $\mathcal{A}(t)$  and  $\mathcal{A}_{BB}(t)$  coincide. This result has consequences in terms of controllability and the nature of trajectories that join two points of the state space in minimal time.

Moreover, it is customary to consider optimal control problems under constraints. These problems usually contain challenging difficulties – especially when it comes to applied areas – and the literature is wide and various, depending on the formulation of the different kinds of constraints and the forms of necessary or sufficient optimality conditions. A certain amount of cases is listed in detail in [13].

We shall consider in this paper non-autonomous regional affine controlled systems, defined by a family of controlled affine systems

$$\dot{X} = f_i(t, X) := A_i(t)X + B(t)u + C_i(t), \quad i \in \mathcal{I}$$

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*Email addresses:* [ruben.chenevat@inrae.fr](mailto:ruben.chenevat@inrae.fr) (Ruben Chenevat), [bruno.cheviron@inrae.fr](mailto:bruno.cheviron@inrae.fr) (Bruno Cheviron), [sebastien.roux@inrae.fr](mailto:sebastien.roux@inrae.fr) (Sebastien Roux), [alain.rapaport@inrae.fr](mailto:alain.rapaport@inrae.fr) (Alain Rapaport).

over a partition of  $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^n$  (where  $\Omega$  may be corresponding to the constraint time-state space), and whose dynamics is given by one of these linear systems on each subset of the partition. Such a dynamics is in general discontinuous and the question of well-posedness of solutions raises (see for example [15], [19], [1]). In particular sliding modes [25] or Filippov solutions [11] may be considered. Controllability of such systems has been well investigated in the literature (see [6,9,16,21,26]). In the present work, we shall assume that the dynamics are continuous with respect to  $X$  (but not necessarily differentiable), which does not bring any issue to define unique solutions as for any Lipschitz continuous differential equations. There is surprisingly much less works dedicated to this precise setting excepted for instance [24] that investigates the controllability of such systems.

For practical interest one may wonder if the bang-bang principle still holds in this simple setting, and if not where the control has to be a non extreme point. The underlying question is to know if it is worth anticipating the change of region with non bang-bang solution in the interior of the regions. To the best of our knowledge, this question has not been addressed specifically in the literature, although the problem of regional optimal control has got recent interests in the literature (see [2–4]). For instance in [4], a minimal time problem with piecewise affine dynamics is presented, but whose dynamics is discontinuous. Note that considering a family of linear systems with the continuity assumption appears to be very restrictive (as it would impose the common boundary of two domains  $P_i, P_j$  to belong to the kernel of  $A_i - A_j$ ). However in many applications one deals with saturation induced by min or max operators. This is why we consider families of affine rather than linear systems. Let us underline that applying directly the bang-bang principle for trajectories joining two points in a same subset  $P_i$  does not guarantee that the trajectory with extreme controls remains in the same set  $P_i$ . This is precisely the point investigated in the present work that leads to extend the classical bang-bang principle. We will then explore through examples the possible existence of singular arcs, also noticing that a constrained arc in this framework will be seen as a singular arc.

The paper is organized as follows. In the next section, we make precise our notations and assumptions, making distinctions between linear and affine bang-bang principle for the minimal time control problem. In Section 3, we give our main result. Section 4 is devoted to examples.

## 2 Settings and assumptions

In this section we present the framework of spatially-temporally regional affine systems, and highlight their similarities and differences with classical linear systems.

We consider an non-autonomous system that is linear with respect to  $u$  :

$$\dot{X} = F(t, X) + B(t)u \quad (2)$$

where  $X \in \mathbb{R}^n$  and  $u \in \mathcal{U}$ , with  $\mathcal{U}$  the set of measurable functions taking values in  $U \subset \mathbb{R}^r$ . Consider the following assumptions.

**Assumption 1** *Given the set  $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^n$  such that  $\overset{\circ}{\Omega}$  is non-empty, connected,  $\text{proj}_t \Omega = \mathbb{R}^+$ , and there exists a partition*

$$\Omega = \bigcup_{i \in \mathcal{I}} \overline{P_i}$$

*where the  $\overline{P_i}$  are pairwise disjoint, non-empty and connected subsets for any  $i \in \mathcal{I}$  (where  $\mathcal{I}$  is countable), such that*

$$F(t, X) = A_i(t)X + C_i(t), \quad (t, X) \in P_i$$

*where  $A_i$  and  $C_i$  are  $n \times n$ - and  $n \times 1$ - time measurable matrix-valued functions,  $B$  is a  $n \times r$ - time measurable matrix-valued function. Moreover  $F$  is continuous with respect to  $X$  for any  $t$ .*

Let us stress that we do not impose any regularity of the boundaries of the sets  $P_i$ .

**Assumption 2** *The set  $U$  is convex compact.*

We consider ( $\mathcal{P}$ ) the minimum time problem to reach a compact target set  $\Gamma \subset \mathbb{R}^n$  among controls in  $\mathcal{U}$ , from an initial state  $(t_0, X(t_0)) \in \Omega$  that belongs to the attainability set of  $\Gamma$ , while fulfilling the constraint  $(t, X(t)) \in \Omega$  for any time  $t$ .

In the case where there is only one area (ie  $\mathcal{I} = \{\alpha\}$ ), the dynamics is linear (ie  $C_\alpha = 0$ ), and there is no constraints (ie  $\Omega = \mathbb{R}^+ \times \mathbb{R}^n$ ), we have in fact a classical linear system

$$\dot{X}(t) = A_\alpha(t)X(t) + B(t)u(t) \quad (3)$$

For this problem, the classical bang-bang principle is often stated for linear dynamics, as follows.

**Theorem 3 (Classical bang-bang principle [14])** *If  $X_0$  belongs to the attainability set of  $\Gamma$ , then there exists a control function  $u(\cdot)$  that takes values in  $E(U)$  for which the associated trajectory of (3) reaches  $\Gamma$  in minimal time from the initial condition  $X(t_0) = X_0$ .*

We recall that this result can be stated as above for a linear system, or similarly for an affine system as in [18]. In fact these two statements are equivalent, as detailed in Appendix A.

We may now introduce the main issue of this work. Considering a time-optimal problem subject to the dynamics (2), we will discuss about an extension of the bang-bang principle for such settings. It is well known that for hybrid dynamical systems or switched systems the optimal time trajectory may lie on the locus of discontinuity of the dynamics, and it is called a sliding mode. In this work, the systems we consider might be seen as spatially hybrid systems (with a non-autonomous regionalization), but with an additional condition of continuity at the boundaries. One may wonder if such sliding mode behaviour is still possible with continuous piecewise affine dynamics. A straightforward application of the classical bang-bang principle would fail in the framework of Assumption 1, since we can not ensure that the obtained bang-bang trajectory doesn't cross any boundary  $\partial P_i$  (especially since these boundaries might be moving), which would thus modify the dynamics driving the system. Therefore we have to be a bit more cautious and examine the evolution of the system in order to locally apply the classical bang-bang principle, and then build a candidate bang-bang control step by step.

**Remark 4** *We underline the fact that this work does not focus on the question of the controllability of our systems, and we will always assume that our problems are feasible, i.e. the target is attainable in finite time.*

### 3 Extended bang-bang principle

Let us now establish the statement and the proof of the main issue of this work.

Let  $u^*(\cdot) \in \mathcal{U}$  be an optimal solution of the time-optimal problem ( $\mathcal{P}$ ) driven by the dynamics (2). Denote  $X^*(\cdot)$  the associated optimal trajectory and  $T^* \in \mathbb{R}_+^*$  the corresponding optimal time. Assume that there exists a time interval  $[a, b] \subset [0, T^*]$  such that  $(b, X^*(b)) \in \partial P_i$  where  $P_i$  is a fixed region of the partition, and  $(t, X^*(t)) \in \overset{\circ}{P}_i$  for all  $t \in [a, b)$ .

First we are going to restrict our analysis on a segment strictly inside  $\overset{\circ}{P}_i$ .

For any  $[t_d, t_f] \subset [a, b)$ , we study the evolution of the system over  $[t_d, t_f]$ , thus following the dynamics of the region  $P_i$  :

$$\dot{X}(t) = A_i(t)X(t) + B(t)u(t) + C_i(t) \quad (4)$$

We aim to apply the classical bang-bang principle for affine systems, but we need to ensure that all the trajectories undergo the same dynamics. To this aim we will have to study the local behaviour of trajectories and we will define an appropriate neighbourhood of  $\zeta_{[t_d, t_f]} = \{(t, X^*(t)), t \in [t_d, t_f]\}$  where the classical bang-bang principle may be applied.

**Lemma 5** *With the notations above, there exists a bang-bang control over the time interval  $[t_d, t_f] \subset [a, b)$  such that its associated trajectory  $Z(\cdot)$  verifies :*

$$\begin{aligned} Z(t_d) &= X^*(t_d) \quad , \quad Z(t_f) = X^*(t_f) \\ (t, Z(t)) &\in \overset{\circ}{P}_i \quad \text{for all } t \in [t_d, t_f] \end{aligned}$$

**PROOF.** In order to evaluate the distance from  $\zeta_{[t_d, t_f]}$  to the boundary  $\partial P_i$ , we define the distance  $d$  between  $(t, x)$  and  $(t', x')$  in  $\mathbb{R}^+ \times \mathbb{R}^n$  by :

$$d((t, x), (t', x')) = \left( |t - t'|^2 + \|x - x'\|^2 \right)^{1/2}$$

where  $\|\cdot\|$  denotes the euclidian norm. Then we have  $r = d(\zeta_{[t_d, t_f]}, \partial P_i) > 0$ , and we define  $\nu_r = \bigcup_{t \in [t_d, t_f]} \mathcal{B}((t, X^*(t)), r)$ .

*Step 1 : Assessing the deviation of any trajectory*

Let us fix  $\hat{t} \in [t_d, t_f]$ . Consider the problem defined by the dynamics of (4) with the initial condition  $X(\hat{t}) = X^*(\hat{t})$ , and select an admissible control  $u(\cdot)$ . We want to evaluate the difference between the associated trajectory  $X(\cdot)$  and the reference trajectory  $X^*(\cdot)$ . In particular, we are looking for a time  $t^+$  such that for all  $t \in [\hat{t}, t^+]$  we have  $\|X(t) - X^*(t)\| < r$ . For  $t \geq \hat{t}$  we have :

$$\begin{aligned} X(t) - X^*(t) &= X(t) - X(\hat{t}) - (X^*(t) - X^*(\hat{t})) \\ &= \int_{\hat{t}}^t A_i(s)(X(s) - X^*(s)) + B(s)(u(s) - u^*(s)) ds \end{aligned}$$

Then :

$$\begin{aligned} \|X(t) - X^*(t)\| &\leq 2M(t - \hat{t}) \sup_{s \in [a, b]} \|B(s)\| \\ &\quad + \int_{\hat{t}}^t \|A_i(s)\| \|X(s) - X^*(s)\| ds \end{aligned}$$

By the Grönwall lemma, we get :

$$\begin{aligned} \|X(t) - X^*(t)\| &\leq 2M(t - \hat{t}) K_{[a, b]}^B \exp \left( \int_{\hat{t}}^t \|A_i(s)\| ds \right) \\ &\leq 2M(t - \hat{t}) K_{[a, b]}^B \exp \left( (t - \hat{t}) K_{[a, b]}^A \right) \end{aligned}$$

where  $K_{[a, b]}^A = \sup_{s \in [a, b]} \|A_i(s)\| < +\infty$  and  $K_{[a, b]}^B = \sup_{s \in [a, b]} \|B(s)\| < +\infty$ . Then there exists  $t^+ \in (\hat{t}, t_f]$  such that :

$$\|X(t) - X^*(t)\| < r, \quad \forall t \in [\hat{t}, t^+]$$

Moreover, the duration  $\delta = |t^+ - \hat{t}|$  only depends on  $r$ .

*Step 2 : Splitting the trajectory to restrain the deviation*

Then there exists  $N \in \mathbb{N}^*$  such that  $t_d + (N - 1)\delta < t_f \leq t_d + N\delta$ , and we define a finite sequence  $(\theta_k)_{0 \leq k \leq N}$  by :

$$\begin{aligned} \theta_k &= t_d + k\delta, \quad \text{for } 0 \leq k \leq N - 1 \\ \theta_N &= t_f \end{aligned}$$

Therefore from above we deduce that for any  $0 \leq k \leq N - 1$ , all the trajectories  $Y_k(\cdot)$  following the dynamics (6) with initial condition  $Y_k(\theta_k) = X^*(\theta_k)$  satisfy :

$$\|Y_k(t) - X^*(t)\| < r, \quad \forall t \in [\theta_k, \theta_{k+1}]$$

In particular, we have :

$$(t, Y_k(t)) \in \mathcal{B}((t, X^*(t)), r), \quad \forall t \in [\theta_k, \theta_{k+1}]$$

and thus the trajectory  $\mathcal{S} = \{(t, Y(t)), t \in [\theta_k, \theta_{k+1}]\}$  satisfies :

$$\mathcal{S} \subset \bigcup_{t \in [\theta_k, \theta_{k+1}]} \mathcal{B}((t, X^*(t)), r) \subset \hat{P}_i$$

Step 3 : Classical bang-bang principle on each division

As a consequence, we now may apply the Bang-Bang principle as follows. Consider the dynamics of (4) over the whole space  $\mathbb{R}^n$ , with the initial condition  $Y_k(\theta_k) = X^*(\theta_k)$ . By assumption, the target  $\{X^*(\theta_{k+1})\}$  is attainable at the time  $\theta_{k+1}$ , thus the Bang-Bang principle for linear time-optimal problems indicates that there exists a Bang-Bang control  $\bar{u}_k(\cdot)$  such that the corresponding trajectory  $\bar{Y}_k(\cdot)$  reaches  $X^*(\theta_{k+1})$  at the time  $\theta_{k+1}$ . Furthermore, we know from the well-chosen construction detailed above that this trajectory verifies :

$$\{(t, \bar{Y}(t)), t \in [\theta_k, \theta_{k+1}]\} \subset \mathring{P}_i$$

Step 4 : Conclusion for the segment inside  $\mathring{P}_i$

Let us now define  $v(\cdot)$  as the concatenation of the  $\bar{u}_k(\cdot)$  for  $0 \leq k \leq N - 1$ , and denote  $Z(\cdot)$  the corresponding trajectory (in  $\mathbb{R}^n$ ) as we can see in Fig. 1. Finally, we showed that for any  $[t_d, t_f] \subset [a, b]$  the obtained control  $v(\cdot)$  is a Bang-Bang control defined over  $[t_d, t_f]$ , and the associated trajectory  $Z(\cdot)$  satisfies  $Z(t_d) = X^*(t_d)$ ,  $Z(t_f) = X^*(t_f)$  and the trajectory  $\mathcal{T} = \{(t, Z(t)), t \in [t_d, t_f]\}$  fulfills :

$$\mathcal{T} \subset \bigcup_{0 \leq k \leq N-1} \left( \bigcup_{t \in [\theta_k, \theta_{k+1}]} \mathcal{B}((t, X^*(t)), r) \right) \subset \mathring{P}_i$$

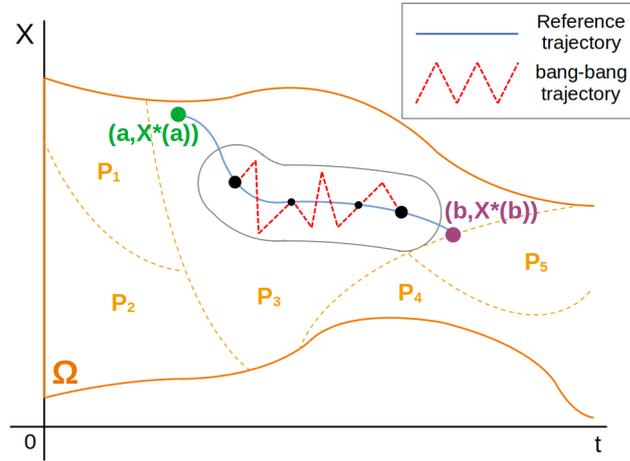


Fig. 1. Optimal bang-bang trajectory  $(\cdot, Z(\cdot))$  (red) built from the reference trajectory  $(\cdot, X^*(\cdot))$  (blue) and the intermediate points  $(\theta_k, X^*(\theta_k))$  over  $[t_d, t_f]$ , included in the neighbourhood  $\nu_r$

Let us now deal with the complete trajectory of an optimal solution to a minimum-time problem subject to the settings (2).

**Theorem 6 (Extended bang-bang principle)** *Considering the minimum-time control problem  $(\mathcal{P})$  subject to the non-autonomous regional affine settings (2), if  $X_0$  belongs to the attainability set of the target  $\Gamma$ , then there exists an optimal solution composed of a succession of bang-bang arcs over time interval where the trajectory belongs to a region  $\mathring{P}_i$ , and possible singular arcs along the boundaries  $\partial P_i$ .*

**PROOF.** A trajectory going through regions  $P_i$  successively, it is sufficient to deal with the case of a part of trajectory in one region  $P_i$ . Then we use again the previous notations, and we make our analysis over the time interval  $[a, b]$ .

To prove the statement, it suffices to apply the Lemma 5 on well-chosen time sub-intervals. We set a sequence of times  $(\sigma_f^j)_{j \geq 0}$  by :

$$\sigma_f^j = b - (b - a)/2^j \quad , \text{ for } j \in \mathbb{N}$$

We get a corresponding family of radiuses  $r_j = d(\zeta_{[\sigma_f^j, \sigma_f^{j+1}]}, \partial P_i) > 0$ , and the previous reasoning provides a family of bang-bang controls  $(v^j)_{j \geq 0}$ , and their associated trajectories  $(Z^j)_{j \geq 0}$ , defined on each portion of time interval by :

$$\begin{aligned} v^0 : [a, \sigma_f^1] &\longrightarrow U \quad \text{given by Lemma 5 on } [a, \sigma_f^1] \\ &\text{with } Z^0(a) = X^*(a), Z^0(\sigma_f^1) = X^*(\sigma_f^1) \\ &\text{and } \{(t, Z^0(t)), t \in [a, \sigma_f^1]\} \subset \overset{\circ}{P}_i \\ v^j : [\sigma_f^j, \sigma_f^{j+1}] &\longrightarrow U \quad \text{given by Lemma 5 on } [\sigma_f^j, \sigma_f^{j+1}] \\ &\text{with } Z^j(\sigma_f^j) = X^*(\sigma_f^j), Z^j(\sigma_f^{j+1}) = X^*(\sigma_f^{j+1}) \\ &\text{and } \{(t, Z^j(t)), t \in [\sigma_f^j, \sigma_f^{j+1}]\} \subset \overset{\circ}{P}_i \end{aligned}$$

Finally, the control defined by a concatenation of the controls  $v^j$  above is :

$$V(t) = v^j(t) \quad \text{for } t \in [\sigma_f^j, \sigma_f^{j+1})$$

Thus  $V$  is well-defined over  $[a, b)$ , is bang-bang, and its corresponding trajectory  $\xi(\cdot)$  satisfies :

$$\begin{aligned} \xi(a) = X^*(a) \quad , \quad \xi(t) &\xrightarrow[t \rightarrow b]{} X^*(b) \\ \{(t, \xi(t)), t \in [a, b)\} &\subset \overset{\circ}{P}_i \end{aligned}$$

Hence the existence of an optimal solution composed of bang-bang arcs in the regions where the dynamics is differentiable.

**Remark 7** *We may have constructed an optimal bang-bang solution that has an infinite (countable) number of commutations. However it would be of interest to look for regularity conditions on the boundaries (such as for example the interior sphere condition in [20]) for which we could be able to show that we have a finite number of commutations.*

**Corollary 8** *Consider an optimal control problem subject to the non-autonomous regional affine settings (2) with  $\Omega = \mathbb{R}^+ \times \mathbb{R}^n$ . We fix a terminal time  $T > 0$  and we seek to minimize the integral cost*

$$J(u(\cdot)) = \int_0^T l(t, X(t), u(t)) dt$$

*such that the function  $l : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$  is piecewise affine, continuous with respect to  $X$ , and affine with respect to  $u$ . Then there exists an optimal solution composed of a succession of bang-bang arcs over time interval where the trajectory belongs to a region where the dynamics and the function  $l$  are differentiable, and possible singular arcs at the loci of non-differentiability of the dynamics or the integrand  $l$ .*

**Remark 9** *The results depicted in this section are expected to be still valid (or at least adapted) for a broader class of problems where the dynamics is discontinuous, for example by investigating the augmentation technique (see [10, 5]) in order to reduce to the present case.*

#### 4 Examples of trajectories with the extension

In this section we are going to present two examples of problems fulfilling the settings described in this work, and we illustrate the result of Theorem 6 by presenting complete optimal trajectories composed of both bang-bang and singular arcs. The first one is a simple autonomous system driving an optimal time problem where the geometry of the (constant) partition of the space gives the intuition of the existence of a singular arc. The second one is from a crop irrigation model [8], and is written as a non-autonomous system driving an optimization problem with a fixed terminal time, an integral cost, and constraints over the input and the state.

#### 4.1 The ridge path

Let us consider the following minimum-time control problem to go from  $(x_0, y_0) = (0, 0)$  to  $(x_f, y_f) = (0, 1)$  :

$$(R) : \begin{cases} \dot{x}(t) = u(t) \\ \dot{y}(t) = D(x(t), y(t)) \end{cases}$$

where :

$$D(x, y) = \min(d_1(x, y), d_2(x, y), d_3(x, y), d_4(x, y))$$

with :

$$\begin{aligned} d_1(x, y) &= x + y & , & \quad d_2(x, y) = -x + y + 1 \\ d_3(x, y) &= x - y + 1 & , & \quad d_4(x, y) = -x - y + 2 \end{aligned}$$

This systems  $(R)$  respects the framework of piecewise affine dynamics and continuous with respect to the state, with the following partition :

$$D = \begin{cases} d_1(x, y) & , \text{if } (x, y) \in ] - \infty, \frac{1}{2}] \times ] - \infty, \frac{1}{2}] = P_1 \\ d_2(x, y) & , \text{if } (x, y) \in [\frac{1}{2}, +\infty[ \times ] - \infty, \frac{1}{2}] = P_2 \\ d_3(x, y) & , \text{if } (x, y) \in ] - \infty, \frac{1}{2}] \times [\frac{1}{2}, +\infty[ = P_3 \\ d_4(x, y) & , \text{if } (x, y) \in [\frac{1}{2}, +\infty[ \times [\frac{1}{2}, +\infty[ = P_4 \end{cases}$$

We set the initial conditions  $x(0) = y(0) = 0$ , and the target is the singleton  $(x_f, y_f) = (0, 1)$ . Moreover the control  $u(\cdot)$  is a measurable function with values constrained in the compact  $U = [-1, 1]$ . We seek to minimize the time to reach the target from the initial point.

By definition, the magnitude of  $\dot{y}$  is locally higher when  $y$  lies on a boundary, then the intuition would say that the trajectory will tend to use the boundaries as much as possible, in order to move as quickly as possible. We are going to show that this trajectory (as illustrated in Fig. 2) is indeed optimal.

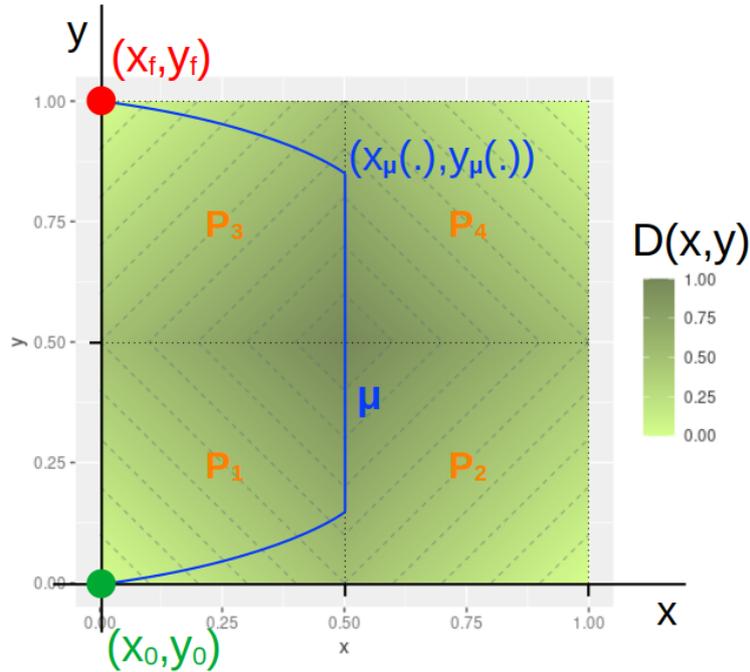


Fig. 2. Reference trajectory on the heatmap of the dynamics  $D$  for the minimum time problem  $(R)$

Consider an optimal control  $u^*(\cdot)$ , with its associated trajectory  $(x^*(\cdot), y^*(\cdot))$  and the corresponding optimal time  $T^*$ . By symmetry, there exists an optimal solution following a symmetrical trajectory. Moreover, denoting  $t_{1/2}$  the first time when this trajectory crosses the line  $y = 1/2$ , we deduce that its symmetrization reaches  $(x_f, y_f)$  at the

time  $2t_{1/2}$  and that the trajectory crosses the line  $y = 1/2$  only once. Indeed, if there existed another time  $t'_{1/2}$  when the trajectory would cross  $y = 1/2$ , then it would be reaching  $(x_f, y_f)$  at a time strictly larger than  $2t_{1/2}$ , which is therefore non optimal. Furthermore it is then optimal to cross the line  $y = 1/2$  at the time  $t_{1/2}$ .

We would like to show that the strategy of going to the boundary  $\{x = 1/2\}$  as fast as possible (with  $u = 1$ ) and then staying on this boundary (with  $u = 0$ ) until reaching  $(\frac{1}{2}, \frac{1}{2})$  is time optimal. We define  $\mu$  the corresponding symmetrical orbit with respect to  $\{y = 1/2\}$  (as we can see in Fig. 2), and we denote  $(x_\mu, y_\mu)$  its associated trajectory, thus obtained when first solving the differential equation with the controls  $u = 1$  while  $x_\mu < 1/2$  and  $u = 0$  until  $y_\mu \leq 1/2$  :

$$(x_\mu, y_\mu)(t) = \begin{cases} (t, e^t - 1 - t) & , \text{if } t \in [0, \frac{1}{2}] \\ (\frac{1}{2}, \frac{\sqrt{e}-1}{\sqrt{e}}e^t - \frac{1}{2}) & , \text{if } t \in [\frac{1}{2}, \ln(\frac{\sqrt{e}}{\sqrt{e}-1})] \end{cases}$$

and then concatenating with its symmetric with respect to  $y = \frac{1}{2}$  for  $t \in [\ln(\frac{\sqrt{e}}{\sqrt{e}-1}), 2\ln(\frac{\sqrt{e}}{\sqrt{e}-1})]$ .

Suppose that the optimal trajectory  $(x^*(\cdot), y^*(\cdot))$  is different than the reference trajectory  $(x_\mu, y_\mu)$  on  $[0, t_{1/2}]$ . Then denote  $J = \{t \in [0, t_{1/2}], (x^*(t), y^*(t)) \neq (x_\mu(t), y_\mu(t))\}$ , which is a set of non-null measure. We have for all  $t \in J$  :

$$\begin{cases} x^*(t) < x_\mu(t) & , \text{if } (x^*(t), y^*(t)) \in P_1 \\ 1 - x^*(t) < 1 - \frac{1}{2} & , \text{if } (x^*(t), y^*(t)) \in P_2 \end{cases}$$

Therefore we can compare  $\dot{y}^*$  and  $\dot{y}_\mu$  over  $J$ , and by integrating between 0 and  $t_{1/2}$  we get :

$$y^*(t_{1/2}) < y_\mu(t_{1/2})$$

Since  $y_\mu$  is continuous, this would contradict the fact that  $t_{1/2}$  is the optimal time for reaching the line  $y = 1/2$ .

To conclude, we know that  $(x^*(\cdot), y^*(\cdot)) = (x_\mu(\cdot), y_\mu(\cdot))$  over the time interval  $[0, t_{1/2}]$ , and the time for crossing the line  $y = 1/2$  is in fact  $t_{1/2} = \ln(\frac{\sqrt{e}}{\sqrt{e}-1})$ . By symmetry, the whole trajectory  $(x_\mu, y_\mu)$  is optimal. As a consequence, the minimum time to reach  $(x_f, y_f)$  from  $(x_0, y_0)$  is  $T^* = 2\ln(\frac{\sqrt{e}}{\sqrt{e}-1})$ .

Finally, the corresponding optimal control is "bang-singular" bang" as follows :

$$u^*(t) = \begin{cases} 1 & , \text{for } t \in [0, \frac{1}{2}] \\ 0 & , \text{for } t \in [\frac{1}{2}, 2\ln(\frac{\sqrt{e}}{\sqrt{e}-1}) - \frac{1}{2}] \\ -1 & , \text{for } t \in [2\ln(\frac{\sqrt{e}}{\sqrt{e}-1}) - \frac{1}{2}, 2\ln(\frac{\sqrt{e}}{\sqrt{e}-1})] \end{cases}$$

It takes extreme values when the trajectory is inside a region  $\mathring{P}_i$ , and it is singular when the trajectory follows a boundary.

#### 4.2 Example from an Irrigation model

Let us consider the following optimal control problem, denoted (*Irrig*), taken from a crop irrigation model (see [8]), and evolving over a fixed time interval  $[0, T]$ .

$$\begin{cases} \dot{S}(t) = k_1 \left( -\varphi(t)K_S(S(t)) \right. \\ \qquad \qquad \qquad \left. -(1 - \varphi(t))K_R(S(t)) + k_2u(t) \right) \\ \dot{V}(t) = u(t) \end{cases}$$

where  $S$  represents the soil moisture,  $u$  is the irrigation and  $V$  measures the water consumed. Set the initial conditions  $S(0) = S_0 \in (S^*, 1]$  and  $V(0) = 0$ .

The functions  $K_S$  and  $K_R$  are continuous piecewise affine, as follows :

$$K_S(S) = \begin{cases} 0 & , S \in [0, S_w] \\ K_M \frac{S - S_w}{S_M - S_w} & , S \in [S_w, S_M] \\ K_M \frac{S - S_w}{S_M - S_w} + \frac{S - S_M}{S^* - S_M} & , S \in [S_M, S^*] \\ 1 & , S \in [S^*, 1] \end{cases}$$

$$K_R(S) = \begin{cases} 0 & , S \in [0, S_h] \\ \frac{S - S_h}{1 - S_h} & , S \geq S_h \end{cases}$$

with  $0 < S_h < S_w < S_M < S^* < 1$  and  $K_M \in (0, 1)$  such that the function  $K_S$  is concave on  $[S_w, 1]$ , as in Fig. 3. These functions can typically be written as a combination of min and max of affine functions.

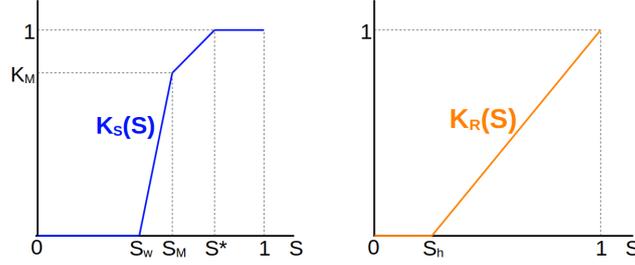


Fig. 3. Graphs of the functions  $K_S$  (left) and  $K_R$  (right)

The control  $u(\cdot)$  takes values in  $[0, 1]$ , and the constants verify  $k_1 > 0$  and  $k_2 > 1$  (local controllability condition). The function  $\varphi$  is supposed to be  $C^1$ , increasing, and taking values in  $[0, 1]$ . The optimization problem consists in fixing a terminal time  $T > 0$  and aiming at maximizing the following integral cost :

$$\int_0^T \varphi(t) K_S(S(t)) dt$$

under some target constraint  $V(T) = V_{init}$ .

The problem formulation in [8] can be improved by adding a time-varying state constraint that models an important objective for crop irrigation : maintaining high soil moisture level for some specific stage of plant growth. For instance, we consider a tolerance threshold  $S_{tol}$  and a humidity level  $S_{flo}$  suitable during the flowering period, such that  $S_w < S_{tol} < S_{flo} < S^*$ . Then we introduce an inequality state constraint as  $S_{tol} + F(t) \leq S(t) \leq 1$  for all  $t \in [0, T]$ , with the function  $F$  defined by :

$$F(t) = \begin{cases} S_{flo} - S_{tol} & , t \in [f_a, f_b] \text{ (flowering period)} \\ 0 & , \text{otherwise} \end{cases}$$

This system (*Irrig*) is a quite complex constrained optimization problem, and does not fulfill the framework of piecewise affine time-optimal problems yet. Let us assume that there exists an optimal solution of the problem (*Irrig*). Define  $B(t) = \int_0^t \varphi(\tau) K_S(S(\tau)) d\tau$ . In this case, there is a corresponding optimal terminal cost  $B^* > 0$  and a final state  $S(T) = \tilde{S}$ . We then consider that it gives in fact an additional terminal condition corresponding to the additional state  $B(\cdot)$ , ie.  $B(T) = B^*$ . Now we may replace all the terminal conditions by a target :

$$(S_f, V_f, B_f) = (\tilde{S}, V_{init}, B^*)$$

With all these transformations, we now have a problem written as a non-autonomous regional affine system with continuous dynamics with respect to the state, and the previous optimization is thus by definition equivalent to a minimum time problem to reach the target  $(S_f, V_f, B_f)$ . Indeed, if there exists  $\hat{t} < T$  such that  $B(\hat{t}) = B^*$  and  $V(\hat{t}) = V_{init}$ , then the control  $u = 0$  over  $[\hat{t}, T]$  would still get  $V(T) = V_{init}$ , and the dynamics of  $B$  would imply  $B(T) > B(\hat{t})$ , which would contradict the optimality of  $B^*$  in the initial formulation of the problem.

The extended bang-bang principle (Theorem 6) thus states that there exists an optimal solution consisting in bang arcs over time intervals when the trajectory belongs to a differentiable area, and possible singular arcs on boundaries. We illustrate this result on our irrigation system with a numerical resolution computed via Bocop (see [23]), using  $\varphi(t) = (t/T)^\alpha$  and the parametrization in Table 1 and Table 2. We highlight in Fig. 4 the partition of the time-state set (only showing the state  $S$ , the partition being constant with respect to the other states), by  $\Omega = P_1 \cup P_2 \cup P_3 \cup P_4$ .

Table 1  
Moisture thresholds for (*Irrig*)

$S_h$	$S_w$	$S_M$	$K_M$	$S^*$	$S_{tol}$	$S_{flo}$
0.2	0.4	0.55	0.8	0.75	0.45	0.6

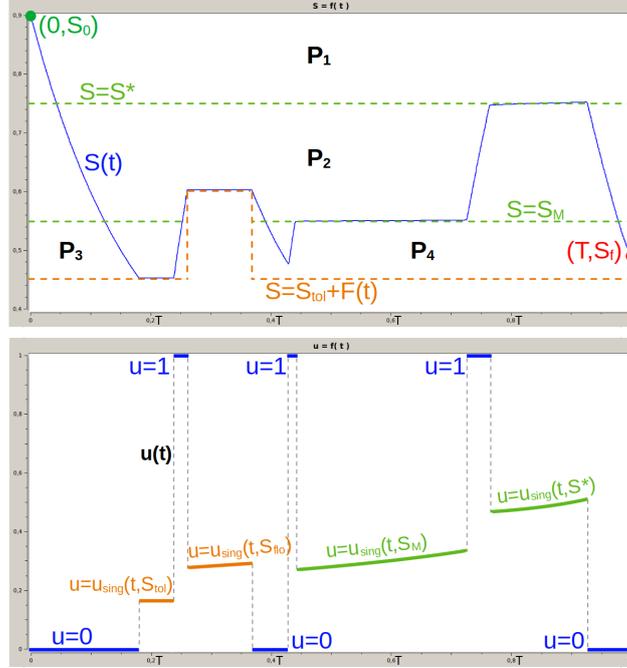


Fig. 4. Optimal mixed bang, constrained and singular trajectory  $S(\cdot)$  (top) and corresponding control  $u(\cdot)$  (bottom) for the problem (*Irrig*)

**Remark 10** The definition of the problem (*Irrig*) implies that the singular arcs occur only at discrete values of  $S$ , then the corresponding singular controls are easy to compute using the equation of  $\dot{S}$ . For a singular arc at the level  $\hat{S}$ , we obtain the following singular control :

$$u_{sing}(t, \hat{S}) = \frac{1}{k_2} \left( \varphi(t) K_S(\hat{S}) + (1 - \varphi(t)) K_R(\hat{S}) \right)$$

## 5 Conclusions

We focused our study here on extending the classical bang-bang principle for minimum time problems with non-autonomous regional affine continuous dynamics and under constraints. As we showed, an optimal solution to reach a target can always be written as a succession of bang arcs inside the areas of the partition of the time-state set, and eventual singular arcs where the dynamics is non-differentiable, as illustrated by our examples.

We expect that the result of optimal bang arcs in the interior of the regions holds true in a more general framework where the dynamics is discontinuous, which would require redefining the notion of solution suited to such framework.

Table 2  
Parametrization of (*Irrig*)

$T$	$S_0$	$k_1$	$k_2$	$f_a$	$f_b$	$\alpha$	$V_{init}$
150	0.9	0.03	1.9	0.26T	0.37T	2	49.5

For future perspectives, we may wonder about the conditions of occurrence of singular arcs, and we can discuss about the possible difficulty to compute the value of such singular controls. Moreover, in order to numerically search in an efficient way optimal solutions of an optimal control problem in our framework, we may consider to adapt the multiple shooting method [7].

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## A From affine to linear

**Lemma 11** Consider the affine system in  $\mathbb{R}^n$

$$\dot{Y}(t) = L(t)Y(t) + M(t)u(t) + N(t) \quad (5)$$

where  $u(t) \in U$ , a convex compact set of  $\mathbb{R}^p$ , and  $L$ ,  $M$  and  $N$  are  $n \times n$ -,  $n \times r$ - and  $n \times 1$ - time measurable matrix-valued functions. Then, the conclusion of Theorem 3 holds for dynamics (5).

**PROOF.** The proof only consists in rewriting the affine system into an equivalent linear one in higher dimension, and we explicitly construct it as follows :

$$\dot{\tilde{Y}}(t) = \tilde{L}(t)\tilde{Y}(t) + \tilde{M}(t)u(t) \quad (6)$$

where  $\tilde{L}$  and  $\tilde{M}$  are  $(n+1) \times (n+1)$ - and  $(n+1) \times r$ - matrix-valued functions. Let us set the additional state variable :

$$\dot{y}_{n+1}(t) = 0 \quad , y_{n+1}(0) = 1$$

Then we define  $\tilde{Y} \in \mathbb{R}^{n+1}$  such that the  $j$ -th coefficient of  $\tilde{Y}$  is  $\tilde{Y}_j = Y_j$  for all  $1 \leq j \leq n$ , and  $\tilde{Y}_{n+1} = y_{n+1}$ . We posit  $\tilde{L} \in \mathcal{M}_{n+1, n+1}(\mathbb{R})$  such that the coefficients  $(k, j)$  are  $\tilde{L}_{k,j} = L_{k,j}$  for all  $1 \leq k, j \leq n$ , the coefficients  $(k, n+1)$  are  $\tilde{L}_{k, n+1} = N_k$  for all  $1 \leq k \leq n$ , and the coefficients  $(n+1, j)$  are  $\tilde{L}_{n+1, j} = 0$  for all  $1 \leq j \leq n+1$ . We similarly define  $\tilde{M} \in \mathcal{M}_{n+1, r}(\mathbb{R})$  such that the coefficients  $(k, j)$  are  $\tilde{M}_{k,j} = M_{k,j}$  for all  $1 \leq k \leq n$  and  $1 \leq j \leq r$ , and the coefficients  $(n+1, j)$  are  $\tilde{M}_{n+1, j} = 0$  for all  $1 \leq j \leq r$ . With such a construction, we easily check that the system (6) describes the same dynamics as the system (5).

## References

- [1] N. Augier and A. Yabo. Time-optimal control of piecewise affine bistable gene-regulatory networks. *International Journal of Robust and Nonlinear Control*, 54(5):205–210, January 2022.
- [2] G. Barles, A. Briani, and E. Chasseigne. A bellman approach for two-domains optimal control problems in  $\mathbb{R}^n$ . *ESAIM: COCV*, 19(3):710–739, 2013.
- [3] G. Barles, A. Briani, and E. Chasseigne. A Bellman Approach for Regional Optimal Control Problems in  $\mathbb{R}^N$ . *SIAM Journal on Control and Optimization*, 52(3):1712–1744, 2014.
- [4] G. Barles, A. Briani, and E. Trélat. Value function for regional control problems via dynamic programming and Pontryagin maximum principle. *Mathematical Control and Related Fields*, 8(3&4):509–533, 2018.
- [5] T. Bayen, A. Bouali, and L. Bourdin. The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a pontryagin maximum principle for  $1^1$ -local solutions. *SIAM Journal on Control and Optimization*, 62(4):2412–2432, 2024.
- [6] A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Transactions on Automatic Control*, 45(10):1864–1876, 2000.
- [7] H.G. Bock and K.J. Plitt. A multiple shooting algorithm for direct solution of optimal control problems\*. *IFAC Proceedings Volumes*, 17(2):1603–1608, 1984. 9th IFAC World Congress: A Bridge Between Control Science and Technology, Budapest, Hungary, 2-6 July 1984.

- [8] K. Boumaza, N. Kalboussi, A. Rapaport, S. Roux, and C. Sinfort. Optimal control of a crop irrigation model under water scarcity. *Optimal Control Applications and Methods*, 42(6):1612–1631, 2021.
- [9] M. Kanat Camlibel, W. P. M. H. Heemels, and J. M. Schumacher. Algebraic necessary and sufficient conditions for the controllability of conewise linear systems. *IEEE Transactions on Automatic Control*, 53(3):762–774, 2008.
- [10] A.V. Dmitruk and A.M. Kaganovich. The hybrid maximum principle is a consequence of pontryagin maximum principle. *Systems & Control Letters*, 57(11):964–970, 2008.
- [11] A. F. Filippov. *Differential equations with discontinuous righthand sides*, volume 18. Kluwer Academic Publishers Group, Dordrecht, 1988.
- [12] H. Halkin. A generalization of lasalle’s “bang-bang” principle. *SIAM J. Control.*, 2(2):199–202, 1964.
- [13] R. F. Hartl, S. P. Sethi, and R. G. Vickson. *A survey of the maximum principles for optimal control problems with state constraints*, volume 37. 1995.
- [14] H. Hermes and J.P. La Salle. *Functional Analysis and Time Optimal Control*. Academic Press. John Wiley & sons, inc., New York, 1969.
- [15] J. Imura and A. van der Schaft. Characterization of well-posedness of piecewise-linear systems. *IEEE Transactions on Automatic Control*, 45(9):1600–1619, 2000.
- [16] M. Johansson. *Piecewise Linear Control Systems: A Computational Approach*. Lecture Notes in Control and Information Sciences. Springer Berlin, 2003.
- [17] J.P. Lasalle. The ‘bang-bang’ principle. *IFAC Proceedings Volumes*, 1(1):503–507, 1960. 1st International IFAC Congress on Automatic and Remote Control, Moscow, USSR, 1960.
- [18] E.B. Lee and L. Markus. *Foundations of optimal control theory*. SIAM series in applied mathematics. John Wiley & sons, inc., New York, 1967.
- [19] H. Lin and P. Antsaklis. *Hybrid Dynamical Systems: Fundamentals and Methods*. Advanced Textbooks in Control and Signal Processing. Springer, 2022.
- [20] C. Nour, R.J. Stern, and J. Takche. Proximal smoothness and the exterior sphere condition. *Journal of Convex Analysis*, 16:501–514, 01 2009.
- [21] L.A. Rodrigues. Stability analysis of piecewise-affine systems using controlled invariant sets. *Syst. Control. Lett.*, 53:157–169, 2004.
- [22] L. M. Sonneborn and F. S. Van Vleck. The bang-bang principle for linear control systems. *SIAM J. Control.*, 2(2):151–159, 1964.
- [23] Inria Saclay Team Commands. Bocop: an open source toolbox for optimal control. <http://bocop.org>, 2017.
- [24] L. Q. Thuan and M. K. Camlibel. Controllability and stabilizability of a class of continuous piecewise affine dynamical systems. *SIAM Journal on Control and Optimization*, 52(3):1914–1934, 2014.
- [25] V. Utkin. *Sliding Modes in Control and Optimization*. Communications and Control Engineering Series. Springer, 1992.
- [26] V. Veliov and M. Krastanov. Controllability of piecewise linear systems. *Systems & Control Letters*, 7(5):335–341, 1986.