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# About the Bang-Bang Principle for spatially-temporally regional affine dynamics under constraints

Ruben Chenevat<sup>1</sup>, Bruno Cheviron<sup>2</sup>, Sébastien Roux<sup>1</sup> and Alain Rapaport<sup>1‡</sup>

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## Abstract

We show that for spatially-temporally regional optimal control problems with affine constrained dynamics, the Bang-Bang Principle does not hold even when the dynamics is continuous with respect to the state variable. However, we show that there exist minimal time trajectories with extreme controls at the loci where the dynamics is differentiable. We give two examples which exhibit singular arcs exactly at the loci of non-differentiability, ie at the boundaries of the regions.

## 1 INTRODUCTION

The Bang-Bang Principle is a pillar of the theory of linear controlled systems (see [12, 15, 9, 13]). Consider a linear dynamical system

$$\dot{X} = A(t)X + B(t)u, X \in \mathbb{R}^n, u \in U \subset \mathbb{R}^r \quad (1)$$

where  $A(\cdot)$ ,  $B(\cdot)$  are time-measurable matrices of adequate dimensions, and  $U$  is a convex compact set. Denote  $E(U)$  the set of extreme points of the set  $U$ . For any initial condition  $(t_0, X_0)$  in  $\mathbb{R}^+ \times \mathbb{R}^n$ , the attainability set  $\mathcal{A}(t)$ , resp.  $\mathcal{A}_{BB}(t)$ , is defined as the set of  $X(t)$  where  $X(\cdot)$  is the solution of (1) for the initial condition  $(t_0, X_0)$  and any time-measurable controls  $u(\cdot)$  that takes values in  $U$ , resp. in  $E(U)$ . The Bang-Bang Principle states that  $\mathcal{A}(t)$  and  $\mathcal{A}_{BB}(t)$  coincide. This result has consequences in terms of controllability and the nature of trajectories that join two points of the state space in minimal time.

Moreover, it is customary to consider optimal control problems under constraints. These problems usually contain challenging difficulties – especially when it comes to applied areas – and the literature is wide and various, depending on the formulation of the different kinds of constraints and the forms of necessary or sufficient optimality conditions. A certain amount of cases is listed in detail in [10].

We shall consider in this paper spatially-temporally regional affine controlled systems, defined by a family of controlled affine systems

$$\dot{X} = f_i(t, X) := A_i(t)X + B(t)u + C_i(t), \quad i \in \mathcal{I}$$

over a partition of  $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^n$  (where  $\Omega$  may be corresponding to the constraint time-state space), and whose dynamics is given by one of these linear systems on each subset of the partition. Such a dynamics is in general discontinuous and the question of well-posedness of solutions raises. In particular sliding modes [18] or Filippov solutions [8] may be considered. Controllability of such systems has been well investigated in the literature (see [4, 6, 14, 19]). In the present work, we shall assume that the dynamics are continuous with respect to  $X$  (but not necessarily differentiable), which does not pose any problem to define unique solutions as for any Lipschitz continuous differential equations. There is much less works dedicated to this precise setting [17].

One may wonder if the Bang-Bang Principle still holds in this simple setting, and if not where the control has to be an non extreme point. It seems that this question has not addressed specifically in the literature, although the problem

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of regional optimal control has get recent interests in the literature (see [1, 2, 3]). For instance in [3], a minimal time problem with piecewise affine dynamics is presented, but whose dynamics is discontinuous. Note that considering a family of linear systems with the continuity assumption is very restrictive (as it imposes the common boundary of two domains  $P_i, P_j$  to belongs to the kernel of  $A_i - A_j$ ). This is why we consider families of affine rather than linear systems. This allows in particular to consider dynamics with saturation. Let us underline that applying directly the Bang-Bang Principle for trajectories joining two points in a same subset  $P_i$  does not guarantee that the trajectory with extreme controls remains in the same set  $P_i$ . This is precisely the first matter investigated in the present work. We will then explore through examples the possible existence of singular arcs, also noticing that a constrained arc in this framework will be seen as a singular arc.

The paper is organized as follows. In the next section, we make precise our notations and assumptions, making distinctions between linear and affine Bang-Bang Principle for the minimal time control problem. In Section 3, we give our main result. Section 4 is devoted to examples.

## 2 Preliminaries

In this section we present the framework of spatially-temporally regional affine systems, and highlight their similarities and differences with classical linear systems.

We consider an non-autonomous system that is linear with respect to  $u$  :

$$\dot{X} = F(t, X) + B(t)u \quad (2)$$

where  $X \in \mathbb{R}^n$  and  $u \in U \subset \mathbb{R}^r$  under the following assumptions.

**Assumption 2.1.** *Given the time-state space  $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^n$  such that  $\dot{\Omega}$  is non-empty, connected,  $\text{proj}_t \Omega = \mathbb{R}^+$ , and there exists a partition*

$$\Omega = \bigcup_{i \in \mathcal{I}} \bar{P}_i$$

where the  $\bar{P}_i$  are pairwise disjoint, non-empty and connected sets for any  $i \in \mathcal{I}$  (where  $\mathcal{I}$  is countable), such that

$$F(t, X) = A_i(t)X + C_i(t), \quad (t, X) \in P_i$$

where  $A_i$  and  $C_i$  are  $n \times n$ - and  $n \times 1$ - time measurable matrix-valued functions,  $B$  is a  $n \times r$ - time measurable matrix-valued function. Moreover  $F$  is continuous with respect to  $X$  for any  $t$ .

Let us stress that we do not impose any regularity of the boundaries of the sets  $P_i$ .

**Assumption 2.2.**  *$U$  is a convex compact set.*

We consider the minimum time problem to reach a compact target set  $\Gamma \subset \mathbb{R}^n$  from an initial state  $(t_0, X(t_0)) \in \Omega$  that belongs to the attainability set of  $\Gamma$ , while fulfilling the constraint  $(t, X(t)) \in \Omega$  for any time  $t$ . For this problem, the Bang-Bang Principle is often stated for linear dynamics [11], as follows.

In the case where there is only one arc (ie  $\mathcal{I} = \{\alpha\}$ ), the dynamics is linear (ie  $C_\alpha = 0$ ), and there is no constraints (ie  $\Omega = \mathbb{R}^+ \times \mathbb{R}^n$ ), we have in fact a classical linear system

$$\dot{X}(t) = A_\alpha(t)X(t) + B(t)u(t) \quad (3)$$

**Theorem 2.1.** *If  $X_0$  belongs to the attainability set of  $\Gamma$ , then there exists a control function  $u(\cdot)$  that takes values in  $E(U)$  for which the associated trajectory of (3) reaches  $\Gamma$  in minimal time from the initial condition  $X(t_0) = X_0$ .*

We recall that this statement holds true also for affine systems.

**Lemma 2.2.** *Consider the affine system in  $\mathbb{R}^n$*

$$\dot{Y}(t) = L(t)Y(t) + M(t)u(t) + N(t) \quad (4)$$

where  $u(t) \in U$ , a convex compact set of  $\mathbb{R}^p$ , and  $L, M$  and  $N$  are  $n \times n$ -,  $n \times r$ - and  $n \times 1$ - time measurable matrix-valued functions. Then, the conclusion of Theorem 2.1 holds for dynamics (4).

*Proof.* The proof only consists in rewriting the affine system into an equivalent linear one in higher dimension, and we explicitly construct it as follows :

$$\dot{\tilde{Y}}(t) = \tilde{L}(t)\tilde{Y}(t) + \tilde{M}(t)u(t) \quad (5)$$

where  $\tilde{L}$  and  $\tilde{M}$  are  $(n+1) \times (n+1)$ - and  $(n+1) \times r$ - matrix-valued functions. Let us set the additional state variable :

$$\dot{y}_{n+1}(t) = 0 \quad , y_{n+1}(0) = 1$$

Then we define  $\tilde{Y} \in \mathbb{R}^{n+1}$  such that the  $j$ -th coefficient of  $\tilde{Y}$  is  $\tilde{Y}_j = Y_j$  for all  $1 \leq j \leq n$ , and  $\tilde{Y}_{n+1} = y_{n+1}$ . We posit  $\tilde{L} \in \mathcal{M}_{n+1, n+1}(\mathbb{R})$  such that the coefficients  $(k, j)$  are  $\tilde{L}_{k,j} = L_{k,j}$  for all  $1 \leq k, j \leq n$ , the coefficients  $(k, n+1)$  are  $\tilde{L}_{k, n+1} = N_k$  for all  $1 \leq k \leq n$ , and the coefficients  $(n+1, j)$  are  $\tilde{L}_{n+1, j} = 0$  for all  $1 \leq j \leq n+1$ . We similarly define  $\tilde{M} \in \mathcal{M}_{n+1, r}(\mathbb{R})$  such that the coefficients  $(k, j)$  are  $\tilde{M}_{k,j} = M_{k,j}$  for all  $1 \leq k \leq n$  and  $1 \leq j \leq r$ , and the coefficients  $(n+1, j)$  are  $\tilde{M}_{n+1, j} = 0$  for all  $1 \leq j \leq r$ . With such a construction, we easily check that the system (5) describes the same dynamics as the system (4).  $\square$

We may now introduce the main issue of this work. Considering a time-optimal problem subject to the dynamics (2), we will discuss about a revisit of the Bang-Bang principle for such settings. It is well known that for hybrid dynamical systems or switched systems the optimal time trajectory may lie on the locus of discontinuity of the dynamics, and it is called a sliding mode. In this work, the systems we consider might be seen as spatially hybrid systems (with a non-autonomous regionalization), but with an additional condition of continuity at the boundaries. One may wonder if such sliding mode behaviour is still possible with continuous piecewise affine dynamics. The Lemma 2.2 suggests that the Theorem 2.1 should remain relevant where the dynamics of (2) is affine. However a direct application of the Bang-Bang principle would fail, since we can not ensure that the obtained Bang-Bang trajectory doesn't cross any boundary (especially since these boundaries might be moving), which would thus modify the dynamics driving the system. Therefore we have to be a bit more cautious and judiciously examine the evolution of the system in order to locally apply the Bang-Bang principle, and then build a candidate Bang-Bang control step by step.

In the end, we still show that there exists an optimal solution composed of a succession of Bang-Bang arcs over time intervals where the time-state trajectory belongs to a region  $\mathring{P}_i$ , but an optimal trajectory may contain a portion located in a boundary  $\partial P_i$ , thus consisting in singular arcs. We shall present in the Section IV examples where there is indeed singular arcs (or even constrained-singular arcs), showing this way that the Bang-Bang principle is not true globally for spatially-temporally regional affine problems, even when the dynamics is continuous with respect to the state.

**Remark 2.3.** *We insist on the fact that this work does not focus on the question of the controllability of our systems, and we will always assume that our problems are feasible.*

### 3 Main results

**Proposition 3.1.** *For a minimum-time control problem subject to the spatially-temporally regional affine settings (2), If  $X_0$  belongs to the attainability set of the target  $\Gamma$ , then there exists an optimal solution composed of a succession of Bang-Bang arcs over time interval where the trajectory belongs to a region  $\mathring{P}_i$ .*

Let  $u^*(\cdot) \in \mathcal{U}$  be an optimal solution of the time-optimal problem (2). Denote  $X^*(\cdot)$  the associated optimal trajectory and  $T^* \in \mathbb{R}_+$  the corresponding optimal time. Assume that there exists a time interval  $[a, b] \subset [0, T^*]$  such that  $(b, X^*(b)) \in \partial P_i$  where  $P_i$  is a fixed region of the partition, and  $(t, X^*(t)) \in \mathring{P}_i$  for all  $t \in [a, b]$ . For any  $[t_d, t_f] \subset [a, b]$ , we study the evolution of the system over  $[t_d, t_f]$ , thus following the dynamics of the region  $P_i$  :

$$\dot{X}(t) = A_i(t)X(t) + B(t)u(t) + C_i(t) \quad (6)$$

We aim to apply the Bang-Bang principle for affine systems, but we need to ensure that all the trajectories undergo the same dynamics. To this aim we will have to study the local behaviour of trajectories and we will define an appropriate neighbourhood of  $\zeta_{[t_d, t_f]} = \{(t, X^*(t)), t \in [t_d, t_f]\}$  where the principle may be applied.

**Lemma 3.2.** *With the notations above, for any  $[t_d, t_f] \subset [a, b]$ , there exists a Bang-Bang control such that its associated trajectory  $Z(\cdot)$  verifies :*

$$Z(t_d) = X^*(t_d) \quad , \quad Z(t_f) = X^*(t_f) \quad , \quad (t, Z(t)) \in \mathring{P}_i \quad \text{for all } t \in [t_d, t_f]$$

*Proof.* In order to evaluate the distance from  $\zeta_{[t_d, t_f]}$  to the boundary  $\partial P_i$ , we define the distance  $d$  by :

$$\forall t, t' \in \mathbb{R}^+, \forall x, x' \in \mathbb{R}^n, d((t, x), (t', x')) = \left( |t - t'|^2 + \|x - x'\|^2 \right)^{1/2}$$

where  $\|\cdot\|$  denotes the euclidian norm. Then we have  $r = d(\zeta_{[t_d, t_f]}, \partial P_i) > 0$ . Let us fix  $\hat{t} \in [t_d, t_f]$ . Consider the problem defined by the dynamics of (6) with the initial condition  $X(\hat{t}) = X^*(\hat{t})$ , and select an admissible control  $u(\cdot)$ . We want to evaluate the difference between the associated trajectory  $X(\cdot)$  and the reference trajectory  $X^*(\cdot)$ . In particular, we are looking for a time  $t^+$  such that for all  $t \in [\hat{t}, t^+]$  we have  $\|X(t) - X^*(t)\| < r$ . For  $t \geq \hat{t}$  we have :

$$\begin{aligned} X(t) - X^*(t) &= X(t) - X(\hat{t}) - (X^*(t) - X^*(\hat{t})) \\ &= \int_{\hat{t}}^t A_i(s)(X(s) - X^*(s)) + B(s)(u(s) - u^*(s)) ds \end{aligned}$$

Then :

$$\begin{aligned} \|X(t) - X^*(t)\| &\leq 2M(t - \hat{t}) \sup_{s \in [a, b]} \|B(s)\| \\ &\quad + \int_{\hat{t}}^t \|A_i(s)\| \|X(s) - X^*(s)\| ds \end{aligned}$$

By the Grönwall lemma, we get :

$$\begin{aligned} \|X(t) - X^*(t)\| &\leq 2M(t - \hat{t}) K_{[a, b]}^B \exp\left(\int_{\hat{t}}^t \|A_i(s)\| ds\right) \\ &\leq 2M(t - \hat{t}) K_{[a, b]}^B \exp((t - \hat{t}) K_{[a, b]}^A) \end{aligned}$$

where  $K_{[a, b]}^A = \sup_{s \in [a, b]} \|A_i(s)\| < +\infty$  and  $K_{[a, b]}^B = \sup_{s \in [a, b]} \|B(s)\| < +\infty$ . Then there exists  $t^+ \in (\hat{t}, t_f]$  such that :

$$\|X(t) - X^*(t)\| < r, \quad \forall t \in [\hat{t}, t^+]$$

Moreover, the duration  $\delta = |t^+ - \hat{t}|$  only depends on  $r$ . Then there exists  $N \in \mathbb{N}^*$  such that  $t_d + (N - 1)\delta < t_f \leq t_d + N\delta$ , and we define a finite sequence  $(\theta_k)_{0 \leq k \leq N}$  by :

$$\begin{aligned} \theta_k &= t_d + k\delta, \quad \text{for } 0 \leq k \leq N - 1 \\ \theta_N &= t_f \end{aligned}$$

Therefore from above we deduce that for any  $0 \leq k \leq N - 1$ , all the trajectories  $Y_k(\cdot)$  following the dynamics (6) with initial condition  $Y_k(\theta_k) = X^*(\theta_k)$  satisfy :

$$\|Y_k(t) - X^*(t)\| < r, \quad \forall t \in [\theta_k, \theta_{k+1}]$$

In particular, we have :

$$(t, Y_k(t)) \in B((t, X^*(t)), r), \quad \forall t \in [\theta_k, \theta_{k+1}]$$

and thus

$$\{(t, Y(t)), t \in [\theta_k, \theta_{k+1}]\} \subset \bigcup_{t \in [\theta_k, \theta_{k+1}]} B((t, X^*(t)), r) \subset \mathring{P}_i$$

As a consequence, we now may apply the Bang-Bang principle as follows. Consider the dynamics of (6) over the whole space  $\mathbb{R}^n$ , with the initial condition  $Y_k(\theta_k) = X^*(\theta_k)$ . By assumption, the target  $\{X^*(\theta_{k+1})\}$  is attainable at the time  $\theta_{k+1}$ , thus the Bang-Bang principle for linear time-optimal problems indicates that there exists a Bang-Bang control  $\bar{u}_k(\cdot)$  such that the corresponding trajectory  $\bar{Y}_k(\cdot)$  reaches  $X^*(\theta_{k+1})$  at the time  $\theta_{k+1}$ . Furthermore, we know from the well-chosen construction detailed above that this trajectory verifies :

$$\{(t, \bar{Y}(t)), t \in [\theta_k, \theta_{k+1}]\} \subset \mathring{P}_i$$

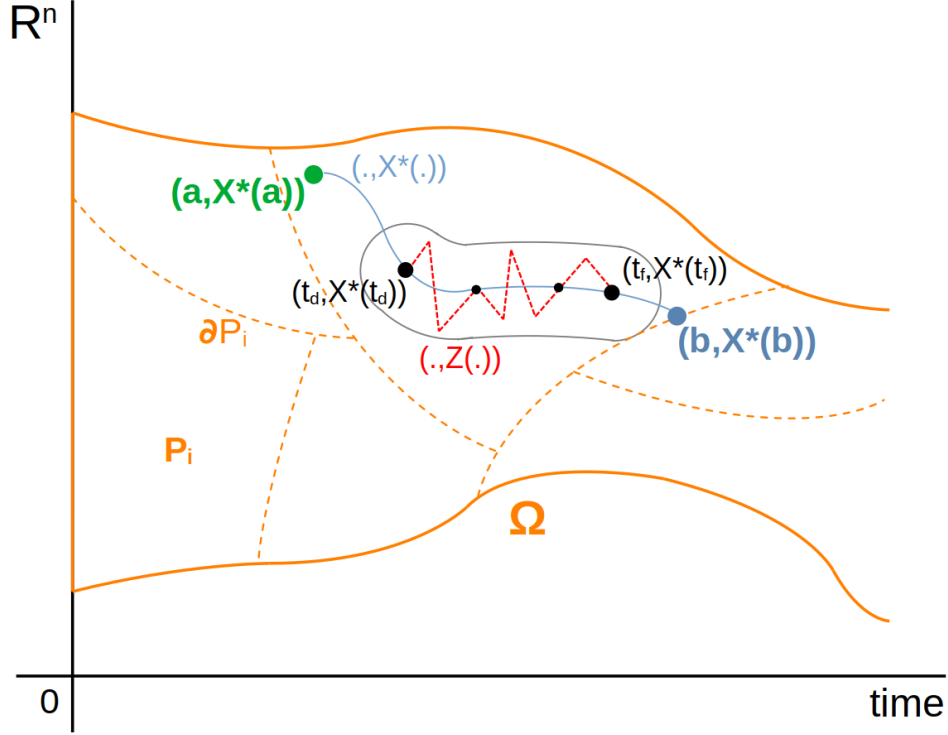


Figure 1: Optimal Bang-Bang trajectory constructed from  $X^*(\cdot)$  on  $[t_d, t_f]$

Let us now define  $v(\cdot)$  as the concatenation of the  $\bar{u}_k(\cdot)$  for  $0 \leq k \leq N-1$ , and denote  $Z(\cdot)$  the corresponding trajectory (in  $\mathbb{R}^n$ ) as we can see in Fig. 1.

Finally, we showed that for any  $[t_d, t_f] \subset [a, b]$  the obtained control  $v(\cdot)$  is a Bang-Bang control defined over  $[t_d, t_f]$ , and the associated trajectory  $Z(\cdot)$  satisfies  $Z(t_d) = X^*(t_d)$ ,  $Z(t_f) = X^*(t_f)$  and :

$$\{(t, Z(t)), t \in [t_d, t_f]\} \subset \bigcup_{0 \leq k \leq N-1} \left( \bigcup_{t \in [\theta_k, \theta_{k+1}]} B((t, X^*(t)), r) \right) \subset \dot{P}_i$$

□

*Proof.* (of Proposition 3.1) To conclude for the main statement, it then suffices to apply the previous Lemma 3.2 on well-chosen time intervals. We set a sequence of times  $(\sigma_f^j)_{j \geq 0}$  by :

$$\sigma_f^j = b - (b - a)/2^j \quad , \text{for } j \in \mathbb{N}$$

We get a corresponding family of radiuses  $r_j = d(\zeta_{[\sigma_f^j, \sigma_f^{j+1}]}, \partial P_i) > 0$ , and the previous reasoning provides a family of Bang-Bang controls  $(v^j)_{j \geq 0}$ , and their associated trajectories  $(Z^j)_{j \geq 0}$ , defined on each portion of time interval by :

$$\begin{aligned} v^0 : [a, \sigma_f^1] &\longrightarrow U \quad \text{given by Lemma 3.2 over } [a, \sigma_f^1] \\ &\text{with } Z^0(a) = X^*(a), Z^0(\sigma_f^1) = X^*(\sigma_f^1) \\ &\text{and } \{(t, Z^0(t)), t \in [a, \sigma_f^1]\} \subset \dot{P}_i \\ v^j : [\sigma_f^j, \sigma_f^{j+1}] &\longrightarrow U \quad \text{given by Lemma 3.2 over } [\sigma_f^j, \sigma_f^{j+1}] \\ &\text{with } Z^j(\sigma_f^j) = X^*(\sigma_f^j), Z^j(\sigma_f^{j+1}) = X^*(\sigma_f^{j+1}) \\ &\text{and } \{(t, Z^j(t)), t \in [\sigma_f^j, \sigma_f^{j+1}]\} \subset \dot{P}_i \end{aligned}$$

Finally, the control defined by a concatenation of the controls  $v^j$  above is :

$$V(t) = v^j(t) \quad \text{for } t \in [\sigma_f^j, \sigma_f^{j+1})$$

Thus  $V$  is well-defined over  $[a, b)$ , is Bang-Bang, and its corresponding trajectory  $\xi(\cdot)$  satisfies :

$$\xi(a) = X^*(a) \quad , \quad \xi(t) \xrightarrow[t \rightarrow b]{} X^*(b) \quad , \quad \{(t, \xi(t)), t \in [a, b)\} \subset \mathring{P}_i$$

Hence the existence of an optimal solution composed of Bang-Bang arcs in the regions where the dynamics is differentiable.  $\square$

**Remark 3.3.** *Since the classical Bang-Bang principle does not give information about the number of commutations, we may have constructed an optimal Bang-Bang solution that has an infinite (countable) number of commutations.*

**Corollary 3.4.** *Consider a minimum-time control problem subject to the spatially-temporally regional affine settings (2) with  $\Omega = \mathbb{R}^+ \times \mathbb{R}^n$ . Suppose that the system is subject to some state constraints  $\Phi_k(t, X(t)) \leq 0$  for any time  $t$  and for  $1 \leq k \leq q$ . If the set  $\{(t, X) \in \mathbb{R}^+ \times \mathbb{R}^n, \Phi_k(t, X) \leq 0, 1 \leq k \leq q\}$  is non-empty, connected, then there exists an optimal solution composed of a succession of Bang-Bang arcs over time interval where the trajectory belongs to a region (where the dynamics is differentiable), and possible singular and constraint arcs where the dynamics is non-differentiable.*

**Corollary 3.5.** *Consider an optimal control problem subject to the spatially-temporally regional affine settings (2) with  $\Omega = \mathbb{R}^+ \times \mathbb{R}^n$ . We fix a terminal time  $T > 0$  and we seek to minimize the integral cost*

$$J(u(\cdot)) = \int_0^T l(t, X(t), u(t)) dt$$

*such that the function  $l : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$  is piecewise affine, continuous with respect to  $X$ , and linear with respect to  $u$ . Then there exists an optimal solution composed of a succession of Bang-Bang arcs over time interval where the trajectory belongs to a region where the dynamics and the function  $l$  are differentiable, and possible singular arcs at the loci of non-differentiability of the dynamics or  $l$ .*

## 4 Examples of non Bang-Bang trajectories

In this section we are going to present two examples. The first one is a very simple autonomous system driving an optimal time problem where the geometry of the (constant) partition of the space gives the intuition of the existence of a singular arc. The second one is from a crop irrigation model, and is written as a non-autonomous system driving an optimization problem with a fixed terminal time, an integral cost, and constraints over the input and the state. Therefore it is much more difficult to study the optimality of solutions, but since we can rewrite the system as a minimal time problem with piecewise affine dynamics, we illustrate that for some setting there exists an optimal solution composed of Bang, singular and constrained arcs.

### 4.1 Turnpike-like Example

Let us consider the following time-optimal control problem :

$$(P1) : \begin{cases} \dot{x}(t) = u(t) \\ \dot{y}(t) = F(x(t)) \end{cases} \quad \text{where } F(x) = \begin{cases} x & , x \leq 1 \\ 2-x & , x \geq 1 \end{cases}$$

This systems (P1) respects the framework of piecewise affine dynamics and continuous with respect to the state, the states space being partitioned in two pieces separated by the line  $x = 1$ . We set the initial conditions  $x(0) = y(0) = 0$ , and the target is the singleton  $(x_f, y_f) = (2, 3)$ . Moreover the control  $u(\cdot)$  is a measurable function with values constrained in the compact  $U = [-1, 1]$ . We seek to minimize the time to reach the target from the initial point.

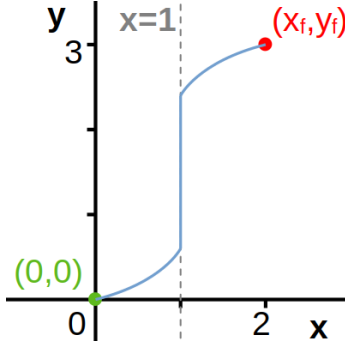


Figure 2: Minimum time "Bang-Singular-Bang" trajectory for (P1)

By definition, the magnitude of  $\dot{y}$  is maximum when  $x = 1$ , then the intuition would say that the trajectory will tend to use this line as much as possible, in order to move as quickly as possible. We are going to show that this strategy (as illustrated in Fig. 2) is indeed optimal.

Consider an optimal control  $u^*(\cdot)$ , with its associated trajectory  $(x^*(\cdot), y^*(\cdot))$  and the corresponding optimal time  $T^*$ . By continuity, the trajectory has to cross the boundary  $x = 1$ . Define  $W = \{t \geq 0, \text{ st } : x^*(t) = 1\}$ , which is non-empty. Denote  $t^- = \inf W$  and  $t^+ = \sup W$ . By definition of the dynamics of  $x$ , we have  $t^- \geq 1$  and  $t^+ \leq T^* - 1$ .

Suppose that  $y^*(t^-) > 1/2$ . Denoting  $\mathcal{C} = \{(x, y), y = x^2/2\}$ , we define  $\Gamma = \{t \in [0, t^-], \text{ st } : (x^*(t), y^*(t)) \in \mathcal{C}\}$ . The set  $\Gamma$  is non-empty since  $y^*(0) = (0, 0)$ , and we define  $\hat{t} = \sup \Gamma \in [0, t^-]$ . Then, for all  $t \in (\hat{t}, t^-)$ , we have either  $y^*(t) \geq 1/2$  and  $x^*(t) < 1$ , or  $y^*(t) < 1/2$  and  $y^*(t) - x^*(t)^2/2 > 0$ . In the first case, we directly have  $\dot{y}^*(t) < 1$ , and in the second case, we have  $x^*(t) < \sqrt{2y^*(t)}$ , and the point  $(\sqrt{2y^*(t)}, y^*(t)) \in \mathcal{C}$  is attainable from the point  $(x^*(\hat{t}), y^*(\hat{t}))$ . As a consequence, we have  $\dot{y}^*(t) < \sqrt{2y^*(t)}$ . Starting from the point  $(x^*(\hat{t}), y^*(\hat{t}))$  at time  $\hat{t}$ , we define a trajectory  $(\tilde{x}, \tilde{y})$  driven by the control  $\tilde{u}(t) = 1$  for  $t \in [\hat{t}, \hat{t} + 1 - x^*(\hat{t})]$ , and  $\tilde{u}(t) = 0$  for  $t \in [\hat{t} + 1 - x^*(\hat{t}), \hat{t} + 1 - x^*(\hat{t}) + y^*(t^-) - 1/2]$ . This trajectory follows the curb  $\mathcal{C}$  until it reaches the point  $(1, 1/2)$ , then it goes straight up on the boundary  $x = 1$  until the point  $(1, y^*(t^-))$ . According to what was said above, we then have for all  $t \in (\hat{t}, \hat{t} + 1 - x^*(\hat{t}) + y^*(t^-) - 1/2)$ ,  $\dot{y}^*(t) < \dot{\tilde{y}}(t)$ . We deduce that the trajectory  $(\tilde{x}, \tilde{y})$  reaches the point  $(x^*(t^-), y^*(t^-))$  at a time  $\tilde{t} < t^-$ , which contradicts the optimality of the trajectory  $(x^*(\cdot), y^*(\cdot))$ . Hence  $y^*(t^-) \leq 1/2$ .

Moreover if  $y^*(t^-) < 1/2$ , a brief study of reachability of  $(1, y^*(t^-))$  shows that the trajectory of  $y$  has to decrease, and hence the trajectory of  $x$  has to go in the negative values. This way, it would exist a time  $s \in (0, t^-)$  such that  $x^*(s) = 0$ ,  $y^*(s) < 0$ , and since  $s > 1/2 - y^*(t^-)$  (since the speed of movement of  $y$  is smaller around  $x = 0$ , and getting  $|\dot{y}| = 1$  in the area  $x < 0$  would require to waste more than  $s = 1/2$  of time). As a consequence, the time  $\sigma$  to reach  $(1, y^*(t^-))$  has to verify  $\sigma \geq 1 + s > 3/2 - y^*(t^-)$ . Thus either the trajectory stays on the boundary  $x = 1$  until the point  $(1, 1/2)$ , or the quantity  $x$  grows. In the first case, the trajectory starting from  $(0, 0)$  and following  $\mathcal{C}$  would be strictly faster to reach  $(1, 1/2)$ , and in the second case, we could make a similar argument as before by defining  $\mathcal{D} = \{(x, y), y = -x^2/2 + 2x + 1\}$  and considering  $\tilde{t} = \inf \Lambda$  where  $\Lambda = \{t \in (\sigma, T^*), \text{ st } : (x^*(t), y^*(t)) \in \mathcal{D}\}$ . In any case, we conclude that it would contradict the optimality of the trajectory  $(x^*(\cdot), y^*(\cdot))$ . Hence  $y^*(t^-) = 1/2$ .

By mirroring these arguments for the end of the trajectory, we can show that  $y^*(t^+) = 5/2$ . Finally, any trajectory  $(x(\cdot), y(\cdot))$  that does not stay on  $x(t) = 1$  for all  $t \in [t^-, t^+]$  will necessarily present an interval  $I$  of non-null measure where  $\dot{y}(t) < 1$  for all  $t \in I$ . Therefore it will not reach the point  $(1, 5/2)$  as fast as the trajectory remaining on  $x = 1$ . Between  $(1, 1/2)$  and  $(1, 5/2)$ , the singular control  $u^* = 0$  is then optimal.

To conclude, we necessarily have  $t^- = 1$ ,  $u^*(t) = 1$  for all  $t \in [0, 1)$ ,  $t^+ = 3$ ,  $u^*(t) = 0$  for all  $t \in [1, 3)$ , and  $u^*(t) = 1$  for all  $t \in [3, T^*]$ . Furthermore, the optimal time is then  $T^* = 4$ .



## 4.2 Example from an Irrigation model

Let us define the following optimal control problem, denoted (P2), taken from a crop irrigation model (see [5]), and evolving over a fixed time interval  $[0, T]$ .

$$\begin{cases} \dot{S}(t) = k_1(-\varphi(t)K_S(S(t)) - (1 - \varphi(t))K_R(S(t)) + k_2u(t) \\ \dot{V}(t) = u(t) \end{cases}$$

with initial conditions  $S(0) = S_0 \in (S^*, 1]$ ,  $B(0) = 0$  and  $V(0) = 0$ , and where the functions  $K_S$  and  $K_R$  are continuous piecewise affine, as follows :

$$K_S(S) = \begin{cases} 0 & , S \in [0, S_w] \\ \frac{S-S_w}{S^*-S_w} & , S \in [S_w, S^*] \\ 1 & , S \geq S^* \end{cases}$$

$$K_R(S) = \begin{cases} 0 & , S \in [0, S_h] \\ \frac{S-S_h}{1-S_h} & , S \geq S_h \end{cases}$$

with  $0 < S_h < S_w < S^* < 1$ . The control  $u(\cdot)$  takes values in  $[0, 1]$ , and the constants verify  $k_1 > 0$  and  $k_2 > 1$  (local controllability condition). The function  $\varphi$  is supposed to be  $C^1$ , increasing, and taking values in  $[0, 1]$ . The optimization problem consists in fixing a terminal time  $T > 0$  and aiming at a maximization of the integral cost

$$\int_0^T \varphi(t)K_S(S(t))dt$$

under some target constraint  $V(T) = V_{init}$ , and some state inequality constraints  $S_{tol} + F(t) \leq S(t) \leq 1$  for all  $t \in [0, T]$ , where  $F$  corresponds to a stronger constraint on the soil moisture ( $S(t) \geq S_{flo} > S_{tol}$ ) during the flowering period, and it is defined by the following continuous function :

$$F(t) = \begin{cases} S_{flo} - S_{tol} & , t \in [0.4T, 0.6T] \\ (S_{flo} - S_{tol}) \frac{t-0.39T}{0.4T-0.39T} & , t \in [0.39T, 0.4T] \\ (S_{flo} - S_{tol}) \frac{t-0.61T}{0.6T-0.61T} & , t \in [0.6T, 0.61T] \\ 0 & , \text{otherwise} \end{cases}$$

This system (P2) is a quite complex constrained optimization problem, and does not fulfill the framework of piecewise affine time-optimal problems yet. In order to solve this issue, we first replace the integral cost by a terminal cost through a new state variable :

$$\dot{B}(t) = \varphi(t)K_S(S(t)) \quad , \quad B(0) = 0$$

The optimization problem thus consists in maximizing  $B(T)$ . Then we cope with the state constraints. Following an idea in [10], we replace the constraint by two additional state equations :

$$\dot{Y}(t) = \min(0, S(t) - (S_{tol} + F(t))) \quad , \quad \dot{Z}(t) = \max(0, S(t) - 1)$$

with complementary initial and final conditions :

$$Y(0) = Z(0) = 0 \quad , \quad Y(T) = Z(T) = 0$$

We obtain a system of five equations and additional initial and final conditions. The functions  $S \mapsto \min(0, S - (S_{tol} + F(t)))$  and  $S \mapsto \max(0, S - 1)$  are piecewise affine and continuous with respect to the state. Moreover, the dynamics of  $Y$  represents a time dependent partition whose boundary moves continuously over time.

Finally, let us assume that there exists an optimal solution of the problem (P2). In this case, there is a corresponding optimal terminal cost  $B^* > 0$  and a final state  $S(T) = \tilde{S}$ . We then consider that it gives in fact an additional terminal condition for the state  $B(\cdot)$ , ie.  $B(T) = B^*$ . Now we may replace all the terminal conditions by a target :

$$(S_f, V_f, B_f, Y_f, Z_f) = (\tilde{S}, V_{init}, B^*, 0, 0)$$

With all these transformations, we now have a problem written as a spatially-temporally regional affine system with continuous dynamics with respect to the state, and the previous optimization is thus by definition equivalent to a minimum time problem to reach the target. Indeed, if there exists  $\hat{t} < T$  such that  $B(\hat{t}) = B^*$  and  $V(\hat{t}) = V_{init}$ , then the control  $u = 0$  over  $[\hat{t}, T]$  would still get  $V(T) = V_{init}$ , and the dynamics of  $B$  would imply  $B(T) > B(\hat{t})$ , which would contradict the optimality of  $B^*$  in the initial formulation of the problem.

We refer to [7] for a sketch of proof for optimality of some irrigation structures while undergoing fixed state constraints. The present case (with a continuously moving state constraint) can then be deduced by adapting the method. We illustrate an example where the numerical resolution has been verified and computed via Bocop (see [16]), as shown in Fig. 3.

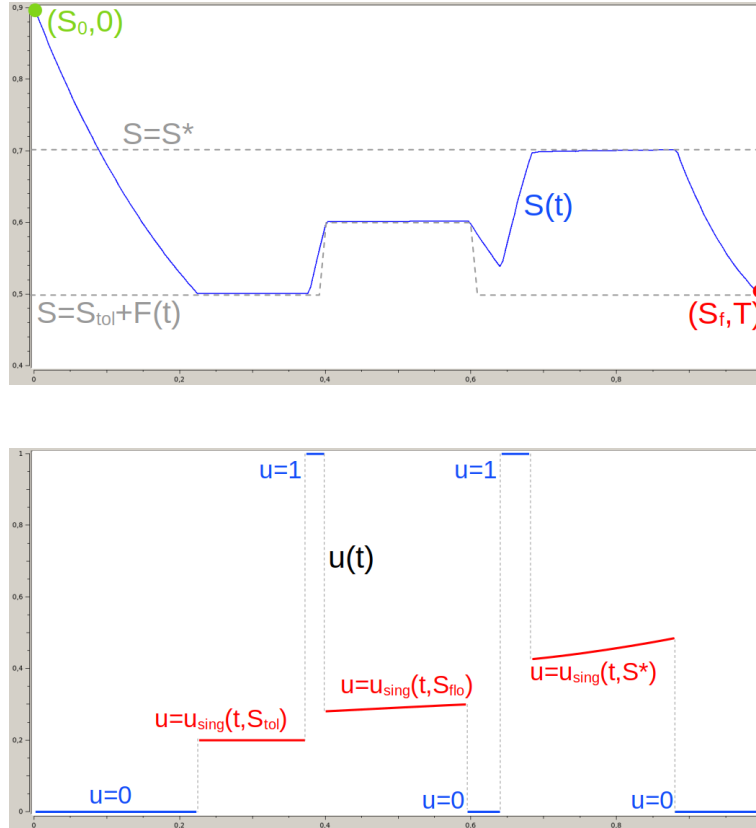


Figure 3: Optimal mixed Bang and singular trajectory and corresponding control for  $(P2)$

## 5 CONCLUSIONS

We focused our study here on a discussion about the validity of the Bang-Bang principle in a spatially-temporally regional affine framework, under constraints. As we saw, an optimal solution may in general be composed of Bang arcs inside the areas of the partition of the time-state space, and eventual singular arcs on the boundaries. For future perspectives, we may wonder about deriving the non-smooth Pontryagin principle for a continuous piecewise affine system, especially we may interest in the conditions of occurrence of such singular arcs, and we can discuss about the possible difficulty to compute the value of such singular controls. Moreover, generalized gradients may arise from the derivation of the Hamiltonian, and the behaviour of the adjoints over time intervals where the state runs through a non-differentiable locus of the dynamics is therefore non-trivial. A more precise description of this situation may be considered as the next step for the study of continuous piecewise affine systems.

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